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A note on polynomial profiles of placement games

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The polynomial profile of a placement game enumerates the number of different positions. For a subclass of placement games, the polynomial profile is the independence polynomial of a related graph. For several important games, we generate the profiles when the board is a path; in the process, we discover some relationships between them.

1. Introduction

A natural enumeration question for combinatorial games is: "How many legal positions are possible in a game?" Surprisingly, few have actually considered this problem. Farr [7; 8], and Tromp and Farnebäck [20] consider the problem of "counting the number of end positions in GO." Similar enumeration questions are addressed by Hetyei [12], who analyses a game where the number of $\mathcal{P}$-positions (second player win positions) of length n is related to the n-th Bernoulli number of the second kind, and in [17], where it is shown that for the game of TIMBER, on paths, the number of $\mathcal{P}$-positions of length n is related to the Catalan and Fine numbers.

In Section 3, we enumerate the positions of several well-known games. A natural subset of combinatorial games, which we call placement games, are those that consist of placing pieces on a board until the board "fills" and there are no further moves. For each game, except NOGO, we find an auxiliary graph for which a position in the game corresponds to an independent set in the auxiliary graph.

In Section 4, we exhibit bijections between the games with identical generating functions.

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2. Background

A *placement game* can be abstractly represented as a game on a graph, with the following properties.

- The game begins on a graph that contains no pieces.
- A move is to place a piece on one (or more) vertices subject to the rules of the particular game.
- The rules must imply that if a piece can be placed in a certain position on the board then it was legal to place it in that position at any time earlier in the game.
- Once played, a piece remains on the graph; it is never moved or removed from the graph.

Placement games were first identified during the seminar which led to this paper and have become of interest because of their properties; see [13; 6; 5; 16]. Some known examples of placement games are DOMINEERING [1], COL [2], SNORT [2], NOGO [4] and NODE KAYLES [3; 10]. (Rules for the games considered in this paper are given below.) CHESS and CHECKERS are not in this class of games because pieces are moved and removed and the starting position is not empty. The game of GO is likewise not a placement game since, while pieces are placed and not moved around, they can be removed from the board.

In this paper, we consider several placement games that appear in the literature. In all, a vertex that has not been played on will be called *empty* and at the start all vertices are empty. A move by Left is to place a *blue* piece on one or more (depending upon the rules) uncolored vertices. Similarly, Right places a *red* piece. No two pieces can share the same vertex. The *size* of a piece refers to the number of vertices it occupies when it is placed.

- CIS: Both blue and red pieces have size 1 and no two pieces can be adjacent.
- O12[1]: A blue has size 1 and red piece size 2; pieces are allowed to be adjacent.
- SNORT [2]: Both blue and red pieces have size 1; a blue piece and a red piece cannot be adjacent but two pieces of the same color can be adjacent.
- COL [2]: Both blue and red pieces have size 1; no two blue vertices can be adjacent, neither can two red, but a red can be adjacent to a blue.
- NOGO (also known as *Anti-Atari Go*) [4]: Both blue and red pieces have size 1. Every maximal connected group of blue pieces must include a vertex adjacent to an empty vertex, similarly for any maximal group of red pieces.

[1] O12 on a strip is a partizan octal game (see [15]) and another generalization is NODE KAYLES [3].

game	approx. # positions on P_n	generating function
CIS	2^n	$\frac{3}{1-t-2t^2}$
O12	2.414^n	$\frac{1}{1-2t-t^2}$
COL	2.414^n	$\frac{1+t}{1-2t-t^2}$
SNORT	2.414^n	$\frac{1+t}{1-2t-t^2}$
NOGO	2.769^n	$\frac{(1+t)(1-2t)t}{(1-t+t^2-2t^3)(1-t)}$

Table 1. Rates of growth.

In enumerating positions, we found that there is a bijection between the positions of certain games and the independent sets of an associated graph. These games are examples of *independent placement games*. CIS, COL, SNORT, and O12 are independent placement games, but NOGO is not since it has a "hyperedge" constraint. Enumerating independent sets was considered by earlier researchers. Prodinger and Tichy [19], as well as others, showed that the number of independent sets of a path with n vertices is the $(n+2)$-nd Fibonacci number. They coined the term the *Fibonacci number of a graph* G to mean the number of independent sets of G. For a placement game, placing a piece prevents another piece from occupying the same vertex. For many of the games the number of legal positions is the Fibonacci number of an auxiliary graph. Of interest is not only the Fibonacci number but also the *independence polynomial* of a graph, which is defined as

$$I_G(x) = \sum_{i=0} f_i x^i,$$

where f_i is the number of independent sets of cardinality i (see [14] for example).

Table 1 contains the summary of our findings for the number of legal positions on a path P_n (i.e., n vertices). Note that O12, COL and SNORT have the same generating functions.

Rather than just enumerating all legal positions, a finer measure of a game is to count the number of legal positions with a total of k pieces. Surprisingly, even this is not sufficient to distinguish between the games of COL and SNORT on a bipartite graph (see Theorem 4.2). For full generality, in a hope to distinguish between games, we define a bivariate polynomial. Let P be a placement game played on a board (graph) G. Let n be the number of vertices of G. The *polynomial profile* of P on G is the bivariate polynomial

$$P_{P,G}(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{k} f_{j,k-j} x^j y^{k-j},$$

where $f_{j,k-j}$ is the number of legal positions of P on G which have j Left pieces and $k-j$ Right pieces. Note that $f_{0,0} = 1$ since there is one position with no pieces. Putting $x = y$ gives $P_{P,G}(x, x) = \sum_{i=0}^{n} c_i x^i$, which we will shorten to $P_{P,G}(x)$. This is the polynomial in which c_i is the number of positions with exactly i pieces. Finally, putting $x = y = 1$ counts the total number of positions. We find that these last two objects sometimes give rise to sequences that are listed in the Online Encyclopedia of Integer Sequences (OEIS) [18]. These we note as they occur.

For example, in the game of COL, we have

$$P_{\mathrm{COL},P_3}(x, y) = 1 + 3x + 3y + 6xy + x^2 + y^2 + x^2y + xy^2;$$
$$P_{\mathrm{COL},P_3}(x) = 1 + 6x + 8x^2 + 2x^3; \quad P_{\mathrm{COL},P_3}(1) = 17.$$

We construct generating functions for the polynomial profiles of these games on a strip. We present only one explicit calculation since the others are similar. (See [9, §1.3 and 1.4] or [11, Chapter 2] for some of the many possible calculation methods.) For the game P, we define

$$GF_P(e, x, y, t) = \sum_{n \geq 0} t^n \sum_{h+i+j=n} f_{h,i,j} e^h x^i y^j,$$

where n is the number of vertices in the path P_n, and $f_{h,i,j}$ is the number of positions with h empty vertices, i Left pieces and j Right pieces. Note, as in O12, i and j may not be the same as the number of colored vertices. In $GF_P(e, x, y, t)$ the coefficient of t^n gives the polynomial that describes all the positions. In practice, we are not interested in h, so we can set $e = 1$ to get

$$GF_P(1, x, y, t) = \sum_{n=0} P_{G,P_n}(x, y) t^n.$$

Several questions suggest themselves.

Question 2.1. For a given independent placement game P, does the closure of the set of roots over all graphs of $P_{P,G}(x)$ cover the complex plane?

Question 2.2. Let P be an independent placement game. Are the coefficients of $P_{P,G}(x)$ unimodal for all graphs G?

Games P and Q are called $\mathcal{G}$-*doppelgänger* if $P_{P,G}(x) = P_{Q,G}(x)$ for all graphs $G \in \mathcal{G}$. We show that COL and SNORT are dopplegänger on bipartite graphs. Under a restricted class of rules, it is shown in [13] that there are no doppelgänger.

Question 2.3. Do there exist placement games P and Q which are doppelgänger for all graphs? Or for any other subclass of graphs other than bipartite? Are there games for which $P_{P,G}(x, y) = P_{Q,G}(x, y)$ for some class of graphs?

3. Profiles of COL, SNORT, CIS, O12, and NOGO on paths

As mentioned in the Introduction, the methods for all the games are similar. We give a proof for COL and omit the others since they similar. We note when the sequences are related to known sequences.

We will be interested in the situation where the board is a strip or path of n vertices which we will denote by P_n. Throughout, we use B to represent a blue (Left) piece and R a red (Right) piece, except in O12 when we'll use RR. In context of the game under consideration, let $f_E(n)$, $f_R(n)$, and $f_B(n)$ be the bivariate polynomials that count the number of positions with, respectively, an uncolored, a red, and a blue rightmost vertex.

3.1. ***The game of* COL.** Given a graph G, with vertices $\{x_1, x_2, \ldots, x_n\}$, we define the auxiliary graph G_{COL} with $V(G_{\text{COL}}) = \{x_1, x_2, \ldots, x_n\} \times \{1, 2\}$. Vertices (x_i, p) and (x_j, q) are adjacent if $x_i \sim x_j$ and $p = q$ or if $i = j$ and $p \neq q$. That is, G_{COL} is the Cartesian product of G and K_2.

In a position, a blue vertex x_i is identified with $(x_i, 1)$ and a red vertex x_j with $(x_j, 2)$ and the reverse identification for an independent set of G_{COL}. This is a bijection between the positions in COL and independent sets of G_{COL} which forms the proof of the result.

Theorem 3.1. *Let G be a graph then $P_{\text{COL},G}(x) = I_{G_{\text{COL}}}(x)$ and thus* COL *is an independent placement game.*

Now we restrict the board to be a path. First we generate the recurrence relations for the positions on P_{n+1}.

Since these are the only three ways a path can end we see that

$$P_{\text{COL},P_{n+1}}(x, y) = f_E(n+1) + f_B(n+1) + f_R(n+1).$$

If a position on P_{n+1} ends with an empty vertex at the right end, the other n vertices can form any legal COL position on P_n; thus, $f_E(n+1) = P_{\text{COL},P_n}(x, y)$. If it ends in a blue (red) vertex then the other n vertices form a legal position that does not end with a blue (red) vertex; therefore,

$$f_B(n+1) = x(f_R(n) + f_E(n)) = x P_{\text{COL},P_n}(x, y) - x f_B(n),$$

n	$P_{\text{COL},P_n}(x,y)$	$P_{\text{COL},P_n}(x)$	$P_{\text{COL},P_n}(1)$
0	1	1	1
1	$1+x+y$	$2x+1+2x$	3
2	$1+2x+2y+2xy$	$2x^2+4x+1$	7
3	$1+3x+3y+6xy+x^2+y^2+x^2y+xy^2$	$1+6x+8x^2+2x^3$	17

Table 2. The first 3 COL polynomials.

likewise,

$$f_R(n+1)=y(f_B(n)+f_E(n))=yP_{\text{COL},P_n}(x,y)-yf_R(n),$$

and so

$$\begin{aligned}P_{\text{COL},P_{n+1}}(x,y)&=f_B(n+1)+f_R(n+1)+f_E(n+1)\\&=(1+x+y)P_{\text{COL},P_n}(x,y)-xf_B(n)-yf_R(n).\end{aligned}$$

In the case $x=y$, we have $f_B(n)=f_R(n)$ so that

$$\begin{aligned}P_{\text{COL},P_{n+1}}(x)&=(1+2x)P_{\text{COL},P_n}(x,y)-x(f_B(n)-f_R(n))\\&=(1+2x)P_{\text{COL},P_n}(x)-x(P_{\text{COL},P_n}(x,y)-f_E(n))\\&=(1+x)P_{\text{COL},P_n}(x)+xP_{\text{COL},P_{n-1}}(x).\end{aligned}$$

Putting $x=1$ gives the number of positions, i.e., $P_{\text{COL},P_{n+1}}(1)=2P_{\text{COL},P_n}(1)+P_{\text{COL},P_{n-1}}(x,y)$. The first seven coefficients are 1, 3, 7, 17, 41, 99, 239. In [18], this is sequence A001333 *Numerators of continued fraction convergents to sqrt(2)*.

Theorem 3.2. *The bivariate generating function for the number of* COL *positions on a path is obtained from*

$$GF_{\text{COL}}(1,x,y,t)=\frac{(1+xt)(1+yt)}{1-(xyt^2+t(1+xt)(1+yt))};$$

the univariate generating function is obtained from

$$GF_{\text{COL}}(1,x,x,t)=\frac{(1+xt)}{1-((1+x)t+xt^2)};$$

and the total number of positions on P_n is $c_n=(1+\sqrt{2})^n+o(1)\simeq 2.414^n$.

Proof. In COL, no adjacent vertices can be colored the same. Therefore, on a path, a position

(1) starts with zero or more empty vertices;

(2) repeated patterns taken from

- B followed by zero or more occurrences of RB followed by at least one E; or
- R followed by zero or more occurrences of BR followed by at least one E; or
- BR followed by zero or more occurrences of BR followed by at least one E; or
- RB followed by zero or more occurrences of RB followed by at least one E;

(3) ends with nothing added;

- B followed by zero or more occurrences of RB; or
- R followed by zero or more occurrences of BR; or
- BR followed by zero or more occurrences of BR; or
- RB followed by zero or more occurrences of RB.

This gives the regular expression

$$E^*((R(BR)^* \mid B(RB)^* \mid RB(RB)^* \mid BR(BR)^*)EE^*)^* \cdot (R(BR)^* \mid B(RB)^* \mid RB(RB)^* \mid BR(BR)^* \mid \epsilon),$$

where ϵ is the empty word. The term $R(BR)^* \mid B(RB)^* \mid RB(RB)^* \mid BR(BR)^*$ occurs twice, and the corresponding expression in the generating function is

$$\frac{xt}{1-xyt^2}+\frac{yt}{1-xyt^2}+\frac{2xyt^2}{1-xyt^2}=\frac{xt+yt+2xyt^2}{1-xyt^2}.$$

The generating function for COL is

$$GF_{\text{COL}}(e,x,y,t)=\left(\frac{1}{1-et}\right)\left(\frac{1}{1-\dfrac{xt+yt+2xyt^2}{1-xyt^2}\dfrac{et}{1-et}}\right)\left(\frac{xt+yt+2xyt^2}{1-xyt^2}+1\right),$$

which gives

$$GF_{\text{COL}}(1,x,y,t)=\frac{(1+xt)(1+yt)}{1-(xyt^2+t(1+xt)(1+yt))};$$

$$GF_{\text{COL}}(1,x,x,t)=\frac{(1+xt)}{1-((1+x)t+xt^2)};$$

and

$$GF_{\text{COL}}(1,1,1,t) = \left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\left(\frac{(1)}{1-(1+\sqrt{2})t}\right)+\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)\left(\frac{(1)}{1-(1-\sqrt{2})t}\right).$$

From the latter equation we see that the coefficient of t^n in $GF_{\text{COL}}(1, 1, 1, t)$ is

$$c_n = \frac{1-\sqrt{2}}{2}(1-\sqrt{2})^n + \frac{1+\sqrt{2}}{2}(1+\sqrt{2})^n.$$

Since $|1-\sqrt{2}| < 1$ this contribution of this term goes to 0 and so $c_n = (1 + \sqrt{2})^n + o(1) \simeq 2.414^n$. □

3.2. *The game of* SNORT. We already know that $P_{\text{SNORT},P_n}(x) = P_{\text{COL},P_n}(x)$ but the bivariate polynomials are different.

We define the auxiliary graph G_{SNORT} with $V(G_{\text{SNORT}}) = \{x_1, x_2, \ldots, x_n\} \times \{1, 2\}$. Vertices (x_i, p) and (x_j, q) are adjacent if $i = j$ and $p \neq q$ or if both $x_i \sim x_j$ and $p \neq q$. Another description is that G_{SNORT} is the categorical product of G and K_2 together with the matching edges $((x_i, 1), (x_i, 2))$, $i = 1, 2, \ldots, n$.

In a position, a blue vertex x_i is identified with $(x_i, 1)$ and a red vertex x_j with $(x_j, 2)$ and the reverse identification for an independent set of G_{SNORT}. This is a bijection between the positions in SNORT on G and independent sets of G_{SNORT} which forms the proof of the result.

Theorem 3.3. *Let G be a graph then* $P_{\text{SNORT},G}(x) = I_{G_{\text{SNORT}}}(x)$ *and thus* SNORT *is an independent placement game.*

In SNORT, we would like to build a position on P_{n+1}. Not surprisingly, we have a similar construction as that for COL:

$$f_B(n+1) = x(f_B(n) + f_E(n)) = xP_{\text{SNORT},P_n}(x, y) - xf_R(n);$$
$$f_R(n+1) = y(f_R(n) + f_E(n)) = yS_{x,y}(n) - yf_L(n).$$

Since $f_B(1) = x$, $f_R(1) = y$, $f_E(1) = 1$ then

$$P_{\text{SNORT},P_{n+1}}(x, y) = f_B(n+1) + f_R(n+1) + f_E(n+1) \quad (1)$$
$$= (1 + x + y)P_{\text{SNORT},P_n}(x, y) - yf_B(n) - xf_R(n). \quad (2)$$

In the next table, the last two columns are the same as Table 2.

Theorem 3.4. *The bivariate generating function for the number of* SNORT *positions on a path is given by*

$$GF_{\text{SNORT}}(1, x, y, t) = \frac{(1 - xyt^2)}{1 - (xt + yt + xyt^2 + t(1 - xyt^2))};$$

the univariate polynomial is given by

$$GF_{\text{SNORT}}(1, x, x, t) = \frac{(1 + xt)}{1 - ((1 + x)t + xt^2)};$$

and the total number of positions on P_n *is* $c_n = (1 + \sqrt{2})^n + o(1) \simeq 2.414^n$.

3.3. *The game of* CIS. If only one colored piece were being played then every independent set would correspond to a legal position and, as reported in [19] and other papers, the number of independent sets on a path with n vertices is the $(n+2)$-nd Fibonacci number.

Given a graph (board) G with $V(G) = \{a_1, a_2, \dots, a_n\}$ we construct the auxiliary graph G_{CIS} where $V(G_{\text{CIS}}) = \{a_1, a_2, \dots, a_n\} \times \{1, 2\}$ and $((b, c), (d, e)) \in E(G_{\text{CIS}})$ if b is adjacent to d. This is also known as the strong product of G and K_2.

Theorem 3.5. *Let G be a graph then* $P_{\text{CIS},G}(x) = I_{G_{\text{CIS}}}(x)$ *where* $I_{G_{\text{CIS}}}(x)$ *is the independence polynomial of* G_{CIS}; *thus*, CIS *is an independent placement game.*

Proof. We construct a bijection between the legal CIS positions on G and the independent subsets of $V(G_{\text{CIS}})$. A position with i blue pieces on $B = \{a_{b_1}, a_{b_2}, \dots, a_{b_i}\}$ and $k-i$ red pieces on $R = \{a_{r_1}, a_{r_2}, \dots, a_{r_{k-i}}\}$ is paired with the set of vertices

$$BR = \{(b, 1) : b \in \text{B}\} \cup \{(r, 2) : r \in \text{R}\}$$

in G_{CIS}. Since $B \cup R$ is independent, so is BR. Any independent set in G_{CIS} can be partitioned into two sets: those with coordinate 1 and those with coordinate 2. The first set is the set of blue pieces and the other is the set of red pieces; the combined set of vertices is an independent set so this is a legal position. □

The recurrence relations are

$$P_{\text{CIS},P_{n+1}}(x, y) = P_{\text{CIS},P_n}(x, y) + (x + y)P_{\text{CIS},P_{n-1}}(x, y).$$

Note that $P_{\text{CIS},P_0}(x, y) = 1$ and $P_{\text{CIS},P_1}(x, y) = 1 + x + y$.

Theorem 3.6. *The bivariate generating function for the number of* CIS *positions is obtained from*

$$GF_{\text{CIS}}(1, x, y, t) = \frac{(1 + x + y)}{1 - t - xt^2 - yt^2};$$

the univariate polynomial is obtained from

$$GF_{\text{CIS}}(1, x, x, t) = \frac{(1 + 2x)}{1 - t - 2xt^2};$$

and the number of positions on P_n *is* $\frac{1}{3}(4 \times 2^n + (-1)^{n+1})$.

In particular, the sequence $\{c_n\} = \{1, 3, 5, 11, 21, 43, \dots\}$ is the Jacobsthal numbers (see A001045 in [18]).

The generating function for the game k-CIS, where there are pieces of k different colors, can be found in a similar fashion.

Corollary 3.7. *In k-CIS (that is, CIS played with k colors), the generating function is*

$$GF_{k\text{-CIS}}(1, 1, 1, t) = \frac{1+kt}{1-t-kt^2}.$$

We leave the proof to the reader, but note that for $k = 3, 4, 5, 6, 7, 8$ these are the sequences A006130, A006131, A015440, A015441, A015442, and A015443 respectively in [18].

3.4. *The game of* O12. We can define an auxiliary graph G_{O12} for a graph G. Let $V(G_{\text{O12}}) = V(G) \cup E(G)$ and $(a, b) \in E(G_{\text{O12}})$ if one of the following holds:

(i) $a \in V(G)$, $b \in E(G)$, and $a \in b$;

(ii) $a, b \in E(G)$ and $a \cap b \neq \varnothing$.

In other words, G_{O12} is the line graph of G plus the vertices of G where a vertex of G is adjacent to all its incident edges.

Theorem 3.8. *Let G be a graph. Then $P_{\text{O12},G}(x) = I_{G_{\text{O12}}}(x)$, and thus* O12 *is an independent placement game.*

On a general graph, the profile is a symmetric polynomial.

Theorem 3.9. *Let G be a graph on n vertices then $P_{\text{O12},G}(x) = \sum_{i=0}^{n} c_i x^i$ is symmetric, that is, $c_i = c_{n-i}$. If $n = 2m$ and then c_m has the same parity as the number of perfect matchings in G. Moreover, in $P_{\text{O12},G}(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{i,j} x^i y^j$ we also have $c_{i,j} = c_{n-i-2j,\,j}$.*

Proof. The proof of all the statements comes from one observation. Let P be a position on G with j Right dominoes and $i - j$ Left pieces and, consequently, $n - (i - j) - 2j = n - i - j$ empty vertices. Interchange empty vertices and Left pieces to get a position with j Right dominoes and $n - i - j$ Left pieces, that is, a position with $n - i$ pieces. Moreover, this is a bijection except, possibly, for the position in which all the vertices of G are occupied by all Right dominoes (the dominoes form a perfect matching and $i = 0$, $j = \frac{1}{2}n$) which is matched to itself. Therefore $c_{i,j} = c_{n-i-2j,j}$. Now

$$c_k = \sum_{j=0}^{k} c_{k-j,j} = \sum_{j=0}^{k} c_{n-k-j,j} = c_{n-k}.$$

If G has $2m$ vertices, then every perfect matching is matched to itself and all the other positions with m pieces are paired off, so the parity of c_m is the same as that of the number of perfect matchings of G. □

Considering just paths, the recurrence relation is

$$P_{\text{O12},P_{n+1}}(x, y) = (1 + x) P_{\text{O12},P_n}(x, y) + y P_{\text{O12},P_{n-1}}(x, y).$$

The coefficients of $P_{\mathrm{O12},P_n}(x)$ (i.e., 1, 1, 1, 1, 3, 1, 1, 5, 5, ...) are the Delannoy numbers; see A008288 [18].

Theorem 3.10. *The bivariate generating function for the number of* O12 *positions on a path is obtained from*

$$GF_{\mathrm{O12}}(1, x, y, t) = \frac{1}{1 - ((x+1)t + yt^2)};$$

the univariate polynomials are obtained from

$$GF_{\mathrm{O12}}(1, x, x, t) = \frac{1}{1 - ((x+1)t + xt^2)};$$

and the total number of positions on P_n is $c_n = (\sqrt{2}+1)^{n+1}/(2\sqrt{2}) + o(1)$.

The sequence of numbers is 1, 2, 5, 12, 29, 70, 169, etc., which is the sequence of Pell Numbers, A000129 in [18]. When played on K_n the number of positions is the sequence A005425 in [18], which is related to the Hermite polynomials.

3.5. ***The game of* NOGO.** The empty-vertex-adjacency constraint is a hyperedge condition and so there is no auxiliary graph whose independent sets correspond to the positions in the games. Consequently, NOGO is not an independent placement game.

The recurrence relations for NOGO positions are trickier to generate via considering the last vertex because they do not always arise out of a smaller legal position. These exceptions can be easily identified though.

Consider a position on P_{n+1}. If this position ends with an unoccupied vertex at the right end, the other n vertices can

- form any legal NOGO position on P_n,
- be n blue pieces,
- be n red pieces,
- be i blue vertices which is then followed by legal position on P_{n-i}, $1 \le i \le n-2$ that starts with a red vertex, or
- as in the previous but with interchanging blue and red.

Thus

$$P_{\mathrm{NOGO},P_{n+1}}(x, y) = f_E(n+1) + f_R(n+1) + f_B(n+1).$$

It follows that

$$f_E(n+1) = P_{\mathrm{NOGO},P_n}(x, y) + x^n + y^n + \sum_{i=1}^{n-2} (f_R(n-i)x^i + f_B(n-i)y^i);$$

$$f_B(n+1) = x(f_B(n) + f_E(n)) = x(P_{\mathrm{NOGO},P_n}(x, y) - f_R(n));$$

and likewise

$$f_R(n+1) = y(P_{\text{NOGO},P_n}(x,y) - f_B(n)).$$

Thus

$$P_{\text{NOGO},P_{n+1}(x,y)} = x(P_{\text{NOGO},P_n}(x,y) - f_E(n)) + y(P_{\text{NOGO},P_n}(x,y) - f_E(n)) + P_{\text{NOGO},P_n}(x,y) + x^n + y^n + \sum_{i=1}^{n-2}(f_R(n-i)x^i + f_B(n-i)y^i).$$

Putting $y = x$ gives

$$P_{\text{NOGO},P_{n+1}}(x) = (2x+1)P_{\text{NOGO},P_n}(x) - 2f_E(n) + 2x^n + \sum_{i=1}^{n-2} x^i(P_{\text{NOGO},P_{n-i}}(x) - f_B(n-i)).$$

The total number of positions, 1, 5, 15, 41, 113, 313, 867, 2401, ..., was already known to Tromp and Farnebäck [20] and the sequence is A102620 in [18].

Theorem 3.11. *The bivariate generating function for the number of* NOGO *positions on a path is obtained from*

$$GF_{\text{NOGO}}(1,x,y,t) = \frac{t(1-xyt^2)(1-xt-yt)}{((1-xt)(1-yt)-t-xyt^3)(1-xt)(1-yt)};$$

the univariate polynomial is obtained from

$$GF_{\text{NOGO}}(1,x,x,t) = \frac{(1+xt)(1-2xt)t}{((1-xt)^2-t-2xt^3)(1-xt)};$$

and the total number of positions on P_n *is* $c_n = 2.769296^n + o(1)$.

4. Relationships between games

As mentioned in the Introduction, if games have the same profile there is the possibility of a bijection between the positions.

4.1. *Relationship between* COL *and* SNORT. We now show that the enumeration of positions for COL and SNORT on bipartite graphs are equal.

Lemma 4.1. *Let G be a bipartite graph and let k be a nonnegative integer. The number of legal* COL *positions with k pieces on G is the same as the number of legal* SNORT *positions with k pieces on G.*

Proof. Number the vertices of G with distinct but not necessarily consecutive positive integers such that one color class consists of even numbers and the other odd numbers. We define two transformations.

A : (SNORT $\rightarrow$ COL) Let S_k be a SNORT position with k pieces. Let H be the subgraph of vertices occupied by a piece, and let H' be a connected component of H (necessarily all the vertices of H' are occupied by blue pieces or all by red pieces). Let $x \in V(H')$ be the least numbered vertex in H'. If x is even then in H' interchange red and blue pieces on all the odd numbered vertices. This component forms a legal COL position since no two adjacent vertices are occupied by the same colored piece. Do this for each component and we have a legal COL position of k pieces in G.

B : (SNORT $\rightarrow$ COL) Let C_k be a COL position with k pieces. Let H be the subgraph of vertices occupied by a piece, and let H' be a connected component of H. In H' all vertices occupied by B will be in one color class (i.e., odd or even) and the vertices in the other will be occupied by R. Let $x \in V(H')$ be the least numbered vertex in H'. If x is even then change the pieces in all the odd numbered vertices to the same as that occupying x, and leave the others as they are. If x is odd then change nothing. This component forms a legal SNORT position since no two adjacent vertices have different colored pieces. Do this for each component and we have a legal COL position of k pieces in G.

Let S_k be a SNORT position with k pieces then $B(A(S_k)) = S_k$. Let C_k be a COL position of k pieces then $A(B(C_k)) = S_k$. Therefore we have a bijection between the positions of k pieces and the lemma is proved. □

This result gives the following.

Theorem 4.2. *If G is a bipartite graph then* $P_{\text{COL},G}(x) = P_{\text{SNORT},G}(x)$. *In particular*, $P_{\text{COL},G}(1) = P_{\text{SNORT},G}(1)$, *i.e., the number of positions on G is the same for* COL *and* SNORT.

4.2. *The relationship between* COL *and* O12.

Theorem 4.3. *Let n be a positive integer; then*

$$P_{\text{COL},P_{n+1}}(x) = 2x\,P_{\text{O12},P_n}(x) + P_{\text{COL},P_n}(x),$$
$$P_{\text{O12},P_{n+1}}(x) = x\,P_{\text{O12},P_n}(x) + P_{\text{COL},P_n}(x).$$

Proof. For each equality, we give a bijection between the positions.

First, we prove $P_{\text{COL},P_{n+1}}(x) = 2x\,P_{\text{O12},P_n}(x) + P_{\text{COL},P_n}(x)$.

The COL positions on P_{n+1} that start with an empty vertex are paired with the COL positions on P_n (i.e., P_{n+1} minus the first vertex).

Consider the COL positions with k pieces on P_{n+1} that start with a blue piece. We will transform this in to an O12 position with $k-1$ pieces on P_n by starting just after the beginning blue piece and converting the pieces as we progress to the other end of the path by the following rules. For ease of translation, we will change the O12 colors: *aqua* replaces blue and *crimson* replaces red.

(1) A blue piece becomes aqua.

(2) A red piece preceded by a blue piece becomes aqua.

(3) An empty vertex followed by a red piece are both replaced by a crimson domino.

(4) An empty vertex not followed by a red piece is left empty.

For the reverse, starting from a O12 position with $k-1$ pieces on P_n:

(1) Start with a new blue piece.

(2) An aqua piece which now is preceded by an empty or a red piece becomes blue, otherwise it is replaced by a red piece.

(3) A crimson domino is replaced by an empty vertex followed by a red piece.

(4) An empty vertex is left empty.

It is clear that two "blue-start" COL positions map to different O12 positions and that two different O12 positions map to different "blue-start" COL positions. Also, one piece in one game is mapped to one piece in the other. So the two sets have the same cardinality and the COL position has one extra piece.

This leaves the COL positions that start with a red piece. For these, in the previous transformation rules interchange "blue" and "red".

Now we prove $P_{\mathrm{O12},P_{n+1}}(x) = x\,P_{\mathrm{O12},P_n}(x) + P_{\mathrm{COL},P_n}(x)$.

The O12 positions on P_{n+1} that start with an aqua piece correspond to all the positions of O12 positions on P_n. Using the transformations, the O12 positions with k pieces on P_{n+1} that start with

(a) a crimson piece correspond to all the COL positions with k pieces on P_n that start with a red piece;

(b) those that start with an empty-aqua pair of vertices correspond to a COL positions that starts with a blue piece; and

(c) those that start with an empty-empty pair of vertices correspond to a COL positions that starts with an empty vertex. □

In [18], it is mentioned that Clark Kimberling in (Mar 09 2012) showed that A008288 is jointly generated with A035607 via an array of coefficients of polynomials $u(n,x)$. Initially, $u(1,x) = v(1,x) = 1$, for $n > 1$, $u(n,x) = xu(n-1,x) + v(n-1)$ and $v(n,x) = 2xu(n-1,x) + v(n-1,x)$. These are the same recursions as in the theorem. No proof is referenced in [18] but our proof gives a combinatorial game theory explanation.

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