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Replica analysis of the one-dimensional KPZ equation

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In the last few years several exact solutions have been obtained for the one-dimensional KPZ equation, which describes the dynamics of growing interfaces. In particular the computations based on replica method have allowed to study fine fluctuation properties of the interface for various initial conditions including the narrow wedge, flat and stationary cases. In addition, an interesting aspect of the replica analysis of the KPZ equation is that the calculations are not only exact but also "almost rigorous". In this article we give a short review of this development.

1. Introduction

The one-dimensional Kardar–Parisi–Zhang (KPZ) equation,

$$\frac{\partial h(x,t)}{\partial t} = \frac{\lambda}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + \nu \frac{\partial^2 h(x,t)}{\partial x^2} + \eta(x,t), \tag{1}$$

is a well known prototypical equation which describes a growing interface [Kardar et al. 1986; Barabási and Stanley 1995]. Here $h(x,t)$ represents the height of the surface at position $x \in \mathbb{R}$ and time $t \geq 0$. The first term represents a nonlinearity effect and the second term describes a smoothing mechanism. The parameters λ and ν measure the strengths of these effects. The last term $\eta(x,t)$ indicates the existence of randomness in our description of surface growth. For the standard KPZ equation it is taken to be the Gaussian white noise with covariance,

$$\langle \eta(x,t)\eta(x',t')\rangle = D\delta(x-x')\delta(t-t'). \tag{2}$$

Here and in the remainder of the article, $\langle \cdots \rangle$ indicates an average with respect to the randomness η.

The KPZ equation (1) is a nonlinear stochastic partial differential equation (SPDE), which is difficult to handle in general. But fortunately the KPZ equation has a nice integrable structure which has allowed detailed studies of its properties. In particular the one-point height distribution has been computed explicitly for three different initial conditions: the narrow wedge $h(x,0) = -|x|/\delta, \delta \to 0$

[Sasamoto and Spohn 2010a; Amir et al. 2011], the flat $h(x,0) = 0, x \in \mathbb{R}$ [Calabrese and Le Doussal 2011], and the stationary, $h(x,0) = B(x)$ [Imamura and Sasamoto 2012], cases. Here $B(x), x \in \mathbb{R}$, represents the (two-sided) one-dimensional Brownian motion with $B(0) = 0$. From the narrow wedge initial condition the surface grows to a shape of parabola macroscopically, which is representative of a curved surface. The flat case has been the most typical initial condition for Monte Carlo simulation studies, whereas the stationary case is regarded as one of the most important situation from the point of view of nonequilibrium statistical mechanics. The macroscopic shape from the flat and the BM cases are both flat but the fluctuations are different.

Historically the narrow wedge case was first “solved” by using a fact that the KPZ equation can be regarded as a certain weakly asymmetric limit of the asymmetric simple exclusion process (ASEP) [Sasamoto and Spohn 2010a; 2010b; 2010c; Amir et al. 2011]. Soon after the same result was rederived by using a replica method, which subsequently allowed the analysis of the other two cases as well [Calabrese and Le Doussal 2011; Imamura and Sasamoto 2012]. So the replica method has the advantage of being suited for various generalizations but it also has a disadvantage related to the analytic continuation about the replica number. In this article we explain and discuss a few aspects of the application of the replica method to the KPZ equation.

2. Cole–Hopf transformation

By a set of scalings of space, time and height,

$$x \to \alpha^2 x, \quad t \to 2\nu\alpha^4 t, \quad h \to \frac{\lambda}{2\nu} h,$$

with $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$, we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$. Applying the Cole–Hopf transformation,

$$Z(x,t) = e^{h(x,t)}, \tag{3}$$

(1) is linearized as

$$\frac{\partial Z(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t)Z(x,t). \tag{4}$$

Without the last term η, the second term is absent and this is simply the diffusion equation which can be solved easily by Fourier analysis. For the KPZ equation, however, there remains the second term which has η as a multiplicative factor.

One should regard the noise to be the cylindrical Brownian motion. Then (4) written in the form of a stochastic differential equation is well defined and one can *define* the solution of the KPZ equation to be $h(x,t) = \log Z(x,t)$. This is called the Cole–Hopf solution.

A merit of this transformation is that this equation can be regarded as the imaginary time Schrödinger equation for a single particle under a random potential η. In other words this can be regarded as a statistical mechanical problem of a directed polymer in a random potential where Z has the meaning of its partition function. In particular one can write down the Feynman–Kac formula for this quantity [Bertini and Cancrini 1995],

$$Z(x,t) = \mathbb{E}_x\left(\exp\left[\int_0^t \eta(b(s), t-s)\, ds\right] Z(b(t), 0)\right), \tag{5}$$

where $\mathbb{E}_x$ represents the averaging over the standard Brownian motion $b(s)$, $0 < s < t$ with $b(0) = x$. The information of the initial condition is given by specifying $Z(x, t = 0)$ in this formula. We take $Z(x, t = 0) = \delta(x)$ for the narrow wedge case, $Z(x = 0, t) = 1$ for the flat case, and

$$Z(x, t = 0) = e^{B(x)} \tag{6}$$

for the stationary case.

3. Replica method

Originally we were interested in the distribution of the height $h(x,t)$ of the solution for the KPZ equation. After the Cole–Hopf transformation in the previous section, it is equivalent to the distribution of the logarithm of the partition function, $\log Z(x,t)$, of the directed polymer. But considering this quantity directly seems very difficult. Within the replica method, we instead compute N-th replica partition function $\langle Z^N(x,t)\rangle$ and try to retrieve the information about $\log Z$ from it. Z^N means that one is considering N copies of directed polymer systems with the same randomness and hence the name "replica".

The replica method is widely used when studying systems with randomness. For example, in spin glass theory, one considers a Hamiltonian like

$$H = \sum_{\langle ij\rangle} J_{ij} s_i s_j. \tag{7}$$

Here i is a site on a d-dimensional hypercube, the summation is taken over all nearest neighbor pairs of sites and $s_i = \pm 1$ is an Ising spin at the site i. The coupling constant $J_{i,j}$ is taken to be random, e.g., Bernoulli distributed independently for all $\langle ij\rangle$s. This is the Ising model with randomness. For low enough temperature in $d \geq 3$, there appears the spin glass phase in which the spin is frozen randomly [Nishimori 2001]. The quantity of main interest is the averaged free energy $\langle \log Z \rangle$, where $Z = \sum_{s_i=\pm 1} e^{-H}$ is the partition function. To study this one often resorts to the following identity,

$$\langle \log Z \rangle = \lim_{n \to 0} \frac{\langle Z^n \rangle - 1}{n}. \tag{8}$$

This is somehow true if n is taken to be a complex number. By writing $\langle Z^n \rangle = \langle e^{n \log Z} \rangle$, for a pure imaginary n one can consider $\langle Z^n \rangle$ as a characteristic function of $\log Z$ which exists in general. Furthermore when $\langle |\log Z| \rangle$ exists, $\langle \log Z \rangle$ is given by its first derivative:

$$\langle \log Z \rangle = \left. \frac{\partial \langle Z^n \rangle}{\partial n} \right|_{n=0}. \tag{9}$$

In many cases, however, one can compute the replica partition function only for integer $n = 1, 2, \ldots$. Then using the identity (8) implies that one is assuming the analytic continuation with respect to n. There is a theorem due to Carlson for this kind of situation but unfortunately in many cases of physical interest the assumption of the theorem does not hold and hence the application of (8) is not justified in general [Tanaka 2007]. But one can still try to utilize (8) which might give the correct answer. In fact this has been accepted as a very powerful techniques to study systems with randomness but when using it one always has to be careful about the pitfalls. The computation of the averaged free energy $\langle \log Z \rangle$ using this procedure is called the replica trick.

For the case of the KPZ equation, we are interested not only in the average but also in the full distribution of $\log Z$. We compute their generating function $G_t(s)$ of $\langle Z^N(x,t) \rangle$ defined as

$$G_t(s) = \sum_{N=0}^{\infty} \frac{(-e^{-\gamma_t s})^N}{N!} \langle Z^N(x,t) \rangle e^{N \frac{\gamma_t^3}{12}}, \tag{10}$$

with $\gamma_t = (t/2)^{1/3}$. Formally one can recover the probability density by inverting the generating function. Of course there is a problem of the uniqueness as discussed above. But this implies the possibility that one can recover the distribution of $\log Z$ by way of the computations of moments. In fact for the KPZ equation with narrow wedge initial condition, one can check that the correct distribution is obtained in this way [Calabrese et al. 2010; Dotsenko 2010]. It gives us a strong motivation to study the KPZ equation with other initial conditions by using the replica method.

4. Replica Bethe ansatz for KPZ equation: δ-Bose gas

Using the Feynman path integral representation of Z and remembering that the noise η is Gaussian, one can take the average with respect to the random potential

η for the replica partition function. As a result it is written as

$$\langle Z^N(x,t)\rangle = \langle x|e^{-H_N t}|\Phi\rangle. \tag{11}$$

More details about this procedure can be found in [Imamura and Sasamoto 2011b]. Here H_N is a nonrandom Hamiltonian of N particles, $\langle x|$ represents the state with all N particles being at the position x and the $|\Phi\rangle$ the initial state. For the KPZ equation, the Hamiltonian H_N turns out to be that of the delta-function Bose gas (δ-Bose gas) with attractive interaction [Kardar 1987]:

$$H_N = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^{N}\delta(x_j - x_k). \tag{12}$$

The eigenvalues and eigenfunctions can be constructed by using the Bethe ansatz [Lieb and Liniger 1963; McGuire 1964; Dotsenko 2010; Calabrese et al. 2010]. In particular the ground state is a bound state whose wave function is

$$\langle x_1,\ldots,x_N \mid \Psi_z\rangle = Ce^{-\sum_{i,j=1}^{N}|x_i - x_j|}, \tag{13}$$

where C is a normalization constant and the corresponding energy is given by $E = -\frac{1}{24}(N^3 - N)$. Kardar [1987] argued that the N^3 term is responsible for the KPZ exponent $\frac{1}{3}$.

Fortunately, for the δ-Bose gas, one can give a description of all the eigenfunctions and eigenvalues. So at least formally one can expand $\langle Z^N(x,t)\rangle$ in terms of the eigenstates of the Hamiltonian and the corresponding eigenvalues. It had been anticipated for a long time that this might lead to more detailed information on h beyond the scaling exponent (see, for instance, [Dotsenko 2001]), but performing the summation over excited states is very involved and it was only very recently that this program was performed successfully to give an expression for the height distribution.

Now we give a description of the eigenstate and its eigenvalues. Let $|\Psi_z\rangle$ and E_z be the eigenstate and its eigenvalue of H_N:

$$H_N|\Psi_z\rangle = E_z|\Psi_z\rangle. \tag{14}$$

By the Bethe ansatz, they are given as follows. For a set of quasimomenta z_js, the eigenfunction is given by

$$\langle x_1,\ldots,x_N \mid \Psi_z\rangle = C_z \sum_{P\in S_N} \operatorname{sgn} P \prod_{1\le j<k\le N}\bigl(z_{P(j)} - z_{P(k)} + i\,\operatorname{sgn}(x_j - x_k)\bigr) \times \exp\left(i\sum_{l=1}^{N} z_{P(l)}x_l\right), \tag{15}$$

where S_N is the set of permutations with N elements and C_z is the normalization constant,

$$C_z = \left(\frac{\prod_{\alpha=1}^{M} n_\alpha}{N!} \prod_{1 \le j < k \le N} \frac{1}{|z_j - z_k - i|^2} \right)^{1/2}, \tag{16}$$

taken to be a positive real number. The corresponding eigenvalue is simply given by $E_z = \frac{1}{2} \sum_{j=1}^{N} z_j^2$. For the δ-Bose gas with attractive interaction, the quasimomenta z_j $(1 \le j \le N)$ are in general complex numbers. They are divided into M groups $(1 \le M \le N)$ and the α-th group consists of n_α quasimomenta which share the common real part q_α. With this notation, the eigenvalue E_z is given by [Dotsenko 2010]

$$E_z = \tfrac{1}{2} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 - \tfrac{1}{24} \sum_{\alpha=1}^{M} (n_\alpha^3 - n_\alpha). \tag{17}$$

Note that for $N = M$ and $q_\alpha = 0, 1 \le \alpha \le N$, this gives the ground state energy $-\frac{1}{24}(N^3 - N)$ mentioned above.

We expand the replica partition function $\langle Z^N(x,t) \rangle$ (11) by the eigenstates as

$$\langle Z^N(x,t) \rangle = \sum_z e^{-E_z t} \langle x \mid \Psi_z \rangle \langle \Psi_z \mid \Phi \rangle. \tag{18}$$

For the case of the narrow wedge initial condition, $|\Phi\rangle$ is simply $|0\rangle$ and hence one only needs to take the summation over z. But in general we would write as

$$\langle Z^N(x,t) \rangle = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \langle x | e^{-H_N t} | y_1, \dots, y_N \rangle \langle y_1, \dots, y_N \mid \Phi \rangle. \tag{19}$$

Expanding the propagator $\langle x | e^{-H_N t} | y_1, \dots, y_N \rangle$ by the Bethe eigenstates of the δ-Bose gas (15), we have

$$\langle Z^N(x,t) \rangle = \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N} \times e^{-E_z t} \langle x \mid \Psi_z \rangle \langle \Psi_z \mid y_1, \dots, y_N \rangle \langle y_1, \dots, y_N \mid \Phi \rangle. \tag{20}$$

Here we want to perform the integrations over y_j $(1 \le j \le N)$ and write

$$\langle Z^N(x,t) \rangle = \sum_{M=1}^{N} \frac{N!}{M!} \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N} \times e^{-E_z t} \langle x \mid \Psi_z \rangle \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \langle \Psi_z \mid y_1, \dots, y_N \rangle \langle y_1, \dots, y_N \mid \Phi \rangle. \tag{21}$$

At this point this is not allowed in general because in some cases like the stationary situation the integrations over $q_\alpha,\ (1 \le \alpha \le M)$ must be performed before those over $y_j,\ (1 \le j \le N)$. But here with this remark in mind we will write the last factor as $\langle \Psi_z | \Phi \rangle$ and then

$$\langle Z^N(x,t)\rangle = \sum_{M=1}^{N} \frac{N!}{M!} \biggl(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \biggr) \delta_{\sum_{\beta=1}^{M} n_\beta, N} e^{-E_z t} \times \langle x \mid \Psi_z \rangle \langle \Psi_z \mid \Phi \rangle. \quad (22)$$

5. Narrow wedge

For the narrow wedge case, $|\Phi\rangle = |0\rangle$. The wave function (15) with $x_l = x$ can be simplified by applying a combinatorial identity:

$$\sum_{P \in S_N} \operatorname{sgn} P \prod_{1 \le j < k \le N} \bigl(w_{P(j)} - w_{P(k)} + i f(j,k) \bigr) = N! \prod_{1 \le j < k \le N} (w_j - w_k). \quad (23)$$

This holds for any complex variables $w_j\ (1 \le j \le N)$ and $f(j,k)$ and was derived as Lemma 1 in [Prolhac and Spohn 2011a]. We find

$$\begin{aligned}
\langle x \mid \Psi_z \rangle \langle \Psi_z \mid \Phi \rangle &= N! \prod_{\alpha=1}^{M} \frac{(n_\alpha!)^2}{n_\alpha} \prod_{1 \le j < k \le N} \frac{|z_j - z_k|^2}{|z_j - z_k - i|^2} \prod_{l=1}^{N} e^{i z_l x} \\
&= N! \prod_{\alpha < \beta} \frac{|q_\alpha - q_\beta - \frac{i}{2}(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - \frac{i}{2}(n_\alpha + n_\beta)|^2} \prod_{\alpha=1}^{M} \frac{e^{i n_\alpha q_\alpha x}}{n_\alpha} \\
&= 2^M N! \prod_{\alpha=1}^{M} \biggl(\int_0^\infty d\omega_\alpha \biggr) \\
&\qquad \times \det\bigl(e^{i n_j q_j x - n_j(\omega_j + \omega_k) - 2 i q_j (\omega_j - \omega_k)} \bigr)_{j,k=1}^{M}, \quad (24)
\end{aligned}$$

where in the last equality we use the Cauchy determinantal formula.

Using this we get the expression for the generating function. Taking $x = 0$ for simplicity, we have

$$G_t(s) = \sum_{M=0}^{\infty} \frac{(-1)^M}{M!} \prod_{k=1}^{M} \int_0^\infty d\omega_k \det(A_{j,k})_{j,k=1}^{M}, \quad (25)$$

where

$$A_{j,k} = \sum_{n=1}^{\infty} (-1)^{n-1} \int_{\mathbb{R}} \frac{dq}{\pi} e^{-n(\omega_j + \omega_k) - 2iq(\omega_j - \omega_k) - \gamma_t^3 n q^2 + \frac{\gamma_t^3}{12} n^3 - \gamma_t n s}.$$

Here notice that the summation over n is divergent because of a factor $e^{\gamma_t^3 n^3/12}$. This is a serious difficulty of the replica analysis of the KPZ equation. But for the moment let us proceed by using a formula,

$$e^{n^3} = \int_{\mathbb{R}} \mathrm{Ai}(y) e^{ny}\, dy, \tag{26}$$

where Ai is the standard Airy function. This linearizes n^3 in the exponent and then one can take the geometric series to arrive at an expression,

$$G_t(s) = \det(1 - P_0 K_{t,s} P_0), \tag{27}$$

where det means the Fredholm determinant, P_0 is the projection to $[0, \infty)$ and the kernel is

$$K_{t,s}(\xi_j, \xi_k) = \int_{-\infty}^{\infty} dy \mathrm{Ai}(\xi_j + y) \mathrm{Ai}_\Gamma(\xi_k + y) \frac{e^{\gamma_t y}}{e^{\gamma_t y} + e^{\gamma_t s}}. \tag{28}$$

By inverting this one gets an expression for the height distribution, which agrees with the expression which has been derived by unarguably correct methods [Sasamoto and Spohn 2010a; Amir et al. 2011]. This may be a posteriori evidence that the replica method is useful for studying the KPZ equation. In addition, this can be considered as a "singular" limit of a rigorous analysis for a discrete model; see the remark in 7.2. The computations are involved and one awaits full details of their derivations.

6. Flat and stationary case

6.1. ***Flat.*** The state $|\Phi\rangle$ corresponding to the flat initial condition is constant. Hence one has to perform the y integration in (21) which is already not easy. Calabrese and Le Doussal [2011] found a formula for this case using the idea of studying a half-infinite system at first.

6.2. ***Stationary.*** In [Imamura and Sasamoto 2012], in order to take the average over the BM initial condition, we employed the strategy that we first consider a generalized initial condition,

$$h(x,0) = \begin{cases} B_{-,v_-}(-x) := \tilde{B}(-x) + v_- x, & x < 0, \\ B_{+,v_+}(x) \;\; := B(x) - v_+ x, & x > 0, \end{cases} \tag{29}$$

where $B(x)$, $\tilde{B}(x)$ are independent standard BMs and $v_\pm$ are the strengths of the drifts. The point is that once this generalized case is solved, one can study the stationary case by taking the $v_\pm \to 0$ limit.

Because the Brownian motion is a Gaussian process, one can perform the average over the initial distribution (29) and the dependence of $|\Phi\rangle$ on $x_1, \ldots, x_N$

can be explicitly calculated. For the region where

$$x_1 < \cdots < x_l < 0 < x_{l+1} < \cdots < x_N, \quad 1 \le l \le N,$$

one finds

$$\begin{aligned}&\langle x_1, \dots, x_N \mid \Phi\rangle \\ &= e^{v_- \sum_{j=1}^{l} x_j - v_+ \sum_{j=l+1}^{N} x_j} \prod_{j=1}^{l} e^{\frac{1}{2}(2l-2j+1)x_j} \prod_{j=1}^{N-l} e^{\frac{1}{2}(N-l-2j+1)x_{l+j}}. \quad (30)\end{aligned}$$

At this point one has to put the conditions $v_\pm > 0$ to have the wave function decaying at infinity. Since we are considering a bosonic system, this should be symmetrized with respect to $x_1, \dots, x_N$. Using this together with another combinatorial identity, one can compute $\langle \Psi_z \mid \Phi\rangle$ as

$$\langle \Psi_z \mid \Phi\rangle = N!\, C_z \frac{\prod_{m=1}^{N}(v_+ + v_- - m) \prod_{1\le j<k\le N}(z_j^* - z_k^*)}{\prod_{m=1}^{N}(-iz_m^* + v_- - 1/2)(-iz_m^* - v_+ + 1/2)}. \quad (31)$$

Then we can proceed in a fairy similar way as for the narrow wedge case. After some computation, we get an expression for the generating function,

$$\begin{aligned}G_t(s) = &\sum_{N=0}^{\infty} \prod_{l=1}^{N}(v_+ + v_- - l) \sum_{M=1}^{N} \frac{(-e^{-\gamma_t s})^N}{M!} \prod_{\alpha=1}^{M}\left(\int_0^\infty d\omega_\alpha \sum_{n_\alpha=1}^{\infty}\right)\delta_{\sum_{\beta=1}^{M} n_\beta, N} \\ &\times \det\left(\int_{\mathbb{R}-ic} \frac{dq}{\pi} \frac{e^{-\gamma_t^3 n_j q^2 + \frac{\gamma_t^3}{12} n_j^3 - n_j(\omega_j+\omega_k) - 2iq(\omega_j-\omega_k)}}{\prod_{r=1}^{n_j}(-iq + v_- + \frac{1}{2}(n_j - 2r))(iq + v_+ + \frac{1}{2}(n_j-2r))}\right)_{j,k=1}^{M}, \quad (32)\end{aligned}$$

with c taken large enough. A big difference from the narrow wedge [Sasamoto and Spohn 2010a; 2010b; 2010c; Amir et al. 2011] and the half BM initial condition [Imamura and Sasamoto 2011a] is that this generating function itself is not a Fredholm determinant because of the existence of the factor $\prod_{l=1}^{N}(v_+ + v_- - l)$. But this difficulty can be overcome by considering a further generalization of the initial condition in which the initial overall height is distributed as the inverse gamma distribution. After some computation, we obtain the height distribution for the initial condition (29) given by

$$F_{v_\pm,t}(s) = \frac{\Gamma(v_+ + v_-)}{\Gamma(v_+ + v_- + \gamma_t^{-1} d/ds)}\left[1 - \int_{-\infty}^{\infty} du\, e^{-e^{\gamma_t(s-u)}} \nu_{v_\pm,t}(u)\right]. \quad (33)$$

Here $\nu_{v_\pm,t}(u)$ is expressed as a difference of two Fredholm determinants,

$$\nu_{v_\pm,t}(u) = \det\bigl(1 - P_u(B_t^\Gamma - P_{\mathrm{Ai}}^\Gamma)P_u\bigr) - \det(1 - P_u B_t^\Gamma P_u), \quad (34)$$

where P_s represents the projection onto (s, ∞),

$$P_{\mathrm{Ai}}^{\Gamma}(\xi_1, \xi_2) = \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_1, \frac{1}{\gamma_t}, v_-, v_+\right)\mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_2, \frac{1}{\gamma_t}, v_+, v_-\right), \tag{35}$$

$$B_t^{\Gamma}(\xi_1, \xi_2) = \int_{-\infty}^{\infty} dy \frac{1}{1 - e^{-\gamma_t y}} \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_1 + y, \frac{1}{\gamma_t}, v_-, v_+\right) \times \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_2 + y, \frac{1}{\gamma_t}, v_+, v_-\right), \tag{36}$$

and

$$\mathrm{Ai}_{\Gamma}^{\Gamma}(a, b, c, d) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{d}{b}}} dz e^{iza + i\frac{z^3}{3}} \frac{\Gamma(ibz + d)}{\Gamma(-ibz + c)}, \tag{37}$$

where Γ_{z_p} represents the contour from $-\infty$ to ∞ and, along the way, passing below the pole at $z = id/b$. Note the similarity of our formulas with the narrow wedge case.

Once this generalized case is solved, it is not difficult to find a formula for the height distribution for the stationary situation. Furthermore by a simple generalization one can also study the stationary two point correlation function. For more details see [Imamura and Sasamoto 2012; 2013].

7. A few remarks

7.1. ***Multipoint distribution.*** Prolhac and Spohn [2011a; 2011b] applied the replica analysis to study the distribution at more than one point. Unfortunately a summation which appears in the computation seems impossible to perform. But they showed that if one introduces an "factorization approximation", one can proceed further and that in the scaling limit it tends to the Airy process which is the expected limiting process.

7.2. ***Replica analysis for discretized models.*** Recently, Borodin and Corwin [2014] introduced a new discrete model called the q-TASEP. In a certain limit this model reduces to the KPZ equation. On the other hand, for this model, the series become convergent and the replica computation can be made rigorous. In this sense, one could say that the replica method for the KPZ equation is "almost rigorous".

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