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Convolution symmetries of integrable hierarchies, matrix models and τ-functions

J. HARNAD AND A. YU. ORLOV

Generalized convolution symmetries of integrable hierarchies of KP and 2D-Toda type act diagonally on the Hilbert space $\mathcal{H} = L^2(S^1)$ in the standard monomial basis. The induced transformations on the Hilbert space Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ may be viewed as symmetries of these hierarchies, acting upon the Sato–Segal–Wilson τ-functions, and thereby generating new solutions of the hierarchies. The corresponding transformations of the associated fermionic Fock space are also diagonal in the standard orthonormal basis, labeled by integer partitions. The Plücker coordinates of the image under the Plücker map of the element $W \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ defining the initial point under the commuting KP flows are the coefficients in the single and double Schur function expansions of the associated τ-functions. These are therefore multiplied by the eigenvalues of the convolution action in the fermionic representation. Applying such transformations to standard matrix model integrals, we obtain new matrix models of externally coupled type whose partition functions are thus also seen to be KP or 2D-Toda τ-functions. More general multiple integral representations of tau functions are similarly obtained, as well as finite determinantal expressions for them.

1. Introduction: convolution symmetries of τ-functions

Solutions of integrable hierarchies of KP and 2D-Toda type are determined by their τ-functions [Sato 1981; Sato and Sato 1983; Segal and Wilson 1985]. Infinite sequences of such KP τ-functions $\{\tau(N,\boldsymbol{t})\}_{N\in\mathbb{Z}}$, depending on the infinite set of commuting flow parameters $\boldsymbol{t} = (t_1, t_2, \dots)$ and an integer lattice label N, may be associated in a standard fashion (see references just cited) to elements of a "universal phase" space, viewed as an infinite Grassmann manifold or flag manifold. These satisfy the Hirota bilinear equations of the KP hierarchy

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and also, in certain cases (e.g., exponential flows of matrix model integrals induced by trace invariants), the equations of the Toda lattice hierarchy.

The τ-functions may be expanded as infinite series in a basis of Schur functions $s_\lambda(\boldsymbol{t})$, labelled by integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$

$$\tau(N, \boldsymbol{t}) = \sum_\lambda \pi_N(\lambda) s_\lambda(\boldsymbol{t}). \tag{1-1}$$

In the approach of Sato and Segal–Wilson [Sato 1981; Segal and Wilson 1985], the coefficients $\pi_N(\lambda)$ are interpreted as Plücker coordinates of the image $\mathcal{P}(W)$ of an element W of a Hilbert space Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ under the Plücker map

$$\mathcal{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbb{P}(\mathcal{F}) \tag{1-2}$$

into the projectivization of the semiinfinite exterior space $\mathcal{F} := \Lambda\mathcal{H}$ (the Fermionic Fock space). In [Segal and Wilson 1985], the Hilbert pace $\mathcal{H}$ is chosen as the square integrable functions $L^2(S^1)$ on the unit circle in the complex z-plane and the elements of $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ are subspaces of $\mathcal{H} = L^2(S^1)$ that are "commensurable" with the subspace $\mathcal{H}_+ \subset \mathcal{H}$ of functions admitting a holomorphic extension to the interior disk.

The image $\mathcal{P}(\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}))$ of the Grassmannian under the Plücker map consists of all decomposable elements of $\Lambda\mathcal{H}$, which is the intersection of the infinite set of quadrics defined by the Plücker relations. The latter are equivalent to the infinite set of Hirota bilinear differential relations [Jimbo and Miwa 1983; Sato 1981; Sato and Sato 1983] for $\tau(N, \boldsymbol{t})$, which are the defining property of τ-functions. Through the Sato formula for the Baker–Akhiezer function

$$\Psi_N(z, \boldsymbol{t}) = e^{\sum_{i=1}^\infty t_i z^i} \frac{\tau(N, \boldsymbol{t} - [z^{-1}])}{\tau(N, \boldsymbol{t})}, \quad [z^{-1}] := (z^{-1}, 2z^{-2}, 3z^{-3}, \dots), \tag{1-3}$$

these equations are equivalent to the KP hierarchy and their associated Lax equations.

The 2D-Toda hierarchy [Jimbo and Miwa 1983; Ueno and Takasaki 1984] can similarly be expressed in terms of τ-functions depending on N, $\boldsymbol{t}$ and a further infinite sequence of flow parameters $\tilde{\boldsymbol{t}} = (\tilde{t}_1, \tilde{t}_2, \dots)$. These admit double Schur function expansions [Takasaki 1984]

$$\tau^{(2)}(N, \boldsymbol{t}, \tilde{\boldsymbol{t}}) = \sum_\lambda \sum_\mu B_N(\lambda, \mu) s_\lambda(\boldsymbol{t}) s_\mu(\tilde{\boldsymbol{t}}), \tag{1-4}$$

in which the coefficients $B_N(\lambda, \mu)$ have a similar interpretation in terms of Plücker coordinates. They also satisfy an infinite set of bilinear differential Hirota-type relations in both sequences of flow variables and difference-differential equations relating different lattice points. For fixed N, they include the KP Hirota equations of the KP hierarchy in each of the two sets of flow variables, so we refer to them as 2D-Toda tau functions.

Starting with any given τ-function of KP-Toda or 2D-Toda type, it will be shown in the following that new τ-functions can be constructed, satisfying the same sets of bilinear relations, having the following Schur function expansions:

$$\widetilde{C}_\rho(\tau)(N, \boldsymbol{t}) = \sum_\lambda r_\lambda(N)\pi_N(\lambda)s_\lambda(\boldsymbol{t}), \tag{1-5}$$

$$\widetilde{C}^{(2)}_{\rho,\tilde{\rho}}(\tau^{(2)})(N, \boldsymbol{t}, \tilde{\boldsymbol{t}}) = \sum_\lambda \sum_\mu r_\lambda(N) B_N(\lambda, \mu)\tilde{r}_\mu(N)s_\lambda(\boldsymbol{t})s_\mu(\tilde{\boldsymbol{t}}), \tag{1-6}$$

where the factors $r_\lambda(N)$, $\tilde{r}_\lambda(N)$ are defined in terms of a given pair of infinite sequences of nonvanishing constants $\{r_i\}_{i\in\mathbb{Z}}$, $\{\tilde{r}_i\}_{i\in\mathbb{Z}}$ through the formulae

$$r_\lambda(N) := c_r(N) \prod_{(i,j)\in\lambda} r_{N-i+j}, \quad \tilde{r}_\mu(N) := c_{\tilde{r}}(N) \prod_{(k,l)\in\mu} \tilde{r}_{N-k+l}. \tag{1-7}$$

Here the products are over pairs of positive integers $(i, j) \in \lambda$ and $(k, l) \in \mu$ that lie within the matrix locations represented by the Young diagrams of the partitions λ and μ, respectively,

$$c_r(N) := \prod_{i=1}^{\infty} \frac{\rho_{N-i}}{\rho_{-i}}, \tag{1-8}$$

and

$$r_i = \frac{\rho_i}{\rho_{i-1}}. \tag{1-9}$$

The sequence of nonvanishing parameters $\{\rho_i\}$ may be viewed as Fourier coefficients of a function $\rho(z)$ on the unit circle, or a distribution. It will be shown (Proposition 3.1) that, in terms of the elements of the subspace $W \subset L^2(S^1)$ corresponding to a point of the Grassmannian, the transformations (1-5), (1-6) mean taking a generalized convolution product with $\rho(z)$ (and similarly for $\tilde{r}_i$). These will therefore be referred to as (generalized) *convolution symmetries*.

With the usual 2D-Toda flow parameters $(\boldsymbol{t}, \tilde{\boldsymbol{t}})$ fixed at some specific values, such transformations extend to an infinite abelian group of commuting flows whose parameters determine the ρ_i's. This has been used to generate new classes of solutions of integrable hierarchies [Bettelheim et al. 2007; Orlov 2006; Orlov and Scherbin 2001]. In the present work, they are studied rather as individual transformations, for fixed values of the parameters ρ_i which, when applied to a given KP-Toda or 2D-Toda τ-function, produce a new one. Particular cases that implicitly use such transformations as symmetries have found applications, for example, as generating functions for topological invariants related to Riemann surfaces, such as Gromov–Witten invariants and Hurwitz numbers [Okounkov 2000; Okounkov and Pandharipande 2006].

As an immediate application, we may start with an integral over $N \times N$ Hermitian matrices:

$$Z_N(\boldsymbol{t}) = \int_{M\in\mathbb{H}^{N\times N}} d\mu(M)\, e^{\operatorname{tr}\sum_{i=1}^{\infty} t_i M^i}, \tag{1-10}$$

where $d\mu$ is a suitably defined $U(N)$ conjugation invariant measure on the space $\mathbb{H}^{N\times N}$ of Hermitian $N\times N$ matrices,. This is known to be a KP-Toda τ-function [Kharchev et al. 1991]. Applying a convolution symmetry (1-5) with $\rho(z)$ taken essentially as the exponential function e^z on the unit disc, and evaluating at flow parameter values

$$t_i = \frac{1}{i}\operatorname{tr}(A^i), \quad \boldsymbol{t} = [A] = (t_1, t_2, \dots), \tag{1-11}$$

for a fixed $N\times N$ Hermitian matrix A we obtain, within a constant multiplicative factor, the externally coupled matrix model integral (Proposition 4.1):

$$Z_{N,\text{ext}}([A]) := \int_{M\in\mathbb{H}^{N\times N}} d\mu(M) e^{\operatorname{tr} AM} = \left(\prod_{i=1}^{N-1} i!\right) \widetilde{C}_\rho(Z_N)([A]). \tag{1-12}$$

Such integrals arise in a number contexts, such as the Kontsevich–Witten generating function [Kontsevich 1992], the Brézin–Hikami model [Brézin and Hikami 1996; Zinn-Justin 1998; 2002] and the complex Wishart ensemble [Silverstein and Bai 1995; Wang 2009]. More general choices for the function $\rho(z)$ are shown in Proposition 4.2 to also determine KP-Toda τ-functions as externally coupled matrix integrals. It is further shown, in Proposition 4.3, that these matrix model τ-functions can be expressed as finite $N\times N$ determinants.

Similarly, Hermitian two-matrix integrals with exponential coupling of Itzykson–Zuber type [Itzykson and Zuber 1980]

$$Z_N^{(2)}(\boldsymbol{t}, \tilde{\boldsymbol{t}}) = \int_{M_1\in\mathbb{H}^{N\times N}} d\mu(M_1) \int_{M_2\in\mathbb{H}^{N\times N}} d\tilde{\mu}(M_2)\, e^{\operatorname{tr}(\sum_{i=1}^{\infty}(t_i M_1^i + \tilde{t}_i M_2^i) + M_1 M_2)} \tag{1-13}$$

are known to be 2D-Toda τ-functions [Adler and van Moerbeke 1999; Harnad and Orlov 2002; Harnad and Orlov 2003; Orlov 2004]. Applying the convolution symmetry (1-6) to (1-13) gives an externally coupled two-matrix integral (Proposition 4.4).

$$\widetilde{C}^{(2)}_{\rho,\tilde{\rho}}(Z_N^{(2)})([A],[B]) = \int_{M_1\in\mathbb{H}^{N\times N}} d\mu(M_2) \int_{M_2\in\mathbb{H}^{N\times N}} d\tilde{\mu}(M_2) \times \tau_r(N,[A],[M_2])\tau_{\tilde{r}}(N,[B],[M_2]) e^{\operatorname{tr}(M_1 M_2)}, \tag{1-14}$$

where $[A]$ and $[B]$ signify the sequences $\left\{\frac{1}{i}\operatorname{tr}(A^i)\right\}_{i\in\mathbb{N}^+}$ and $\left\{\frac{1}{i}\operatorname{tr}(B^i)\right\}_{i\in\mathbb{N}^+}$ of trace invariants for the pair of Hermitian matrices A and B and

$$\tau_r(N,[A],[M_1]) = \sum_{\lambda} r_\lambda(N)s_\lambda([A])s_\lambda([M_1]), \tag{1-15}$$

$$\tau_{\tilde{r}}(N,[B],[M_2]) = \sum_{\lambda} \tilde{r}_\lambda(N)s_\lambda([B])s_\lambda([M_2]). \tag{1-16}$$

This doubly externally coupled two-matrix model τ-function can also be expressed in a finite $N\times N$ determinantal form (eq. (4-37), Proposition 4.5).

This approach can also be extended to more general 2D-Toda τ-functions admitting multiple integral representations of the form (4-47). Applying the convolution symmetry (1-6) then gives a new 2D-Toda τ-function expressible either as a multiple integral (eq. (4-48), Proposition 4.6) or as a finite determinant (eq. (4-50), Proposition 4.7).

The key to understanding these constructions, and further results following from them, is the interpretation of the Sato τ-function as a vacuum state expectation value of products of exponentials of bilinear combinations of fermionic creation and annihilation operators [Sato 1981; Jimbo and Miwa 1983; Ueno and Takasaki 1984]. This well-known construction will be summarized in the next section.

2. Fermionic construction of τ-functions

We recall here the approach to the construction of τ-functions for integrable hierarchies of the KP and Toda types due to Sato [Sato 1981; Sato and Sato 1983], the Kyoto school [Date et al. 1981/82; 1983; Jimbo and Miwa 1983; Ueno and Takasaki 1984] and Segal and Wilson [1985].

2.1. *Hilbert space Grassmannian and fermionic Fock space.*

We begin with the "first quantized" Hilbert space $\mathcal{H}$, which will be identified, as in [Segal and Wilson 1985], with the space of square integrable functions on the unit circle

$$\mathcal{H} = L^2(S^1) = \mathcal{H}_+ + \mathcal{H}_-, \tag{2-1}$$

decomposed as the direct sum of the subspaces $\mathcal{H}_+ = \operatorname{span}\{z^i\}_{i\in\mathbb{N}}$ and $\mathcal{H}_- = \operatorname{span}\{z^{-i}\}_{i\in\mathbb{N}^+}$ consisting of functions that admit holomorphic extensions, respectively, to the interior and exterior of the unit circle S^1 in the complex z-plane, with the latter vanishing at $z=\infty$. For consistency with other conventions, the monomial (orthonormal) basis elements $\{e_i\}_{i\in\mathbb{Z}}$ of $\mathcal{H}$ will be denoted by

$$e_i := z^{-i-1}, \quad i\in\mathbb{Z}. \tag{2-2}$$

Two infinite abelian groups act on $\mathcal{H}$ by multiplication:

$$\Gamma_+ := \left\{\gamma_+(\boldsymbol{t}) := e^{\sum_{i=1}^{\infty} t_i z^i}\right\} \quad \text{and} \quad \Gamma_- := \left\{\gamma_-(\boldsymbol{t}) := e^{\sum_{i=1}^{\infty} t_i z^{-i}}\right\}, \tag{2-3}$$

where $\boldsymbol{t} := (t_1, t_2, \dots)$ is an infinite sequence of (complex) flow parameters corresponding to the one-parameter subgroups. More generally, we have the general linear group $\mathrm{GL}(\mathcal{H})$ consisting of invertible endomorphisms connected to the identity with well defined determinants. (See [Segal and Wilson 1985] for more detailed definitions of this and what follows.)

We consider the Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ of subspaces $W \subset \mathcal{H}$ that are *commensurable* with $\mathcal{H}_+ \subset \mathcal{H}$ (in the sense of Segal and Wilson, namely that orthogonal projection $\pi_+ : W \to \mathcal{H}_+$ to $\mathcal{H}_+$ is a Fredholm operator while projection $\pi_- : W \to \mathcal{H}_-$ to $\mathcal{H}_-$ is Hilbert–Schmidt). The connected components of $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, denoted $\mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$, $N \in \mathbb{Z}$, consist of those $W \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ for which the Fredholm index of $\pi_+ : W \to \mathcal{H}_+$ is N. These are the $\mathrm{GL}(\mathcal{H})$ orbits of the subspaces

$$\mathcal{H}_+^N := z^{-N}\mathcal{H}_+ \subset \mathcal{H}, \tag{2-4}$$

whose elements are denoted $W_{g,N} = g(\mathcal{H}_+^N) \in \mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$. The solutions to the KP hierarchy are given by the τ-function $\tau_{N,g}(\boldsymbol{t})$ as defined below, which determines the orbit of $W_{g,N}$ in $\mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$ under Γ_+ through its Plücker coordinates. In the terminology of Segal and Wilson, the index N is called the "virtual dimension" of the elements $W_{g,N} \in \mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$; i.e., their dimension *relative* to the those in the component $\mathrm{Gr}_{\mathcal{H}_+^0}(\mathcal{H})$ containing $\mathcal{H}_+$.

The *Fermionic Fock space* is the exterior space $\mathcal{F} := \Lambda\mathcal{H}$ consisting of (a completion of) the span of the semiinfinite wedge products:

$$|\lambda, N\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots, \tag{2-5}$$

where $\{l_j\}_{j\in\mathbb{N}^+}$ is a strictly decreasing sequence of integers that saturates, for sufficiently large j, to a descending sequence of consecutive integers. This is equivalent to requiring that there be an associated pair (λ, N) consisting of an integer N and a partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$ of length $\ell(\lambda)$ and weight $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$, where the parts λ_i are a weakly decreasing sequence of nonnegative integers that are positive for $i \le \ell(\lambda)$, and zero for $i > \ell(\lambda)$, such that the sequence $\{l_j\}_{j\in\mathbb{N}^+}$ is given by

$$l_j := \lambda_j - j + N. \tag{2-6}$$

In particular, for the trivial partition $\lambda = (0)$, we have the "charge N vacuum" vector

$$|0, N\rangle = e_{N-1} \wedge e_{N-2} \wedge \cdots, \tag{2-7}$$

which will henceforth be denoted $|N\rangle$. The full Fock space $\mathcal{F}$ thus admits a decomposition as an orthogonal direct sum of the subspaces $\mathcal{F}_N$ of states with charge N

$$\mathcal{F} = \bigoplus_{N\in\mathbb{Z}} \mathcal{F}_N. \tag{2-8}$$

Denoting by $\{\tilde{e}^i\}_{i\in\mathbb{Z}}$ the basis for $\mathcal{H}^*$ dual to the monomial basis $\{e_i\}_{i\in\mathbb{Z}}$ for $\mathcal{H}$, we define the Fermi creation and annihilation operators ψ_i and $\psi_i^\dagger$ on an arbitrary vector $v \in \mathcal{F}$ by exterior and interior multiplication, respectively:

$$\psi_i v = e_i \wedge v, \quad \psi_i^\dagger v := i_{\tilde{e}^i} v, \quad v \in \mathcal{F}. \tag{2-9}$$

These satisfy the standard canonical anticommutation relations generating the Clifford algebra on $\mathcal{H} + \mathcal{H}^*$ with respect to the natural corresponding quadratic form

$$[\psi_i, \psi_j]_+ = [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}. \tag{2-10}$$

The basis states $|\lambda, N\rangle$ may be expressed in terms of creation and annihilation operators acting upon the charge N vacuum vector as follows [Harnad and Orlov 2007]

$$|\lambda, N\rangle = (-1)^{\sum_{i=1}^k \beta_i} \prod_{i=1}^k \psi_{N+\alpha_i} \psi^\dagger_{N-\beta_i-1} |N\rangle, \tag{2-11}$$

where $(\alpha_1, \dots \alpha_k | \beta_1, \dots, \beta_k)$ is the Frobenius notation (see [Macdonald 1995]) for the partition λ; i.e., α_i is the number of boxes in the corresponding Young diagram to the right of the i-th diagonal element and β_i the number below it.

The Plücker map $\mathfrak{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbb{P}(\mathcal{F})$ takes the subspace

$$W = \mathrm{span}(w_1, w_2, \dots) \tag{2-12}$$

into the projectivization of the exterior product of its basis elements:

$$\mathfrak{P} : \mathrm{span}(w_1, w_2, \dots) \mapsto [w_1 \wedge w_2 \wedge \cdots], \tag{2-13}$$

and may be lifted to a map from the bundle $\mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ of frames on $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ to $\mathcal{F}$:

$$\hat{\mathfrak{P}} : \mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathcal{F}, \quad (w_1, w_2, \dots) \mapsto w_1 \wedge w_2 \wedge \cdots. \tag{2-14}$$

These interlace the lift of the action of the abelian group $\Gamma_+ \times \mathcal{H} \to \mathcal{H}$ to $\mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ or $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ with the following representation of Γ_+ on $\mathcal{F}$ (and its projectivization):

$$\gamma_+(\boldsymbol{t}) : v \mapsto \hat{\gamma}_+(\boldsymbol{t}) v, \quad \hat{\gamma}_+(\boldsymbol{t}) := e^{\sum_{i=1}^\infty t_i H_i}, \quad v \in \mathcal{F}, \tag{2-15}$$

where

$$H_i := \sum_{n\in\mathbb{Z}} \psi_n \psi^\dagger_{n+i}, \quad i \in \mathbb{Z},\ i \neq 0, \tag{2-16}$$

and $\boldsymbol{t} = (t_1, t_2, \dots)$ is the infinite sequence of flow parameters. Similarly, the Plücker maps $\hat{\mathfrak{P}}$ and $\mathfrak{P}$ interlace the lift of the action of the abelian group $\Gamma_- \times \mathcal{H} \to \mathcal{H}$ to $\mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ or $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ with the following representation of Γ_- on $\mathcal{F}$ (and its projectivization):

$$\gamma_-(\boldsymbol{t}) : v \mapsto \hat{\gamma}_-(\boldsymbol{t})v, \quad \hat{\gamma}_-(\boldsymbol{t}) := e^{\sum_{i=1}^\infty t_i H_{-i}}, \quad v \in \mathcal{F}. \tag{2-17}$$

Remark 2.1. Note that the image under the Plücker map of the virtual dimension N component $\mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$ of the Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is the $\mathrm{GL}(\mathcal{H})$ orbit of the charged vacuum state $|N\rangle$, consisting of all decomposable elements of $\mathcal{F}_N$.

The KP-Toda τ-function $\tau_g(N, \boldsymbol{t})$ corresponding to the element $W_{g,N} \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is given, within a nonzero multiplicative constant, by applying the group elements $\gamma_+(\boldsymbol{t})$ to $W_{g,N}$, to obtain the Γ_+ orbit

$$\{W_{g,N}(\boldsymbol{t}) := \gamma_+(\boldsymbol{t})(W_{g,N})\}, \tag{2-18}$$

and taking the linear coordinate (within projectivization) of the image under the Plücker map corresponding to projection along the basis element $|N\rangle$

$$\tau_g(N, \boldsymbol{t}) = \langle N|\hat{\mathfrak{P}}(W_{g,N}(\boldsymbol{t}))\rangle. \tag{2-19}$$

If the group element $g \in \mathrm{GL}(\mathcal{H})$ is interpreted, relative to the monomial basis $\{e_i\}_{i\in\mathbb{Z}}$, as an infinite matrix exponential $g = e^{\boldsymbol{A}}$ of an element of the Lie algebra $A \in \mathfrak{gl}(\mathcal{H})$ with matrix elements A_{ij}, then the corresponding representation of $\mathrm{GL}(\mathcal{H})$ on $\mathcal{F}$ is given by

$$\hat{g} := e^{\sum_{i,j\in\mathbb{Z}} A_{ij} :\psi_i \psi_j^\dagger:}, \tag{2-20}$$

where $:\ :$ denotes normal ordering (i.e., annihilation operators $\psi_j^\dagger$ appearing to the right when $j \geq 0$ and creation operators ψ_i to the right when $i < 0$). This gives the following expression for $\tau_{N,g}(\boldsymbol{t})$ as a charge N vacuum state expectation value of a product of exponentiated bilinears in the Fermi creation and annihilation operators

$$\tau_g(N, \boldsymbol{t}) = \langle N|\hat{\gamma}_+(\boldsymbol{t})\hat{g}|N\rangle. \tag{2-21}$$

The equations of the KP hierarchy are then equivalent to the well-known infinite system of Hirota bilinear equations [Jimbo and Miwa 1983; Sato 1981; Sato and Sato 1983] which, in turn, are just the Plücker relations for the decomposable element $\mathfrak{P}(W_{g,N}(\boldsymbol{t})) \in \mathbb{P}(\mathcal{F}_N)$.

Similarly, we may define a 2-Toda sequence of double KP τ-functions associated to the group element $\hat{g}$

$$\tau_g^{(2)}(N,\boldsymbol{t},\tilde{\boldsymbol{t}}) = \langle N|\hat{\gamma}_+(\boldsymbol{t})\hat{g}\hat{\gamma}_-(\tilde{\boldsymbol{t}})|N\rangle, \tag{2-22}$$

where $\tilde{\boldsymbol{t}} = (\tilde{t}_1, \tilde{t}_2, \dots)$ is a second infinite set of flow parameters. This may similarly be shown to satisfy the Hirota bilinear relations of the 2D-Toda hierarchy.

2.2. *Schur function expansions.*

Evaluating the matrix elements of $\hat{\gamma}_+(\boldsymbol{t})$ and $\hat{\gamma}_-(\boldsymbol{t})$ between the states $|N\rangle$ and $|\lambda, N\rangle$ gives the Schur function

$$\langle N|\hat{\gamma}_+(\boldsymbol{t})|\lambda, N\rangle = \langle\lambda, N|\hat{\gamma}_-(\boldsymbol{t})|N\rangle = s_\lambda(\boldsymbol{t}), \tag{2-23}$$

(cf. [Sato 1981; Sato and Sato 1983; Harnad and Orlov 2003; 2006] which is determined through the Jacobi–Trudy formula

$$s_\lambda(\boldsymbol{t}) = \det(h_{\lambda_i - i + j}(\boldsymbol{t}))|_{1\le i,j\le \ell(\lambda)} \tag{2-24}$$

in terms of the complete symmetric functions $h_i(\boldsymbol{t})$, defined by

$$e^{\sum_{i=1}^{\infty} t_i z^i} = \sum_{i=0}^{\infty} h_i(\boldsymbol{t}) z^i. \tag{2-25}$$

Inserting a sum over a complete set of intermediate states in Equations (2-21), (2-22), we obtain the single and double Schur function expansions

$$\tau_g(N,\boldsymbol{t}) = \sum_\lambda \pi_{N,g}(\lambda) s_\lambda(\boldsymbol{t}), \tag{2-26}$$

$$\tau_g^{(2)}(N,\boldsymbol{t},\tilde{\boldsymbol{t}}) = \sum_\lambda \sum_\mu B_{N,g}(\lambda,\mu) s_\lambda(\boldsymbol{t}) s_\mu(\tilde{\boldsymbol{t}}). \tag{2-27}$$

Here the sum is over all partitions λ and μ and

$$\pi_{N,g}(\lambda) = \langle\lambda, N|\hat{g}|N\rangle \tag{2-28}$$

is the Plücker coordinate of the image of the element $g(\mathcal{H}_+^N) \in \mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$ under the Plücker map $\mathfrak{P}$ along the basis direction $|\lambda, N\rangle$ in the charge N sector $\mathcal{F}_N$ of the Fock space. Similarly,

$$B_{N,g}(\lambda,\mu) = \langle\lambda, N|\hat{g}|\mu, N\rangle \tag{2-29}$$

may be viewed as the $|\lambda, N\rangle$ Plücker coordinate of the image of the element $g(w_{\mu,N}) \in \mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H})$, where

$$w_{\mu,N} := \mathrm{span}\{e_{\mu_i - i + N}\} \in \mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H}). \tag{2-30}$$

In particular, choosing g to be the identity element $\mathbb{1}$, and using Wick's theorem (or equivalently, the Cauchy–Binet identity in semiinfinite form), we obtain [Harnad and Orlov 2003]

$$\tau^{(2)}_{\mathbb{1}}(N,\boldsymbol{t},\tilde{\boldsymbol{t}}) = \langle N|\hat{\gamma}_+(\boldsymbol{t})\hat{\gamma}_-(\tilde{\boldsymbol{t}})|N\rangle = \sum_{\lambda} s_\lambda(\boldsymbol{t})s_\lambda(\tilde{\boldsymbol{t}}) = e^{\sum_{i=1}^{\infty} i t_i \tilde{t}_i}, \tag{2-31}$$

where the last equality is the Cauchy–Littlewood identity (cf. [Macdonald 1995]).

3. Convolution symmetries

3.1. *Convolution action on* $\mathcal{H}$ *and* $\mathbf{Gr}_{\mathcal{H}_+}(\mathcal{H})$.

Consider now an infinite sequence of complex numbers $\{T_i\}_{i\in\mathbb{Z}}$, and define

$$\rho_i := e^{T_i}, \quad r_i := \frac{\rho_i}{\rho_{i-1}}, \quad i \in \mathbb{Z}. \tag{3-1}$$

In the following, we will assume that the series $\sum_{i=1}^{\infty} T_{-i}$ converges and that

$$\lim_{i\to\infty} |r_i| = r \le 1 \tag{3-2}$$

(although, for some purposes, the latter condition may be weakened). It follows that the two series

$$\rho_+(z) = \sum_{i=0}^{\infty} \rho_i z^i \quad \text{and} \quad \rho_-(z) = \sum_{i=1}^{\infty} \rho_{-i} z^{-i} \tag{3-3}$$

are absolutely convergent in the interior and exterior of the unit circle $|z| = 1$, respectively, defining analytic functions $\rho_\pm(z)$ in these regions and that

$$R_\rho := \prod_{i=1}^{\infty} \rho_{-i} \tag{3-4}$$

converges to a finite value. If the inequality (3-2) is strict, $\rho_+(z)$ extends to the unit circle, defining a function in $L^2(S^1)$. Henceforth, we denote the pair (ρ_+, ρ_-) by ρ, where the latter can be viewed as a sum $\rho_- + \rho_+$ in the sense of distributional convolutions, as defined below.

If $w \in L^2(S^1)$ has the Fourier series decomposition

$$w(z) = \sum_{i=-\infty}^{\infty} w_i e_i = \sum_{i=-\infty}^{\infty} w_i z^{-i-1} = w_+(z) + w_-(z), \tag{3-5}$$

where

$$w_+(z) := \sum_{i=0}^{\infty} w_{-i-1} z^i, \quad w_-(z) := \sum_{i=1}^{\infty} w_i z^{-i-1}, \tag{3-6}$$

(note the different labelling conventions in (3-3) and (3-6)), we can define a bounded linear map $C_\rho : L^2(S^1) \to L^2(S^1)$ that has the effect of multiplying each Fourier coefficient w_i by the factor ρ_i, and hence each basis element e_i by ρ_i:

$$C_\rho(w)(z) = \sum_{i=-\infty}^{\infty} \rho_i w_i z^{-i-1} = C_\rho(w)_+ + C_\rho(w)_-. \tag{3-7}$$

This can be interpreted as taking a convolution product with the function (or distribution)

$$\tilde{\rho}(z) = \tilde{\rho}_+(z) + \tilde{\rho}_-(z), \tag{3-8}$$

where

$$\tilde{\rho}_+(z) := z^{-1}\rho_-(z^{-1}) = \sum_{i=0}^{\infty} \rho_{-i-1} z^i, \tag{3-9}$$

$$\tilde{\rho}_-(z) := z^{-1}\rho_+(z^{-1}) = \sum_{i=0}^{\infty} \rho_i z^{-i-1}, \tag{3-10}$$

$$C_\rho(w)_+(z) := \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \oint_{|\zeta|=1-\epsilon} \tilde{\rho}_+(\zeta) w_+(z/\zeta)\zeta^{-1}\, d\zeta, \tag{3-11}$$

$$C_\rho(w)_-(z) := \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \oint_{|\zeta|=1+\epsilon} \tilde{\rho}_-(\zeta) w_-(z/\zeta)\zeta^{-1}\, d\zeta \tag{3-12}$$

(with the contour integrals taken counterclockwise).

If $\rho_-(z)$ extends analytically to S^1, eq. (3-11) is an ordinary convolution product on the circle (in exponential variables). In the examples detailed below, all but a finite number of the T_{-i} values vanish for $i > 0$, and hence the infinite product (3-4) is really finite, but $\rho_-(z)$ is rational with a pole at $z = 1$ and the convolution product (3-12) may be understood on S^1 only in the sense of distributions.

Remark 3.1. Note that the class of generalized convolution mappings defined by (3-7)–(3-12) only forms a semigroup since, although they may be invertible, their inverse does not generally belong to the same class. It may be extended to a group by dropping the condition (3-2), or restricted to one by requiring $r = 1$, but this will not be needed in the sequel. The linear maps $C_\rho : \mathcal{H} \to \mathcal{H}$ may nevertheless be interpreted as elements of $\mathrm{GL}(\mathcal{H})$, and are simply represented in the monomial basis $\{e_i\}$ by the diagonal matrix $\mathrm{diag}\{\rho_i\}$. They thus belong to the abelian subgroup of $\mathrm{GL}(\mathcal{H})$ consisting of invertible elements that are diagonal in the monomial basis.

Remark 3.2. Since the Baker–Akhiezer function (1-3), evaluated at all values of the parameters $\boldsymbol{t} = (t_1, t_2, \dots)$, spans the shifted element $z^N(W_{g,N}) \in \mathrm{Gr}_{\mathcal{H}^0_+}(\mathcal{H})$

in the zero virtual dimension component of the Grassmannian, the convolution action (3-11), (3-12), lifted to the Grassmannian, may be obtained by applying its conjugate $z^N \circ C_\rho \circ z^{-N}$ under the shift map

$$z^{-N} : \mathrm{Gr}_{\mathcal{H}_+^0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{H}_+^N}(\mathcal{H}) \tag{3-13}$$

to $\Psi_N(z, \boldsymbol{t})$. But note that, at fixed values of the flow parameters $\boldsymbol{t}$, this does not equal the value of the Baker–Akhiezer function corresponding to the transformed τ-function as defined below; only the subspaces of $\mathcal{H}$ that they span, varying over the $\boldsymbol{t}$ values, will coincide. This fact will not be used explicitly in the following, but it underlies the geometrical meaning of generalized convolutions as symmetries of KP-Toda and 2D-Toda hierarchies.

3.2. *Convolution action on Fock space.*

We now consider the action $\widehat{C} \times \mathcal{F} \to \mathcal{F}$ of the abelian subgroup of $\mathrm{GL}(\mathcal{H})$ consisting of diagonal elements in the monomial basis, and associate an element $\widehat{C}_\rho \in \widehat{C}$ to each sequence $\{\rho_i\}_{i\in\mathbb{Z}}$ defined as above, such that the Plücker map $\widehat{\mathfrak{P}}$ intertwines the $\widehat{C}_\rho$ action with that of C_ρ, lifted to the bundle $\mathrm{Fr}_{\mathcal{H}_+}(\mathcal{H})$ of frames over $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, and is equivariant with respect to group multiplication in $\widehat{C}$.

To do this, we first introduce the abelian algebra generated by the operators

$$K_i := :\psi_i \psi_i^\dagger: = \begin{cases} \psi_i \psi_i^\dagger & \text{if } i \geq 0, \\ -\psi_i^\dagger \psi_i & \text{if } i < 0, \end{cases} \tag{3-14}$$

$$[K_i, K_j] = 0, \quad i, j \in \mathbb{Z}. \tag{3-15}$$

For $\{\rho_i = e^{T_i}\}_{i\in\mathbb{Z}}$ as above, define the operator

$$\widehat{C}_\rho := e^{\sum_{i=-\infty}^{\infty} T_i K_i}. \tag{3-16}$$

Definition 3.1. For each pair (λ, N), where $N \in \mathbb{Z}$, and λ is a partition which, expressed in Frobenius notation, is $(\alpha_1 \cdots \alpha_k \mid \beta_1 \cdots \beta_k)$, let

$$r_\lambda(N) := c_r(N) \prod_{(i,j)\in\lambda} r_{N-i+j} = c_r(N) \left(\prod_{i=1}^{k} \frac{\rho_{N+\alpha_i}}{\rho_{N-\beta_i-1}} \right), \tag{3-17}$$

$$c_r(N) := \begin{cases} \prod_{i=0}^{N-1} \rho_i & \text{if } N > 0, \\ 1 & \text{if } N = 0, \\ \prod_{i=N}^{-1} \rho_i^{-1} & \text{if } N < 0. \end{cases} \tag{3-18}$$

Here the inclusion $(i, j) \in \lambda$ is understood to mean that the matrix location (i, j) corresponds to a box within the Young diagram of the partition λ; that is, $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$. The second equality in (3-17) follows from the definition (3-1).

It follows that $\widehat{C}_\rho$ acts diagonally in the basis $\{|\lambda, N\rangle\}$, with eigenvalues $r_\lambda(N)$.

Lemma 3.1.
$$\widehat{C}_\rho|\lambda, N\rangle = r_\lambda(N)|\lambda, N\rangle. \tag{3-18}$$

Proof. Since the Fock space basis element $|\lambda, N\rangle$ is an infinite wedge product

$$|\lambda, N\rangle = e_{l_1} \wedge e_{l_2} \wedge \cdots = (-1)^{\sum_{i=1}^k \beta_i} \prod_{i=1}^{k} \psi_{N+\alpha_i} \psi^\dagger_{N-\beta_i-1}|N\rangle, \tag{3-20}$$
$$l_j := \lambda_j - j + N, \quad j \in \mathbb{N}^+, \tag{3-21}$$

it follows from the definition (2-9) and the normal ordering in (3-14) that the effect of the action of $e^{T_i K_i}$ on $|\lambda, N\rangle$ is to introduce a multiplicative factor ρ_i if $i \geq 0$ and e_i is present in the wedge product (3-20) or ρ_i^{-1} if $i < 0$ and it is absent, and otherwise no factor. Therefore

$$\begin{aligned}
\widehat{C}_\rho|\lambda, N\rangle &= \widehat{C}_\rho(-1)^{\sum_{i=1}^k \beta_i} \prod_{i=1}^{k} \psi_{N+\alpha_i} \psi^\dagger_{N-\beta_i-1}|N\rangle \\
&= \frac{\prod_{i=1}^\infty \rho_{N-i}}{\prod_{i=1}^\infty \rho_{-i}} \left(\prod_{i=1}^{k} \frac{\rho_{N+\alpha_i}}{\rho_{N-\beta_i-1}} \right)|\lambda, N\rangle \\
&= c_r(N)\left(\prod_{i=1}^{k} \frac{\rho_{N+\alpha_i}}{\rho_{N-\beta_i-1}} \right)|\lambda, N\rangle \\
&= r_\lambda(N)|\lambda, N\rangle.
\end{aligned} \tag{3-22}$$

□

Now let

$$W = \operatorname{span}\{w_i(z) \in L^2(S^1)\}_{i \in \mathbb{N}^+} \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \tag{3-23}$$

and view $\{w_i\}_{i \in \mathbb{N}^+}$ as a frame for W.

Lemma 3.2. *The Plücker map $\widehat{\mathfrak{P}}$ intertwines the convolution action* (3-7) *and the $\widehat{C}$-action on $\mathcal{F}$*

$$\widehat{\mathfrak{P}}(\{C_\rho(w_i)\}_{i \in \mathbb{N}^+}) = R_\rho \widehat{C}_\rho(\widehat{\mathfrak{P}}\{w_i\}_{i \in \mathbb{N}^+}), \tag{3-24}$$

with multiplicative factor $R_\rho := \prod_{i=1}^{\infty} \rho_{-i}$.

Proof. Applying C_ρ to each element $w_i \in L^2(S^1)$ defining the frame for $W \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ just multiplies its Fourier coefficients by the factors ρ_j as in (3-7). It follows that the basis element $|\lambda, N\rangle$ is multiplied by the product of the factors ρ_{l_j} corresponding to the terms e_{l_j} it contains, as in (3-18). Equation (3-24) then follows from the definition of the Plücker map $\widehat{\mathfrak{P}}$ and linearity. □

Example 3.1. Choose

$$\rho_+(z) = e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}, \qquad |z| \le 1 \tag{3-25}$$

$$\rho_-(z) = \frac{1}{z-1} = \sum_{i=1}^{\infty} z^{-i}, \quad |z| > 1, \tag{3-26}$$

so

$$\rho_i = \begin{cases} 1/i! & \text{if } i \ge 1, \\ 1 & \text{if } i \le 0, \end{cases} \tag{3-27}$$

$$r_i = \begin{cases} 1/i & \text{if } i \ge 1 \\ 1 & \text{if } i \le 0, \end{cases} \tag{3-28}$$

$$r_\lambda(N) = \frac{1}{\left(\prod_{i=1}^{N-1} i!\right)(N)_\lambda} \quad \text{if } \ell(\lambda) \le N, \tag{3-29, 3-30}$$

where

$$(N)_\lambda := \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N - i + j) \tag{3-31}$$

is the extended Pochhammer symbol.

Example 3.2. Choose

$$\rho_+(z) = \frac{1}{(1-\zeta z)^a} = \sum_{i=0}^{\infty} (a)_i \frac{(\zeta z)^i}{i!}, \quad |\zeta| < 1,\ |z| \le 1, \tag{3-32}$$

and $\rho_-(z)$ again as in (3-26), so

$$\rho_i = \begin{cases} (a)_i \zeta^i / i! & \text{if } i \ge 1, \\ 1 & \text{if } i \le 0, \end{cases} \tag{3-33}$$

$$r_i = \begin{cases} (a-1+i)\zeta / i & \text{if } i \ge 1, \\ 1 & \text{if } i \le 0, \end{cases} \tag{3-34}$$

$$r_\lambda(N) = \left(\prod_{i=0}^{N-1} \frac{(a)_i}{i!}\right) \frac{\zeta^{|\lambda| + \frac{1}{2}N(N-1)} (a-1+N)_\lambda}{(N)_\lambda} \quad \text{if } \ell(\lambda) \le N. \tag{3-35}$$

3.3. *Convolutions and Schur function expansions of τ-functions.*

We now consider the KP-Toda tau function

$$\tau_{C_\rho g}(N, \boldsymbol{t}) = \langle N|\hat{\gamma}_+(\boldsymbol{t})\widehat{C}_\rho \hat{g}|N\rangle, \tag{3-36}$$

obtained by replacing the group element g in (2-21) by $C_\rho g$. Such a τ-function, obtained from τ_g by applying a convolution symmetry will be denoted

$$\tau_{C_\rho g} =: \widetilde{C}_\rho(\tau_g). \tag{3-37}$$

Introducing a second pair $(\tilde{\rho}_+, \tilde{\rho}_-)$, defined as in (3-3), with the Fourier coefficients ρ_i replaced by $\tilde{\rho}_i$, we also consider the 2-Toda tau function

$$\tau^{(2)}_{C_\rho g C_{\tilde{\rho}}}(N, \boldsymbol{t}, \tilde{\boldsymbol{t}}*) = \langle N|\hat{\gamma}_+(\boldsymbol{t})\widehat{C}_\rho \hat{g}\widehat{C}_{\tilde{\rho}}\hat{\gamma}_-(\tilde{\boldsymbol{t}})|N\rangle, \tag{3-38}$$

obtained by replacing the group element g in (2-22) by $C_\rho g C_{\tilde{\rho}}$, and denote this transformed 2-Toda τ-function

$$\tau^{(2)}_{C_\rho \hat{g} C_{\tilde{\rho}}} =: \widetilde{C}^{(2)}_{(\rho,\tilde{\rho})}(\tau^{(2)}_g). \tag{3-39}$$

Inserting sums over complete sets of intermediate orthonormal basis states in (3-36) and (3-38), and defining $\tilde{r}_\lambda(N)$ as in (3-17), with the factors ρ_i replaced by $\tilde{\rho}_i$, we obtain the following form for the Schur function expansions (2-26), (2-27).

Proposition 3.1. *The effect of the convolution actions* (3-37), (3-39) *is to multiply the coefficients in the Schur function expansions of* $\tau_{C_\rho g}(N,\boldsymbol{t})$ *and* $\tau^{(2)}_{C_\rho \hat{g} C_{\tilde{\rho}}}(N,\boldsymbol{t},\tilde{\boldsymbol{t}})$ *by the diagonal factors* $r_\lambda(N)$ *and* $\tilde{r}_\mu(N)$:

$$\tau_{C_\rho g}(N, \boldsymbol{t}) = \sum_\lambda r_\lambda(N)\pi_{N,g}(\lambda)s_\lambda(\boldsymbol{t}), \tag{3-40}$$

$$\tau^{(2)}_{C_\rho g C_{\tilde{\rho}}}(N, \boldsymbol{t}, \tilde{\boldsymbol{t}}) = \sum_\lambda \sum_\mu r_\lambda(N) B_{N,g}(\lambda,\mu)\tilde{r}_\mu(N)s_\lambda(\boldsymbol{t})s_\mu(\tilde{\boldsymbol{t}}). \tag{3-41}$$

The Plücker coordinates for the modified Grassmannian elements $C_\rho g(\mathcal{H}^N_+)$ *and* $C_\rho g C_{\tilde{\rho}}(w_{\mu,N})$ *are thus*

$$\pi_{N,C_\rho g}(\lambda) = r_\lambda(N)\pi_{N,g}(\lambda), \tag{3-42}$$

$$B_{N,C_\rho g C_{\tilde{\rho}}}(\lambda,\mu) = r_\lambda(N) B_{N,g}(\lambda,\mu)\tilde{r}_\mu(N). \tag{3-43}$$

Proof. This follows immediately from the diagonal form (3-18) of the $\widehat{C}$ action in the orthonormal basis $\{|\lambda, N\rangle\}$, substituted into the expansions (2-26), (2-27), using the definitions (2-28) and (2-29) of the Plücker coordinates $\pi_{N,C_\rho g}(\lambda)$ and $B_{N,C_\rho g C_{\tilde{\rho}}}(\lambda,\mu)$. □

In particular, setting $g = C_{\tilde\rho} = \mathbb{1}$, in (3-41) we obtain

$$\tau^{(2)}_{C_\rho}(N,\boldsymbol{t},\tilde{\boldsymbol{t}}) = \sum_\lambda r_\lambda(N) s_\lambda(\boldsymbol{t}) s_\lambda(\tilde{\boldsymbol{t}}) =: \tau_r(N,\boldsymbol{t},\tilde{\boldsymbol{t}}), \tag{3-44}$$

where $\tau_r(N,\boldsymbol{t},\tilde{\boldsymbol{t}})$ is defined by the second equality. Such τ-functions have been studied as generalizations of hypergeometric functions in [Orlov and Scherbin 2001; Orlov 2006]. (See also [Harnad and Orlov 2003; 2006], where the notation differs slightly due to the presence of the normalization factor $c_r(N)$ in the definition (3-17) of $r_\lambda(N)$.)

In the following, the infinite sequence of parameters $\boldsymbol{t} = (t_1, t_2, \dots)$ will often be chosen as the trace invariants of some square matrix M. The sequence so formed will be denoted

$$\boldsymbol{t} = [M] = \left\{ \frac{1}{i} \operatorname{tr}(M^i) \right\} \Big|_{i\in\mathbb{N}^+}, \quad [M]_i := \frac{1}{i} \operatorname{tr}(M^i). \tag{3-45}$$

If $\boldsymbol{t}$ and $\tilde{\boldsymbol{t}}$ in (3-44) are replaced by $[A]$ and $[B]$, respectively, where A and B are a pair of diagonal matrices

$$A = \operatorname{diag}(a_1, \dots, a_N), \quad B = \operatorname{diag}(b_1, \dots, b_N), \tag{3-46}$$

with distinct eigenvalues, and

$$\Delta(A) := \prod_{1\le i<j}^{n} (a_i - a_j), \qquad \Delta(B) := \prod_{1\le i<j}^{n} (b_i - b_j) \tag{3-47}$$

denote the Vandermonde determinants in the variables $\{a_i\}$ and $\{b_i\}$, we obtain a simple $N \times N$ determinantal expression for $\tau_r(N,[A],[B])$ (cf. [Harnad and Orlov 2006; Orlov 2004]).

Lemma 3.3. *Choosing $\rho_-(z)$ as in* (3-26) (*i.e.*, $\rho_{-i} = 1$ *for* $i < 1$), *we have*

$$\tau_r(N,[A],[B]) = \sum_{\ell(\lambda)\le N} r_\lambda(N) s_\lambda([A]) s_\lambda([B]) \tag{3-48}$$

$$= \frac{\det(\rho_+(a_i b_j)\big|_{1\le i,j\le N}}{\Delta(A)\Delta(B)}. \tag{3-49}$$

Remark 3.3. Although various proofs of this result may be found elsewhere (see [Harnad and Orlov 2006], for example), we provide a detailed version here, based on the Cauchy–Binet identity in semiinfinite form, since it involves some useful further relations. An equivalent way is to use the fermionic form of Wick's theorem, which is really just the Cauchy–Binet identity expressed in terms of fermionic operators and matrix elements.

Proof of Lemma 3.3. The Cauchy–Binet identity in semiinfinite form may be expressed by considering two N-dimensional framed subspaces $\mathrm{span}\{F_i\}_{1\le i\le N}$ and $\mathrm{span}\{G_i\}_{1\le i\le N}$ of the complex Euclidean vector space $\ell^2(\mathbb{N}) = \mathrm{span}\{e_i\}_{i\in\mathbb{N}}$, identified with $\mathcal{H}_+ \subset \mathcal{H} = L^2(S^1)$, by choosing the monomials $\{z^i\}_{i\in\mathbb{N}}$ as orthonormal basis. The vectors F_i and G_j are thus identified with elements $F_i(z), G_j(z) \in \mathcal{H}_+$ defined by

$$F_i(z) := \sum_{j=0}^{\infty} F_{ji} z^j, \quad G_i(z) := \sum_{j=0}^{\infty} G_{ji} z^j. \tag{3-50}$$

(Note that, to avoid needless use of negative indices, we are not using the same labelling conventions here for the basis elements $\{e_i\}$ as in (2-2).) The complex inner product $(\ ,\ )$ is defined by integration

$$(F, G) := \frac{1}{2\pi i}\oint_{z\in S^1} F(z)G(z^{-1})\frac{dz}{z}. \tag{3-51}$$

The Cauchy–Binet identity can then be expressed as

$$\det(F_i, G_j)|_{1\le i,j\le N} = \sum_{\ell(\lambda)\le N} \det(F_{\lambda_i - i+N,j})\det(G_{\lambda_i-i+N,j}), \tag{3-52}$$

where

$$F_i = \sum_{j\in\mathbb{Z}} F_{ji} e_j, \quad G_i = \sum_{j\in\mathbb{Z}} G_{ji} e_j, \tag{3-53}$$

and the sum is over all partitions λ of length $\ell(\lambda) \le N$, completed so that the $N\times N$ submatrices $F_{\lambda_i-i+N,j}$ and $G_{\lambda_i-i+N,j}$ are defined by setting $\lambda_i = 0$ for $i > \ell(\lambda)$. Since all expressions in the sum will be polynomials in the parameters (a_i, b_i) there is no loss of generality in assuming that these lie within the unit disc. We define

$$F_i(z) := \rho_+(a_i z), \quad G_i(z) := (1-b_i z)^{-1}, \tag{3-54}$$

and hence

$$F_{ij} = \rho_i(a_j), \quad G_{ij} = (b_j)^i. \tag{3-55}$$

From the character formula

$$s_\lambda([A]) = \frac{\det(a_i^{\lambda_j - j+N})}{\Delta(A)}, \quad s_\lambda([B]) = \frac{\det(b_i^{\lambda_j-j+N})}{\Delta(B)}, \tag{3-56}$$

it follows that the determinant factors on the right side of (3-52) are

$$\det(F_{\lambda_i-i+N,j}) = \det(a_j^{\lambda_i-i+N}\rho_{\lambda_i-i+N}) = \left(\prod_{i=1}^{N}\rho_{\lambda_i-i+N}\right)s_\lambda([A])\Delta([A]), \tag{3-57}$$

$$\det(G_{\lambda_i-i+N,j}) = \det(b_j^{\lambda_i-i+N}) = s_\lambda([B])\Delta(B). \tag{3-58}$$

From the definitions (3-17) and (3-18), it follows that

$$\left(\prod_{i=1}^{N}\rho_{\lambda_i-i+N}\right) = r_\lambda(N), \tag{3-59}$$

so the right side of the Cauchy–Binet identity (3-52) is just the right side of (3-48) multiplied by $\Delta([A])\Delta([B])$. On the other hand, from (3-51), the left side of (3-52) is

$$\det(F_i, G_j) = \det\left(\frac{1}{2\pi i}\oint_{z\in S^1}\frac{\rho_+(a_i z)}{z-b_j}\frac{dz}{z}\right) = \det(\rho_+(a_i b_j)), \tag{3-60}$$

which is just the expression (3-49) multiplied by $\Delta([A])\Delta([B])$. ☐

Remark 3.4. Note that, for the case of Example 3.1, (3-49) becomes the key identity (cf. [Harnad and Orlov 2006; Zinn-Justin 2002])

$$\sum_{\ell(\lambda)\le N}\frac{1}{(N)_\lambda}s_\lambda([A])s_\lambda([B]) = \left(\prod_{k=1}^{N-1}k!\right)\frac{\det(e^{a_i b_j})\big|_{1\le i,j\le N}}{\Delta(A)\Delta(B)}, \tag{3-61}$$

which, together with the character integral [Macdonald 1995]

$$d_{\lambda,N}\int_{U\in U(N)} d\mu_H(U)s_\lambda([AUXU^\dagger]) = s_\lambda([A])s_\lambda([X]), \tag{3-62}$$

(where $d\mu_H(U)$ is the Haar measure on $U(N)$), implies the Harish-Chandra–Itzykson–Zuber (HCIZ) integral [Itzykson and Zuber 1980]

$$\int_{U\in U(N)} d\mu_H(U)e^{\mathrm{tr}(AUXU^\dagger)} = \left(\prod_{k=1}^{N-1}k!\right)\frac{\det(e^{a_i x_j})}{\Delta(A)\Delta(X)}. \tag{3-63}$$

Remark 3.5. The condition that the eigenvalues $\{a_i\}$ and $\{b_i\}$ of A and B be distinct can be eliminated simply by taking limits in which some or all of these are made to coincide. In the resulting determinantal formulae, like (3-49), and those appearing in subsequent sections, in which a Vandermonde determinant $\Delta(A)$ or $\Delta(B)$ appears in the denominator, the only modification is that the terms in the numerator determinants depending on the a_i's and b_i's are replaced by their derivatives with respect to these parameters, taken to the same degree as the degeneracy of their values, while the denominator Vandermonde determinants are correspondingly replaced by their lower dimensional analogs. This will not be further developed here, but will be considered elsewhere, in connection with correlation kernels for externally coupled matrix models. All formulae below in which no Vandermonde determinant factors $\Delta(A)$ or $\Delta(B)$ appear in the denominator remain valid in the case of degenerate eigenvalues.

4. Applications to matrix models

We now consider $N \times N$ matrix Hermitian integrals that are τ-functions, and show how the application of convolution symmetries leads to new matrix models of the externally coupled type. In the following, let $d\mu(M)$, be a measure on the space of $N \times N$ Hermitian matrices $M \in \mathbb{H}^{N\times N}$ that is invariant under conjugation by unitary matrices, and such that the reduced measure, projected to the space of eigenvalues by integration over the group $U(N)$, is a product of N identical measures $d\mu_0$ on $\mathbb{R}$, times the Jacobian factor $\Delta^2(X)$,

$$\int_{U\in U(N)} d\mu(UXU^\dagger) = \prod_{a=1}^{N} d\mu_0(x_a)\Delta^2(X), \tag{4-1}$$

where $X = \mathrm{diag}(x_1, \dots, x_N)$.

4.1. *Convolution symmetries, externally coupled Hermitian matrix models and τ-functions as finite determinants.*

It is well known that Hermitian matrix integrals of the form

$$Z_N(\boldsymbol{t}) = \int_{M\in\mathbb{H}^{N\times N}} d\mu(M)\, e^{\mathrm{tr}\sum_{i=1}^{\infty} t_i M^i} \tag{4-2}$$

$$= \prod_{a=1}^{N} \int_{\mathbb{R}} d\mu_0(x_a) e^{\sum_{i=1}^{\infty} t_i x_a^i} \Delta^2(X) \tag{4-3}$$

are KP-Toda τ-functions [Kharchev et al. 1991]. The Schur function expansion is

$$Z_N(\boldsymbol{t}) = \sum_{\ell(\lambda)\le N} \pi_{N,d\mu}(\lambda) s_\lambda(\boldsymbol{t}), \tag{4-4}$$

where the coefficients $\pi_{N,d\mu}(\lambda)$ are expressible as determinants in terms of the matrix of moments [Harnad and Orlov 2002; 2003; 2006]

$$\pi_{N,d\mu}(\lambda) = \prod_{a=1}^{N} \left(\int_{\mathbb{R}} d\mu_0(x_a) \right) \Delta^2(X) s_\lambda([X]) \tag{4-5}$$

$$= (-1)^{\frac{1}{2}N(N-1)} N!\ \det(\mathcal{M}_{\lambda_i+N-i,j-1})|_{1\le i,j\le N}, \tag{4-6}$$

$$\mathcal{M}_{ij} := \int_{\mathbb{R}} d\mu_0(x) x^{i+j}. \tag{4-7}$$

Now consider the externally coupled matrix model integral (cf. [Brézin and Hikami 1996; Wang 2009; Zinn-Justin 1998; 2002])

$$Z_{N,\mathrm{ext}}(A) := \int_{M \in \mathbb{H}^{N\times N}} d\mu(M) e^{\mathrm{tr}(AM)}, \tag{4-8}$$

where $A \in \mathbb{H}^{N\times N}$ is a fixed $N \times N$ Hermitian matrix. This can be obtained by simply applying a convolution symmetry transformation of the type given in Example 3.1 to the τ-function defined by the matrix integral (4-3).

Proposition 4.1. *Applying the convolution symmetry* $\widetilde{C}_\rho$ *to the* τ*-function* $Z_N(\boldsymbol{t})$, *where* $\rho_+(z)$ *and* $\rho_-(z)$ *are defined as in* (3-25), (3-26), *and choosing the KP flow parameters as* $\boldsymbol{t} = [A]$ *gives*, *within a multiplicative constant*, *the externally coupled matrix integral* (4-8)

$$\widetilde{C}_\rho(Z_N)([A]) = \left(\prod_{i=1}^{N-1} i!\right)^{-1} Z_{N,\mathrm{ext}}(A). \tag{4-9}$$

Proof. Substituting the expansion [Harnad and Orlov 2006]

$$e^{\mathrm{tr}\, AM} = \sum_{\ell(\lambda)\le N} \frac{d_{\lambda,N}}{(N)_\lambda} s_\lambda([AM]) \tag{4-10}$$

into (4-8), where

$$d_{\lambda,N} = s_\lambda(\mathbb{1}_N) \tag{4-11}$$

is the dimension of the irreducible $\mathrm{GL}(N)$ tensor representation of symmetry type λ, and expressing M in diagonalized form as

$$M = UXU^\dagger, \tag{4-12}$$

where $U \in U(N)$ and $X = \mathrm{diag}(x_1, \dots x_N)$, gives

$$Z_{N,\mathrm{ext}}(A) = \sum_{\ell(\lambda)\le N} \int_{U\in U(N)} d\mu_H(U) \prod_{a=1}^{N} \int_{\mathbb{R}} d\mu_0(x_a) e^{\sum_{i=1}^{\infty} t_i x_a^i} \times \Delta^2(X) \frac{d_{\lambda,N}}{(N)_\lambda} s_\lambda([AUXU^\dagger]). \tag{4-13}$$

Evaluating the character integral (3-62) and using (4-5), it follows that

$$\begin{aligned} Z_{N,\text{ext}}(A) &= \sum_{\ell(\lambda)\le N} \prod_{a=1}^{N} \int_{\mathbb{R}} d\mu_0(x_a) e^{\sum_{i=1}^{\infty} t_i x_a^i} \Delta^2(X) \frac{1}{(N)_\lambda} s_\lambda([A]) s_\lambda([X]) \\ &= \sum_{\ell(\lambda)\le N} \frac{1}{(N)_\lambda} \pi_{N,d\mu}(\lambda) s_\lambda([A]) \\ &= \sum_{\ell(\lambda)\le N} \left(\prod_{i=1}^{N-1} i!\right) r_\lambda(N) s_\lambda([A]) \\ &= \left(\prod_{i=1}^{N-1} i!\right) \tilde{C}_\rho(Z_N)\big|_{\boldsymbol{t}=[A]}, \end{aligned} \tag{4-14}$$

where the third line follows from the expression (3-30) for $r_\lambda(N)$ in Example 3.1 and the last from Proposition 4.1, 3.1. □

More generally, given an arbitrary function $\rho_+(z)$, analytic on the interior of S^1 and choosing $\rho_-(z)$ as in (3-26), we may define a new externally coupled matrix integral

$$Z_{N,\rho}(A) := \int_{M\in\mathbb{H}^{N\times N}} d\mu(M)\, \tau_r(N, [AM]), \tag{4-15}$$

in which $e^{\operatorname{tr} AM}$ is replaced by

$$\tau_r(N,[M]) := \tau_r(N,[\mathbb{1}_N],[M]) = \sum_{\ell(\lambda)\le N} d_{\lambda,N} r_\lambda(N) s_\lambda([M]). \tag{4-16}$$

Then by the same calculation as above, it follows that $Z_{N,\rho}(A)$ is again just the τ-function obtained by applying the convolution symmetry $\tilde{C}_\rho$ to Z_N, evaluated at the parameter values $\boldsymbol{t} = [A]$.

Proposition 4.2. *Applying the convolution symmetry $\tilde{C}_\rho$ to Z_N gives*

$$\tilde{C}_\rho(Z_N)([A]) = Z_{N,\rho}(A). \tag{4-17}$$

In particular, if we take (ρ_+, ρ_-) as in Example 3.2 above, we obtain (cf. [Harnad and Orlov 2006])

$$Z_{N,\rho}(A) = \left(\prod_{i=0}^{N-1} \frac{(a)_i}{i!}\right) \zeta^{\frac{1}{2}N(N-1)} \int_{M\in\mathbb{H}^{N\times N}} d\mu(M)\, \det(1-\zeta AM)^{-a-N+1}, \tag{4-18}$$

showing that this also is a KP-Toda τ-function evaluated at parameter values $\boldsymbol{t} = [A]$.

Returning to the general case, a finite determinantal formula for $Z_{N,\rho}(A)$ is given by the following.

Proposition 4.3.

$$Z_{N,\rho}(A) = \frac{(-1)^{\frac{1}{2}N(N-1)}N!}{\Delta(A)} \det(G_{ij}(\rho, A))\Big|_{1\le i,j\le N}, \tag{4-19}$$

where

$$G_{ij}(\rho, A) := \int_{\mathbb{R}} d\mu_0(x) x^{i-1} \rho_+(a_j x). \tag{4-20}$$

Proof. Applying the character integral identity (3-62) to (4-15) gives

$$Z_{N,\rho}(A) = \int_{M\in\mathbb{H}^{N\times N}} d\mu(M) \sum_{\ell(\lambda)\le N} r_\lambda(N) s_\lambda([A]) s_\lambda([M]) \tag{4-21}$$

$$= \frac{1}{\Delta(A)} \int d\mu_0(X) \Delta(X) \det(\rho_+(a_i x_j))\Big|_{1\le i,j\le N} \tag{4-22}$$

$$= \frac{(-1)^{\frac{1}{2}N(N-1)}N!}{\Delta(A)} \det(G_{ij}(\rho, A))\Big|_{1\le i,j\le N}, \tag{4-23}$$

with $G_{ij}(\rho, A)$ defined by (4-20). Here, the integration over the $U(N)$ group has been performed and Lemma 3.3 has been used in (4-22). Equation (4-23) follows from (4-22) by applying the Andréief identity [Andréief 1886] in the form

$$\left(\prod_{m=1}^{N} \int d\mu_0(x_m)\right) \det(\phi_i(x_j)) \det(\psi_k(x_l))\Big|_{\substack{1\le i,j\le N\\ 1\le k,l\le N}} = N! \det\left(\int \phi_i(x)\psi_j(x)\right)\Big|_{1\le i,j\le N}, \tag{4-24}$$

with

$$\phi_i(x) = x^{N-i}, \quad \psi_j(x) := \rho_+(a_j x), \tag{4-25}$$

since

$$\Delta(X) = \det(\phi_i(x_j)). \tag{4-26}$$

□

4.2. *Externally coupled two-matrix models.*

We now turn to the case of two-matrix models. For simplicity, we only consider Itzykson–Zuber exponential coupling [1980], although the same double convolution transformations may be applied to all the couplings considered in [Harnad and Orlov 2006]. Using the HCIZ identity (3-63) to evaluate the integrals over the unitary groups $U(N)$, we obtain

$$Z_N^{(2)}(\boldsymbol{t},\tilde{\boldsymbol{t}}) = \int_{M_1\in\mathbb{H}^{N\times N}} d\mu(M_1)\int_{M_2\in\mathbb{H}^{N\times N}} d\tilde{\mu}(M_2) \times e^{\operatorname{tr}(\sum_{i=1}^{\infty}(t_i M_1^i+\tilde{t}_i M_2^i)+M_1M_2)} \tag{4-27}$$

$$= \prod_{k=1}^{N} k!\prod_{a=1}^{N}\biggl(\int_{\mathbb{R}} d\mu_0(x_a)\int_{\mathbb{R}} d\tilde{\mu}_0(y_a)\, e^{\sum_{i=1}^{\infty}(t_i x_a^i+\tilde{t}_i y_a^i+x_a y_a)}\biggr) \times \Delta(X)\Delta(Y),$$

where $Y = \operatorname{diag}(y_1,\dots,y_N)$. This is known to be a 2D-Toda τ-function [Adler and van Moerbeke 1999; 2005; Harnad and Orlov 2002; 2003; 2006; Orlov and Shiota 2005], with double Schur function expansion

$$Z_N^{(2)}(\boldsymbol{t},\tilde{\boldsymbol{t}}) = \sum_{\lambda}\sum_{\mu} B_{N,d\mu,d\tilde{\mu}}(\lambda,\mu)s_\lambda(\boldsymbol{t})s_\mu(\tilde{\boldsymbol{t}}), \tag{4-28}$$

where the coefficients $B_{N,d\mu,d\tilde{\mu}}(\lambda,\mu)$ are $N\times N$ determinants of submatrices in terms of the matrix of bimoments

$$B_{N,d\mu,d\tilde{\mu}}(\lambda,\mu) = \prod_{k=1}^{N} k!\prod_{a=1}^{N}\biggl(\int_{\mathbb{R}} d\mu_0(x_a)\int_{\mathbb{R}} d\tilde{\mu}_0(y_a)e^{x_a y_a}\biggr) \times \Delta(X)\Delta(Y)s_\lambda([X])s_\mu([Y])$$

$$= (N!)\prod_{k=1}^{N} k!\, \det(\mathcal{B}_{\lambda_i-i+N,\,\mu_j-j+N})|_{1\le i,j\le N}, \tag{4-29}$$

$$\mathcal{B}_{ij} := \int_{\mathbb{R}} d\mu_0(x_a)\int_{\mathbb{R}} d\tilde{\mu}_0(y_a)e^{x_a y_a}x^i y^j. \tag{4-30}$$

Now, choosing a pair of elements $(\rho,\tilde{\rho})$, with both ρ_- and $\tilde{\rho}_-$ as in (3-26), we may define a family of externally coupled two-matrix models, by

$$Z_{N,\rho,\tilde{\rho}}^{(2)}(A,B) := \int_{M_1\in\mathbb{H}^{N\times N}} d\mu(M_1)\int_{M_2\in\mathbb{H}^{N\times N}} d\tilde{\mu}(M_2) \times \tau_r(N,[A],[M_1])\tau_{\tilde{r}}(N,[B],[M_2])e^{\operatorname{tr}(M_1M_2)}, \tag{4-31}$$

where A, B are hermitian $N\times N$ matrices. This class may be obtained as the 2D-Toda τ-function resulting from applying the convolution symmetry $\widetilde{C}_{\rho,\tilde{\rho}}$ to $Z_N^{(2)}$.

Proposition 4.4. *Applying the convolution symmetry $\widetilde{C}_{\rho,\tilde{\rho}}$ to $Z_N^{(2)}$ and evaluating at the parameter values $\boldsymbol{t}=[A]$, $\tilde{\boldsymbol{t}}=[B]$ gives the externally coupled matrix integral* (4-31)

$$\widetilde{C}_{\rho,\tilde{\rho}}^{(2)}(Z_N^{(2)})([A],[B])) = Z_{N,\rho,\tilde{\rho}}^{(2)}(A,B). \tag{4-32}$$

Proof. Because of the $U(N)\times U(N)$ invariance of the measures $d\mu$ and $d\tilde{\mu}$ in (4-31) and all factors in the integrand, except for the coupling term $e^{\mathrm{tr}(M_1M_2)}$, we may carry out the two $U(N)$ integrations, using the HCIZ identity (3-63), to obtain a reduced integral over the diagonal matrices $X=\mathrm{diag}(x_1,\dots,x_N)$, $Y=\mathrm{diag}(y_1,\dots,y_N)$ of eigenvalues of M_1 and M_2:

$$\begin{aligned} Z^{(2)}_{N,\rho,\tilde{\rho}}(A,B) &= \prod_{k=1}^{N} k! \prod_{a=1}^{N}\left(\int_{\mathbb{R}} d\mu_0(x_a)\int_{\mathbb{R}} d\tilde{\mu}_0(y_a)\, e^{x_a y_a}\right)\Delta(X)\Delta(Y) \qquad (4\text{-}33)\\ &\qquad\qquad \times \tau_{C_\rho}(N,[A],[X])\tau_{C_{\tilde{\rho}}}(N,[B],[Y]) \\ &= \sum_{\ell(\lambda)\le N}\sum_{\ell(\mu)\le N} r_\lambda(N) B_{N,d\mu,d\tilde{\mu}}(\lambda,\mu)\tilde{r}_\lambda(N) s_\lambda([A]) s_\mu([B]) \qquad (4\text{-}34)\\ &= \widetilde{C}^{(2)}_{\rho,\tilde{\rho}}(Z^{(2)}_N)([A],[B])). \qquad (4\text{-}35) \end{aligned}$$

where the second equality follows from (3-44) and the last from Proposition 3.1, (3-41). □

Since the dependence on A and B is $U(N)\times U(N)$ conjugation invariant we may choose, without loss of generality, A and B to be diagonal matrices

$$A=\mathrm{diag}(a_1,\dots,a_N),\quad B=\mathrm{diag}(b_1,\dots,b_N). \qquad (4\text{-}36)$$

We then obtain, as in the one-matrix case, a finite determinantal formula for the 2D-Toda τ-function $Z^{(2)}_{N,\rho,\tilde{\rho}}(A,B)$.

Proposition 4.5.

$$Z^{(2)}_{N,\rho,\tilde{\rho}}(A,B) = \frac{N!\left(\prod_{k=1}^{N} k!\right)}{\Delta(A)\Delta(B)} \det(G_{ij}(\rho,\tilde{\rho},A,B)\big|_{1\le i,j\le N}, \qquad (4\text{-}37)$$

where

$$G_{ij}(\rho,\tilde{\rho},A,B) := \int_{\mathbb{R}} d\mu_0(x)\int_{\mathbb{R}} d\tilde{\mu}_0(y) e^{xy}\rho_+(a_i x)\tilde{\rho}_+(b_j y). \qquad (4\text{-}38)$$

Proof.

$$\begin{aligned} Z^{(2)}_{N,\rho,\tilde{\rho}}(A,B) &= \int_{M_1\in\mathbb{H}^{N\times N}} d\mu(M_1)\int_{M_2\in\mathbb{H}^{N\times N}} d\tilde{\mu}(M_2) e^{\mathrm{tr}(M_1M_2)} \qquad (4\text{-}39)\\ &\quad \times \sum_{\ell(\lambda)\le N} r_\lambda(N)s_\lambda([A])s_\lambda([M_1]) \sum_{\ell(\mu)\le N}\tilde{r}_\mu(N)s_\mu([B])s_\mu([M_2]) \end{aligned}$$

$$= \frac{\left(\prod_{k=1}^{N} k!\right)}{\Delta(A)\Delta(B)} \int d\mu(X) \int d\tilde{\mu}(Y)\, e^{\sum_{i=1}^{N} x_i y_i} \tag{4-40}$$

$$\times \det(\rho_+(a_k x_l))\big|_{1\le k,l\le N} \det(\tilde{\rho}_+(b_m y_n))\big|_{1\le m,n\le N} \tag{4-41}$$

$$= \frac{N!\left(\prod_{k=1}^{N} k!\right)}{\Delta(A)\Delta(B))} \det(G_{ij}(\rho,\tilde{\rho},A,B)\big|_{1\le i,j\le N}. \tag{4-42}$$

In (4-41), we have used the HCIZ identity (3-63), antisymmetry of the determinants in the integrand with respect to permutations in the integration variables $(x_1,\dots,x_N)$ and $(y_1,\dots,y_N)$ and Lemma 3.3 twice, while in (4-42), we have used the Andréief identity [1886] in the form

$$\left(\prod_{m=1}^{N} \int d\mu(x_m, y_m)\right) \det(\phi_i(x_j)) \det(\psi_k(y_l))\Big|_{\substack{1\le i,j\le N\\ 1\le k,l\le N}} \tag{4-43}$$

$$= N! \det\left(\int d\mu(x,y)\phi_i(x)\psi_j(y)\right)\Big|_{1\le i,j\le N}. \quad \square$$

As the simplest example of a 2D-Toda τ-function obtained through Propositions 4.4 and 4.5, consider the case when the measures $d\mu_0(x)$ and $d\mu_0(y)$ are both Gaussian, and ρ_+ and $\tilde{\rho}_+$ are both taken as the exponential function.

Example 4.1.

$$d\mu_0(x) = e^{-\sigma x^2} dx, \quad d\mu_0(y) = e^{-\sigma y^2} dy, \quad \rho_+(x) = e^x, \quad \tilde{\rho}_+(y) = e^y. \tag{4-44}$$

Evaluating the Gaussian integrals gives

$$G_{ij} = \frac{2\pi}{\sqrt{1+4\sigma^2}} \exp \frac{\sigma(a_i^2 + b_j^2) - a_i b_j}{4\sigma^2 - 1}, \tag{4-45}$$

and hence

$$Z_{N,\rho}(A) = \frac{(2\pi)^N N! \prod_{k=1}^{N} k!}{(1+4\sigma^2)^{N/2}\Delta(A)\Delta(B)} \exp\left(\frac{\sigma}{4\sigma^2-1}\sum_{i=1}^{N}(a_i^2 + b_i^2)\right)$$

$$\times \det\left(\exp \frac{\sigma a_i b_j}{1-4\sigma^2}\right). \tag{4-46}$$

The exponential factor on the first line of (4-46) is a linear exponential in terms of the 2KP flow variable t_2 and $\tilde{t}_2$ and hence, through the Sato formula (1-3), produces just a gauge factor multiplying the Baker–Akhiezer function [Segal and Wilson 1985]. Therefore (4-46) is just a rescaled, gauge transformed version of the 2KP τ-function of hypergeometric type appearing in the integrand of the Itzykson–Zuber coupled two-matrix model [Itzykson and Zuber 1980].

4.3. *More general 2D-Toda τ-functions as multiple integrals.*

We may extend the above results to more general 2KP-Toda τ-functions expressed as multiple integrals and finite determinants. To begin with, the following multiple integral

$$\tau^{(2)}_{d\mu}(N,\boldsymbol{t},\tilde{\boldsymbol{t}}) = \prod_{a=1}^{N}\left(\int_{\Gamma}\int_{\widetilde{\Gamma}} d\mu(x_a,y_a)e^{\sum_{i=1}^{\infty}(t_i x_a^i+\tilde{t}_i y_a^i)}\right)\Delta(X)\Delta(Y), \quad (4\text{-}47)$$

where Γ, $\widetilde{\Gamma}$ are curves in the complex x- and y-planes and $d\mu(x,y)$ is a measure on $\Gamma\times\widetilde{\Gamma}$, is a 2D-Toda τ-function [Harnad and Orlov 2006] for a large class of measures $d\mu_0(x,y)$. Applying a double convolution symmetry $\widetilde{C}_{\rho,\tilde{\rho}}$, with ρ_- and $\tilde{\rho}_-$ the same as in (3-26), gives a new 2D-Toda τ-function, also having a multiple integral representation.

Proposition 4.6.

$$\begin{aligned}\widetilde{C}^{(2)}_{\rho,\tilde{\rho}}&(\tau^{(2)}_{d\mu})(N,\boldsymbol{t},\tilde{\boldsymbol{t}})\\ &= \prod_{a=1}^{N}\left(\int_{\Gamma}\int_{\widetilde{\Gamma}} d\mu(x_a,y_a)\right)\Delta(X)\Delta(Y)\tau_r(N,\boldsymbol{t},[X])\tau_{\tilde{r}}(N,\tilde{\boldsymbol{t}},[Y]). \quad (4\text{-}48)\end{aligned}$$

Proof. This is proved similarly to Proposition 4.4, using the Cauchy–Littlewood identity (2-31) twice in the form

$$\prod_{a=1}^{N} e^{\sum_{i=1}^{\infty}(t_i x_a^i+\tilde{t}_i y_a^i)} = \sum_{\ell(\lambda)\le N} s_\lambda(\boldsymbol{t})s_\lambda([X]) \sum_{\ell(\mu)\le N} s_\mu(\tilde{\boldsymbol{t}})s_\mu([Y]). \quad (4\text{-}49)$$

□

Evaluating at parameter values $\boldsymbol{t}=[A]$ and $\tilde{\boldsymbol{t}}=[B]$ and applying Lemma 3.3 again gives the τ-function of (4-48) in $N\times N$ determinantal form.

Proposition 4.7.

$$\widetilde{C}^{(2)}_{\rho,\tilde{\rho}}(\tau^{(2)}_{d\mu})([A],[B]) = \frac{N!}{\Delta(A)\Delta(B)}\det(G_{ij}(\rho,\tilde{\rho},A,B)\big|_{1\le i,j\le N}, \quad (4\text{-}50)$$

where

$$G_{ij}(\rho,\tilde{\rho},A,B) := \int_{\Gamma}\int_{\widetilde{\Gamma}} d\mu(x,y)\rho_+(a_j x)\tilde{\rho}_+(b_j y). \quad (4\text{-}51)$$

Proof.

$$\begin{aligned}\widetilde{C}^{(2)}_{\rho,\tilde\rho}(\tau^{(2)}_{d\mu})([A],[B]) &= \frac{1}{\Delta(A)\Delta(B)}\prod_{a=1}^{N}\left(\int_\Gamma\int_{\widetilde\Gamma} d\mu(x_a,y_a)\right)\\ &\qquad\times\det(\rho_+(a_k x_l))\big|_{1\le k,l\le N}\det(\tilde\rho_+(b_m y_n))\big|_{1\le m,n\le N}\\ &= \frac{N!}{\Delta(A)\Delta(B)}\det(G_{ij}(\rho,\tilde\rho,A,B))\big|_{1\le i,j\le N},\end{aligned} \tag{4-52}$$

where again we have used the Lemma 3.3 twice and the Andréief identity in the form (4-43). □

This therefore provides a new class of 2D-Toda τ-functions expressible in such a finite determinantal form, associated to any pair of curves Γ, $\widetilde\Gamma$, together with a measure $d\mu$ on their product, and a pair of functions $\rho_+(x)$ and $\tilde\rho_+(y)$, such that the integrals in (4-51) are well defined and convergent.

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harnad@crm.umontreal.ca
Centre de recherches mathématiques,
Université de Montréal,
C. P. 6128, Succ. centre-ville, Montréal H3C 3J7, Canada
Department of Mathematics and Statistics,
Concordia University, 1455 de Maisonneuve Boulevard W.,
Montreal, Quebec, Canada H3G 1M8

orlovs55@mail.ru
Nonlinear Wave Processes Laboratory, Oceanlogy Institute,
36 Nakhimovskii Prospect, Moscow, 117851, Russia