CLOSED 1-FORMS WITH AT MOST ONE ZERO

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Abstract. We prove that in any nonzero cohomology class \( \xi \in H^1(M; \mathbb{R}) \) there always exists a closed 1-form having at most one zero.

1. Statement of the result

Let \( M \) be a closed connected smooth manifold. By Hopf’s theorem, there exists a nowhere zero tangent vector field on \( M \) if and only if \( \chi(M) = 0 \). If \( \chi(M) \neq 0 \) one may find a tangent vector field on \( M \) vanishing at a single point \( p \in M \). A Riemannian metric on \( M \) determines a one-to-one correspondence between vectors and covectors; therefore on any closed connected manifold \( M \) there exists a smooth 1-form \( \omega \) vanishing at most at one point \( p \in M \). The question we address in this note is whether the 1-form \( \omega \) which is nonzero on \( M - \{p\} \) can be chosen to be closed, \( d\omega = 0 \)?

The Novikov theory [8] gives bounds from below on the number of distinct zeros which have closed 1-forms \( \omega \) lying in a prescribed cohomology class \( \xi \in H^1(M; \mathbb{R}) \). However the Novikov theory imposes an additional requirement that all zeros of \( \omega \) are non-degenerate in the sense of Morse. The number of zeros is then at least the sum \( \sum_j b_j(\xi) \) of the Novikov numbers \( b_j(\xi) \).

If \( \omega \) is a closed 1-form representing the zero cohomology class then \( \omega = df \) where \( f : M \to \mathbb{R} \) is a smooth function; in this case \( \omega \) must have at least \( \text{cat}(M) \) geometrically distinct zeros, according to the classical Lusternik-Schnirelman theory [1].

Our goal in this paper is to show that in general, with the exception of two situations mentioned above, there are no obstructions for constructing closed 1-forms possessing a single zero. We prove the following statement:

**Theorem 1.** Let \( M \) be a closed connected \( n \)-dimensional smooth manifold, and let \( \xi \in H^1(M; \mathbb{R}) \) be a nonzero real cohomology class. Then there exists a smooth closed 1-form \( \omega \) in the class \( \xi \) having at most one zero.

This result suggests that “the Lusternik-Schnirelman theory for closed 1-forms” (see [3, 4] and Chapter 10 of [5]) has a new character which is quite distinct from both the classical Lusternik-Schnirelman theory of functions and the Novikov theory of closed 1-forms.

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Theorem 1 was proven in [3] under an additional assumption that the class \( \xi \) is integral, \( \xi \in H^1(M; \mathbb{Z}) \). See also [5], Theorem 10.1. This essentially covers all rank 1 cohomology classes \( \xi \in H^1(M; \mathbb{R}) \) since any such class is a multiple of an integral class.

Theorem 1 has interesting implications in the theory of symplectic intersections, compare [9], [2]. Y. Eliashberg and M. Gromov mention in [2] that a statement in the spirit of Theorem 1 was made by Yu. Chekanov at a seminar talk in 1996. No written account of his work is available.

Let us mention briefly a similar question. We know that if \( \chi(M) = 0 \) then there exists a nowhere zero 1-form \( \omega \) on \( M \). Given \( \chi(M) = 0 \), one may ask if it is possible to find a nowhere zero 1-form \( \omega \) on \( M \) which is closed \( d\omega = 0 \)? The answer is negative in general. For example, vanishing of the Novikov numbers \( b_j(\xi) = 0 \) is a necessary condition for the class \( \xi \) to be representable by a closed 1-form without zeros. The full list of necessary and sufficient conditions (in the case \( \dim M > 5 \)) is given by the theorem of Latour [6].

2. Preliminaries

Here we recall some basic terminology. We refer to [5] for more detail.

A smooth 1-form \( \omega \) is a smooth section \( x \mapsto \omega_x, x \in M \) of the cotangent bundle \( T^*(M) \to M \). A zero of \( \omega \) is a point \( p \in M \) such that \( \omega_p = 0 \).

If \( \omega \) is a closed 1-form on \( M \), i.e. \( d\omega = 0 \), then in any simply connected domain \( U \subset M \) there exists a smooth function \( f : U \to \mathbb{R} \) such that \( \omega|_U = df \). Zeros of \( \omega \) are precisely the critical points of \( f \). A zero \( p \in M, \omega_p = 0 \) is said to be Morse type iff \( p \) is a Morse type critical point for \( f \).

The homomorphism of periods

\[
\text{Per}_\xi : H_1(M) \to \mathbb{R}
\]

is defined by

\[
\text{Per}_\xi(\gamma) = \int_\gamma \omega \in \mathbb{R}.
\]

Here \( \xi = [\omega] \in H^1(M; \mathbb{R}) \) is the de Rham cohomology class of \( \omega \) and \( \gamma \) is a closed loop in \( M \); the symbol \( [\gamma] \in H_1(M) \) denotes the homology class of \( \gamma \).

The image of the homomorphism of periods (1) is a finitely generated free abelian subgroup of \( \mathbb{R} \); it is called the group of periods. Its rank is denoted \( \text{rk}(\xi) \) – the rank of the cohomology class \( \xi \in H^1(M; \mathbb{R}) \).

A closed 1-form \( \omega \) with Morse zeros determines a singular foliation \( \omega = 0 \) on \( M \). It is a decomposition of \( M \) into leaves: two points \( p, q \in M \) belong to the same leaf if there exists a path \( \gamma : [0, 1] \to M \) with \( \gamma(0) = p, \gamma(1) = q \) and \( \omega(\gamma(t)) = 0 \) for all \( t \). Locally, in a simply connected domain \( U \subset M \), we have \( \omega|_U = df \), where \( f : U \to \mathbb{R} \); each connected component of the level set \( f^{-1}(c) \) lies in a single leaf. If \( U \) is small enough and does not contain the zeros of \( \omega \), one may find coordinates \( x_1, \ldots, x_n \) in \( U \) such that \( f \equiv x_1 \); hence the leaves in \( U \) are the sets \( x_1 = c \). Near such points the singular foliation
\( \omega = 0 \) is a usual foliation. On the contrary, if \( U \) is a small neighborhood of a zero \( p \in M \) of \( \omega \) having Morse index \( 0 \leq k \leq n \), then there are coordinates \( x_1, \ldots, x_n \) in \( U \) such that \( x_i(p) = 0 \) and the leaves of \( \omega = 0 \) in \( U \) are the level sets \(-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 = c.\) The leaf \( L \) with \( c = 0 \) contains the zero \( p.\) It has a singularity at \( p: \) a neighborhood of \( p \) in \( L \) is homeomorphic to a cone over the product \( S^{k-1} \times S^{n-k-1}.\) There are finitely many singular leaves, i.e. the leaves containing the zeros of \( \omega.\)

We are particularly interested in the singular leaves containing the zeros of \( \omega \) having Morse indices 1 and \( n - 1.\) Removing such a zero \( p \) locally disconnects the leaf \( L.\) However globally the complement \( L - p \) may or may not be connected.

The singular foliation \( \omega = 0 \) is co-oriented: the normal bundle to any leaf at any nonsingular point has a specified orientation.

We shall use the notion of a weakly complete closed 1-form introduced by G. Levitt [7]. A closed 1-form \( \omega \) is called weakly complete if it has Morse type zeros and for any smooth path \( \sigma : [0, 1] \to M^* \) with \( \int_\sigma \omega = 0 \) the endpoints \( \sigma(0) \) and \( \sigma(1) \) lie in the same leaf of the foliation \( \omega = 0 \) on \( M^*.\)

Here \( M^* \) denotes \( M - \{p_1, \ldots, p_m\} \) where \( p_j \) are the zeros of \( \omega.\)

A weakly complete closed 1-form with \( \xi = [\omega] \neq 0 \) has no zeros with Morse indices 0 and \( n.\) According to Levitt [7], any nonzero real cohomology class \( \xi \in H^1(M; \mathbb{R}) \) can be represented by a weakly complete closed 1-form.

The plan of our proof of Theorem 1 is as follows. We start with a weakly complete closed 1-form \( \omega \) lying in the prescribed cohomology class \( \xi \in H^1(M; \mathbb{R}), \xi \neq 0.\) We show that assuming \( \text{rk}(\xi) > 1 \) all leaves of the singular foliation \( \omega = 0 \) are dense (see §3). We perturb \( \omega \) such that the resulting closed 1-form \( \omega' \) has a single singular leaf (see §4). After that we apply the technique of Takens [10] allowing us to collide the zeros in a single (highly degenerate) zero. We first prove Theorem 1 assuming that \( n = \dim M > 2; \) the special case \( n = 2 \) is treated separately later.

### 3. Density of the leaves

In this section we show that if \( \omega \) is weakly complete and \( \text{rk}(\xi) > 1 \) then the leaves of \( \omega = 0 \) are dense.

Note that in general the assumption \( \text{rk}(\xi) > 1 \) alone does not imply that the leaves are dense, see the examples in §9.3 of [5].
Let $\omega$ be a weakly complete closed 1-form in class $\xi$. Consider the covering map $\pi: \tilde{M} \to M$ corresponding to the kernel of the homomorphism of periods $\text{Per}_\xi: H_1(M) \to \mathbb{R}$, where $\xi = [\omega] \in H^1(M; \mathbb{R})$. Let $H \subset \mathbb{R}$ be the group of periods. The rank of $H$ equals $\text{rk}(\xi)$; since we assume that $\text{rk}(\xi) > 1$, the group $H$ is dense in $\mathbb{R}$. The group of periods $H$ acts on the covering space $\tilde{M}$ as the group of covering transformations. We have $\pi^*\omega = dF$ where $F: \tilde{M} \to \mathbb{R}$ is a smooth function. The leaves of the singular foliation $\omega = 0$ are the images under the projection $\pi$ of the level sets $F^{-1}(c)$; this property follows from the weak completeness of $\omega$, see [7], Proposition II.1. For any $g \in H$ and $x \in \tilde{M}$ one has

$$F(gx) - F(x) = g \in \mathbb{R}. \quad (3)$$

Let $L = \pi(F^{-1}(c))$ be a leaf and let $x \in M$ be an arbitrary point. Our goal is to show that $x$ lies in the closure $\bar{L}$ of $L$. Let $U \subset M$ be a neighborhood of $x$. We shall assume that $U$ is “small” in the following sense: $\xi|_U = 0$.

Consider a lift $\tilde{x} \in \tilde{M}$, $\pi(\tilde{x}) = x$. Let $\tilde{U}$ be a neighborhood of $\tilde{x}$ which is mapped by $\pi$ homeomorphically onto $U$. We claim that the set of values $F(\tilde{U}) \subset \mathbb{R}$ contains an interval $(a - \epsilon, a + \epsilon)$ where $a = F(\tilde{x})$ and $\epsilon > 0$.

This claim is obvious if $\tilde{x}$ is not a critical point of $F$ since in this case one may choose the coordinates $x_1, \ldots, x_n$ around $\tilde{x}$ such that $F(x) = a + x_1$. In the case when $\tilde{x}$ is a critical point of $F$, one may choose the coordinates $x_1, \ldots, x_n$ near the point $\tilde{x} \in \tilde{M}$ such that $F(x)$ is given by $a \pm x_1^2 \pm x_2^2 + \cdots \pm x_n^2$ and our claim follows since we know that the Morse index is distinct from 0 and $n$.

Because of the density of the group of translations $H \subset \mathbb{R}$ one may find $g \in H$ such that the real number $F(g\tilde{x}) = F(\tilde{x}) + g = a + g$ lies in the interval $(c - \epsilon, c + \epsilon)$. Then we obtain

$$c \in (a + g - \epsilon, a + g + \epsilon) \subset g + F(\tilde{U}) = F(g\tilde{U}). \quad (4)$$

Hence we see that the sets $F^{-1}(c)$ and $g\tilde{U}$ have a nonempty intersection. Therefore the neighborhood $U = \pi(g\tilde{U})$ intersects the leaf $L = \pi(F^{-1}(c))$ as claimed.
An obvious modification of the above argument proves a slightly more precise statement:

*Given a point \( x \in M \) and a leaf \( L \subset M \) of the singular foliation \( \omega = 0 \), there exist two sequences of points \( x_k \in L \) and \( y_k \in L \) such that*

\[
(5) \quad x_k \to x \quad \text{and} \quad y_k \to x,
\]

*and, moreover,*

\[
(6) \quad \int_x^{x_k} \omega > 0, \quad \text{while} \quad \int_x^{y_k} \omega < 0.
\]

The integrals in (6) are calculated along an arbitrary path lying in a small neighborhood of \( x \).

This can also be expressed by saying that the leaf \( L \) approaches \( x \) from both the positive and the negative sides.

### 4. Modification

Our next goal is to replace \( \omega \) by a Morse closed 1-form \( \omega' \) which has the property that all its zeros lie on the same singular leaf of the singular foliation \( \omega' = 0 \). In this section we assume that \( n = \dim M > 2 \).

Let \( \omega \) be a weakly complete Morse closed 1-form in class \( \xi \) where \( \text{rk}(\xi) > 1 \). Let \( p_1, \ldots, p_m \in M \) be the zeros of \( \omega \). For each \( p_j \) choose a small neighborhood \( U_j \ni p_j \) and local coordinates \( x_1, \ldots, x_n \) in \( U_j \) such that \( x_i(p_j) = 0 \) for \( i = 1, \ldots, n \) and

\[
(7) \quad \omega|_{U_j} = df_j, \quad \text{where} \quad f_j = -x_1^2 - \cdots - x_{m_j}^2 + x_{m_j+1}^2 + \cdots + x_n^2.
\]

Here \( m_j \) denotes the Morse index of \( p_j \). We assume that the ball \( \sum_{i=1}^n x_i^2 \leq 1 \) is contained in \( U_j \) and that \( U_j \cap U_{j'} = \emptyset \) for \( j \neq j' \). Denote by \( W_j \) the open ball \( \sum_{i=1}^n x_i^2 < 1 \).

Let \( \phi: [0, 1] \to [0, 1] \) be a smooth function with the following properties:

(a) \( \phi \equiv 0 \) on \([3/4, 1]\); (b) \( \phi \equiv \epsilon > 0 \) on \([0, 1/2]\); (c) \( -1 < \phi' \leq 0 \). Such a function exists assuming that \( \epsilon > 0 \) is small enough. (a), (b), (c) imply that

\[
(8) \quad \phi'(r) > -2r, \quad \text{for} \quad r > 0.
\]

We replace the closed 1-form \( \omega \) by

\[
(9) \quad \omega' = \omega - \sum_{j=1}^m \mu_j \cdot dg_j
\]
where \( g_j : M \to \mathbb{R} \) is a smooth function with support in \( U_j \). In the coordinates \( x_1, \ldots, x_n \) of \( U_j \) (see above) the function \( g_j \) is given by \( g_j(x) = \phi(||x||) \). The parameters \( \mu_j \in [-1, 1] \) appearing in (9) are specified later.

One has \( \omega \equiv \omega' \) on \( M - \cup_j U_j \) and near the zeros of \( \omega \). Let us show that \( \omega' \) has no additional zeros. We have \( \omega'|_{U_j} = d(f_j - \mu_j g_j) \) (where \( f_j \) is defined in (7)) and

\[
\frac{\partial}{\partial x_i} (f_j - \mu_j g_j) = \pm 2x_i - \mu_j \phi'(||x||) \frac{x_i}{||x||}
\]

If this partial derivative vanishes and \( x_i \neq 0 \) then \( \phi'(r) = \pm 2r \mu_j^{-1} \) which may happen only for \( r = ||x|| = 0 \) according to (8).

We now show how to choose the parameters \( \mu_j \) so that the closed 1-form \( \omega' \) given by (9) has a unique singular leaf. Let \( L \) be a fixed nonsingular leaf of \( \omega = 0 \). Since \( L \) is dense in \( M \) (see §3) for any \( j = 1, \ldots, m \) the intersection \( L \cap U_j \) contains infinitely many connected components approaching \( p_j \) from below and from above and the function \( f_j \) is constant on each of them. We say that a subset \( T_c \subset L \cap W_j \) is a level set if \( T_c = f_j^{-1}(c) \cap W_j \) for some \( c \in \mathbb{R} \). Note that \( f_j(p_j) = 0 \). The level set \( c = 0 \) contains the zero \( p_j \); it is homeomorphic to the cone over the product \( S^{m_j-1} \times S^{n-m_j-1} \). Each level set \( T_c \) with \( c < 0 \) is diffeomorphic to \( S^{m_j-1} \times D^{n-m_j} \) and each level set \( T_c \) with \( c > 0 \) is diffeomorphic to \( D^{m_j} \times S^{n-m_j-1} \). Recall that \( m_j \) denotes the Morse index of \( p_j \).

Let \( \mathcal{V}_j = f_j(L \cap W_j) \subset \mathbb{R} \) denote the set of values of \( f_j \) on different level sets belonging to the leaf \( L \). The zero \( 0 \) does not lie in \( \mathcal{V}_j \) since we assume that the leaf \( L \) is nonsingular. However, according to the result proven in §3, the zero \( 0 \in \mathbb{R} \) is a limit point of \( \mathcal{V}_j \) and, moreover, the closure of either of the sets \( \mathcal{V}_j \cap (0, \infty) \) and \( \mathcal{V}_j \cap (-\infty, 0) \) contains \( 0 \in \mathbb{R} \).

For the modification \( \omega' \) (given by (9)) one has \( \omega'|_{U_j} = dh_j \) where \( h_j = f_j - \mu_j g_j \). The level sets \( T_c' \) for \( h_j \) are defined as \( h_j^{-1}(c) \cap W_j \). Clearly \( T_c' \) is given by the equation

\[
f_j(x) = \mu_j \phi(||x||) + c, \quad x \in W_j.
\]

Hence for \( ||x|| \geq 3/4 \) this is the same as \( T_c \); for \( ||x|| \leq 1/2 \) the level set \( T_c' \) coincides with \( T_{c+\mu_j} \). In the ring \( 1/2 \leq ||x|| \leq 3/4 \) the level set \( T_c' \) is homeomorphic to a cylinder.
The following figure illustrates the distinction between the level sets $T_c$ and $T'_c$ in the case $\mu_j > 0$.

Examine the changes which undergoes the leaf $L$ when we replace $\omega$ by $\omega'$. Here we view $L$ with the leaf topology; it is the topology induced on $L$ from the covering $\tilde{M}$ using an arbitrary lift $L \to \tilde{M}$. First, let us assume that: (1) the Morse index $m_j$ satisfies $m_j < n - 1$; (2) the coefficient $\mu_j > 0$ is positive; (3) the number $-\epsilon \mu_j$ lies in the set $V_j$. Then the complement

$$L - \bigcup_{c \in V_j} T_c$$

is connected and it lies in a single leaf $L'$ of the singular foliation $\omega' = 0$. We see that the new leaf $L'$ is obtained from $L$ by infinitely many surgeries. Namely, each level set $T_c \subset L$, where $c \in V_j$ satisfies $-\epsilon \mu_j < c < 0$, is removed and replaced by a copy of $D^{m_j} \times S^{n-m_j-1}$; besides, the set $T_c \subset L$ where $c = -\epsilon \mu_j$, is removed and gets replaced by a cone over the product $S^{m_j-1} \times S^{n-m_j-1}$. Hence the new leaf $L'$ contains the zero $p_j$.

Let us now show how one may modify the above construction in the case $m_j = n - 1$. Since $n > 2$ we have in this case $n - m_j - 1 < n - 2$; hence removing the sphere $S^{n-m_j-1}$ from the leaf $L$ does not disconnect $L$. We shall assume that the coefficient $\mu_j$ is negative and that the number $-\epsilon \mu_j$ lies in $V_j \subset \mathbb{R}$. The complement

$$L - \bigcup_{c \in V_j} T_c$$

is connected and it lies in a single leaf $L'$ of the singular foliation $\omega' = 0$. Clearly, $L'$ is obtained from $L$ by removing the level sets $T_c$ where $c \in V_j$ satisfies $0 < c < -\epsilon \mu_j$ (each such $T_c$ is diffeomorphic to $D^{m_j} \times S^{n-m_j-1}$) and by replacing them by copies of $S^{m_j-1} \times D^{n-m_j}$. In addition, the set
$T_c \subset L$ where $c = -\epsilon \mu_j$, is removed and is replaces by a cone over the product $S^{m_j-1} \times S^{n-m_j-1}$.

We see that $L'$ is a leaf of the singular foliation $\omega' = 0$ containing all the zeros $p_1, \ldots, p_m$.

5. Proof of Theorem 1

Below we assume that $\text{rk}(\xi) > 1$. The case $\text{rk}(\xi) = 1$ is covered by Theorem 2.1 from [3].

The results of the preceding sections allow to complete the proof of Theorem 1 in the case $n = \dim M > 2$. Indeed, we showed in §4 how to construct a Morse closed 1-form $\omega'$ lying in the prescribed cohomology class $\xi$ such that all zeros of $\omega'$ are Morse and belong to the same singular leaf $L'$ of the singular foliation $\omega' = 0$. Now we may apply the colliding technique of F. Takens [10], pages 203–206. Namely, we may find a piecewise smooth tree $\Gamma \subset L'$ containing all the zeros of $\omega'$. Let $U \subset M$ be a small neighborhood of $\Gamma$ which is diffeomorphic to $\mathbb{R}^n$. We may find a continuous map $\Psi : M \to M$ with the following properties:

- $\Psi(\Gamma)$ is a single point $p \in \Gamma$;
- $\Psi |_{\Gamma - \Gamma}$ is a diffeomorphism onto $M - p$;
- $\Psi(U) = U$;
- $\Psi$ is the identity map on the complement of a small neighborhood $V \subset M$ of $\Gamma$ where the closure $\overline{V}$ is contained in $U$.

Consider a smooth function $f : U \to \mathbb{R}$ such that $df = \omega'|_U$; it exists and is unique up to a constant. The function $g = f \circ \Psi^{-1} : U \to \mathbb{R}$ is well-defined (since $f|_\Gamma$ is constant). $g$ is continuous by the universal property of the quotient topology. Moreover, $g$ is smooth on $M - p$. Applying Theorem 2.7 from [10], we see that we can replace $g$ by a smooth function $h : U \to \mathbb{R}$ having a single critical point at $p$ and such that $h = f$ on $U - \overline{V}$.

Let $\omega''$ be a closed 1-form on $M$ given by

$$\omega'' |_{M - \Gamma} = \omega'|_{M - \Gamma} \quad \text{and} \quad \omega'' |_U = dh.$$  

Clearly $\omega''$ is a smooth closed 1-form on $M$ having no zeros in $M - \{p\}$. Moreover, $\omega''$ lies in the cohomology class $\xi = [\omega']$ (since any loop in $M$ is homologous to a loop in $M - V$).

Now we prove Theorem 1 the case $n = 2$. We shall replace the construction of §4 (which requires $n > 2$) by a direct construction. The final argument using the Takens’ technique [10] remains the same.

Let $M$ be a closed surface and let $\xi \in H^1(M; \mathbb{R})$ be a nonzero cohomology class. We can split $M$ into a connected sum

$$M = M_1 \# M_2 \# \cdots \# M_k$$

where each $M_j$ is a torus or a Klein bottle and such that the cohomology class $\xi_j = [\omega_j] \in H^1(M_j; \mathbb{R})$ is nonzero. Let $\omega_j$ be a closed 1-form on $M_j$ lying in the class $\xi_j$ and having no zeros; obviously such a form exists. §9.3.2 of [5] describes the construction of connected sum of closed 1-forms
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on surfaces. Each connecting tube contributes two zeros. In fact there are three different ways of forming the connected sum, they are denoted by A, B, C on Figure 9.8 in [5]. In the type C connected sum the zeros lie on the same singular leaf. Hence by using the type C connected sum operation we get a closed 1-form $\omega$ on $M$ having $2k - 2$ zeros which all lie on the same singular leaf of the singular foliation $\omega = 0$. The colliding argument based on the technique of Takens [10] applies as in the case $n > 2$ and produces a closed 1-form with at most one zero lying in class $\xi$.

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