MSRI GRADUATE STUDENT WORKSHOP, JUNE 2011: NEW DIRECTIONS IN COMMUTATIVE ALGEBRA

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A detailed description of Daniel Erman's lectures: Boij and Söderberg recently proposed the radical notion that the numerics of graded free resolutions are easier to understand if one works "up to scalar multiplication" [BS08b]. The subsequent proof of their conjectures [EFW08], [ES09], [BS08a], [ES10] represents a breakthrough in our understanding of the structure of graded free resolutions, and has already led to a number of applications. The resulting Boij–Söderberg theory is an exciting subject for a graduate summer school, as it is a new tool which is already important in the study of graded free resolutions. In addition, the theory is so different from previous approaches to studying free resolutions, that there are a large number of opportunities for graduate students to find accessible research problems.

The lectures would roughly be organized as follows:

(1) Why study graded free resolutions? Applications of graded free resolutions in commutative algebra, algebraic geometry, and combinatorics.
(2) Overview of Boij–Söderberg theory, and discussion of pure resolutions.
(3) The main theorem: decomposition into pure diagrams, and the simplicial structure.
(4) How to apply Boij–Söderberg theory.
(5) The other side of Boij–Söderberg theory: vector bundles on $\mathbb{P}^n$, and the proofs of the main results.
(6) What we don't know about Boij–Söderberg theory. Open questions, suggestions for experiments, etc.

A detailed description of Amelia Taylor's lectures: Algebraic statistics in its current form might be argued to have started in 1993 when [DS98] was first circulated as a manuscript, with slow growth until the book by Pistone, Riccomagno, and Wynn [PRW01] was published in 2001. Today it is one of the fastest growing fields related to commutative algebra with applications to biology, including reverse engineering and phylogenetics, and statistics, including experimental design and distribution theory. One exciting aspect of this field is that there are questions about how to use commutative algebra effectively in statistical questions as well as interesting questions in pure commutative algebra that are motivated by statistics. For example, there has been recent interest in the primary decomposition of binomial ideals corresponding to conditional independence relations in statistical models [Fin09], [HHH$^+$10], (and Irena and I have preliminary work as well). The breadth of applications and myriad of connections between questions in algebra and statistics present many opportunities for questions accessible to graduate students. My lectures will have a distinctly commutative algebra focus with the goal of exposing the students to a variety of different connections between commutative algebra and statistics.

My lectures would roughly be organized as follows:

(1) What is a statistical model and the probabilistic language we need to get started. Why study algebraic statistics, including a brief introduction to some of the applications.

(2) Markov bases and the many different ways of thinking about binomial ideals (including Gröbner bases, toric ideals, and related topics).
(3) Conditional independence models and graphical models.
(4) Reverse engineering of biological systems.

A detailed description of Irena Swanson's lectures: Whereas Daniel Erman and Amelia Taylor would give expert lectures on Boij–Söderberg theory and on algebraic statistics, I will lecture on more basic background subjects in commutative algebra. This includes material on free resolutions, cellular resolutions [BS98], primary decompositions, monomial and binomial ideals [ES96] and integral closure with explanations on the symbolic computability of these concepts. In the second week, these topics will be central to many of the student projects, thus tying together the three sets of lectures as much as possible.

My lectures would roughly be organized as follows:

(1) Overview of (graded) free resolutions and of cellular resolutions.
(2) Overview of primary decomposition, specifically of monomial and binomial ideals.
(3) Computability of resolutions and of primary decompositions.
(4) Integral closure.
(5) Computability of integral closure. The Seidenberg–Stolzenberg procedure [Sei70], [Sei75], [Sto68], [HS06].

A sample of group projects for the students: Binomial ideals are definitely used in algebraic statistics, but they have not been looked at from the point of view of Boij–Söderberg theory. We envision at least one student group project to do exactly that, namely to work on the free resolutions of certain binomial ideals or monomial ideals and see how the Betti diagrams change with certain parameters. Other projects for the groups would be to establish primary decompositions of certain (binomial) ideals arising from specific problems in algebraic statistics. Another project, to go along with the computational lectures, would be to implement the Seidenberg–Stolzenberg procedure for the computation of integral closure — this had never been done before, as, for one thing, the procedure is not clearly algorithmic.

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