Random Matrices
MSRI Publications
Volume **65**, 2014

Fluctuations and large deviations of some perturbed random matrices

MYLÈNE MAÏDA

We review joint results with Benaych-Georges and Guionnet (*Electron. J. Probab.* **16**:60 (2011), 1621–1662 and *Prob. Theory Rel. Fields* **154**:3–4 (2012), 703–751) about fluctuations and large deviations of some spiked models, putting them in the perspective of various works of the last years on extreme eigenvalues of finite-rank deformations of random matrices.

1. Introduction

General statement of the problem. The following algebraic problem is very classical: *Let A and B be two Hermitian matrices of the same size. Assume we know the spectrum of each of them. What can be said about the spectrum of their sum $A+B$?* The problem was posed by Weyl [1912]. He gave a series of necessary conditions, known as Weyl's interlacing inequalities: if $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$, $\lambda_1(B) \geq \cdots \geq \lambda_n(B)$ and $\lambda_1(A+B) \geq \cdots \geq \lambda_n(A+B)$ are the spectra of A, B and $A+B$, then

$$\lambda_{i+j}(A+B) \leq \lambda_{i+1}(A) + \lambda_{j+1}(B),$$

whenever $0 \leq i,\ j,\ i+j < n$. These inequalities have been very fruitful in various fields.

After that, it took a long time to get necessary and sufficient conditions. Horn in the sixties formulated the right conjecture, but the final answer was only given in the late nineties in a series a papers [Klyachko 1998; Helmke and Rosenthal 1995; Knutson and Tao 2001].

If we now look at the problem asymptotically, namely when the size of both matrices goes to infinity, an important breakthrough was made by free probability theory with the notion of asymptotic freeness. This property can be roughly stated as follows: *If A and B are large-dimensional and in generic position relative to one another, the limiting spectrum of their sum depends only on their respective spectra and is given by the free convolution of the two spectra.* We won't go

This work was supported by the Agence Nationale de la Recherche, grant ANR-08-BLAN-0311-03.

further into free probability theory, but we refer the reader to [Emery et al. 2000] for general background on free probability.

In this review paper, we will address the problem of the asymptotic spectrum of the sum of two Hermitian matrices in a particular framework: in the case when one of the matrix is of finite rank, fixed and independent of the size of the matrices. The problem can be stated as follows: take your favorite ensemble of matrices. You know very well the global and local behavior of the spectrum (asymptotic spectral measure, convergence and fluctuations of extreme eigenvalues etc). Add to your random matrix a finite-rank perturbation. How is the spectrum affected by this perturbation?

By Weyl's interlacing inequalities, it is not hard to check that the global behavior is not changed at the macroscopic level. Only extreme eigenvalues can be substantially affected. This review paper discusses almost sure limits, fluctuations and large deviations for these extreme eigenvalues in various models of this type. There exist also a few results on eigenvectors that we won't review here; see [Benaych-Georges and Nadakuditi 2011], for example.

The BBP transition. Before giving a more panoramic view of the literature on these models, we will describe in detail the first set of rigorous results in this direction. This seminal work is due to Baik, Ben Arous and Péché [Baik et al. 2005], and the phenomenon we will describe is therefore often called the BBP transition.

They considered the following model: let G_n be an $n \times m$ matrix whose column vectors are centered Gaussian with covariance matrix Σ and let $S_n = (1/m)G_n G_n^*$ be the corresponding sample covariance matrix. Assume $n/m \to c \in (0, 1)$ and Σ is a perturbation of the identity in the sense that it has at most r eigenvalues different from 1. Let $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r \geq 1, \ldots, 1$ be the eigenvalues of Σ.

Let us first recall the unperturbed case. If Σ is the identity matrix, the model is known as the Laguerre unitary ensemble (LUE). The following results are now classical:

0. The limiting spectral measure is Marchenko–Pastur, being supported in $\left[(1-\sqrt{c})^2, (1+\sqrt{c})^2\right]$.
1. For any k fixed, the k largest eigenvalues converge to the edge of the bulk, $(1+\sqrt{c})^2$.
2. They have fluctuations in the scale $n^{-2/3}$ with Gaussian unitary ensemble (GUE) Tracy–Widom laws.

As mentioned above, it is easy to check that the global regime won't be affected: property 0 remains valid in every regime and we won't repeat it. The interesting thing is the behavior of the largest eigenvalues.

The results of [Baik et al. 2005] are as follows:

- If $\ell_1 < 1+\sqrt{c}$, properties (1) and (2) remain unchanged: we are in the subcritical regime.
- If $\ell_1 = \cdots = \ell_k > 1+\sqrt{c}$, we are in the supercritical regime.
  (1) The largest eigenvalue converges outside the bulk to $\ell_1 + c\ell_1/(\ell_1 - 1)$.
  (2) Its fluctuations are in the scale $n^{-1/2}$ with the law of the largest eigenvalue of a matrix from the GUE of size $k \times k$.

In the sequel, we will refer to this phenomenon as the BBP transition.

For completeness, we mention that [Baik et al. 2005] also treats the critical case, when there exists k such that $\ell_1 = \cdots = \ell_k = 1+\sqrt{c}$, but this critical behavior is hard to generalize; it seems that only a case-by-case analysis is pertinent for critical parameters and we won't dwell on this behavior.

Our last remark about this model is that the perturbation is multiplicative, whereas we are more interested in additive perturbations. But in fact an additive perturbation gives the same kind of behavior: for example, Péché [2006] showed the same kind of transition for a matrix from the GUE perturbed (additively) by a matrix of finite rank.

A quick review of the literature about fluctuations (see also Section 5)**.** Since the appearance of [Baik et al. 2005], there has been quite a lot of work on fluctuations of extreme eigenvalues of different variant of those spiked models.

A large part of the literature has been devoted to applications, more specifically statistical applications of those spiked models. The seminal work in this direction is probably [Johnstone 2001], dealing with applications to principal component analysis (see also [El Karoui 2005]). Indeed, in many papers, the finite-rank matrix is seen as the signal (with a fixed number of significant parameters) and the unperturbed random matrix as the noise. The general question addressed is to know whether the observation of the eigenvalues of "signal plus noise" can give access to the parameters of interest. The results on fluctuations that we will expound below allow us to construct statistical tests on the parameters. All these applied results are a subject in themselves and we won't review them here; we will stick to theoretical results on those spiked models.

As pointed out above, Baik et al. [2005] and Péché [2006] dealt with models with Gaussian entries (perturbed LUE and GUE). There quickly followed attempts to extend it beyond the Gaussian case.

The first results were in the direction of a generalization of the BBP transition. Féral and Péché [2007] showed, by using combinatorics of moments techniques, that the model $W_n + A_n$, where W_n is a Wigner matrix with independent, identically distributed (iid) complex entries with a law having sub-Gaussian moments and A_n is a rank-one perturbation such that $(A_n)_{ij} = \theta/n$ for all i, j, exhibits all the features of the BBP transition. Bai and Yao [2008] studied in more detail the fluctuations of the eigenvalues converging out of the bulk for quite general spiked models and showed that these fluctuations, as in the BBP case, were in the scale $n^{-1/2}$ and governed by small GUE (or GOE in the case of real entries) matrices with sizes the multiplicities of the supercritical eigenvalues of the perturbation.

Then, in 2009, there appeared what we can call the first non-BBP features in such models. Capitaine, Donati-Martin and Féral [Capitaine et al. 2009] showed, among other results, that the fluctuations of the eigenvalues converging out of the bulk are not universal. If again, W_n is a Wigner matrix with complex iid entries with a nice symmetric law μ and A_n has rank one but this time is of the form $A_n = \text{diag}(\theta, 0, \dots, 0)$, with θ large enough, then the fluctuations of the largest eigenvalue are not Gaussian anymore, the law being rather the convolution of a Gaussian measure together with the law μ.

In [Capitaine et al. 2009], the reason for this nonuniversality remained a bit mysterious, but in [Capitaine et al. 2012] the same authors showed that the crucial feature is whether the perturbation has delocalized eigenvectors (as in [Féral and Péché 2007]), in which case the BBP transition occurs, or localized eigenvectors (as in [Capitaine et al. 2009]), in which case the fluctuations may depend on the law of the entries of the unperturbed matrix.

We emphasize that in the sequel, we are going to work only in the framework of perturbations with delocalized eigenvectors. The main object of this review paper is to present the results of two joint papers of the author with Benaych-Georges and Guionnet [Benaych-Georges et al. 2011; 2012].

2. Fluctuations of extreme eigenvalues of spiked models

Presentation of the deterministic version of the models. As pointed out in the introduction, many versions of the spiked model have been studied in the literature. Let us detail the precise models we have been studying. We first present and detail the case when the unperturbed part is deterministic and we will then explain how the results can be easily generalized to the usual ensembles of matrices (Wigner, Wishart, etc.)

Let X_n be deterministic self-adjoint with eigenvalues $\lambda_1^n \geq \cdots \geq \lambda_n^n$.

We make the following hypothesis on the spectrum of X_n: as n goes to infinity,

$$\text{(H1)} \qquad \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^n} \to \mu_X, \quad \lambda_1^n \to a, \quad \lambda_n^n \to b,$$

with μ_X a compactly supported probability measure and a and b are respectively the left and right edges of support of μ_X.

We then add to X_n a finite-rank perturbation R_n and consider the perturbed matrix $\widetilde{X}_n = X_n + R_n$. The eigenvalues of $\widetilde{X}_n$ will be denoted by $\tilde{\lambda}_1^n \geq \cdots \geq \tilde{\lambda}_n^n$. The finite-rank perturbation R_n will have the following form:

$$R_n = \sum_{j=1}^{r} \theta_i G_i^n (G_i^n)^*,$$

with

$$\theta_1 \geq \cdots \geq \theta_{r_0} > 0 > \theta_{r_0+1} \geq \cdots \geq \theta_r$$

fixed and independent of n, and the G_i^n such that $\sqrt{n} G_i^n$ are vectors with iid entries with law ν satisfying a log-Sobolev inequality. This latter hypothesis is technical; it allows us to use some concentration properties for the quantities we will be interested in.

One can also consider R_n of the form

$$R_n = \sum_{j=1}^{r} \theta_i U_i^n (U_i^n)^*,$$

where the U_i^n are obtained from the vectors $\sqrt{n} G_i^n$ by a Gram–Schmidt orthonormalization procedure.

In particular, we stress that in our model the eigenvectors of the perturbation are delocalized.

Almost sure convergence of extreme eigenvalues. Before getting to our results on fluctuations themselves, let us first look at the convergence of those extreme eigenvalues.

We set

$$G_{\mu_X}(z) := \int \frac{1}{z-x} d\mu_X(x),$$

and define

$$\frac{1}{\underline{\theta}} = \lim_{z \to a^-} G_{\mu_X}(z), \qquad \frac{1}{\bar{\theta}} = \lim_{z \to b^+} G_{\mu_X}(z),$$

$$\rho_\theta := \begin{cases} G_{\mu_X}^{-1}(1/\theta) & \text{if } \theta \in (-\infty, \underline{\theta}) \cup (\bar{\theta}, +\infty), \\ a & \text{if } \theta \in [\underline{\theta}, 0), \\ b & \text{if } \theta \in (0, \bar{\theta}]. \end{cases}$$

Theorem 2.1 [Benaych-Georges and Nadakuditi 2011]. *Let* $r_0 \in \{0, \ldots, r\}$ *be such that*

$$\theta_1 \geq \cdots \geq \theta_{r_0} > 0 > \theta_{r_0+1} \geq \cdots \geq \theta_r.$$

The largest eigenvalues have the following behavior:

$$\tilde{\lambda}_i^n \xrightarrow{\text{a.s.}} \rho_{\theta_i} \quad \textit{for all } i \in \{1, \dots, r_0\},$$
$$\tilde{\lambda}_i^n \xrightarrow{\text{a.s.}} b \quad \textit{for } i > r_0.$$

Similarly, *for the smallest eigenvalues*:

$$\tilde{\lambda}_{n-r+i}^n \xrightarrow{\text{a.s.}} \rho_{\theta_i} \quad \textit{for all } i \in \{r_0+1, \dots, r\},$$
$$\tilde{\lambda}_{n-i}^n \xrightarrow{\text{a.s.}} a \quad \textit{for all } i \geq r - r_0.$$

In the sequel we will state only the part of the results concerning the largest eigenvalues, the part concerning the smallest eigenvalues being very similar.

Before moving to the fluctuations, we emphasize that our model exhibits the first feature of the BBP transition: if the perturbation is small, extreme eigenvalues stick to the bulk; if it is strong enough, they converge out of the bulk.

Gaussian fluctuations outside the bulk. The second feature of the BBP transition is the fact that the fluctuations of the eigenvalues converging outside the bulk are in the scale $n^{-1/2}$ and are "of Gaussian type" — in fact, governed by a small matrix from the Gaussian unitary or orthogonal ensemble (GUE/GOE).

Under the additional hypothesis that

(H2) $$\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^n} \to \mu_X \text{ converges at least as fast as } 1/\sqrt{n},$$

our model exhibits this second feature. More precisely, we have the following result. Let $\alpha_1 > \cdots > \alpha_q > 0$ be the different values of the θ_i such that $\rho_{\theta_i} > b$.

For each j, let I_j be the set of indices i such that $\theta_i = \alpha_j$. Set $k_j = |I_j|$.

Theorem 2.2. *Under hypotheses* (H1) *and* (H2), *if the fourth cumulant* $\kappa_4(\nu)$ *of the law* ν *is zero*, *the random vector*

$$\big(\gamma_i := \sqrt{n}(\tilde{\lambda}_i^n - \rho_{\theta_i}),\, i \in I_j\big)_{1 \leq j \leq q}$$

converges in law to the eigenvalues of $(c_j M_j)_{1 \leq j \leq q}$ *with independent matrices* $M_j \in$ GUE / GOE *of size* $k_j \times k_j$ *and* c_j *is an explicit constant depending only on* μ_X *and* α_j.

We have a similar result if $\kappa_4(\nu)$ is not zero, only the limiting law will be a bit different. We refer the reader to [Benaych-Georges et al. 2011] for details.

Nonuniversality of the fluctuations near the bulk. In the BBP transition mentioned in the introduction, it turned out that the fluctuations of extreme eigenvalues that sticks to the bulk at the level of almost sure convergence exhibited the same fluctuations as the unperturbed model (in the LUE case, it happened to be governed by Tracy–Widom laws).

In our model, we also addressed the question of the so-called "sticking eigenvalues". Their study happened to be much more delicate than the study of the fluctuations of eigenvalues outside the bulk.

As mentioned in the introduction, we won't address the problem of critical parameters. Our result can be roughly stated as follows:

Theorem 2.3. *Under additional hypotheses on X_n, if none of the θ_i is critical, with overwhelming probability, the eigenvalues of $\widetilde{X}_n$ converging to a or b remain at distance at most $n^{-1+\epsilon}$ of the extreme eigenvalues of X_n, for some $\epsilon > 0$.*

We therefore say that the fluctuations of the eigenvalues near the bulk are nonuniversal, in the sense that they follow the fluctuations of the eigenvalues of X_n, that could be in any scale and according to any probability law.

Before giving a more precise statement of the hypotheses and the theorem, one can give a rough explanation of the phenomenon: for fixed values of the θ_i, we have a repulsion phenomenon from the eigenvalues (ev) of X_n at the edge.

- If the repulsion is very strong, the extreme ev of $\widetilde{X}_n$ converge away from the bulk.
- If the repulsion is milder, the extreme ev of $\widetilde{X}_n$ stick to the edge of the bulk.
- If the repulsion is even milder, the extreme ev of $\widetilde{X}_n$ stick to the extreme ev of X_n even at the level of fluctuations.

For the repulsion to be very mild, we need the spacings of the eigenvalues of X_n at the edge to stay small, in the following sense:

(H3)[p, α] There exists a sequence m_n of positive integers tending to infinity with $m_n = O(n^\alpha)$, and constants $\eta_2 > 0$ and $\eta_4 > 0$ such that for any $\delta > 0$ and n large enough,

$$\sum_{i=m_n+1}^{n} \frac{1}{(\lambda_p^n - \lambda_i^n)^2} \leq n^{2-\eta_2},$$

$$\frac{1}{n}\sum_{i=m_n+1}^{n} \frac{1}{\lambda_p^n - \lambda_i^n} \geq \frac{1}{\underline{\theta}} - \delta,$$

$$\sum_{i=m_n+1}^{n} \frac{1}{(\lambda_p^n - \lambda_i^n)^4} \leq n^{4-\eta_4}$$

The fact that the eigenvalues of the unperturbed matrix are sufficiently spread at the edges to insure the above hypothesis allows the eigenvalues of the perturbed matrix to be very close to them, as stated in the following theorem.

Theorem 2.4. *Let I_b be the set of indices corresponding to the eigenvalues $\widetilde{\lambda}_i^n$ converging to the upper bound of the support of μ_X. Suppose* (H1) *and* (H3)[r, α]

hold. Then for any $\alpha' > \alpha$, we have, for all $i \in I_b$,

$$\min_{1\le k\le i+r-r_0} |\tilde{\lambda}_i^n - \lambda_k^n| \le n^{-1+\alpha'},$$

with overwhelming probability.[1]

Moreover, in the case where the perturbation has rank one, we can locate exactly in the neighborhood of which eigenvalues of the unperturbed matrix the eigenvalues of the perturbed matrix lie. We won't review this particular case.

Applications to classical models. In comparison with the models presented in the introduction, the model we have chosen till then that is (deterministic + finite-rank random perturbation with delocalized eigenvectors) can seem a bit disappointing. In fact, we can easily extend our theorem to models where the unperturbed matrix is random and therefore generalize some of the results presented in the introduction.

If (X_n) is a sequence of random matrices, we say that it satisfies an hypothesis (H) in probability if the probability that (X_n) satisfies (H) goes to one as n goes to infinity.

To extend our result, we will use the following, which is easy to show.

Theorem 2.5. *Let (X_n) be a sequence of random matrices independent of the column vectors G_i^n or U_i^n of the perturbation.*

(1) *If* (H1) *holds in probability*, *Theorem 2.1 holds.*

(2) *If $\kappa_4(\nu) = 0$ and Hypotheses* (H1) *and* (H2) *hold in probability*, *Theorem 2.2 holds.*

(3) *If none of the θ_i is critical and Hypotheses* (H1) *and* (H3) *hold in probability*, *Theorem 2.4 holds "with probability converging to one" instead of "with overwhelming probability".*

This result allows us to treat a lot of classical models. For each of these models, it will be enough to check that the different hypotheses hold in probability. This will allow us to generalize some of the results of [Péché 2006; Féral and Péché 2007; Capitaine et al. 2009]. We detail hereafter some of these generalizations.

Let X_n be one of the following models:

(a) X_n is a Wigner matrix with iid entries (up to symmetry) with zero mean, variance one and finite fourth moment.

(b) X_n is a Wishart matrix of the form $X_n = G_n G_n^*/m$, with G_n a $n \times m$ matrix with (real or complex) iid entries with zero mean, variance one and finite fourth moment, with $n/m \to c \in (0, 1)$.

[1] We say that a sequence of events $(E_n)_{n\in\mathbb{N}}$ holds with overwhelming probability if there exists $C, \eta > 0$ such that for n large enough, $\mathbb{P}(E_n) \ge 1 - Ce^{-n^\eta}$.

Then, for the deformed model $\widetilde{X}_n$:

- Almost sure convergence of extreme eigenvalues is governed by Theorem 2.1.
- The eigenvalues converging out of the bulk have Gaussian fluctuations (see Theorem 2.2, where c_j can be computed explicitly).
- If the entries of X_n in the Wigner case and G_n in the Wishart case have a subexponential decay,
  - for a one dimensional perturbation: if the perturbation is supercritical, the largest eigenvalue converges outside and has Gaussian fluctuations, the p-th largest has the $p-1$-th Tracy–Widom law, if it is subcritical, the p-th largest has the p-th Tracy–Widom law;
  - for a multidimensional perturbation: the sticking eigenvalues of $\widetilde{X}_n$ are at distance negligible with respect to $n^{-2/3}$ to the extreme eigenvalues of X_n.

We also get the same kind of results for perturbed Coulomb gases, that is, when the joint law of eigenvalues of the unperturbed part is of the form

$$dP_n(\lambda_1, \dots, \lambda_n) = (1/Z_n)\, |\Delta(\lambda)|^{\beta} e^{-n\beta \sum_{i=1}^{n} V(\lambda_i)} \prod_{i=1}^{n} d\lambda_i,$$

with V a strictly convex polynomial potential.

3. Large deviations of extreme eigenvalues of spiked models

Introduction. As the spectrum of a matrix is a very complicated function of the entries, usual large deviations theorems, mainly based on independence do not easily apply. There have been only few works dealing with large deviations in the context of random matrices. The first breakthrough in this direction, which played an important role in the development of the theory, appeared in [Ben Arous and Guionnet 1997], which showed a full large deviation principle (LDP) for the empirical spectral law of Gaussian Wigner matrices or more generally in models where the joint law of the eigenvalues is given by a Coulomb gas distribution, as introduced above (see also [Anderson et al. 2010]). Recently, Adrien Hardy [2012] gave some extension of the result of Ben Arous and Guionnet in the case when the potential is weakly confining. For the empirical spectral law of the sum of a Gaussian Wigner matrix and a deterministic self-adjoint matrix, the LDP is also known thanks to [Guionnet and Zeitouni 2002]. We can also mention [Chatterjee and Varadhan 2012] and [Bordenave and Caputo 2012] on Wigner matrices, and [Hardy and Kuijlaars 2013] on noncentered Wishart matrices.

If we now look at the large deviations for extreme eigenvalues, a little bit more is known. The first result in this direction concerns the largest eigenvalue of a matrix from the GOE or GUE and was shown by Ben Arous, Dembo and Guionnet [Ben Arous et al. 2001] (see also [Anderson et al. 2010] for generalizations). Wishart matrices of the form XX^*, with X a $n \times m$ matrix with iid, not necessarily Gaussian, entries, have been studied in [Fey et al. 2008] in the case when the ratio m/n goes to zero.

Hereafter, we study the problem of the large deviations of extreme eigenvalues in spiked models that are of the same form as the models introduced at the start of Section 2. The only previous result in this direction was established in [Maïda 2007] for a rank-one perturbation of a GOE matrix.

Before going into our results, we stress that among the results mentioned above, all of them, except [Chatterjee and Varadhan 2012; Fey et al. 2008], dealt with Gaussian entries or with cases when the joint law of eigenvalues was explicitly known.

Large deviation principle: the statement. In the paper [Benaych-Georges et al. 2012], we consider the same models (iid and orthonormalized) as introduced at the start of Section 2. The perturbation is more or less the same, except, instead of taking $\sqrt{n}G_i^n$ vectors with iid entries with law ν satisfying a log-Sobolev inequality, we assume that $G = (g_1, \dots, g_r)$ is a random vector satisfying $E\big(e^{\alpha \sum |g_i|^2}\big) < \infty$ for some $\alpha > 0$ and $(\sqrt{n}G_1^n, \dots, \sqrt{n}G_r^n)$ are random vectors whose entries are independent copies of G; again $(U_1^n, \dots, U_r^n)$ are obtained from $(\sqrt{n}G_1^n, \dots, \sqrt{n}G_r^n)$ by a Gram–Schmidt orthonormalization procedure.

Theorem 3.1. *If X_n satisfies* (H1), *the law of the r_0 largest eigenvalues of $\widetilde{X}_n$ satisfies a LDP in the scale n with a good rate function L. It has a unique minimizer towards which almost sure convergence holds.*

For the reader that are not familiar with large deviations, we recall that this means that for any open set $O \subset \mathbb{R}^{r_0}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P\big((\tilde{\lambda}_1, \dots, \tilde{\lambda}_{r_0}) \in O\big) \geq -\inf_0 L,$$

and for any closed set $F \subset \mathbb{R}^{r_0}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\big((\tilde{\lambda}_1, \dots, \tilde{\lambda}_{r_0}) \in F\big) \leq -\inf_F L.$$

In particular, in the simplest case when $X_n = 0$, and for the iid model, we are back to the following: if G_n are $n \times r$ matrices whose rows are i.i.d. copies of G and $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_r)$, we study the deviations of the eigenvalues of $W_n = (1/n)G_n^* \Theta G_n$ (see [Fey et al. 2008] where they treat the case $\Theta = \mathrm{Id}$).

Before going any further, we want to mention an important generalization of our theorem that will be crucial in the next application: provided the law of $\frac{G}{\sqrt{n}}$ satisfies a LDP, one can relax the hypothesis (H1) in the sense that we do not need to assume that the extreme eigenvalues of X_n converge to the edges (respectively a and b) of the limiting measure μ_X. We can allow a finite number of eigenvalues, that we call outliers, to have their limit outside the support of μ_X.

Application: LDP for perturbed Coulomb gases. As for the study of fluctuations, the deterministic model (i.e., when the unperturbed part is deterministic) may seem a bit artificial. The real interest is in X_n random. But there isn't much hope to get a LDP for the extreme eigenvalues of the perturbed model if we don't even know the deviations of extreme eigenvalues of the original model. As pointed out in the introduction of this section, there are only a few models for which we know a LDP for extreme eigenvalues.

Here, we consider the case when X_n is random with a law with density proportional to $e^{-ntrV(X)}$. We work with the orthonormalized model. More precisely, we assume the U_i^n to be a family of orthonormal vectors, either deterministic or independent of X_n.

Theorem 3.2. *Under appropriate assumptions on V, for any fixed k, the law of the k largest eigenvalues of $\widetilde{X}_n$ satisfies a large deviation principle with a good rate function.*

The strategy of the proof goes as follows: as we have enough information on the deviations of the eigenvalues of X_n, as only a finite number can deviate, we first condition on these deviations so that conditionally to the positions of the eigenvalues of X_n, we can apply the generalization of Theorem 3.1 to the case with outliers.

4. A sketch of the proofs

Without getting too much into the details, we would like to briefly give a few ideas of the proofs of the theorems stated in Sections 2 and 3.

The starting point is a determinant computation: we recall that if $V_n = (V_1^n, \dots V_r^n)$ is the $n \times r$ matrix with column vectors $(G_1^n, \dots G_r^n)$ in the iid model or $(U_1^n, \dots U_r^n)$ in the orthonormalized model then $R_n = V_n \Theta V_n^*$, with $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_r)$. Now, for any z which is not an eigenvalue of X_n we have

$$\begin{aligned}\det(z - \widetilde{X}_n) &= \det(z - X_n - V_n \Theta V_n^*) \\ &= \det(z - X_n) \det \Theta \det(\Theta^{-1} - V_n^*(z - X_n)^{-1} V_n).\end{aligned}$$

Therefore, the eigenvalues of $\widetilde{X}_n$ that are not eigenvalues of X_n satisfy

$$f_n(z) := \det(\Theta^{-1} - V_n^*(z - X_n)^{-1} V_n) = 0.$$

The fact that f_n is the determinant of an $r \times r$, that is fixed size, matrix will considerably ease its study.

From this, the almost sure limits are easy to determine, as one can show, by concentration of measure arguments, that

$$\big(\langle V_i^n, (z-X_n)^{-1}V_j^n\rangle\big)_{1\le i,j\le r} \simeq \operatorname{diag}\big(G_{\mu_X}(z),\dots,G_{\mu_X}(z)\big),$$

so that the possible limits are the solutions of $G_{\mu_X}(z)=\theta_i^{-1}$, for $i\in\{1,\dots r\}$.

The analysis of the fluctuations out of the bulk consists in looking at the fine asymptotics of $\langle V_i^n, (\rho_n-X_n)^{-1}V_j^n\rangle$ around its limits $G_{\mu_X}(\rho_\alpha)=1/\alpha$ for $\rho_n=\rho_\alpha+x/\sqrt{n}$. The result comes essentially from a precise analysis of the orthonormalization procedure and the use of a central limit theorem for quadratic forms developed in [Bai and Yao 2008].

Then comes the more delicate part which is the analysis of the fluctuations of the eigenvalues sticking to the bulk. The strategy is much more involved as it is hard to distinguish between the eigenvalues of X_n and those of $\widetilde{X}_n$ which are not well separated. The work essentially consists in checking that $f_n(z)$ may vanish only if z is very near to the eigenvalues of X_n.

The starting point to show the LDP is the same as for almost sure convergence and fluctuations, namely the fact that the eigenvalues we are interested in are solutions of $f_n(z)=0$. Now assume that X_n is diagonal, then $f_n(z)$ is a polynomial function of

$$\langle G_i^n, (z-X_n)^{-1}G_j^n\rangle=\frac{1}{n}\sum_{i=1}^n\frac{g_i(k)\overline{g_j(k)}}{z-\lambda_k^n}\quad\text{and}\quad\langle G_i^n, G_j^n\rangle=\frac{1}{n}\sum_{i=1}^n g_i(k)\overline{g_j(k)}.$$

By Cramer's theorem or weighted Cramer's theorem and some abstract arguments that are pretty standard in large deviation theory, one can derive LDPs for those two sums and then for f_n.

The eigenvalues being the zeroes of f_n, one could expect to get easily an LDP for them. In fact, they are not continuous functions of f_n in the topology we are dealing with so it will be quite delicate to get this LDP but *in fine*, we will get the expected rate function.

5. A few concluding remarks

There is still a lot of activity around spiked models and as for concluding remarks, we would like to briefly mention the content of a few very recent works that have appeared in the last years or months on the subject.

A first interesting direction was developed by a group of authors and shed a new light on the links between almost sure convergence of the outliers in various spiked models and free probability theory. They successfully used the notion of subordination to characterize the limits of extreme eigenvalues. Capitaine, Donati-Martin, Féral and Février [Capitaine et al. 2011], and then Capitaine

[2013] could approach the problem of spiked models in the case when the perturbation is not of finite rank anymore. Belinschi, Bercovici, Capitaine and Février [Belinschi et al. 2012], address the problem of the outliers of $A+UBU^*$, when A and B are deterministic, U Haar unitary and A has a finite number of outliers but its limiting distribution may not be the Dirac mass at zero.

Baik and Wang [2011; 2013] studied a model in which the law of the matrices is proportional to

$$e^{-n\,\mathrm{Tr}(V(X_n)+A_nX_n)}\,dX_n,$$

with A_n which is of finite rank. In the physics literature (see, e.g., [Brézin and Hikami 1998]), V is usually called the potential and A_n the external field. Although looking very similar to we called perturbed Coulomb gases, this model turns out to be quite different ; in particular it is strongly anisotropic and exhibits some subtle nonuniversal behavior when the potential V is not convex.

In [Pizzo et al. 2013; Renfrew and Soshnikov 2013], the results of [Capitaine et al. 2009] and [Benaych-Georges et al. 2011] on perturbed Wigner matrices have been extended. Using techniques close to those explained in Section 4, the authors could relax the conditions on the moments of the entries of X_n and consider more general forms of finite-rank perturbation, going back to the dichotomy between localized and delocalized eigenvectors.

To conclude we mention that Bloemendal and Virag [2013; 2011] studied the largest eigenvalue of a sample covariance matrix from a spiked population in a model which is the real counterpart of the perturbed LUE studied in [Baik et al. 2005]. In particular, they proved a conjecture from that paper regarding the law of the fluctuations of the outliers when the perturbation is properly scaled around its critical value. These results were generalized to non-Gaussian models by Knowles and Yin [2013; 2014], relying on isotropic local semicircle law for such models.

Acknowledgements

The author thanks MSRI for its hospitality during the program “Random matrix theory, interacting particle systems and integrable systems” and for giving her the opportunity to present this work at the workshop “Connections for women, an introduction to random matrices”, which took place in September 2010.

References

[Anderson et al. 2010] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics **118**, Cambridge University Press, Cambridge, 2010.

[Bai and Yao 2008] Z. Bai and J.-f. Yao, "Central limit theorems for eigenvalues in a spiked population model", *Ann. Inst. Henri Poincaré Probab. Stat.* **44**:3 (2008), 447–474.

[Baik and Wang 2011] J. Baik and D. Wang, "On the largest eigenvalue of a Hermitian random matrix model with spiked external source, I: rank 1 case", *Int. Math. Res. Not.* **2011**:22 (2011), 5164–5240.

[Baik and Wang 2013] J. Baik and D. Wang, "On the largest eigenvalue of a Hermitian random matrix model with spiked external source, II: higher rank cases", *Int. Math. Res. Not.* **2013**:14 (2013), 3304–3370.

[Baik et al. 2005] J. Baik, G. Ben Arous, and S. Péché, "Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices", *Ann. Probab.* **33**:5 (2005), 1643–1697.

[Belinschi et al. 2012] S. Belinschi, H. Bercovici, M. Capitaine, and M. Fevrier, "Outliers in the spectrum of large deformed unitarily invariant models", 2012. arXiv 1207.5443

[Ben Arous and Guionnet 1997] G. Ben Arous and A. Guionnet, "Large deviations for Wigner's law and Voiculescu's non-commutative entropy", *Probab. Theory Related Fields* **108**:4 (1997), 517–542.

[Ben Arous et al. 2001] G. Ben Arous, A. Dembo, and A. Guionnet, "Aging of spherical spin glasses", *Probab. Theory Related Fields* **120**:1 (2001), 1–67.

[Benaych-Georges and Nadakuditi 2011] F. Benaych-Georges and R. R. Nadakuditi, "The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices", *Adv. Math.* **227**:1 (2011), 494–521.

[Benaych-Georges et al. 2011] F. Benaych-Georges, A. Guionnet, and M. Maida, "Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices", *Electron. J. Probab.* **16** (2011), no. 60, 1621–1662.

[Benaych-Georges et al. 2012] F. Benaych-Georges, A. Guionnet, and M. Maida, "Large deviations of the extreme eigenvalues of random deformations of matrices", *Probab. Theory Related Fields* **154**:3-4 (2012), 703–751.

[Bloemendal and Virág 2011] A. Bloemendal and B. Virág, "Limits of spiked random matrices, II", 2011. arXiv 1109.3704

[Bloemendal and Virág 2013] A. Bloemendal and B. Virág, "Limits of spiked random matrices I", *Probab. Theory Related Fields* **156**:3-4 (2013), 795–825.

[Bordenave and Caputo 2012] C. Bordenave and P. Caputo, "A large deviations principle for Wigner matrices without Gaussian tails", 2012. arXiv 1207.5570

[Brézin and Hikami 1998] E. Brézin and S. Hikami, "Level spacing of random matrices in an external source", *Phys. Rev. E* (3) **58**:6, part A (1998), 7176–7185.

[Capitaine 2013] M. Capitaine, "Additive/multiplicative free subordination property and limiting eigenvectors of spiked additive deformations of Wigner matrices and spiked sample covariance matrices", *J. Theoret. Probab.* **26**:3 (2013), 595–648.

[Capitaine et al. 2009] M. Capitaine, C. Donati-Martin, and D. Féral, "The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations", *Ann. Probab.* **37**:1 (2009), 1–47.

[Capitaine et al. 2011] M. Capitaine, C. Donati-Martin, D. Féral, and M. Février, "Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices", *Electron. J. Probab.* **16** (2011), no. 64, 1750–1792.

[Capitaine et al. 2012] M. Capitaine, C. Donati-Martin, and D. Féral, "Central limit theorems for eigenvalues of deformations of Wigner matrices", *Ann. Inst. Henri Poincaré Probab. Stat.* **48**:1 (2012), 107–133.

[Chatterjee and Varadhan 2012] S. Chatterjee and S. R. S. Varadhan, "Large deviations for random matrices", *Commun. Stoch. Anal.* **6**:1 (2012), 1–13.

[El Karoui 2005] N. El Karoui, "Recent results about the largest eigenvalue of random covariance matrices and statistical application", *Acta Phys. Polon. B* **36**:9 (2005), 2681–2697.

[Emery et al. 2000] M. Emery, A. Nemirovski, and D. Voiculescu, *Lectures on probability theory and statistics*, Lecture Notes in Mathematics **1738**, Springer, Berlin, 2000.

[Féral and Péché 2007] D. Féral and S. Péché, "The largest eigenvalue of rank one deformation of large Wigner matrices", *Comm. Math. Phys.* **272**:1 (2007), 185–228.

[Fey et al. 2008] A. Fey, R. van der Hofstad, and M. J. Klok, "Large deviations for eigenvalues of sample covariance matrices, with applications to mobile communication systems", *Adv. in Appl. Probab.* **40**:4 (2008), 1048–1071.

[Guionnet and Zeitouni 2002] A. Guionnet and O. Zeitouni, "Large deviations asymptotics for spherical integrals", *J. Funct. Anal.* **188**:2 (2002), 461–515.

[Hardy 2012] A. Hardy, "A note on large deviations for 2D Coulomb gas with weakly confining potential", *Electron. Commun. Probab.* **17** (2012), no. 19, 12.

[Hardy and Kuijlaars 2013] A. Hardy and A. B. J. Kuijlaars, "Large deviations for a non-centered Wishart matrix", *Random Matrices Theory Appl.* **2**:1 (2013), 1250016, 40.

[Helmke and Rosenthal 1995] U. Helmke and J. Rosenthal, "Eigenvalue inequalities and Schubert calculus", *Math. Nachr.* **171** (1995), 207–225.

[Johnstone 2001] I. M. Johnstone, "On the distribution of the largest eigenvalue in principal components analysis", *Ann. Statist.* **29**:2 (2001), 295–327.

[Klyachko 1998] A. A. Klyachko, "Stable bundles, representation theory and Hermitian operators", *Selecta Math.* (*N.S.*) **4**:3 (1998), 419–445.

[Knowles and Yin 2013] A. Knowles and J. Yin, "The isotropic semicircle law and deformation of Wigner matrices", *Comm. Pure Appl. Math.* **66**:11 (2013), 1663–1750.

[Knowles and Yin 2014] A. Knowles and J. Yin, "The outliers of a deformed Wigner matrix", *Ann. Probab.* **42**:5 (2014), 1980–2031.

[Knutson and Tao 2001] A. Knutson and T. Tao, "Honeycombs and sums of Hermitian matrices", *Notices Amer. Math. Soc.* **48**:2 (2001), 175–186.

[Maïda 2007] M. Maïda, "Large deviations for the largest eigenvalue of rank one deformations of Gaussian ensembles", *Electron. J. Probab.* **12** (2007), 1131–1150.

[Péché 2006] S. Péché, "The largest eigenvalue of small rank perturbations of Hermitian random matrices", *Probab. Theory Related Fields* **134**:1 (2006), 127–173.

[Pizzo et al. 2013] A. Pizzo, D. Renfrew, and A. Soshnikov, "On finite rank deformations of Wigner matrices", *Ann. Inst. Henri Poincaré Probab. Stat.* **49**:1 (2013), 64–94.

[Renfrew and Soshnikov 2013] D. Renfrew and A. Soshnikov, "On finite rank deformations of Wigner matrices, II: delocalized perturbations", *Random Matrices Theory Appl.* **2**:1 (2013), 1250015, 36.

[Weyl 1912] H. Weyl, "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)", *Math. Ann.* **71**:4 (1912), 441–479.

mylene.maida@math.univ-lille1.fr *Laboratoire Paul Painlevé, Université Lille 1, Bâtiment M3, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex,, France*