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The Geometry and Topology of Three-Manifolds

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This is an electronic edition of the 1980 notes distributed by Princeton University. The text was typed in TEX by Sheila Newbery, who also scanned the figures. Typos have been corrected (and probably others introduced), but otherwise no attempt has been made to update the contents. Genevieve Walsh compiled the index.

Numbers on the right margin correspond to the original edition's page numbers.

Thurston's *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997) is a considerable expansion of the first few chapters of these notes. Later chapters have not yet appeared in book form.

Please send corrections to Silvio Levy at levy@msri.org.

Index

(G, X)-manifold, 27
M_Γ, 179
N_Γ, 180
O_Γ, 180
P_Γ, 180
$\mathrm{PSL}(2, \mathbb{C})$, 92–96
$\mathcal{GL}(S)$, *209*, 215
$\mathcal{G}$-manifold, 27
$\mathcal{ML}(S)$, *209*, 210, 251
$\mathcal{ML}_0(S)$, 209, 251
$\mathcal{PL}(S)$, *209*
$\mathcal{PL}_0(S)$, 210, 263
$\mathcal{H}$-foliation, *see also* foliation
$\mathcal{G}$-orbifold, 301
$\mathcal{N}(O)$, 347
$\mathcal{T}_\gamma$, 243

$\mathcal{N}(O)$, 351

accidental parabolics, 257
action
  discrete orbits, *175*
  properly discontinuous, 174
  wandering, *175*
Ahlfors' Theorem, 180
algebraic hyperbolic manifolds, 168
algebraic limit, *225*
algebraic numbers, 143
Andreev's Theorem, 330

Bass, H, 143
bending measure, 189–191
Bers, 111
billiard table, 298
Borromean rings, 33–34, 105, 300, 322
  commensurable with Whitehead link, 141
boundary mirror, 301
branched cover over a link, 2

circle packing, *330*
  nerve of, 330
commensurability classes, 144
  infinite number of, 150
commensurable
  and cusp structure, 142
  discrete subgroups of $PSL(2, \mathbb{C})$, *140*
  manifolds, *140*
  with complete manifolds, 141
complete
  (G, X)-manifold, *35*
  $\mathcal{H}$-foliation, 64
completeness criteria, 36, 38, 41, 42
completion
  of a hyperbolic 3-manifold, 54–56
  of a hyperbolic surface, 41–42
cone manifolds, 55–56
convergence
  algebraic, *225*
  geometric, *225*
  strong, *226*
convex, *177*
  locally, 177
    implies convex, 177
  strictly, 178
    and homotopy equivalence, 179
convex hull, 171
  boundary, 185
corner reflectors, 309
Coxeter diagram, 326

cusp
  extra, 216

deformations
  of a 3-manifold, 85
    dimension of, 88, 97
    extend to Dehn filling, 103
  of compact convex hyperbolic manifolds, 178
Dehn surgery, *2*, *see also* figure-eight knot
  invariants, *57*
developing map, *35*, 54, 185
  of a (G, X)-orbifold, 309
  of a convex manifold, 177
  of an affine torus, 37
discrete, 64
domain of discontinuity, *174*
  retraction onto convex hull, 174

edge equations, 49–51
elementary group, 171
elliptic
  why distressing, 10
essentially complete, 244
Euclidean triangles, 47
Euler number
  for a fibration, 323
extension of a vector field, 288
  Curl and Div, 292
  direction derivative of, 290

fibration
  over an orbifold, 319, 323
figure-eight knot, 4–7, 29–31, 120
  commensurable with $PSL(2, \mathcal{O}_3)$, 149
  complete hyperbolic structure, 54
  Dehn surgery on, 58–61, 70
    yields hyperbolic manifold, 61
  fundamental group, 172
  gluing diagram, 4
  incompressible surfaces in, 72–83
  limit set, 172
  parametrization space of complement, 52
  volume of complement, 164
foliation
  developing map for, 63
  hyperbolic, 62, 64
Fricke space, 92
Fuchsian group, 172, 192
fundamental group
  acts ergodically on S^{n-1}_{∞}, 111

Gauss-Bonnet
  for orbifolds, 312
Gehring, 111
geodesic flow
  conditions for ergodicity, 277
geodesic lamination, *186*, 200, *see also* geometric and measure topology, *see also* lamination
  complete, 196
  ending, *238*
  essentially complete, *243*, 246
  measure on, 207
  near a cusp, 201
  realizable, 208, 211, 214, 240
    criterion for, 261
  train track approximation of, 204, 206, 210, 213
  with compact support, 239
geodesics
  on hyperboloid, 18
geometric limit, *225*
geometric structure, *85*
geometric topology, *225*
  and compactness, 228
  on geodesic laminations, *208*
geometrically finite, *180*, *183*
  and cusps, 182
  hyperbolic 3-manifold, 203
geometrically near, *118*
geometrically tame, *219*, 221, 229
  almost, *230*, 240
  and algebraic convergence, 259
  and geodesic flow, 278
  implies topologically tame, 240
Gieseking, 29
Gromov, 102
Gromov's
  invariant, *123*, 140
    for manifolds with boundary, 134
  norm, *123*, *127*
  Theorem

relative version, 136
strict version, 130

Haken
3-orbifold, *324*, 325
manifold, 71
Haken, W., 72
Hatcher, 101
Heegaard decomposition, 3
Hilbert, 10
holonomy, *35*, 53, 85, 97–100
defines structure, 85
horoball, *39*
horocycles, 40
horospheres, 38
hyperbolic
isometries, 67
line, 10, 13, 14
metric, 11, 13, 17, 39–40
plane, 10
structures on a manifold, 87
hyperbolic Dehn surgery theorem, 104
hyperbolic structure
with nodes, 346
hyperboloid, 17
hyperplane, 13
and dual point, 16, 19

ideal tetrahedra, 45–48
parametrization of, 48
volume of, 160
ideal triangles, 40
identifying faces of polyhedra, 3
incompressible
suborbifold, *324*
surface, *71*
and algebraic representations, 143
inner product, 18, 21
intersection number, 267, 270
irreducible
3-manifold, *2*
orbifold, *324*

Jørgensen's Theorem, 119–120
first version, 116
Jørgensen, T., 61, 74, 228

Kleinian group, *174*
manifold of, 178
Kleinian structure, 350
knotted Y, 31

lamination, *185*, *see also* geodesic lamination
when isotopic to geodesic lamination, 206
on boundary of convex hull , 186–187
law of cosines
hyperbolic, 22
law of sines
hyperbolic, 25
Lickorish, 2
limit set, *171*
of a closed hyperbolic manifold, 172
link of a vertex, 42
of an ideal tetrahedron, 45–46
links
C_k, 144
D_{2k}, 150
F_n, 154
having isomorphic complements, 149
Lobachevsky, 157

manifold
affine, 27–28
differentiable, 27
elliptic, 28–29
hyperbolic, 29
Margulis lemma, 113
measure topology
on geodesic laminations, *209*
measured lamination space, 210, 251
metric of constant curvature
and patterns of circles, 338
Micky Mouse, 194
minimal set, 172
modular space, 201
Mostow's Theorem, 101–102, 106–112, 129–130
Mumford, 201

nodes, 346

orbifold, *300*
bad, 304, 324

classification of 2-dimensional, 312
covering, *303*, 305, 311, 313
Euler number, 311
fundamental group, *307*
good, 304, 310, 312
hyperbolic structure, 314–318
with boundary, 308

pair of pants, 90
Papakyriakopoulos, 2
parallel, 14
pared manifolds, 259
pleated surface, *see also* uncrumpled surface
Poincaré dodecahedral space, 320
Prasad, 102
projective lamination space, *209*, 210, 263
properly discontinuous action, *174*
pseudo-isometry, 106
pseudogroup, *27*
Pythagorean theorem
hyperbolic, 25

quasi-conformal map, 110
quasi-Fuchsian group, *192*, 215
mapping surfaces into, 194
quasi-isometric vector field, 290
quasi-isometry, 285

rational depth, 252
reflection group, 323
Riley, R., 29, 74, 168

Schottky group, 173
Seifert fibration, 64
singular locus, 302, 308
smear, 127
sphere at infinity, 11
suborbifold, 308
sufficiently large, 71
Sullivan, 277
symmetry
of 2-generator 3-manifold, 94

tangent space
of an orbifold, 318
Teichmüller space, *88*, 89–92
for hyperbolic orbifold, 318
of the boundary of a 3-manifold, 97
thick-thin decomposition, 112
characterization of $M_{(0,\epsilon]}$, 115
for an orbifold, 346
three-punctured sphere, 36
tractrix, 10
train track, *205*
dual, 267, 271
transverse measure, 189

ultraparallel, 14
uncrumpled surface [pleated], *200*, 219, 249, *see also* wrinkling locus
realizing essentially complete lamination, 249–251
unit tangent bundle
of orbifold, 319–323

visual average, 285
volume
and Gromov's invariant, 126–128
goes down after Dehn filling, 138
is a continuous function, 119
is well-ordered, 139
of a straight k-simplex, 124

Waldhausen, 142
Whitehead link, 32–33, 120
commensurable with Borromean rings, 141
volume of complement, 165
wrinkling locus, 201, 209