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Misère games and misère quotients

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These notes are based on a short course offered at the Weizmann Institute of Science in Rehovot, Israel, in November 2006. The notes include an introduction to impartial games, starting from the beginning; the basic misère quotient construction; a proof of the periodicity theorem; and statements of some recent results and open problems in the subject.

Introduction

This course is concerned with *impartial combinatorial games*, and in particular with misère play of such games. Loosely speaking, a *combinatorial game* is a two-player game with no hidden information and no chance elements. We usually impose one of two winning conditions on a combinatorial game: under *normal play*, the player who makes the last move wins, and under *misère play*, the player who makes the last move loses. We will shortly give more precise definitions.

The study of combinatorial games began in 1902, with C. L. Bouton's published solution to the game of NIM [2]. Further progress was sporadic until the 1930s, when R. P. Sprague [17; 18] and P. M. Grundy [6] independently generalized Bouton's result to obtain a complete theory for normal-play impartial games.

In a seminal 1956 paper [8], R. K. Guy and C. A. B. Smith introduced a wide class of impartial games known as *octal games*, together with some general techniques for analyzing them in normal play. Guy and Smith's techniques proved to be enormously powerful in finding normal-play solutions for such games, and they are still in active use today [4].

At exactly the same time (and, in fact, in exactly the same issue of the *Proceedings of the Cambridge Philosophical Society*), Grundy and Smith published a paper on misère games [7]. They noted that misère play appears to be quite difficult, in sharp contrast to the great success of the Guy–Smith techniques.

Despite these complications, Grundy remained optimistic that the Sprague–Grundy theory could be generalized in a meaningful way to misère play. These hopes were dashed in the 1970s, when Conway [3] showed that the Grundy–Smith

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complications are intrinsic. Conway's result shows that the most natural misère-play generalization of the Sprague–Grundy theory is hopelessly complicated, and is therefore essentially useless in all but a few simple cases.[1]

The next major advance occurred in 2004, when Thane Plambeck [10] recovered a tractable theory by *localizing* the Sprague–Grundy theory to various restricted sets of misère games. Such localizations are known as *misère quotients*, and they will be the focus of this course. While some of the ideas behind the quotient construction are present in Conway's work of the 1970s, it was Plambeck who recognized that the construction can be made systematic — in particular, he showed that the Guy–Smith *periodicity theorem* can be generalized to the local setting.

This course is a complete introduction to the theory of misère quotients, starting with the basic definitions of combinatorial game theory and a proof of the Sprague–Grundy theorem. We include a full proof of the periodicity theorem and many motivating examples. The final lecture includes a discussion of major open problems and promising directions for future research.

Lecture 1. Normal play

November 26, 2006 Scribes: Leah Nutman and Dan Kushnir

Impartial combinatorial games — a few examples. A combinatorial game is a two player game with no hidden information (i.e., both players have full information of the game's position) and no chance elements (given a player's move, the next position of the game is completely determined). Let us demonstrate this notion with a few useful examples.

Example (NIM). A position of NIM consists of several strips, each containing several boxes. A move consists of removing one or more boxes from a *single* strip. Whoever takes the last box (from the last remaining strip) wins. A sample game of NIM is illustrated in Figure 1.

Example (KAYLES). A position of KAYLES consists of several strips, each containing several boxes, as in NIM. A move consists of removing one or two *adjacent* boxes from a single strip. If the player takes a box (or two) from the middle of a strip then this strip is split into two *separate* strips. In particular, no future move can affect both sides of the original strip. (See Figure 2 for an illustration of one such move.)

Whoever takes the last box (from the last remaining strip) wins.

[1] Despite its apparent uselessness, Conway's theory is actually quite interesting from a theoretical point of view. We will not say much about it in this course, but it is well worth exploring; see [3] for discussion.

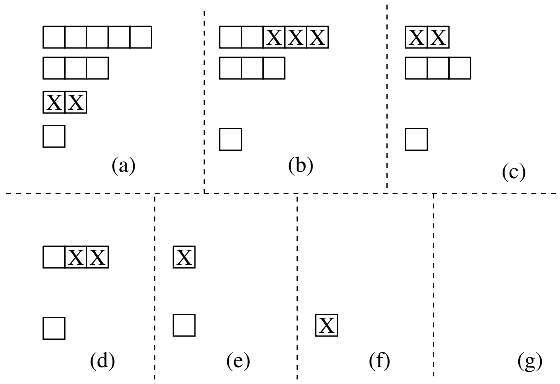


Figure 1. The seven subfigures represent seven consecutive positions in a play of a game of NIM. The last position is the empty one (with no boxes left and thus no more possible moves). The first six minifigures also indicate the move taken next (which transforms the current position into the next position): a box marked with "X" is a box that was selected to be taken by the player whose turn it is to play.

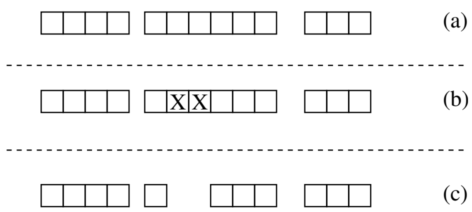


Figure 2. (a) The position before the move, consisting of three strips. (b) The move: the selected boxes are marked with "X". (c) In the new position, the middle strip was split, leaving four strips.

Example (DAWSON'S KAYLES). This game is identical to KAYLES up to two differences:

(1) A move consists of removing *exactly two* adjacent boxes from a single strip.

(2) The winning condition is flipped: whoever makes the last move *loses*.

Winning conditions and the difficulty of a game. All three examples above share some common properties. They are:

Finite: For any given first position, there are only finitely many possible positions that the game may take (throughout its execution).

Loopfree: No position can occur twice in an execution of a game. Once we leave a position, this position will never repeat itself.

Impartial: Both players have the same moves available at all times.

All of the games we will consider in this course have these three properties. As we will further discuss below, the first two properties (finite and loopfree) imply that one of the players must have a perfect winning strategy — that is, a strategy that guarantees a win no matter what his opponent does.

Main Goal. Given a combinatorial game Γ, find an efficient winning strategy for Γ.[2]

We will consider in this course two possible winning conditions for our games:

Normal play: Whoever makes the last move wins.

Misère play: Whoever makes the last move loses.

The different winning conditions of the aforementioned games turn out to have a great effect on their difficulty. NIM was solved in 1902 and KAYLES was solved in 1956. By contrast, the solution to DAWSON'S KAYLES remains an open problem after 70 years. (That is, we still do not know an efficient winning strategy for it.)

What makes DAWSON'S KAYLES so much harder? It is exactly the fact that the last player to move loses. In general, games with misère play tend to be vastly more difficult. The themes for this course are the following:

(1) Why is misère play more difficult?

(2) How can we tackle this difficulty?

Game representations and outcomes. We have mentioned that our goal is to obtain efficient winning strategies for impartial combinatorial games. We will in fact be even more concerned with the structure of individual positions. Therefore, by a "game" G, we will usually mean an individual position in a combinatorial game.

Sometimes we will shamelessly abuse terminology and use the term "game" to refer to a system of rules. It will (hopefully) always be clear from the context which meaning is intended. To help minimize confusion, we will always denote individual positions by roman letters (G, H, ...) and systems of rules by Γ.

One way to formally represent a game is as a tree. For example, the NIM position G which contains three boxes in a single strip can be represented by the tree given in Figure 3. Here 0 represents a strip with zero boxes, from which there are no possible moves.

Definition 1.1. Two games G and H are identical (isomorphic) if they have isomorphic trees. If G and H are identical, we write $G \cong H$.

[2]More precisely, we seek a winning strategy that can be computed in polynomial time (measured against the size of a *succinct description* of a game position). In general, any use of the word "efficient" in this course can be safely interpreted to mean "polynomial-time," though we will be intentionally vague about issues of complexity.

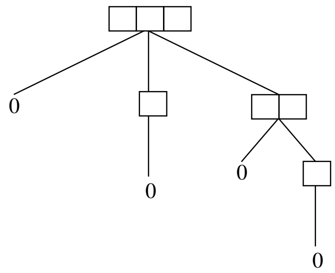


Figure 3. Tree representation of the NIM position G which contains three boxes in a single strip.

We can also think of the NIM position G from Figure 3 as a *set*: $\square\square\square = \{0, \square, \square\square\}$. We call the positions we can move to directly from a game G the *options* of G. So we are identifying G with the set of its options; for example, 0 is identified with the empty set $\{\}$.

We will now introduce some notation that will make it easier to discuss the value of any given position of a game and in particular, the values of NIM positions.

Definition 1.2. For every $n \geq 0$ we denote by $*n$ a strip in NIM of length n. We write 0 and $*$ as shorthand for $*0$ and $*1$, respectively. Formally, we have

$$*n = \{0, *, *2, *3, \ldots, *(n-1)\}.$$

As we mentioned above, every game with the properties we have specified has a well-defined outcome (indicating who will win when both players play perfectly). Assuming both players play perfectly, either

(1) the first player has a winning move, or

(2) any move the first player may make will move to a position where he loses. In this case the second player can win.

Definition 1.3. Let G be a game. The *normal outcome* $o^+(G)$ is defined by

- $o^+(G) = \mathscr{P}$ if second player can win G, assuming normal play;
- $o^+(G) = \mathscr{N}$ if first player can win G, assuming normal play.

Likewise, the *misère outcome* $o^-(G)$ is defined by

- $o^-(G) = \mathscr{P}$ if second player can win G, assuming misère play;
- $o^-(G) = \mathscr{N}$ if first player can win G, assuming misère play.

We say G is a *normal* $\mathscr{P}$*-position* if $o^+(G) = \mathscr{P}$, etc.

Note that o^+ and o^- have simple recursive descriptions:

$$o^+(G) = \mathcal{P} \iff o^+(G') = \mathcal{N} \text{ for every option } G' \text{ of } G.$$
$$o^-(G) = \mathcal{P} \iff G \neq 0 \text{ and } o^-(G') = \mathcal{N} \text{ for every option } G' \text{ of } G.$$

$\mathcal{P}$ and $\mathcal{N}$ are short for *p*revious player and *n*ext player, respectively.

For example, we can consider NIM played with a single strip and see which positions are $\mathcal{P}$-positions and which are $\mathcal{N}$-positions:

- $o^+(0) = \mathcal{P}$: If there are no more boxes, then the previous move was the winning move (the previous player took the last box).
- $o^+(*n) = \mathcal{N}$ for every $n > 0$: When there is only one strip left, the next player can take all the remaining boxes and thus win.

What about the misère outcomes?

- $o^-(0) = \mathcal{N}$: If there are no more boxes, then the previous player took the last box and lost. So the next player is the winning one.
- $o^-(*) = \mathcal{P}$: When there is only one box left, the next player must take it and lose, so the previous player is the winning one.
- $o^-(*n) = \mathcal{N}$ for every $n > 1$: Here the winning move is to take all boxes but one.

We now revise our main goal.

Main Goal (revised). Given a position G in a combinatorial game, find an efficient way to compute the outcome of G.

In all the examples we consider in this course, the two goals are equivalent: efficient methods for computing the outcomes of positions will instantly yield efficient winning strategies.

Disjunctive sums. The positions in each of our examples naturally decompose. In NIM, no single move may affect more than one strip, so each strip is effectively independent. Both KAYLES and DAWSON'S KAYLES exhibit an even stronger form of decomposition: a typical move cuts a strip into two components, and since the components are no longer adjacent, no subsequent move can affect them both.

These observations motivate the following definition.

Definition 1.4. Let G and H be games. The *(disjunctive) sum* of G and H, denoted $G + H$, is the game played as follows. Place copies of G and H side-by-side. A move consists of choosing exactly one component and making a move in that component.

Formally, we can define $G + H$ as the direct sum of the trees for G and H. Or, thinking in terms of sets,

$$G + H = \{G' + H : G' \text{ is an option of } G\} \cup \{G + H' : H' \text{ is an option of } H\}.$$

In combinatorial game theory, it is customary to be lazy in our use of notation and write simply

$$G + H = \{G' + H, G + H'\}.$$

***The strategy for* NIM.** Here is the strategy for NIM, assuming normal play: write the size of each strip in binary, and then do a bitwise XOR. G is a $\mathscr{P}$-position if and only if the result is identically 0. For example, the starting position of Figure 1(a) has strips of sizes 5, 3, 2 and 1, so we can write

$$\begin{array}{cr} & 101 = 5 \\ \oplus & 11 = 3 \\ \oplus & 10 = 2 \\ \oplus & 1 = 1 \\ \hline & 101 \end{array}$$

The result is nonzero, so Figure 1(a) is an $\mathscr{N}$-position (in normal play).

We will shortly prove a stronger statement that implies this strategy.

Equivalence. We would like to regard two games as equivalent if they behave the same way in any disjunctive sum. For now assume normal play.

Definition 1.5. We say G and H are *equal*, and write $G = H$, if and only if

$$o^+(G + X) = o^+(H + X) \text{ for every combinatorial game } X.$$

Note that if $G \cong H$, then necessarily $G = H$, but we will see that nonisomorphic games can be equal.

Proposition 1.6. *$G + 0 = G$ for any game G.*

Proof. Adding 0 does not change the structure of G at all. (In fact, $G+0 \cong G$.) □

Proposition 1.7. *$G + G = 0$ for any game G.*

Proof. We need to show that X and $G + G + X$ have the same outcome, for any X.

First suppose $o^+(X) = \mathscr{P}$. Second player can win $G + G + X$ as follows. Whenever first player moves on X, just use the winning strategy there. If first player ever moves on one of the copies of G, make the identical move on the other copy. Second player will get the last move on X because she is following the winning strategy there, and she will get the last move on $G + G$ by symmetry.

Conversely, if $o^+(X) = \mathcal{N}$, then on $G + G + X$, just make a winning move on X and proceed as before. □

Example. Here is a simple example to show how disjunctive sums can be useful for studying combinatorial games. Consider a NIM position with strips of sizes 19, 23, 16, 45, 23 and 19. By the previous argument, the two strips of size 19 together equal 0, as do the two strips of size 23. So this is equivalent to NIM with strips of sizes 16 and 45.

Exactly the same argument works for KAYLES or DAWSON'S KAYLES.

Proposition 1.8. (a) $=$ *is an equivalence relation.*

(b) *If* $G = H$, *then* $G + K = H + K$.

Proof. Since equality of outcomes is an equivalence relation, (a) is immediate. For (b), if $G = H$ then

$$o^+(G + X) = o^+(H + X) \text{ for all } X,$$

so in particular

$$o^+(G + (K + X)) = o^+(H + (K + X)) \text{ for all } X.$$

Disjunctive sum is associative, so $G + K = H + K$. □

Proposition 1.9. *The following are equivalent, for games* G, H:

(i) $G = H$.

(ii) $o^+(G + H) = \mathcal{P}$.

Proof. (i) ⇒ (ii): If $G = H$, then $G + G = G + H$. But $G + G = 0$, so $o^+(G + H) = o^+(0) = \mathcal{P}$.

(ii) ⇒ (i): By a symmetry argument (just like Proposition 1.7), X and $G+H+X$ have the same outcome, for all X. Therefore $G + H = 0$, so $G + H + H = H$. But $H + H = 0$. □

The Sprague–Grundy theorem.

Theorem 1.10 (Sprague–Grundy). *For any game* G, *there is some* m *such that* $G = *m$.

We will in fact prove the following stronger statement.

Definition 1.11. Let S be a finite set of nonnegative integers. We define the *minimal excludant* of S, denoted $\text{mex}(S)$, to be the least integer not in S.

Theorem 1.12 (mex rule). *Suppose* $G \cong \{*a_1, \ldots, *a_k\}$. *Then* $G = *m$, *where*

$$m = \text{mex}\{a_1, \ldots, a_k\}.$$

Proof. By Proposition 1.9, it suffices to show that $G + *m$ is a $\mathcal{P}$-position. There are two cases.

Case 1: First player moves in G. This leaves the position $*a + *m$, where $*a$ is some option of G. Since $m \notin \{a_1, \dots, a_k\}$, we necessarily have $a \neq m$. If $a > m$, second player can move to $*m + *m$; if $a < m$, she can move to $*a + *a$. In either case, she leaves a $\mathcal{P}$-position.

Case 2: First player moves in $*m$. This leaves $G + *a$, for some $a < m$. Since m is the *minimal* excludant of $\{a_1, \dots, a_k\}$, we must have $a = a_i$ for some i. Therefore second player can move to $*a + *a$, a $\mathcal{P}$-position. □

The Sprague–Grundy theorem follows from one more ingredient.

Exercise (prove the *replacement lemma*). Suppose $G = \{G_1, \dots, G_k\}$ and suppose $G_1 = H$ for some H. Then

$$G = \{H, G_2, \dots, G_k\}.$$

Proof of Sprague–Grundy theorem. Write $G = \{G_1, \dots, G_k\}$. Inductively, we may assume that $G_1 = *a_1, \dots, G_k = *a_k$. By the replacement lemma, $G = \{*a_1, \dots, *a_k\}$, and by the mex rule we are done. □

Lecture 2. Octal games and misère play

November 27, 2006 Scribes: Omer Kadmiel and Shai Lubliner

We introduce a broad class of games known as *octal games*, and then give the definition of misère quotient.

NIM *values.* In the previous lecture we showed the following:

- Assuming *normal* play, if G is any impartial combinatorial game, then $G = *m$ for some m. Moreover, if $G = \{*a_1, \dots, *a_k\}$ then $m = \text{mex}\{a_1, \dots, a_k\}$.
- For any impartial G, H, $o^+(G + H) = \mathcal{P}$ if and only if $G = H$.

We denote by $\mathcal{G}(G)$ the unique integer m such that $G = *m$ in normal play. $\mathcal{G}(G)$ is called the *nim value* of G.

***XOR and a winning strategy for (normal-play)* NIM.** If m, n integers then $m \oplus n$ denotes the binary XOR of m and n.

Theorem 2.1. *Let a, b, c be integers. Then*

$$o^+(*a + *b + *c) = \mathcal{P} \iff a \oplus b \oplus c = 0.$$

Proof (by example). Consider the following example:

$$\begin{array}{rl} & 11101001 \quad a \\ \oplus & 01101111 \quad b \\ \oplus & 00000111 \quad c \\ \hline & 10000001 \end{array}$$

As the XOR of these values $\neq 0$, we must show that this is an $\mathcal{N}$-position. The first player simply finds the most significant bit marked 1 in the XOR and chooses any component in which this bit is a 1. In this example, that component is a. He then makes an appropriate move in a that switches the most significant bit to 0, and sets all lower-order bits as needed to make the sum equal 0. Here the winning move is from a to $a' = 01101000$, changing just the first and last bits; $a' \oplus b \oplus c = 0$, so by induction it is a $\mathcal{P}$-position. □

Corollary 2.2. $*a + *b = *(a \oplus b)$.

Proof. We know $a \oplus b \oplus (a \oplus b) = 0$, so $*a + *b + *(a \oplus b) = 0$. □

Example (DAWSON'S KAYLES). Recall that in DAWSON'S KAYLES, a move consists of removing *exactly* two adjacent boxes. We defined DAWSON'S KAYLES as a misère-play game, but we can just as easily consider it in normal play. Denote by H_n a single strip of length n. Then the moves from H_n are to $H_a + H_{n-2-a}$, where $1 \le a \le n-2$.

We can use the Sprague–Grundy theorem and the NIM addition rule to compute normal-play values of H_n easily:

$$H_0 = \{\} = 0,$$

$$H_1 = \{\} = 0,$$

$$H_2 = \{H_0\} = \{0\} = *,$$

$$H_3 = \{H_1\} = \{0\} = *,$$

$$H_4 = \{H_1 + H_1,\ H_2 + H_0\} = \{0+0,\ *+0\} = \{0,\ *\} = *2,$$

$$H_5 = \{H_2 + H_1,\ H_3 + H_0\} = \{*+0,\ *+0\} = \{*,\ *\} = 0,$$

$$H_6 = \{H_2 + H_2,\ H_3 + H_1,\ H_4 + H_0\} = \{*+*,\ *+0,\ *2+0\} = \{0,\ *,\ *2\} = *3.$$

This rapidly becomes tedious, and it's easily implemented on a computer. The results of a computer calculation are shown in Figure 4. Each row represents a block of 34 NIM values: the first row shows $\mathcal{G}(H_0)$ through $\mathcal{G}(H_{33})$; the next row shows $\mathcal{G}(H_{34})$ through $\mathcal{G}(H_{67})$; etc. The number 34 was obviously not chosen by accident; after a few initial anomalies, a strong regularity quickly emerges with period 34. We now prove a theorem that shows, for a wide class of games,

	0										1										2										3			
	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3
0+	0	0	1	1	2	0	3	1	1	0	3	3	2	2	4	0	5	2	2	3	3	0	1	1	3	0	2	1	1	0	4	5	2	7
34+	4	0	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	2	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7
68+	4	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7
102+	4	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7
136+	4	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7
170+	4	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7

Figure 4. NIM values of DAWSON'S KAYLES in normal play.

that if such periodicity is observed for "sufficiently long" (in a sense to be made precise) then it must continue forever.

Octal games and octal codes.

Definition 2.3. An *octal code* is a sequence of digits $0.d_1d_2d_3\ldots$ where $0 \le d_i < 8$ for all i.

An octal code specifies the rules for a particular *octal game*. An octal game is played with strips of boxes, and the code describes how many boxes may be removed and under what circumstances. The digit d_k specifies the conditions under which k boxes may be removed.

Let us consider the bit representation of each d_k: denote $d_k = \epsilon_0 + \epsilon_1 \cdot 2 + \epsilon_2 \cdot 4$, where each $\epsilon_i = 0$ or 1.

- We can remove an entire strip of length k if and only if $\epsilon_0 = 1$.
- We can remove k boxes from the *end* of a strip (leaving at least one box) if and only if $\epsilon_1 = 1$.
- We can remove k boxes from the *middle* of a strip (leaving at least one box on each end) if and only if $\epsilon_2 = 1$.

Therefore,

- DAWSON'S KAYLES is represented by 0.07 as you have to remove exactly two blocks every time from anywhere in the strip, and you can remove an entire strip of length 2.
- KAYLES is represented by 0.77 as you can remove one or two boxes from a single strip.
- NIM is represented by the infinite sequence 0.3333333... as you are allowed to take any number of boxes from the end or to take an entire strip of any length (but you are not allowed to separate the original strip into two strips).

Guy–Smith periodicity theorem.

Theorem 2.4 (Guy–Smith periodicity theorem). *Consider an octal game with* finitely many *nonzero code digits*, *and let k be largest with $d_k \neq 0$. Denote by H_n a strip of length n. Suppose that for some $n_0 > 0$ and $p > 0$ we have*

$$\mathcal{G}(H_{n+p}) = \mathcal{G}(H_n) \text{ for every } n \text{ with } n_0 \le n < 2n_0 + p + k.$$

Then

$$\mathcal{G}(H_{n+p}) = \mathcal{G}(H_n) \text{ for all } n \ge n_0.$$

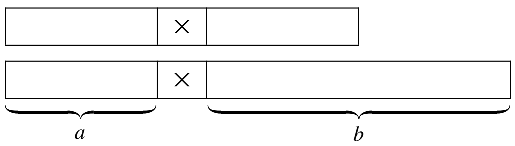
Proof. Note that a move from H_n is always to $H_a + H_b$, where $n - k \le a + b < n$. (In taking a whole strip, or from the end of the strip, we may take one or both of a, b to be 0.)

Now proceed by induction on n. The base case $n < 2n_0 + p + k$ is given by hypothesis, so assume $n \ge 2n_0 + p + k$. A move from H_{n+p} is to $H_a + H_b$ where $a + b \ge n + p - k$.

Since $n \ge 2n_0 + p + k$, we have $n + p - k \ge 2n_0 + 2p$, so without loss of generality $b \ge n_0 + p$. (Since the sum $a + b$ is greater than or equal to $2(n_0 + p)$, at least one of the elements must be at least $n_0 + p$.) By induction $\mathcal{G}(H_{b-p}) = \mathcal{G}(H_b)$, so

$$\mathcal{G}(H_a + H_{b-p}) = \mathcal{G}(H_a) \oplus \mathcal{G}(H_{b-p}) = \mathcal{G}(H_a) \oplus \mathcal{G}(H_b) = \mathcal{G}(H_a + H_b).$$

Here is the picture:



$$H_n \to H_a + H_{b-p}$$

$$H_{n+p} \to H_a + H_b$$

Now $H_a + H_{b-p}$ is an option of H_n, so we conclude that the options of H_{n+p} have exactly the same $\mathcal{G}$-values as those of H_n. Since the $\mathcal{G}$-values of H_{n+p} and H_n both observe the mex rule, we have

$$\mathcal{G}(H_{n+p}) = \mathcal{G}(H_n). \qquad \square$$

When p and n_0 are as small as possible, we say that Γ has (normal-play) *period* p and *preperiod* n_0.

Examples. In normal play:

- KAYLES (0.77) has period 12.
- DAWSON'S KAYLES (0.07) has period 34.
- 0.106 has period 328226140474. (See [4].)
- 0.007 is not known to be periodic.

Open Problem. Does there exist a finite octal code (i.e., an octal code with finitely many nonzero digits) that yields an aperiodic game?

***Misère* NIM.** We now consider NIM in misère play. It is not hard to show the following. If G consists of heaps of sizes $a_1, \ldots, a_k$, then

$$o^-(G) = \mathcal{P} \iff a_1 \oplus a_2 \oplus \cdots \oplus a_k = 0,$$

unless every a_i is equal to 0 or 1. In that case, $o^-(G) = \mathcal{P} \iff a_1 \oplus \cdots \oplus a_k = 1$.

So the strategy for misère NIM is: play exactly like in normal NIM, unless your move would leave only heaps of size 0 or 1. In that case, play to leave an odd number of heaps of size 1.

Misère equality. We now make the exact same definition of equality as before (see Definition 1.5), this time assuming misère play.

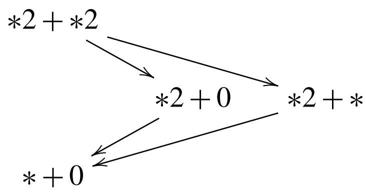
Definition 2.5. $G = H \iff o^-(G+X) = o^-(H+X)$ for all X.

Recall that in normal play any two $\mathcal{P}$-positions are equal (and in particular, any $\mathcal{P}$-position is equal to 0). We shall see that this is not the case in misère play.

In misère play:

- 0 is an $\mathcal{N}$-position.
- $*$ is a $\mathcal{P}$-position.
- $*2$ is an $\mathcal{N}$-position.

This we have already seen. Note that $*2 + *2$ is also a $\mathcal{P}$-position. No matter what first player does, second player can always respond by moving to $*$:



This immediately shows that $*2 + *2 \neq 0$, since $*2 + *2$ is a $\mathcal{P}$-position but 0 is an $\mathcal{N}$-position. In fact, we will now show that $*2 + *2 \neq *$, thus exhibiting two distinct $\mathcal{P}$-positions.

Proposition 2.6. $$* + * = 0.$$

Proof. Whoever can win X can also win $X + * + *$: he follows the winning strategy on X, and if his opponent ever moves on one copy of $*$, he responds by moving on the other. This guarantees that his opponent will make the last move on X, leaving either 0 or $* + *$. But both of these are $\mathcal{N}$-positions. □

Now $* + *2 + *2$ is an $\mathcal{N}$-position, since it has a move to $*2 + *2$. The following proposition therefore shows that $* \neq *2 + *2$.

Proposition 2.7. $*2 + *2 + *2 + *2$ *is a* $\mathcal{P}$*-position.*

Proof. The options are $*2 + *2 + *2 + 0$ and $*2 + *2 + *2 + *$. But these have moves to $*2 + *2 + 0 + 0$ and $*2 + *2 + * + *$, respectively. By the previous proposition, both of these are equal to $*2 + *2$, a $\mathcal{P}$-position. □

In fact, it is possible to show that $*2 + *2 \neq *m$ for any m. So even among sums of NIM-heaps, we have games that are not equivalent to any NIM-heap. This contrasts sharply with the situation in normal play, where *every* game is equivalent to a NIM-heap.

We have seen that $* + * = 0$. There are very few other identities we can establish in misère play. Here are really the only two:

Exercise (misère mex rule). Suppose $G \cong \{*a_1, \dots, *a_k\}$. Then $G = *m$, where

$$m = \text{mex}\{a_1, \dots, a_k\},$$

provided that at least one $a_i = 0$ or 1. (See Theorem 1.12.)

Exercise. For any m, we have $*m + * = *(m \oplus 1)$. (See Corollary 2.2.)

The misère mex rule is spectacularly false if every $a_i \geq 2$. For example, let

$$G = \{*2\},$$

the game whose only option is $*2$. (G is sometimes called $*2_{\#}$.) G is a $\mathcal{P}$-position, so right away we have $G \neq 0$. As an exercise, show that G is not equal to any $*m$. In fact, it is possible to show that G is not equal to any *sum* of NIM-heaps, but we won't do that in this course.

Birthdays. Clearly, things are more complicated in misère play than in normal play. We now state a result that shows just how much worse they are.

Definition 2.8. The *birthday* of a game G is the height of its game tree.

In *normal* play there are just six games with birthday ≤ 5 (modulo equality): $0, *, *2, *3, *4$, and $*5$. In misère play, there are 4171780. On day 6 there are more than $2^{4171779}$.

The theory of misère games modulo $=$ is beautiful and fascinating, but these results suggest that it is not terribly useful: we very quickly run into seemingly intractable complications. We will not say much more about this "global theory" in this course; the interested reader is referred to [3].

Misère quotients. If $G = H$, then $G + X$ and $H + X$ have the same outcomes, for any game X. As we've just observed, this equality relation gives rise to a virtually intractable theory. The problem is that $G = H$ is too strong a relation — we are requiring that G and H behave identically in any context, which is asking a bit too much.

Key Idea. Suppose we just want to know how to play KAYLES (for example). We just need to specify how a KAYLES position G interacts with other positions that actually occur in KAYLES.

With this in mind, fix a set $\mathscr{A}$ of games (usually, $\mathscr{A}$ will be the set of positions that occur in some octal game). Assume that $\mathscr{A}$ is closed under addition.

Definition 2.9. Let $\mathscr{A}$ be a set of games, closed under addition. Then for $G, H \in \mathscr{A}$,

$$G \equiv_{\mathscr{A}} H \iff o^-(G + X) = o^-(H + X) \text{ for all } X \in \mathscr{A}.$$

Compare this to Definition 2.5: we are restricting the domain of games that can be used to distinguish G from H. This coarsens the equivalence and allows us to recover a tractable theory. Very often, the set of equivalence classes modulo $\equiv_{\mathscr{A}}$ is finite, even when $\mathscr{A}$ is infinite. (It is trivial to see that $\equiv_{\mathscr{A}}$ is an equivalence relation, since outcome-equality is an equivalence relation.)

Now, think of normal-play NIM values as elements of the group

$$\mathcal{D} = \bigoplus_{\mathbb{N}} \mathbb{Z}_2,$$

a (countably) infinite direct sum of copies of $\mathbb{Z}_2$ (one for each binary digit). The Sprague–Grundy theory maps each game G to an element of $\mathcal{D}$, thus representing the normal-play structure of G in terms of the group structure of $\mathcal{D}$. We will show that the equivalence classes modulo $\equiv_{\mathscr{A}}$ function as a *localized* misère analogue of the Sprague–Grundy theory.

We will make a slightly stronger assumption on $\mathscr{A}$ than closure under addition.

Definition 2.10. A set of games $\mathscr{A}$ is *hereditarily closed* if, for any $G \in \mathscr{A}$ and any option G' of G, we also have $G' \in \mathscr{A}$.

Definition 2.11. $\mathscr{A}$ is *closed* if it is both hereditarily closed and closed under addition.

Note that if $\mathscr{A}$ is the set of positions that occur in an octal game, then $\mathscr{A}$ is closed. In fact, virtually all sets of games that are interesting to us are closed, so there is little harm in making this assumption.

Example. Let $\mathscr{A} = \{\text{all sums of } * \text{ and } *2\}$, that is,

$$\mathscr{A} = \{m \cdot * + n \cdot *2 : m, n \in \mathbb{N}\}.$$

Let's compute the equivalence classes modulo $\equiv_{\mathscr{A}}$:

- $* \not\equiv_{\mathscr{A}} 0$, since $*$ is a $\mathscr{P}$-position and 0 is an $\mathscr{N}$-position.
- Likewise, $*2 \not\equiv_{\mathscr{A}} *$ since $*2$ is an $\mathscr{N}$-position. Further, $*2 \not\equiv_{\mathscr{A}} 0$: let $X = *2$; then $*2 + X = *2 + *2$ is a $\mathscr{P}$-position, but $0 + X = *2$ is an $\mathscr{N}$-position.
- Finally, $*2 + * \not\equiv_{\mathscr{A}} *$ since it's an $\mathscr{N}$-position; $*2 + * \not\equiv_{\mathscr{A}} 0$, since they're distinguished by $X = *$; and $*2 + * \not\equiv_{\mathscr{A}} *2$, since they're distinguished by $X = *2$.

This gives four equivalence classes:

$$\begin{array}{cccc} [0] & [*] & [*2] & [*2+*] \\ \mathscr{N} & \mathscr{P} & \mathscr{N} & \mathscr{N} \end{array}$$

Are there others? Yes! $*2 + *2$ is a $\mathscr{P}$-position, so it's either equivalent to $*$, or a new equivalence class. But,

- $* + (*2 + *2)$ is an $\mathscr{N}$-position, since it has a move to $*2 + *2$, which is $\mathscr{P}$;
- $*2 + *2 + (*2 + *2)$ is a $\mathscr{P}$-position (Proposition 2.7).

Therefore $* \not\equiv_{\mathscr{A}} *2 + *2$. Similar reasoning shows that $*2 + *2 + *$ gives yet another equivalence class.

So we have six equivalence classes total:

$$\begin{array}{cccccc} [0] & [*] & [*2] & [*2+*] & [*2+*2] & [*2+*2+*] \\ \mathscr{N} & \mathscr{P} & \mathscr{N} & \mathscr{N} & \mathscr{P} & \mathscr{N} \end{array}$$

We now show that these are the only six.

Lemma 2.12. *Let $n \geq 1$. Then $n \cdot *2$ is a $\mathscr{P}$-position if and only if n is even.*

Proof. If n is even, then second player's strategy is to cancel out copies of $*2$ (using the fact that $* + * = 0$) until we get down to $*2 + *2$, which is known to be a $\mathscr{P}$-position.

If n is odd, $n \geq 3$, then first player can win by moving to $(n-1) \cdot *2$.

Finally, if $n = 1$, then first player simply moves to $*$. □

Lemma 2.13. *Let $n \geq 1$. Then $n \cdot *2 + *$ is always an $\mathscr{N}$-position.*

Proof. If n is even, then the winning move is to $n \cdot *2$, which is a $\mathscr{P}$-position by the previous lemma.

If n is odd, $n \geq 3$, then the winning move is to $(n-1) \cdot *2 + * + *$, which again is a $\mathscr{P}$-position, since $* + * = 0$.

Finally, if $n = 1$, then the winning move is to $0 + *$. □

Corollary 2.14. *Suppose $G = m \cdot * + n \cdot *2$ and $X = m' \cdot * + n' \cdot *2$. If $n \geq 1$, then the outcome of $G + X$ depends only on the parities of $m + m'$ and $n + n'$.*

Proof. Follows immediately from the previous two lemmas and the fact that $* + * = 0$. □

Corollary 2.15. *Let* $G = m \cdot * + n \cdot *2$ *and* $H = m' \cdot * + n' \cdot *2$*. If* $n, n' \geq 1$, $m \equiv m' \pmod 2$, *and* $n \equiv n' \pmod 2$, *then* $G \equiv_{\mathscr{A}} H$.

Proof. Follows immediately from the previous corollary. □

Corollary 2.16. *There are exactly six equivalence classes modulo* $\equiv_{\mathscr{A}}$.

Proof. By the previous corollary, every $G \in \mathscr{A}$ is equivalent to $m \cdot * + n \cdot *2$, for some $m < 2$ and $n < 3$. There are only six such possibilities, and we've already shown that all six are mutually inequivalent. □

Warning. We've just shown that $*2 + *2 + *2 \equiv_{\mathscr{A}} *2$. However, equality does *not* hold:

Exercise. Show that $*2 + *2 + *2 \neq *2$. (Hint: try $X = *2_{\#}1$, defined by $*2_{\#}1 = \{*2_{\#}, *\} = \{\{*2\}, *\}$.)

This shows that the equivalence $\equiv_{\mathscr{A}}$ is a genuine coarsening of equality. There exist unequal games that are equivalent modulo $\mathscr{A}$.

This finishes our example. We now return to the general context.

Lemma 2.17. *Let* $\mathscr{A}$ *be any closed set of games and* $G, H \in \mathscr{A}$*. If* $G \equiv_{\mathscr{A}} H$ *and* $K \in \mathscr{A}$*, then* $G + K \equiv_{\mathscr{A}} H + K$.

Proof. For $X \in \mathscr{A}$, we have

$$o^-((G+K)+X) = o^-(G+(K+X)) \text{ and } o^-(H+(K+X)) = o^-((H+K)+X).$$

But $\mathscr{A}$ is closed, so $K + X \in \mathscr{A}$. Since $G \equiv_{\mathscr{A}} H$, we have $o^-(G + (K + X)) = o^-(H + (K + X))$, as needed. □

Moreover, since $\mathscr{A}$ is hereditarily closed, we have $0 \in \mathscr{A}$. So the equivalence class of 0 is an identity, and in fact we have a monoid.

Definition 2.18. A *semigroup* is a set S equipped with an associative binary operation $\cdot$. That is,

- if $x, y \in S$, then $x \cdot y \in S$;
- if $x, y, z \in S$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

A semigroup S is a *monoid* if it has an identity, and *commutative* if its operation is commutative.

We've shown that the equivalence classes of $\mathscr{A}$ modulo $\equiv_{\mathscr{A}}$ form a commutative monoid $\mathbb{Q}$:

$$\mathbb{Q} = \{[G]_{\equiv_{\mathscr{A}}} : G \in \mathscr{A}\}.$$

Furthermore, if $G \equiv_{\mathscr{A}} H$, then since $o^-(G+0) = o^-(H+0)$, we have

$$G \text{ is a } \mathscr{P}\text{-position} \Longleftrightarrow H \text{ is a } \mathscr{P}\text{-position.}$$

So we can define a subset $\mathcal{P} \subset \mathbb{Q}$ by

$$\mathcal{P} = \{[G]_{\equiv_{\mathscr{A}}} : G \in \mathscr{A} \text{ is a } \mathscr{P}\text{-position}\}.$$

Definition 2.19. The structure $(\mathbb{Q}, \mathcal{P})$ is the *misère quotient* of $\mathscr{A}$, and we denote it by $\mathbb{Q}(\mathscr{A})$.

We'll continue to use uppercase letters $G, H, \ldots$ to denote games in the set $\mathscr{A}$, and lowercase letters $a, b, \ldots$ to denote elements of the quotient $\mathbb{Q}(\mathscr{A})$. Likewise, disjunctive sums of games will always be written additively (for example, $G + 2 \cdot H$), whereas the quotient operation will always be written multiplicatively (for example, ab^2).

Example. Let's sketch the structure of $\mathbb{Q}(\mathscr{A})$ for our example:

$$\mathscr{A} = \{\text{sums of } * \text{ and } *2\}.$$

Denote by $\Phi : \mathscr{A} \to \mathbb{Q}$ the quotient map

$$\Phi(G) = [G]_{\equiv_{\mathscr{A}}}.$$

Now $\mathscr{A}$ is generated (as a monoid) by $*$ and $*2$. Put

$$1 = \Phi(0) = [0], \quad a = \Phi(*) = [*], \quad b = \Phi(*2) = [*2].$$

We know that $* + * = 0$, so in fact $a^2 = 1$. Furthermore, we've seen that $*2 + *2 + *2 \equiv_{\mathscr{A}} *2$, so we have $b^3 = b$. But we also know that the six elements

$$\begin{array}{lllllll} \mathscr{A} & [0] & [*] & [*2] & [*2+*] & [*2+*2] & [*2+*2+*] \\ \downarrow & & & & & & \\ \mathbb{Q} & 1 & a & b & ab & b^2 & ab^2 \end{array}$$

are all distinct. Thus $\mathbb{Q} = \{1, a, b, ab, b^2, ab^2\}$ and we have the presentation

$$\mathbb{Q} \cong \langle a, b \mid a^2 = 1,\ b^3 = b \rangle.$$

Since $*$ and $*2 + *2$ are the only $\mathscr{P}$-positions (up to equivalence), we also have $\mathcal{P} = \{a, b^2\}$. This misère quotient is called $\mathcal{T}_2$, and it is the first of many that we will see.

Lecture 3. The periodicity theorem

November 28, 2006 Scribes: Amir Kantor and Gideon Amir

Definition 3.1 (Definitions). Let $\mathscr{A}$ be any set of games. Define

$$\mathrm{hcl}(\mathscr{A}) = \{\text{subpositions of all games in } \mathscr{A}\},$$
$$\mathrm{cl}(\mathscr{A}) = \text{Closure under addition of } \mathrm{hcl}(\mathscr{A}).$$

Remark. To see that $\mathrm{cl}(\mathscr{A})$ is hereditarily closed, let $G = G_1 + G_2 + \cdots + G_k$, where $G_i \in \mathrm{hcl}(\mathscr{A})$. Without loss of generality, $G' = G_1' + G_2 + \cdots + G_k$. We know that $G_1' \in \mathrm{hcl}(\mathscr{A})$ since the latter is hereditarily closed.

Example. $\mathrm{cl}(\{*2\}) = \{\text{sums of } *, *2\} = \{i \cdot * + j \cdot *2 : i, j \in \mathbb{N}\}$.

Exercise.
- If $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B}$ is closed, then $\mathrm{cl}(\mathscr{A}) \subseteq \mathscr{B}$.
- $\mathrm{cl}(\mathrm{cl}(\mathscr{A})) = \mathrm{cl}(\mathscr{A})$.

Definition 3.2. If $\mathscr{A}$ is not closed, $\mathbb{Q}(\mathscr{A}) \triangleq \mathbb{Q}(\mathrm{cl}(\mathscr{A}))$. We sometimes write $\mathbb{Q}(G) \triangleq \mathbb{Q}\,(\mathrm{cl}\,(\{G\}))$.

Example. $$\mathcal{T}_2 \cong \mathbb{Q}(*2).$$

Quotients of octal games. Let's consider the context of a specific octal game, such as KAYLES. Denote by H_n a KAYLES heap of size n and let $\mathscr{A}$ be the set of all KAYLES positions; that is,

$$\mathscr{A} = \mathrm{cl}(H_0, H_1, H_2, H_3, \ldots).$$

Let $(\mathbb{Q}, \mathcal{P})$ be the misère quotient for KAYLES and consider the quotient map $\Phi : \mathscr{A} \to \mathbb{Q}$.

Remark. If we know $\Phi(H_n)$ for all n, then if $G = H_{n_1} + \cdots + H_{n_k}$ we can easily compute $\Phi(G) = \Phi(H_{n_1}) \cdots \Phi(H_{n_k})$. So, in order to specify Φ, it suffices to specify the single-heap values $\Phi(H_n)$.

The *main point* is this:

> Suppose we know $\mathbb{Q}(\mathscr{A})$, together with $\Phi(H_n)$ for all n. If we want to know $o^-(G)$ for $G \in \mathscr{A}$, we can write $G = H_{n_1} + \cdots + H_{n_k}$, compute $\Phi(G) = \Phi(H_{n_1}) \cdots \Phi(H_{n_k})$, and simply look up whether $\Phi(G) \in \mathcal{P}$. If the quotient is finite, we've reduced the problem of finding $o^-(G)$ to a small number of operations on a finite multiplication table. This yields an efficient way to compute $o^-(G)$.

So we direct our energies at computing the values of $\Phi(H_n)$ for all n. In practice, we can construct good algorithms for computing quotients of a finite number of heaps. (We won't have time to discuss them in this course; see [13,

Appendix C].) If we run these algorithms on KAYLES to heap 120, we get the result shown in Figure 5.

Now examine the Φ-values $\Phi(H_n) \in \mathbb{Q}$. We observe that

$$\Phi(H_{n+12}) = \Phi(H_n), \text{ for } 71 \leqslant n \leqslant 120 - 12.$$

This situation is much like the periodicity of $\mathcal{G}$-values that we observed in normal play.

The following notation will be very useful; it applies to KAYLES as well as to an arbitrary octal game Γ. Denote by

- $\mathscr{A}$ the set of all positions, $\mathscr{A} = \mathrm{cl}(H_0, H_1, \ldots)$;
- $\mathbb{Q}(\Gamma) = \mathbb{Q}(\mathscr{A})$;
- $\mathscr{A}_n$ the set of all positions with no heap larger than n, $\mathscr{A}_n = \mathrm{cl}(H_0, \ldots, H_n)$;
- $\mathbb{Q}_n(\Gamma) = \mathbb{Q}(\mathscr{A}_n)$, the n-th *partial quotient* for Γ.

$$\begin{aligned}\mathbb{Q}(H_0, H_1, H_2, \ldots, H_{120})&\\ \cong \langle a, b, c, d, e, f, g \mid{}& a^2 = 1,\ b^3 = b,\ bc^2 = b,\ c^3 = c,\ bd = bc,\\ & cd = b^2,\ d^3 = d,\ be = bc,\ ce = b^2,\ e^2 = de,\\ & bf = ab,\ cf = ab^2c,\ d^2f = f,\ f^2 = b^2,\\ & b^2g = g,\ c^2g = g,\ dg = cg,\\ & eg = cg,\ fg = ag,\ g^2 = b^2\rangle,\end{aligned}$$

$$\mathcal{P} = \{a, b^2, ac, ac^2, d, ad^2, e, ade, adf\}.$$

	1	2	3	4	5	6	7	8	9	10	11	12
0+	a	b	ab	a	c	ab	b	ab^2	d	b	bc	e
12+	ab^2	b	abc	ab^2	d^2e	ab	b	ade	b^2c	bc	abc	b^2c
24+	f	b	g	ab^2c	b^2c	abc	b	ab^2	g	bc	abc	b^2c
36+	ab^2	b	ab	ab^2	b^2c	abc	b	ab^2	g	b	abc	b^2c
48+	ab^2	b	g	ab^2	b^2c	abc	b	ab^2	b^2c	b	abc	b^2c
60+	ab^2	b	g	ab^2	b^2c	abc	b	ab^2	g	bc	abc	b^2c
72+	ab^2	b	g	ab^2	b^2c	abc	b	ab^2	g	b	abc	b^2c
84+	ab^2	b	g	ab^2	b^2c	abc	b	ab^2	g	b	abc	b^2c
96+	ab^2	b	g	ab^2	b^2c	abc	b	ab^2	g	b	abc	b^2c
108+	ab^2	b	g	ab^2	b^2c	abc	b	ab^2	g	b	abc	b^2c

Figure 5. Quotient presentation and pretending function for misère KAYLES to heap 120.

We've computed $\mathbb{Q}_{120}(\text{KAYLES})$, and the quotient map $\Phi_{120} : \mathscr{A}_{120} \longrightarrow \mathbb{Q}_{120}$, and found that it is periodic past a certain point.

A brief digression. In a moment we will state a misère version of the periodicity theorem. We first pause to consider some potential difficulties.

Remark. Suppose we computed $\mathbb{Q}_n$. Now we throw H_{n+1} into the quotient. There might be games $G, K \in \mathscr{A}_n$ such that $G \equiv_{\mathscr{A}_n} K$ but are distinguished by H_{n+1}. When this happens, we have $\Phi_n(G) = \Phi_n(K)$, but $\Phi_{n+1}(G) \neq \Phi_{n+1}(K)$.

This remark shows that we must be careful not to confuse the partial quotients of Γ with its full quotient.

Note that in normal play, there is no such concern. Given a set of games $\mathscr{A}$, it is possible to define *normal equivalence modulo* $\mathscr{A}$ in exactly the same way we've defined misère equivalence modulo $\mathscr{A}$. However, in normal play it will always be the case that $G \equiv_{\mathscr{A}} K$ if and only if $G = K$. That is, in normal play, local and global equivalence coincide. (To see this, observe that if $G \neq K$ in normal play, then G and K must have different NIM values, so $G+G$ and $G+K$ have different outcomes. So if G and K are distinguished by anything, then they must be distinguished locally, by G itself.) So, although the sorts of localizations we're discussing are perfectly applicable to normal play, they don't provide any further resolution (and in a sense, they don't need to, because normal play is simple enough to begin with).

Let us consider another difference between normal play and misère play. Consider a finitely generated set $\mathscr{A}$. In *normal play*, there can be only finitely many $\mathscr{G}$-values represented. To see this, let $H_1, \ldots, H_n$ generate $\mathscr{A}$. Then the $\mathscr{G}$-values represented by $\mathscr{A}$ are bitwise exclusive-or's of $\mathscr{G}(H_1), \ldots, \mathscr{G}(H_n)$, but these are bounded.

What about misère play? Is $\mathbb{Q}(\mathscr{A})$ finite? The answer is: not in general. Later in this course we will see an example of an infinite, finitely generated quotient. Our picture of such quotients is still very hazy. In fact, the following question is still open.

Open Problem. Specify an algorithm to determine whether $\mathbb{Q}(\mathscr{A})$ is infinite, assuming $\mathscr{A}$ is finitely generated.

We'll say more about this later in the course. Finally, now is as good a time as any to interject the following remark:

Remark. All monoids we consider in this course are commutative. Sometimes I will slip and say "monoid" when I really mean "commutative monoid."

Periodicity. We now return to the setting of an octal game Γ with heaps H_n.

Recall the periodicity theorem for normal play:

Let Γ be an octal game with last nonzero code digit k. Suppose there are integers n_0, p such that $\mathscr{G}(H_{n+p}) = \mathscr{G}(H_n)$ for $n_0 \le n < 2n_0 + p + k$. Then in fact

$$\mathscr{G}(H_{n+p}) = \mathscr{G}(H_n) \text{ for all } n \ge n_0.$$

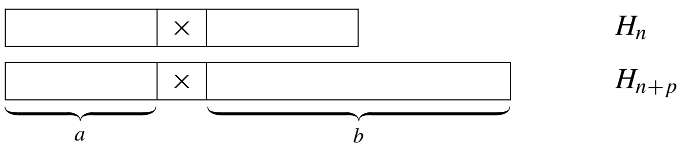
Theorem 3.3 (periodicity theorem for misère play). *Let Γ be an octal game with last nonzero code digit k. Fix n_0, p and let $M = 2n_0 + 2p + k$. Let $(\mathbb{Q}_M, \mathcal{P}_M) = \mathbb{Q}_M(\Gamma)$. Suppose that $\Phi_M : A_M \longrightarrow \mathbb{Q}_M$, and that $\Phi_M(H_{n+p}) = \Phi_M(H_n)$ for $n_0 \le n < 2n_0 + p + k$. Then in fact*

$$\mathbb{Q}(\Gamma) \cong \mathbb{Q}_M(\Gamma),$$

and

$$\Phi(H_{n+p}) = \Phi(H_n) \textit{ for all } n \ge n_0.$$

Proof. Recall the proof for normal play. By induction on n,



$H_{n+p} \longrightarrow H_a + H_b$ is a typical move from H_{n+p}. We chose the upper bound of our induction base case to be large enough that one of $a, b \ge n_0 + p$. Assume without loss of generality that it's b. But then $\mathscr{G}(H_{b-p}) = \mathscr{G}(H_b)$, so $\mathscr{G}(H_a + H_b) = \mathscr{G}(H_a + H_{b-p})$. We conclude that the options of H_n and H_{n+p} represent exactly the same $\mathscr{G}$-values. But $\mathscr{G}$-values are computed by the mex rule, so this implies $\mathscr{G}(H_{n+p}) = \mathscr{G}(H_n)$.

To prove the periodicity theorem for misère play, we can use exactly the same argument to show that the options of H_n, H_{n+p} represent exactly the same Φ_M-values. So the proof now depends only on the following lemma.

Lemma 3.4. *Suppose $\mathscr{A}$ is a closed set of games, and G is a game all of whose options are in $\mathscr{A}$. Assume that, for some $H \in \mathscr{A}$,*

$$\{\Phi(G') : G' \textit{ is an option of } G\} = \{\Phi(H') : H' \textit{ is an option of } H\}.$$

Then $\mathbb{Q}(\mathscr{A} \cup \{G\}) \cong \mathbb{Q}(\mathscr{A})$ and $\Phi(G) = \Phi(H)$.

Assuming Lemma 3.4, the proof of the periodicity theorem is complete, for we can go by induction to show that

$$Q_M(\Gamma) \cong Q_{M+1}(\Gamma) \cong Q_{M+2}(\Gamma) \cong \dots,$$

and that the resulting Φ-values are periodic. □

Bipartite monoids. Although we could prove Lemma 3.4 directly, it will be easier after we introduce a suitable abstraction of the misère quotient construction. Since the abstract setting is also useful in other situations, this is worth the effort.

Definition 3.5. A *bipartite monoid* is a pair $(\mathbb{Q}, \mathcal{P})$ where $\mathbb{Q}$ is a commutative monoid, and $\mathcal{P} \subset \mathbb{Q}$ is some subset. We will usually write bm for bipartite monoid.

Definition 3.6. Let $(\mathbb{Q}, \mathcal{P})$ be a bm; $x, y \in \mathbb{Q}$ are said to be *indistinguishable* if, for all $z \in \mathbb{Q}$,

$$xz \in \mathcal{P} \Longleftrightarrow yz \in \mathcal{P}.$$

Definition 3.7. A bm $(\mathbb{Q}, \mathcal{P})$ is *reduced* if the elements of $\mathbb{Q}$ are pairwise distinguishable. We write rbm for reduced bipartite monoid.

Proposition 3.8. *Every misère quotient is a rbm.*

Proof. Suppose $[G]_{\equiv_{\mathscr{A}}}$ and $[H]_{\equiv_{\mathscr{A}}}$ are indistinguishable. Then for any $X \in \mathscr{A}$,

$$[G] + [X] \in \mathcal{P} \Longleftrightarrow [H] + [X] \in \mathcal{P}.$$

Therefore $o^-(G + X) = o^-(H + X)$ for all $X \in \mathscr{A}$, so $[G] = [H]$. □

Example. If $\mathscr{A}$ is a closed set of games, and $\mathscr{B}$ is the set of misère $\mathscr{P}$-positions of $\mathscr{A}$, then $(\mathscr{A}, \mathscr{B})$ is a bipartite monoid. The same is true if we take $\mathscr{B}$ to be the set of normal $\mathscr{P}$-positions of $\mathscr{A}$.

Definition 3.9. A function $f : (\mathbb{Q}, \mathcal{P}) \to (\mathcal{S}, \mathcal{R})$ is a *bipartite monoid homomorphism* if $f : \mathbb{Q} \to \mathcal{S}$ is a monoid homomorphism, and for every $x \in \mathbb{Q}$, we have $x \in \mathcal{P}$ if and only if $f(x) \in \mathcal{R}$.

Definition 3.10. Let $(\mathbb{Q}, \mathcal{P})$ and $(\mathcal{S}, \mathcal{R})$ be bipartite monoids. $(\mathcal{S}, \mathcal{R})$ is a *quotient* of $(\mathbb{Q}, \mathcal{P})$ if and only if there is a surjective homomorphism $f : (\mathbb{Q}, \mathcal{P}) \to (\mathcal{S}, \mathcal{R})$.

Definition 3.11. Let $(\mathbb{Q}, \mathcal{P})$ be a bm. Define a relation ρ on $\mathbb{Q}$ by $x \rho y$ if and only if x and y are indistinguishable.

Exercise. Show that ρ is an equivalence relation, and that the equivalence classes modulo ρ form a bipartite monoid.

Definition 3.12. The *reduction* of $(\mathbb{Q}, \mathcal{P})$ is the bipartite monoid of equivalence classes modulo ρ. We denote it by $(\mathbb{Q}', \mathcal{P}')$.

Exercise. Show that $(\mathbb{Q}', \mathcal{P}')$ is reduced and is a quotient of $(\mathbb{Q}, \mathcal{P})$.

Example. Let $\mathscr{A}$ be a closed set of games, and let $\mathscr{B}$ be the set of misère $\mathscr{P}$-positions in $\mathscr{A}$. Then the misère quotient $\mathbb{Q}(\mathscr{A})$ is the reduction of $(\mathscr{A}, \mathscr{B})$.

The following proposition is extremely useful.

Proposition 3.13. *Suppose* $(\mathbb{Q}, \mathcal{P})$ *is a bm with reduction* $(\mathbb{Q}', \mathcal{P}')$. *Let* $(\mathcal{S}, \mathcal{R})$ *be any quotient of* $(\mathbb{Q}, \mathcal{P})$, *via* $f : (\mathbb{Q}, \mathcal{P}) \to (\mathcal{S}, \mathcal{R})$, *and let* $(\mathcal{S}', \mathcal{R}')$ *be its reduction. Then there is an isomorphism* $i : (\mathbb{Q}', \mathcal{P}') \to (\mathcal{S}', \mathcal{R}')$ *making the following diagram commute*:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\ f\ } & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathbb{Q}' & \overset{i}{=\!=\!=} & \mathcal{S}' \end{array}$$

Proof. Let ρ be the reduction relation on $\mathbb{Q}$ ($\mathbb{Q}' = \mathbb{Q}/\rho$), and let τ be the reduction relation on $\mathcal{S}$ ($\mathcal{S}' = \mathcal{S}/\tau$).

Now for $x, y \in \mathbb{Q}$, we have

$$\begin{aligned} [x]_\rho = [y]_\rho & \\ &\Longleftrightarrow xz \in \mathcal{P} \Leftrightarrow yz \in \mathcal{P} \text{ for all } z \in \mathbb{Q}; \\ &\Longleftrightarrow f(xz) \in \mathcal{R} \Leftrightarrow f(yz) \in \mathcal{R} \text{ for all } z \in \mathbb{Q}; \\ &\Longleftrightarrow f(x)w \in \mathcal{R} \Leftrightarrow f(y)w \in \mathcal{R} \text{ for all } w \in \mathcal{S} \text{ (since } f \text{ is surjective);} \\ &\Longleftrightarrow [f(x)]_\tau = [f(y)]_\tau. \end{aligned}$$

So we may define the map i by $i([x]_\rho) = [f(x)]_\tau$. We just showed that i is well defined and one-to-one. Since f is surjective, so is i, and it follows that i is an isomorphism. Commutativity of the diagram follows trivially from the definition of i. □

Corollary 3.14. *Every bipartite monoid has exactly one reduced quotient* (*up to isomorphism*).

Let us see why this is important. Let $\mathscr{A}$ be a closed set of games, and $\mathscr{B}$ the set of misère $\mathscr{P}$-positions in $\mathscr{A}$. Then the misère quotient $\mathbb{Q}(\mathscr{A})$ is the reduction of $(\mathscr{A}, \mathscr{B})$. Therefore, suppose we have some putative quotient $(\mathbb{Q}, \mathcal{P})$, and we want to assert that it is $\mathbb{Q}(\mathscr{A})$. We just need to show that

(a) $(\mathbb{Q}, \mathcal{P})$ is reduced; and

(b) $(\mathbb{Q}, \mathcal{P})$ is a quotient of $(\mathscr{A}, \mathscr{B})$.

By Proposition 3.13, these conditions imply that $(\mathbb{Q}, \mathcal{P}) \cong \mathbb{Q}(\mathscr{A})$. We can therefore avoid the exhaustive analysis used to construct $\mathcal{T}_2$ during the previous lecture.

Lecture 4. More examples

November 29, 2006 Scribes: Shai Lubliner and Ohad Manor

Proof of Lemma 3.4. We now prove Lemma 3.4, thus completing the proof of the periodicity theorem.

Definition 4.1. Suppose $\mathscr{A}$ is a set of games, and G is a game all of whose options are in $\mathscr{A}$. Define

$$\Phi''G = \{\Phi(G') : G' \text{ is an option of } G\}.$$

(This definition includes the case when $G \in \mathscr{A}$.)

Lemma 4.2. *Suppose $\mathscr{A}$ is a closed set of games and $(\mathbb{Q}, \mathcal{P})$ a rbm. The following are equivalent*:

(i) $(\mathbb{Q}, \mathcal{P}) \cong \mathbb{Q}(\mathscr{A})$;

(ii) *there exists a surjective monoid homomorphism $\Phi : \mathscr{A} \to \mathbb{Q}$, such that for all $G \in \mathscr{A}$,*

$$\Phi(G) \in \mathcal{P} \iff G \neq 0 \textit{ and } \Phi(G') \notin \mathcal{P} \textit{ for every option } G' \textit{ of } G.$$

Proof. (i) $\Rightarrow$ (ii): Let Φ be the quotient map $\mathscr{A} \to \mathbb{Q}(\mathscr{A})$. We know that, for all G,

$$G \text{ is a } \mathscr{P}\text{-position} \iff G \neq 0 \text{ and every } G' \text{ is an } \mathscr{N}\text{-position.}$$

But since Φ is a homomorphism of bipartite monoids, we have for all $X \in \mathscr{A}$,

$$X \text{ is a } \mathscr{P}\text{-position} \iff \Phi(X) \in \mathcal{P},$$

and the conclusion follows immediately.

(ii) $\Rightarrow$ (i): By Corollary 3.14, $\mathbb{Q}(\mathscr{A})$ is the unique reduced quotient of $(\mathscr{A}, \mathscr{B})$ (where $\mathscr{B}$ is the set of $\mathscr{P}$-positions in $\mathscr{A}$). Thus it suffices to show that Φ is a homomorphism of bipartite monoids, since this implies that $(\mathbb{Q}, \mathcal{P})$ is a quotient of $(\mathscr{A}, \mathscr{B})$. So we must prove the following, for all $G \in \mathscr{A}$:

$$G \text{ is a } \mathscr{P}\text{-position} \iff \Phi(G) \in \mathcal{P}.$$

Now by induction on G (i.e., on the height of the game tree of G), we may assume that

$$G' \text{ is a } \mathscr{P}\text{-position} \iff \Phi(G') \in \mathcal{P},$$

for all options G' of G. But now,

$$\begin{aligned}\Phi(G) \in \mathcal{P} &\iff G \neq 0 \text{ and } \Phi(G') \notin \mathcal{P} \text{ for all } G' \quad \text{(by assumption)}\\ &\iff G \neq 0 \text{ and every } G' \text{ is an } \mathscr{N}\text{-position} \quad \text{(by induction)}\\ &\iff G \text{ is a } \mathscr{P}\text{-position} \quad \text{(by definition of } \mathscr{P}\text{-position).}\end{aligned}$$

This proves the lemma. □

Proof of Lemma 3.4. Assume $\mathscr{A}$ is a closed set of games, all options of G are in $\mathscr{A}$ and $\Phi''G = \Phi''H$ for some $H \in \mathscr{A}$. We must show that $\mathbb{Q}(\mathscr{A} \cup \{G\}) \cong \mathbb{Q}(\mathscr{A})$ and $\Phi(G) = \Phi(H)$.

Define $\Phi^+ : \text{cl}(\mathscr{A} \cup \{G\}) \to \mathbb{Q}$ by

- $\Phi^+(G) = \Phi(H)$;
- $\Phi^+(Y) = \Phi(Y)$ for all $Y \in \mathscr{A}$.

If we regard G as a free generator of the monoid $\mathrm{cl}(\mathscr{A} \cup \{G\})$ over $\mathscr{A}$, then this defines a monoid homomorphism. So we just need to show that Φ^+ satisfies condition (ii) of Lemma 4.2.

Fix $X \in \mathrm{cl}(\mathscr{A} \cup \{G\})$. We can write $X = n \cdot G + Y$ for some $n \geq 0$ and $Y \in \mathscr{A}$. The case $n = 0$ is already known, so we can assume $n \geq 1$. Let $W = n \cdot H + Y$; clearly $\Phi^+(X) = \Phi^+(W)$.

Now consider an option X' of X. We have $X' = n \cdot G + Y'$ or $(n-1) \cdot G + G' + Y$:

- If $X' = n \cdot G + Y'$, then $\Phi^+(X') = \Phi^+(n \cdot H + Y')$, which is an option of W.
- If $X' = (n-1) \cdot G + G' + Y$, then $\Phi^+(X') = \Phi^+((n-1) \cdot H + G' + Y)$. But since $\Phi'' G = \Phi'' H$, there must be some H' with $\Phi^+(H') = \Phi^+(G')$. So $\Phi^+(X') = \Phi^+((n-1) \cdot H + H' + Y)$, again an option of W'.

This shows that $(\Phi^+)'' X \subset (\Phi^+)'' W$, and an identical argument shows that $(\Phi^+)'' W \subset (\Phi^+)'' X$. But since $W \in \mathscr{A}$, we know that

$$\Phi(W) \in \mathcal{P} \iff W \neq 0 \quad \text{and} \quad \Phi(W') \notin \mathcal{P} \text{ for all } W'.$$

Since $\Phi^+(X) = \Phi^+(W)$ and $(\Phi^+)'' X = (\Phi^+)'' W$, we have

$$\Phi^+(X) \in \mathcal{P} \iff W \neq 0 \quad \text{and} \quad \Phi^+(X') \notin \mathcal{P} \text{ for all } X'.$$

This satisfies Lemma 4.2(ii) except for the condition $W \neq 0$. But if either of G, H is identically 0, then both must be, since $\Phi'' G = \varnothing$ if and only if $\Phi'' H = \varnothing$. Therefore $W \neq 0$ if and only if $X \neq 0$, and we are done. ☐

Further examples. The partial quotients of NIM are fundamental examples, and we denote them by $\mathcal{T}_n$:

- $\mathcal{T}_0 = \mathbb{Q}(0)$;
- $\mathcal{T}_1 = \mathbb{Q}(*)$;
- $\mathcal{T}_2 = \mathbb{Q}(*2)$;
- $\mathcal{T}_n = \mathbb{Q}(*2^{n-1})$.

Here are their presentations:

- $\mathcal{T}_0 = \{1\}$; $\mathcal{P} = \varnothing$.
- $\mathcal{T}_1 = \langle a \mid a^2 = 1 \rangle$; $\mathcal{P} = \{a\}$.
- $\mathcal{T}_2 = \langle a, b \mid a^2 = 1,\ b^3 = b \rangle$; $\mathcal{P} = \{a, b^2\}$.
- $\mathcal{T}_3 = \langle a, b, c \mid a^2 = 1,\ b^3 = b,\ c^3 = c,\ b^2 = c^2 \rangle$; $\mathcal{P} = \{a, b^2\}$.

- $\mathcal{T}_n = \langle a, b_1, b_2, \ldots, b_{n-1} \mid a^2 = 1,\ b_i^3 = b_i,\ b_1^2 = b_2^2 = \cdots = b_{n-1}^2 \rangle$; $\mathcal{P} = \{a, b_1^2\}$.

To find $\Phi(*m)$ (in any of the $\mathcal{T}_n$), write m in binary, as $\cdots \epsilon_3 \epsilon_2 \epsilon_1 \epsilon_0$, and we have

$$\Phi(*m) = a^{\epsilon_0} \cdot b_1^{\epsilon_1} \cdot b_2^{\epsilon_2} \cdots b_n^{\epsilon_n}.$$

For example, in $\mathcal{T}_4$, we have

$$\Phi(*4) = b_2, \quad \Phi(*5) = ab_2, \quad \Phi(*6) = b_1 b_2, \quad \Phi(*7) = ab_1 b_2, \quad \Phi(*8) = b_3.$$

Notice that we always have

$$b_1^2 = b_2^2 = \cdots = b_{n-1}^2.$$

Denote this element by z; z represents the sum $*m + *m$, for any NIM-heap with $m \geq 2$. In fact, it represents any NIM position of $\mathcal{G}$-value 0, provided it has at least one heap of size ≥ 2.

The structure of $\mathcal{T}_n$. Let's write out the elements of $\mathcal{T}_3$:

$$\mathcal{T}_3 = \{1, a, b_1, ab_1, b_2, ab_2, b_1 b_2, ab_1 b_2, z, az\}.$$

Consider the subset

$$\mathcal{K} = \{b_1, ab_1, b_2, ab_2, b_1 b_2, ab_1 b_2, z, az\}.$$

Observe that $z \cdot z = z$, $z \cdot b_1 = b_1$, and $z \cdot b_2 = b_2$. Therefore z is an identity of $\mathcal{K}$ and $x^2 = z$ for all $x \in \mathcal{K}$. So $\mathcal{K}$ is a group, and we have

$$\mathcal{K} \cong \mathbb{Z}_2^3.$$

In fact $\mathcal{K}$ behaves just like normal play $\mathcal{G}$-values: it has eight elements, corresponding one-to-one with NIM positions of $\mathcal{G}$-value 0 through 7.

Recall the strategy for misère NIM: play exactly like in normal NIM, unless your move would leave only heaps of size 0 or 1. In that case, play to leave an odd number of heaps of size 1.

$\mathcal{K}$ corresponds to the "exactly like normal NIM" clause of this strategy: it is isomorphic to the normal-play quotient of $*4$. The two elements 1 and a correspond to the "unless": they represent positions with all heaps of size ≤ 1.

Note that every $\mathcal{T}_n$, for $n \geq 2$, can be written as $\mathcal{K} \cup \{1, a\}$, where $\mathcal{K} \cong \mathbb{Z}_2^n$. $\mathcal{K}$ is called the *kernel* of the monoid, and in the next lecture we will see how to generalize it.

In particular, we have

- $|\mathcal{T}_0| = 1$,
- $|\mathcal{T}_1| = 2$,

- $|\mathcal{T}_n| = 2^n + 2$ for all $n \geq 2$.

We can also define the full quotient of NIM:

$$\mathcal{T}_\infty = \mathbb{Q}(0, *, *2, *3, *4, \ldots) \\ \cong \langle a, b_1, b_2 \mid a^2 = 1,\ b_i^3 = b_i,\ b_1^2 = b_2^2 = \cdots \rangle, \quad \mathcal{P} = \{a, b_1^2\}.$$

Remember that normal-play $\mathcal{G}$-values look like

$$\bigoplus_{\mathbb{N}} \mathbb{Z}_2.$$

Well, we can write $\mathcal{T}_\infty = \mathcal{K}_\infty \cup \{1, a\}$ in exactly the same way, and we have $\mathcal{K}_\infty \cong \bigoplus_{\mathbb{N}} \mathbb{Z}_2$.

Tame and wild quotients.

Definition 4.3. A set $\mathscr{A}$ is *tame* if and only if $\mathbb{Q}(\mathscr{A}) \cong \mathcal{T}_n$ for some $n \in \mathbb{N} \cup \{\infty\}$. Otherwise it is *wild*.

Not all quotients are tame.

Example. Let $G = \mathbb{Q}(*2_\#320)$, where $*2_\#320 = \{0, *2, *3, *2_\#\}$ and $*2_\# = \{*2\}$. We have

$$\mathbb{Q}(G) \cong \langle a, b, t \mid a^2 = 1, b^3 = b, t^2 = b^2, bt = b \rangle; \quad \mathcal{P} = \{a, b^2\}.$$

This quotient is called $\mathcal{R}_8$. It is very common; many octal games have quotient $\mathcal{R}_8$, including (for example) 0.75. In fact, it can be shown that $\mathcal{R}_8$ is the smallest quotient except for $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$ (see [16]). The quotient map is given by (writing $z = b^2$, as before)

$$\Phi(*) = a, \quad \Phi(*2) = b, \quad \Phi(*3) = ab, \quad \Phi(*2_\#) = z, \quad \Phi(G) = at.$$

Notice that $\mathcal{R}_8$ is just $\mathcal{T}_2$ with two extra elements:

$$\mathbb{Q} = \{\overbrace{1, a, \underbrace{b, ab, z, az}_{\mathcal{K}}}^{\mathcal{T}_2}, t, at\}.$$

Now $\mathcal{K} \cong \mathbb{Z}_2^2$, and $\{1, a\}$ is a (separate) isomorphic copy of $\mathbb{Z}_2$. But $\{t, at\}$ is not a group, because $t^2 = z \in \mathcal{K}$.

The right picture of $\mathcal{R}_8$ is this: it is the union

$$\mathcal{K} \cup \{1, a\} \cup \{t, at\},$$

where $\mathcal{K}$ and $\{1, a\}$ are two disjoint groups, and $\{t, at\}$ are two extra elements that are "associated" with $\mathcal{K}$. We'll say more about this in the next lecture.

General structure.

Lemma 4.4. *Suppose that $\mathscr{A}$ is hereditarily closed, $\mathscr{A} \neq \varnothing$, and $\mathscr{A} \neq \{0\}$. Then necessarily $* \in \mathscr{A}$.*

Proof. $*$ is the only game whose only option is 0. □

Proposition 4.5. *Let $(\mathbb{Q}, \mathcal{P})$ be any nontrivial misère quotient. Then for all $x \in \mathbb{Q}$, there is some $y \in \mathbb{Q}$ with $xy \in \mathcal{P}$.*

Proof. Write $(\mathbb{Q}, \mathcal{P}) = \mathbb{Q}(\mathscr{A})$ and choose $G \in \mathscr{A}$ with $\Phi(G) = x$. First suppose $G = 0$. Then $x = 1$. By the assumption of nontriviality, we have $\mathscr{A} \neq \{0\}$, so by the previous lemma $* \in \mathscr{A}$. But $\Phi(*) \in \mathcal{P}$ and $1 \notin \mathcal{P}$, so we can take $y = \Phi(*)$.

Now assume $G \neq 0$, and consider $G + G$. If it is a $\mathscr{P}$-position, then we are done, with $y = x$. Otherwise, some option of $G + G$ must be a $\mathscr{P}$-position, say $G + G'$. So we can take $y = \Phi(G')$. □

Proposition 4.6. *For any G and any option G', $\Phi(G) \neq \Phi(G')$.*

Proof. The proof is left as an exercise for the reader. (Hint: use the previous proposition.) □

Proposition 4.7. *If $\mathscr{A}$ is nontrivial and $G \in \mathscr{A}$, then $G \not\equiv_{\mathscr{A}} G + *$.*

Proof. By Proposition 4.5, there is a game $H \in A$ such that $G+H$ is a $\mathscr{P}$-position. But then $G + H + *$ is an $\mathscr{N}$-position, so H distinguishes G from $G + *$. □

Corollary 4.8. *Every nontrivial misère quotient has even order.*

Proof. The proof is left as an exercise for the reader. (Hint: consider the mapping $x \mapsto ax$.) □

In fact, one can prove the following facts:

- $\mathcal{T}_1$ is the only quotient of order 2. (Immediate from Lemma 4.4.)
- There are no quotients of order 4. (Proved in [13].)
- $\mathcal{T}_2$ is the only quotient of order 6. (Also proved in [13].)
- $\mathcal{R}_8$ is the only quotient of order 8. (Much harder to prove; see [16].)

Lecture 5. Further topics

November 30, 2006 Scribes: Shiri Chechik and Menachem Rosenfeld

In this lecture, we will discuss four interesting problems, most of which have not yet been solved completely. We will also discuss the structure of finite commutative monoids.

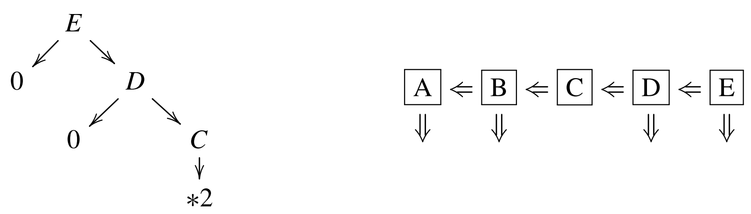


Figure 6. Two representations for the game $E = *(2_{\#}0)0$. Left: the game tree of E. Right: a visual representation of $\mathrm{cl}(E)$.

Four interesting problems.

1. Infinite quotients. We can think of infinite quotients as belonging to either one of two categories: those that are finitely generated, and those that are not. We have already seen one infinite quotient, $\mathcal{T}_\infty = \mathcal{Q}(0, *, *2, \ldots)$. It is not finitely generated. Every one of its finitely generated submonoids is finite, and it is built up from these finite quotients. It is therefore not an interesting quotient to study.

There also exist finitely generated infinite quotients. We can find an example of this by denoting

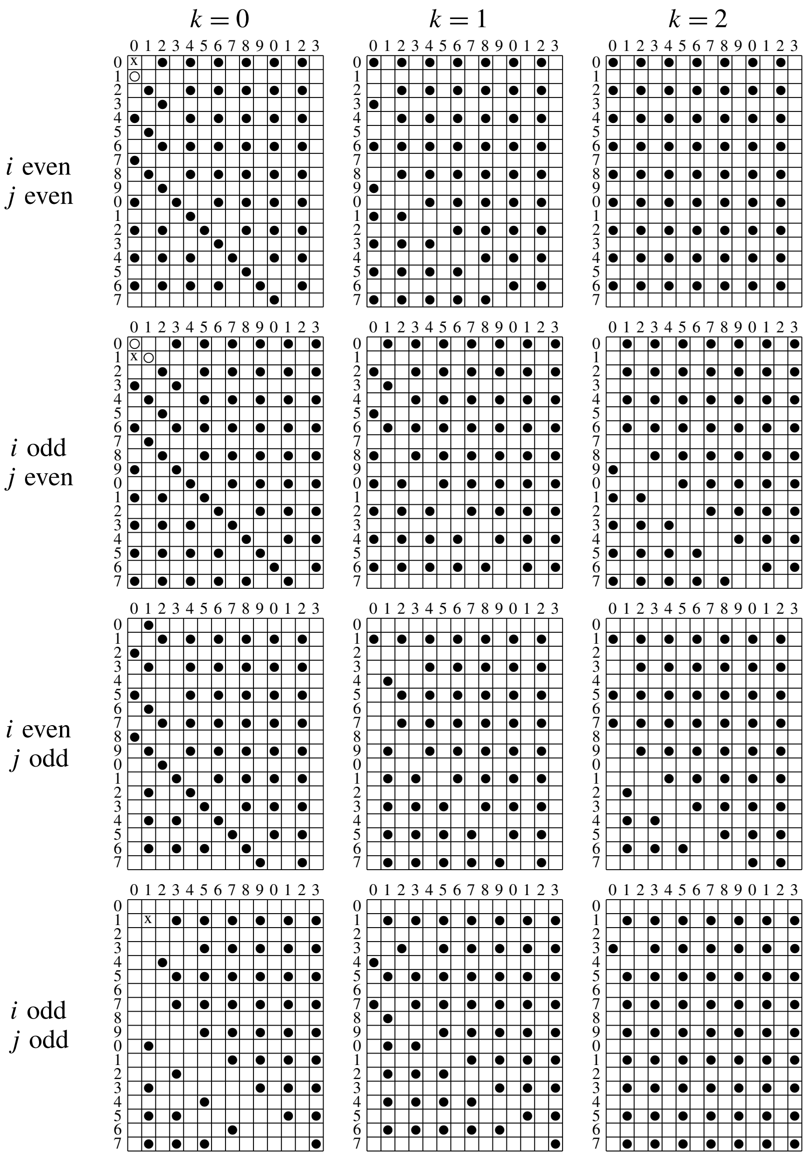
$$A = *,\ B = *2,\ C = \{B\} = *2_{\#},\ D = *2_{\#}0 = \{C, 0\},\ \text{and } E = \{D, 0\} = *(2_{\#}0)0.$$

Figure 6, left, shows the game tree of E.

Denoting $\mathscr{A} = \mathrm{cl}(E)$, a visual way to understand a game in $\mathscr{A}$ is suggested in Figure 6, right; for every game, there are several coins in every box, and a move consists of moving a coin along an arrow (either one step to the left, or from boxes other than C, outside the game board). The last player to move loses.

As it turns out, $|\mathbb{Q}(E)| = \infty$, but every game with a smaller tree has a finite quotient. So E is in some sense the *simplest* game that gives rise to an infinite quotient. To understand why the quotient is infinite, first note that every $X \in \mathscr{A}$ can be written as $X = iA + jB + kC + lD + mE$. In [13, §A.7], we compute the outcome of every such X. It turns out that when $k \geq 3$, the outcomes follow a simple rule: $o^-(X) = \mathscr{P} \iff i + l$ and $j + m$ are both even. However, when $k \leq 2$, the outcomes can be quite erratic. See Figure 7. Each table represents the outcomes for a particular choice of (i, j, k). Within each table, there is a dot at (row m, column l) if and only if $iA + jB + kC + lD + mE$ is a $\mathscr{P}$-position.

Inspecting this figure, we can see that the structure of the $\mathscr{P}$-positions is very complicated. For example, for $i = j = k = l = 0$, $X = mE$ is a $\mathscr{P}$-position $\iff$ $m \in \{1, 4, 7, 10, 12, 14, 16, \ldots\}$.



$\bullet = \mathcal{P}$; x $= \mathcal{P}$ if and only if $j \geq 2$; $\circ = \mathcal{P}$ if and only if $j < 2$

Figure 7. Schematic of the $\mathcal{P}$-positions for $\mathrm{cl}(*(2_\#0)0)$ with $k \leq 2$.

To see that the quotient is infinite, consider the case $i = j = k = 0$. For sufficiently large odd l, we have that $lD + mE$ is a $\mathcal{P}$-position if and only if $m = l + 7$. This means that the lD's are pairwise distinguishable.

It was mentioned in a previous lecture that infinite quotients are still poorly understood. We still cannot solve the following problem.

Open Problem. Specify an algorithm to determine whether a quotient is infinite.

Of course, we'd really like to know much more about $\mathbb{Q}(\mathscr{A})$ than merely *whether* it's infinite. An old theorem about commutative semigroups guarantees that this is possible:

Theorem 5.1 (Rédei). *Every finitely generated commutative semigroup is finitely presented.*

We won't prove Rédei's theorem in this course; see [5; 14]. It makes the following question meaningful.

Open Problem. Specify an algorithm to compute the *presentation* of $\mathbb{Q}(\mathscr{A})$ (even if $\mathbb{Q}$ is infinite), assuming $\mathscr{A}$ is finitely generated.

In particular, the following would be a good start.

Open Problem. Give a presentation for $\mathcal{Q}(E)$.

Note: when we proved the periodicity theorem, at no point did we assume that the partial quotients are finite. Thus the periodicity theorem applies perfectly well to octal games whose partial quotients are infinite. If we could produce an algorithm for computing infinite quotients, then we could (in theory) use the periodicity theorem to provide solutions to games with infinite partial quotients.

2. Algebraic periodicity of octal games. Let Γ be an octal game. Then $\mathcal{Q}(\Gamma)$ is uniquely determined by its sequence of partial quotients,

$$\langle \mathbb{Q}_n(\Gamma) : n \in \mathbb{N} \rangle.$$

We can ask, when is it determined by only finitely many of these partial quotients?

The periodicity theorem is a good start in trying to answer this question — it happens, for instance, when the sequence stabilizes and we have periodicity.

There are intriguing cases in which the sequence does not stabilize but exhibits a strong regularity, which is called *algebraic periodicity*. This phenomenon is not yet understood well enough for a precise definition to be given. The term is derived from *arithmetic periodicity* in normal play, which means that the sequence is periodic but on each period we add a "saltus". For example, if the period is 5 and the saltus is 4, a possible sequence is

$$0, 4, 5, 3, 2, \quad 4, 8, 9, 7, 6, \quad 8, 12, 13, 11, 10, \ldots$$

Theorem 5.2 (see [1]). *No finite octal game* (*that is*, *one with finitely many nonzero digits*) *can be arithmetic periodic* (*with nontrivial saltus*) *in normal play.*

(Remark: NIM is a trivial example of a nonfinite octal game which is arithmetic periodic.)

order (n)	2	4	6	8	10	12	14	16	⋯
# of quotients	1	0	1	1	1	6	9	50	⋯

Table 1. Number of different quotients for every order.

However, algebraic periodicity is manifested in finite octal games with misère play. Several examples are presented in [13].

Several two-digit octal games for which the normal solution is known, have not yet been solved for misère play. Of these, 0.54 is the only one for which the solution seems to be in reach — because it appears to be algebraic periodic, which suggests a solution for it.

Open Problem. Prove the solution for 0.54 presented in [13].

Open Problem. Formulate a suitable general definition of "algebraic periodicity" and prove a theorem that states: if Γ is algebraic periodic for sufficiently long, then it continues this period, and we can compute $\mathcal{Q}(\Gamma)$.

Presumably, this would immediately provide a solution for 0.54, and probably six or eight three-digit octals as well.

3. Generalizations of the mex rule. Suppose we have a quotient map $\Phi : \mathscr{A} \to \mathcal{Q}$. Let G be a game all of whose options are in $\mathscr{A}$. Can we determine, based only on $\Phi''G$, whether $\mathcal{Q}(\mathscr{A} \cup \{G\}) \cong \mathcal{Q}(\mathscr{A})$? If they are isomorphic, can we determine $\Phi(G)$? (Recall that $\Phi''G$ is defined as $\{\Phi(G') : G'$ is an option of $G\}$.)

By asking these questions, we are essentially looking for a way to generalize the mex rule, which solves them for normal play.

The answer to both question is: yes! However, more information is needed than what it contained in $\mathcal{Q}$.

Recall that in the previous lecture, we proved a lemma that answers this question in case there is some $H \in \mathscr{A}$ such that $\Phi''G = \Phi''H$. It turns out that we can get a much stronger result. However, this result is beyond the scope of this lecture; see [13].

4. Classification. How many misère quotients are there of order n (up to isomorphism)? Table 1 displays some of what is known so far. The results for $n = 14$ and 16 are tentative.

A related question: can we identify other interesting classification results? Here is one such result.

It is possible to define the "tame extension" $\mathcal{T}(\mathcal{Q}, \mathcal{P})$ of an arbitrary quotient $(\mathcal{Q}, \mathcal{P})$. See [16] for a precise definition. It turns out that

$$(\mathcal{Q}, \mathcal{P}) \subsetneq \mathcal{T}(\mathcal{Q}, \mathcal{P}),$$

but $\mathcal{T}(\mathcal{Q}, \mathcal{P})$ adds no new $\mathscr{P}$-position types. Furthermore,

$$\mathcal{T}_{n+1} = \mathcal{T}(\mathcal{T}_n).$$

We therefore have two families of quotients,

$$\mathcal{T}_2, \mathcal{T}_3, \dots, \mathcal{T}_\infty$$

and

$$\mathcal{R}_8, \mathcal{T}(\mathcal{R}_8), \mathcal{T}(\mathcal{T}(\mathcal{R}_8)), \dots, \mathcal{T}^\infty(\mathcal{R}_8),$$

all of which have $|\mathcal{P}| = 2$. The following result is proved in [16].

Theorem 5.3. *Every quotient with $|\mathcal{P}| = 2$ is isomorphic to a quotient in one of these two families.*

So we have

$0, \mathcal{T}(0), \mathcal{T}(\mathcal{T}(0)), \dots$	(normal play)
$\mathcal{T}_2, \mathcal{T}(\mathcal{T}_2), \dots, \mathcal{T}^\infty(\mathcal{T}_2)$	(tame misère play)
$\mathcal{R}_8, \mathcal{T}(\mathcal{R}_8), \mathcal{T}^2(\mathcal{R}_8), \dots, \mathcal{T}^\infty(\mathcal{R}_8)$	(“almost tame” misère play)

Can we say anything else along these lines?

The structure of finite commutative monoids. Let $\mathcal{Q}$ be any finite commutative monoid, and let $x, y \in \mathcal{Q}$.

Definition 5.4. If $xz = y$ for some $z \in \mathcal{Q}$, x *divides* y. In this case, we write $x \mid y$.

Definition 5.5. If $x \mid y$ and $y \mid x$, x and y are *mutually divisible* (md).

Example. $$\mathcal{T}_2 = \langle a, b \mid a^2 = 1, b^3 = b \rangle = \{\underbrace{1, a}_{\text{md}}, \underbrace{b, ab, z, az}_{\text{md}}\}.$$

Exercise. Show that md is an equivalence relation.

Definition 5.6. The *mutual divisibility classes* of $\mathcal{Q}$ are the equivalence classes of $\mathcal{Q}$ under the relation md.

Example. The md classes of $\mathcal{R}_8 = \{1, a, b, ab, z, az, t, at\}$ are

$$\{1, a\}, \quad \{b, ab, z, az\} \quad \text{and} \quad \{t, at\}.$$

Definition 5.7. An element $x \in \mathcal{Q}$ is an *idempotent* if $x^2 = x$.

Example. In $\mathcal{T}_2$ (and also $\mathcal{R}_8$), 1 and z are the only idempotents.

Exercise. The md class of an idempotent x is a group with x for an identity.

Exercise. If S is a maximal subgroup of $\mathcal{Q}$ (that is, a group which is not contained in any larger subgroup of $\mathcal{Q}$) then it is the md class of its idempotent.

Let $z_1, z_2, \dots, z_k$ be the idempotents of $\mathcal{Q}$ (since $\mathcal{Q}$ is finite, we can enumerate them all). We write $z = z_1 z_2 \dots z_k$. We then have $z^2 = z_1^2 z_2^2 \dots z_k^2 = z_1 z_2 \dots z_k = z$ and $z z_i = z$.

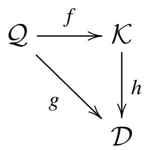
Definition 5.8. The *kernel* of $\mathcal{Q}$ is the md class of z, and is denoted $\mathcal{K}$.

We will soon prove the following theorem.

Theorem 5.9. *The map $x \mapsto zx$ is a surjective homomorphism from $\mathcal{Q}$ onto $\mathcal{K}$.*

The kernel $\mathcal{K}$ can be characterized in two ways:

(1) It is the unique group such that there is a surjective homomorphism $f : \mathcal{Q} \to \mathcal{K}$ with the following property: if $g : \mathcal{Q} \to \mathcal{D}$ is a homomorphism onto a group $\mathcal{D}$, then there exists an $h : \mathcal{K} \to \mathcal{D}$ which makes the following diagram commute:



In other words, any map from $\mathcal{Q}$ onto a group $\mathcal{D}$ factors through f.

(2) $\mathcal{K}$ is the *group of fractions* of $\mathcal{Q}$, that is, it is the group obtained by adjoining formal inverses to $\mathcal{Q}$.

Lemma 5.10. *If $y \in \mathcal{Q}$ then for some $r > 0$, y^r is an idempotent.*

(Note: This does not hold for infinite monoids!)

Proof. Consider the sequence $y, y^2, y^3, y^4, \dots$ Since $\mathcal{Q}$ is finite, there must be some $n > 0$ and some $k > 0$ such that $y^n = y^{n+k}$. We then have for every $t \geq 0$, $y^{n+tk} = y^n$. Let r be the unique integer such that $n \leq r < n+k$ and $r \equiv 0 \pmod{k}$. Then,

$$y^{2r} = y^{r+tk} = y^{n+tk} y^{r-n} = y^n y^{r-n} = y^r.$$

So y^r is an idempotent. □

Note that this idempotent is uniquely determined for any given y. Therefore, for any y, there is a unique idempotent x such that $y^n = x$ for some $n > 0$. This motivates the following definition:

Definition 5.11. For any idempotent $x \in \mathcal{Q}$, the *archimedean component* of x is $\{y \in \mathcal{Q} : \exists n(y^n = x)\}$.

What we have actually shown is that every $y \in \mathcal{Q}$ is a member of a unique archimedean component. Therefore, $\mathcal{Q}$ is partitioned into several archimedean components. For example, $\mathcal{R}_8$ is partitioned into $\{1, a\}$ and $\{b, ab, z, az, t, at\}$.

We complete the picture by defining a natural partial order on idempotents.

Definition 5.12. For idempotents $x, y \in \mathcal{Q}$, $x \leq y \iff xy = x$.

Example. For any idempotent x, $z \leq x \leq 1$.

Theorem 5.13. *The idempotents of $\mathcal{Q}$ form a lattice with respect to the relation $\leq$.*

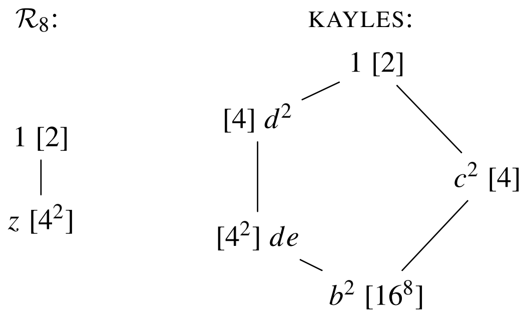
Exercise. Prove this theorem. *Hint*: Define

$$x \wedge y = xy$$

and

$$x \vee y = \prod \{w \in \mathcal{Q} : w \text{ is an idempotent and } w \geq x, y\}.$$

Examples. In these examples, $[a^b]$ denotes an archimedean component with a elements contained in the md class of the idempotent, and b additional elements.



Theorem 5.14. *The map $x \mapsto zx$ is a surjective homomorphism $\mathcal{Q} \to \mathcal{K}$.*

Proof. We must show that $zx \in \mathcal{K}$ for all x; it follows easily that $x \mapsto zx$ is a surjective homomorphism.

Clearly $z \mid zx$, so we must show that $zx \mid z$. Let us take an $n > 0$ such that x^n is an idempotent. Then $zx^n = z$ by the definition of z, so $(zx)x^{n-1} = z$. □

Corollary 5.15. *If $x \in \mathcal{K}$, then $\forall y \in \mathcal{Q}$, $xy \in \mathcal{K}$ (because $x = zx$, so $xy = zxy$).*

Corollary 5.16. *$\mathcal{K} \cap \mathcal{P} \neq \varnothing$ (because by a previous lemma, $\exists x \in \mathcal{Q}$ such that $xz \in \mathcal{P}$, but $xz \in \mathcal{K}$ so $xz \in \mathcal{K} \cap \mathcal{P}$).*

Definition 5.17. $(\mathcal{Q}, \mathcal{P})$ is *normal* if $\mathcal{K} \cap \mathcal{P} = \{z\}$.

(Remark: the smallest known example of an abnormal quotient is of size 420.)

Definition 5.18. $(\mathcal{Q}, \mathcal{P})$ is *regular* if $|\mathcal{K} \cap \mathcal{P}| = 1$.

(Remark: the smallest known example of an irregular quotient is of size over 3000.)

Definition 5.19. A quotient map $\Phi : \mathscr{A} \to \mathcal{Q}$ is *faithful* if, for all $G, H \in \mathscr{A}$,

$$\Phi(G) = \Phi(H) \Rightarrow \mathscr{G}(G) = \mathscr{G}(H).$$

Open Problem. Is every quotient map faithful?

Theorem 5.20. *If* $(\mathcal{Q}, \mathcal{P})$ *is normal and* Φ *is faithful*, *then for all* $G, H \in \mathscr{A}$,

$$z\Phi(G) = z\Phi(H) \Longleftrightarrow \mathscr{G}(G) = \mathscr{G}(H).$$

There is therefore a one-to-one correspondence between elements of $\mathcal{K}$ and normal-play NIM values of games in $\mathscr{A}$. Furthermore, we can compute the mex function in the kernel. This gives us the following strategy for playing a misère octal game Γ: play as if you were playing normal Γ, unless your move would take you outside of $\mathcal{K}$. Then pay attention to the fine structure of the misère quotient.

Example. The octal game 0.414 has not yet been solved for normal play. Nevertheless, we can prove that $\Phi(H_n) \in \mathcal{K}$ for $n > 18$, and we can prove that its quotient is one of

$$\mathcal{Q}_{18}, \mathcal{T}(\mathcal{Q}_{18}), \mathcal{T}(\mathcal{T}(\mathcal{Q}_{18})), \ldots,$$

though we do not know which. The strategy for misère 0.414 is: play as if you were playing normal 0.414, unless your move would leave only heaps of size ≤ 18. Then pay attention to the fine structure of the misère quotient.

One last open problem:

Open Problem. Let $\mathcal{S}$ be an arbitrary maximal subgroup of $\mathcal{Q}$. Must $\mathcal{S} \cap \mathcal{P}$ be nonempty?

Further reading

Misère quotients for impartial games [13], by Plambeck and Siegel, includes most of the material presented in these notes, and a great deal else as well. It is the best resource both for additional examples of misère quotients and for a deeper look at the structure theory. Plambeck's original paper introducing misère quotients [10] includes a proof of the periodicity theorem that is somewhat different from the one presented here. His survey paper [11] provides a nice informal summary of much that is known about misère games. The forthcoming paper [16] dives much more deeply into the structure of misère quotients.

The most current source of information is the *Misére games* website [12], which includes Plambeck's misére games blog. See also [9].

The *canonical* theory is virtually useless in practice, but nonetheless absolutely fascinating. It is (essentially) the "quotient" obtained by taking $\mathscr{A}$ to be the universe of all misère games. See [3] for many results along these lines.

Finally, perhaps the best way to get acquainted with misère quotients is to download a copy of *MiséreSolver* [15] and start experimenting. It can easily reproduce all the examples in this paper, and of course many more as well.

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References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning ways for your mathematical plays*. A. K. Peters, Ltd., Natick, MA, 2nd edition, 2001.

[2] C. L. Bouton. Nim, a game with a complete mathematical theory. *Ann. of Math.*, 3(2):35–39, 1902.

[3] J. H. Conway. *On numbers and games*. A. K. Peters, Ltd., Natick, MA, 2nd edition, 2001.

[4] A. Flammenkamp. Sprague–Grundy values of octal games. http://wwwhomes.uni-bielefeld.de/achim/octal.html.

[5] P. A. Grillet. *Commutative semigroups*. Number 2 in Advances in Mathematics. Springer, 2001.

[6] P. M. Grundy. Mathematics and games. *Eureka*, 2:6–8, 1939.

[7] P. M. Grundy and C. A. B. Smith. Disjunctive games with the last player losing. *Proc. Cambridge Philos. Soc.*, 52:527–533, 1956.

[8] R. K. Guy and C. A. B. Smith. The G-values of various games. *Proc. Cambridge Philos. Soc.*, 52:514–526, 1956.

[9] T. E. Plambeck. Misère games (web pages devoted to problems, computer software, and theoretical results).

[10] T. E. Plambeck. Taming the wild in impartial combinatorial games. *Integers*, 5(#G5), 2005.

[11] T. E. Plambeck. Advances in losing. In M. H. Albert and R. J. Nowakowski, editors, *Games of No Chance 3*, number 56 in MSRI Publications. Cambridge University Press, Cambridge, 2009.

[12] T. E. Plambeck and A. N. Siegel. Misère games on the web. http://www.miseregames.org/.

[13] T. E. Plambeck and A. N. Siegel. Misère quotients for impartial games. *J. Combin. Thy. Ser. A*, 115(5), 2008.

[14] L. Rédei. *The theory of finitely generated commutative semigroups*. Pergamon, 1965.

[15] A. N. Siegel. *MisereSolver*. http://www.cgsuite.org/.

[16] A. N. Siegel. The structure and classification of misère quotients. Forthcoming.

[17] R. P. Sprague. Über mathematische Kampfspiele. *Tôhoku Math. J.*, 41:438–444, 1935–36.

[18] R. P. Sprague. Über zwei Abarten von Nim. *Tôhoku Math. J.*, 43:351–359, 1937.

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