Random Matrices
MSRI Publications
Volume **65**, 2014

# Riemann–Hilbert approach to the six-vertex model

PAVEL BLEHER AND KARL LIECHTY

The six-vertex model, or the square ice model, with domain wall boundary conditions (DWBC) has been introduced and solved for finite $n$ by Korepin and Izergin. The solution is based on the Yang–Baxter equations and it represents the free energy in terms of an $n \times n$ Hankel determinant. Paul Zinn-Justin observed that the Izergin–Korepin formula can be expressed in terms of the partition function of a random matrix model with a nonpolynomial interaction. We use this observation to obtain the large $n$ asymptotics of the six-vertex model with DWBC. The solution is based on the Riemann–Hilbert approach. In this paper we review asymptotic results obtained in different regions of the phase diagram.

## 1. Six-vertex model

The *six-vertex model*, or the model of *two-dimensional ice*, is stated on a square lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. This rule is sometimes called the ice-rule. There are only six possible configurations of arrows at each vertex, hence the name of the model; see Figure 1.

We will consider the *domain wall boundary conditions* (DWBC), in which the arrows on the upper and lower boundaries point into the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the $4 \times 4$ lattice is shown on Figure 2.

The name of the *square ice* comes from the two-dimensional arrangement of water molecules, $H_2O$, with oxygen atoms at the vertices of the lattice and one hydrogen atom between each pair of adjacent oxygen atoms. We place an arrow in the direction from a hydrogen atom toward an oxygen atom if there is a bond between them. Thus, as we already noticed before, there are two in-bound and two out-bound arrows at each vertex.

Bleher is supported in part by the National Science Foundation (NSF) grant DMS-0969254.
*Keywords:* none.

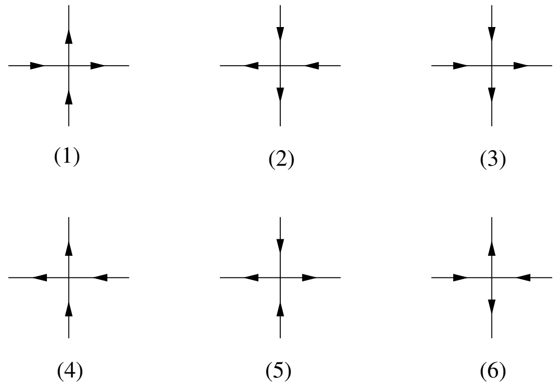


**Figure 1.** The six arrow configurations allowed at a vertex.

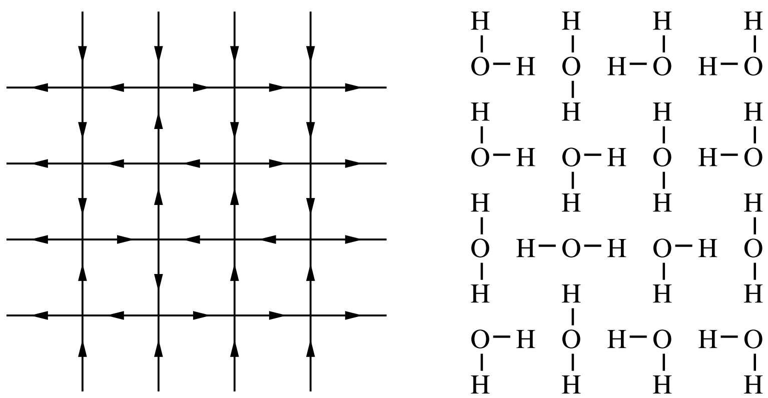

**Figure 2.** An example of a $4 \times 4$ configuration (left) and the corresponding ice crystal (right).

For each possible vertex state we assign a *weight* $w_i$, $i = 1, \dots, 6$, and define, as usual, the *partition function*, as a sum over all possible arrow configurations of the product of the vertex weights,

$$Z_n = \sum_{\substack{\text{arrow} \\ \text{configurations } \sigma}} w(\sigma), \quad w(\sigma) = \prod_{x \in V_n} w_{\sigma(x)} = \prod_{i=1}^{6} w_i^{N_i(\sigma)}, \tag{1-1}$$

where $V_n$ is the $n \times n$ set of vertices, $\sigma(x) \in \{1, \dots, 6\}$ is the vertex configuration of $\sigma$ at vertex $x$ according to Figure 1, and $N_i(\sigma)$ is the number of vertices of type $i$ in the configuration $\sigma$. The sum is taken over all possible configurations obeying the given boundary condition. The *Gibbs measure* is defined then as

$$\mu_n(\sigma) = \frac{w(\sigma)}{Z_n}. \tag{1-2}$$

Our main goal is to obtain the *large n asymptotics* of the partition function $Z_n$.

In general, the six-vertex model has *six parameters*: the weights $w_i$. However, by using some conservation laws we can reduce these to only *two parameters*. Any fixed boundary conditions impose some conservation laws on the six-vertex model. In the case of DWBC, they are

$$N_1(\sigma) = N_2(\sigma), \quad N_3(\sigma) = N_4(\sigma), \quad N_5(\sigma) = N_6(\sigma) + n. \tag{1-3}$$

This allows us to reduce to the case

$$w_1 = w_2 \equiv a, \quad w_3 = w_4 \equiv b, \quad w_5 = w_6 \equiv c. \tag{1-4}$$

Then by using the identity,

$$Z_n(a,a,b,b,c,c) = c^{n^2} Z_n\left(\frac{a}{c},\frac{a}{c},\frac{b}{c},\frac{b}{c},1,1\right), \tag{1-5}$$

we can reduce to the two parameters, $a/c$ and $b/c$. For details on how we make this reduction, see, e.g., [Allison and Reshetikhin 2005; Ferrari and Spohn 2006; Bleher and Liechty 2009a].

## 2. Phase diagram of the six-vertex model

Introduce the parameter

$$\Delta = \frac{a^2+b^2-c^2}{2ab}. \tag{2-1}$$

The *phase diagram* of the six-vertex model consists of the following three regions: the *ferroelectric phase region*, $\Delta > 1$; the *antiferroelectric phase region*, $\Delta < -1$; and, the *disordered phase region*, $-1 < \Delta < 1$ (see, e.g., [Lieb and Wu 1972]). In these three regions we parametrize the weights in the standard way: in the ferroelectric phase region,

$$a = \sinh(t-\gamma), \quad b = \sinh(t+\gamma), \quad c = \sinh(2|\gamma|), \quad 0 < |\gamma| < t; \tag{2-2}$$

in the antiferroelectric phase region,

$$a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t), \quad c = \sinh(2\gamma), \quad |t| < \gamma; \tag{2-3}$$

and in the disordered phase region,

$$a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \quad |t| < \gamma. \tag{2-4}$$

The phase diagram of the model is shown on Figure 3.

The phase diagram and the Bethe-ansatz solution of the *six-vertex model for periodic and antiperiodic boundary conditions* are thoroughly discussed in [Lieb 1967a; 1967b; 1967c; 1967d; Lieb and Wu 1972; Sutherland 1967; Baxter 1989;

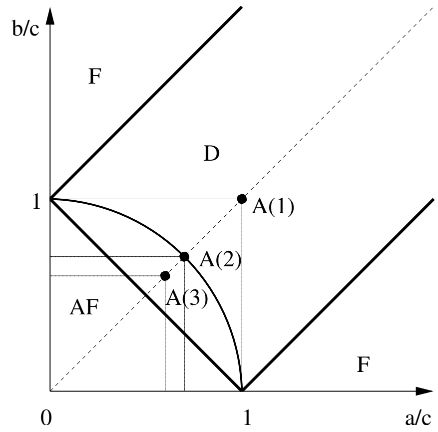


**Figure 3.** Phase diagram of the model. F, AF and D mark the ferroelectric, antiferroelectric, and disordered phases. The circular arc corresponds to the so-called “free fermion” line, where $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices.

Batchelor et al. 1995]. See also [Wu and Lin 1975], in which the Pfaffian solution for the six-vertex model with periodic boundary conditions is obtained on the free fermion line, $\Delta = 0$.

## 3. Izergin–Korepin determinantal formula

The *six-vertex model with DWBC* was introduced by Korepin [1982], who derived an important recursion relation for the partition function of the model. This led to a beautiful *determinantal formula* of Izergin [1987] for the partition function of the six-vertex model with DWBC, known as the *Izergin–Korepin formula*. A detailed proof of this formula and its generalizations are given in the paper of Izergin, Coker and Korepin [Izergin et al. 1992]. When the weights are parametrized according to (2-4), the Izergin–Korepin formula is

$$Z_n = \frac{(ab)^{n^2}}{\left(\prod_{k=0}^{n-1} k!\right)^2}\, \tau_n, \tag{3-1}$$

where $\tau_n$ is the Hankel determinant

$$\tau_n = \det\left(\frac{d^{i+k-2}\phi}{dt^{i+k-2}}\right)_{1\le i,k\le n}, \tag{3-2}$$

and

$$\phi(t) = \frac{c}{ab}. \tag{3-3}$$

Observe that $a, b, c$ have different parametrizations (2-2)–(2-4) in different phase regions. An elegant derivation of the Izergin–Korepin determinantal formula from the *Yang–Baxter equations* is given in [Korepin and Zinn-Justin 2000; Kuperberg 1996].

One of the applications of the determinantal formula is that it implies that the partition function $\tau_n$ solves the *Toda equation*,

$$\tau_N \tau_n'' - {\tau_n'}^2 = \tau_{n+1}\tau_{n-1}, \quad n \geq 1, \quad (\,'\,) = \frac{\partial}{\partial t}; \tag{3-4}$$

cf. [Sogo 1993]. This was used by Korepin and Zinn-Justin [2000] to derive the free energy of the six-vertex model with DWBC, assuming some ansatz on the behavior of subdominant terms in the large $n$ asymptotics of the free energy.

## 4. The six-vertex model with DWBC and a random matrix model

Another application of the Izergin–Korepin determinantal formula is that $\tau_n$ can be expressed in terms of a partition function of a *random matrix model*. The relation to the random matrix model was obtained and used in [Zinn-Justin 2000]. It can be derived as follows. Consider first the disordered phase region.

***Disordered phase region.*** For the evaluation of the Hankel determinant (3-2), it is convenient to use an integral representation of the function

$$\phi(t) = \frac{\sin 2\gamma}{\sin(\gamma - t)\sin(\gamma + t)}; \tag{4-1}$$

namely, to write it in the form of the Laplace transform,

$$\phi(t) = \int_{-\infty}^{\infty} e^{t\lambda} m(\lambda)\, d\lambda, \tag{4-2}$$

where

$$m(\lambda) = \frac{\sinh \frac{\lambda}{2}(\pi - 2\gamma)}{\sinh \frac{\lambda}{2}\pi}. \tag{4-3}$$

Then

$$\frac{d^i\phi}{dt^i} = \int_{-\infty}^{\infty} \lambda^i e^{t\lambda} m(\lambda)\, d\lambda, \tag{4-4}$$

and by substituting this into the Hankel determinant, (3-2), we obtain

$$\begin{aligned}\tau_n &= \int \prod_{i=1}^{n} [e^{t\lambda_i} m(\lambda_i)\, d\lambda_i] \det(\lambda_i^{i+k-2})_{1\leq i,k\leq n} \\ &= \int \prod_{i=1}^{n} [e^{t\lambda_i} m(\lambda_i)\, d\lambda_i] \det(\lambda_i^{k-1})_{1\leq i,k\leq n} \prod_{i=1}^{n} \lambda_i^{i-1}.\end{aligned} \tag{4-5}$$

Consider any permutation $\sigma \in S_n$ of variables $\lambda_i$. From the last equation we have that

$$\tau_n = \int \prod_{i=1}^{n} [e^{t\lambda_i} m(\lambda_i)\, d\lambda_i] (-1)^{\sigma} \det(\lambda_i^{k-1})_{1 \le i,k \le n} \prod_{i=1}^{n} \lambda_{\sigma(i)}^{i-1}. \tag{4-6}$$

By summing over $\sigma \in S_n$, we obtain that

$$\tau_n = \frac{1}{n!} \int \prod_{i=1}^{n} [e^{t\lambda_i} m(\lambda_i) d\lambda_i] \Delta(\lambda)^2, \tag{4-7}$$

where $\Delta(\lambda)$ is the Vandermonde determinant,

$$\Delta(\lambda) = \det(\lambda_i^{k-1})_{1 \le i,k \le n} = \prod_{i<k} (\lambda_k - \lambda_i). \tag{4-8}$$

Equation (4-7) expresses $\tau_n$ in terms of a matrix model integral. Namely, if $m(x) = e^{-V(x)}$, then

$$\tau_n = \frac{\prod_{n=0}^{n-1} n!}{\pi^{n(n-1)/2}} \int dM\, e^{\mathrm{Tr}[tM - V(M)]}, \tag{4-9}$$

where the integration is over the space of $n \times n$ Hermitian matrices. The matrix model integral can be solved, furthermore, in terms of *orthogonal polynomials*.

Introduce monic polynomials $P_k(x) = x^k + \cdots$ orthogonal on the line with respect to the weight

$$w(x) = e^{tx} m(x), \tag{4-10}$$

so that

$$\int_{-\infty}^{\infty} P_j(x) P_k(x) e^{tx} m(x)\, dx = h_k \delta_{nm}. \tag{4-11}$$

Then it follows from (4-7) that

$$\tau_n = \prod_{k=0}^{n-1} h_k. \tag{4-12}$$

The orthogonal polynomials satisfy the three term recurrence relation,

$$x P_k(x) = P_{k+1}(x) + Q_k P_k(x) + R_k P_{k-1}(x), \tag{4-13}$$

where $R_k$ can be found as

$$R_k = \frac{h_k}{h_{k-1}}; \tag{4-14}$$

see, e.g., [Szegő 1975]. This gives

$$h_k = h_0 \prod_{j=1}^{k} R_j, \tag{4-15}$$

where

$$h_0 = \int_{-\infty}^{\infty} e^{tx} m(x)\, dx = \frac{\sin(2\gamma)}{\sin(\gamma+t)\sin(\gamma-t)}. \tag{4-16}$$

By substituting (4-15) into (4-12), we obtain that

$$\tau_n = h_0^n \prod_{k=1}^{n-1} R_k^{n-k}. \tag{4-17}$$

Colomo and Pronko identified the orthogonal polynomials $\{P_k\}$ at some points on the phase diagram with classical orthogonal polynomials, see [Colomo and Pronko 2003; 2004; 2005; 2006]. They showed that on the free fermion line, $\{P_k\}$ are the Meixner–Pollaczek polynomials, at the ice point $(1, 1)$ they are the continuous Hahn polynomials, and at the point $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $\{P_k\}$ can be expressed in terms of the continuous dual Hahn polynomials. The ice point $(1, 1)$ and the point $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ correspond to the 1- and 3-enumerations, respectively, of alternating sign matrices The 2-enumeration of ASMs corresponds to the point on the free fermion line at which $a = b$. The 1-, 2-, and 3-enumerations of ASMs are marked A(1), A(2), and A(3), respectively, in Figure 3, and the full free fermion line is marked there as well. In all these cases the normalizing constants $h_k$ are known explicitly, and formula (4-12) can be used to find the asymptotic behavior of $\tau_n$ as $n \to \infty$. At all other points on the phase diagrams, no reduction to classical orthogonal polynomials is known.

***Ferroelectric phase.*** In the ferroelectric phase, the parameters $a, b$, and $c$ are parametrized by (2-2). We consider the case $\gamma > 0$, which corresponds to the region $b > a + c$ in the phase diagram. The case $\gamma < 0$ is similar, and $a$ and $b$ should be exchanged in that case. The function $\phi$ is the Laplace transform of a discrete measure supported on the positive integers:

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)} = 4\sum_{l=1}^{\infty} e^{-2tl} \sinh(2\gamma l). \tag{4-18}$$

Then, similar to (4-7), we find that

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1,\dots,l_n=1}^{\infty} \Delta(l_i)^2 \prod_{i=1}^{n} [2e^{-2tl_i} \sinh(2\gamma l_i)]. \tag{4-19}$$

This is the partition function for a discrete version of a Hermitian random matrix model, often called a *discrete orthogonal polynomial ensemble* (DOPE), and can also be solved in terms of orthogonal polynomials. The appropriate polynomials in this case are the monic polynomials $P_n(l) = l^n + \cdots$ with the orthogonality

$$\sum_{l=1}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}, \qquad w(l) = 2e^{-2tl} \sinh(2\gamma l) = e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}. \tag{4-20}$$

Then it follows from (4-19) that

$$\tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k. \tag{4-21}$$

***Critical line between disordered and ferroelectric phase.*** When the parameters $a, b$, and $c$ are such that $b - a = c$ (so $\Delta = 1$ in (2-1)), the Izergin–Korepin formula is not directly applicable. However, we may consider a limiting case of the orthogonal polynomial formula (4-21). On the critical line

$$\frac{b}{c} - \frac{a}{c} = 1, \tag{4-22}$$

we fix a point,

$$\frac{a}{c} = \frac{\alpha - 1}{2}, \quad \frac{b}{c} = \frac{\alpha + 1}{2}; \qquad \alpha > 1, \tag{4-23}$$

and consider the partition function

$$Z_n = Z_n\Big(\frac{\alpha-1}{2}, \frac{\alpha-1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+1}{2}, 1, 1\Big). \tag{4-24}$$

Consider the limit of (4-21) as

$$t, \gamma \to +0, \quad \frac{t}{\gamma} \to \alpha. \tag{4-25}$$

Observe that in this limit,

$$\frac{a}{c} = \frac{\sinh(t-\gamma)}{\sinh(2\gamma)} \to \frac{\alpha - 1}{2}, \quad \frac{b}{c} = \frac{\sinh(t+\gamma)}{\sinh(2\gamma)} \to \frac{\alpha + 1}{2}. \tag{4-26}$$

By (1-5), (3-1), and (4-12), we have

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = \left(\frac{2\sinh(t-\gamma)\sinh(t+\gamma)}{\sinh(2\gamma)}\right)^{n^2} \prod_{k=0}^{n-1} \frac{h_k}{(k!)^2}. \tag{4-27}$$

To deal with limit (4-25) we need to rescale the orthogonal polynomials $P_k(l)$. Introduce the rescaled variable

$$x = 2tl - 2\gamma l, \tag{4-28}$$

and the rescaled limiting weight,

$$w_\alpha(x) = \lim_{\substack{t,\gamma\to+0\\ t/\gamma\to\alpha}} (e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}) = e^{-x} - e^{-rx}, \qquad r = \frac{\alpha+1}{\alpha-1} > 1. \tag{4-29}$$

Consider monic orthogonal polynomials $P_j(x;\alpha)$ satisfying the orthogonality condition,

$$\int_0^\infty P_j(x;\alpha) P_k(x;\alpha) w_\alpha(x)\, dx = h_{k,\alpha}\delta_{jk}. \tag{4-30}$$

To find a relation between $P_k(l)$ and $P_k(x;\alpha)$, introduce the monic polynomials

$$\widetilde{P}_k(x) = \delta^k P_k(x/\delta), \tag{4-31}$$

where

$$\delta = 2t - 2\gamma, \tag{4-32}$$

and rewrite orthogonality condition (4-11) in the form

$$\sum_{l=1}^{\infty} \widetilde{P}_j(l\delta)\widetilde{P}_k(l\delta) w_\alpha(l\delta)\delta = \delta^{2k+1} h_k \delta_{jk}, \tag{4-33}$$

which is a Riemann sum for the integral in orthogonality condition (4-30). Therefore,

$$\lim_{\substack{t,\gamma\to+0\\ t/\gamma\to\alpha}} \widetilde{P}_k(x) = P_k(x;\alpha), \qquad \lim_{\substack{t,\gamma\to+0\\ t/\gamma\to\alpha}} \delta^{2k+1} h_k = h_{k,\alpha}. \tag{4-34}$$

Thus, if we rewrite formula (4-27) as

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = \left(\frac{2\sinh(t-\gamma)\sinh(t+\gamma)}{\sinh(2\gamma)\delta}\right)^{n^2} \prod_{k=0}^{n-1} \frac{\delta^{2k+1} h_k}{(k!)^2}, \tag{4-35}$$

we can take limit (4-25). In the limit we obtain that

$$Z_n = Z_n\left(\frac{\alpha-1}{2}, \frac{\alpha-1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+1}{2}, 1, 1\right) = \left(\frac{\alpha+1}{2}\right)^{n^2} \prod_{k=0}^{n-1} \frac{h_{k,\alpha}}{(k!)^2}. \tag{4-36}$$

***Antiferroelectric phase.*** In the antiferroelectric phase, the parameters $a, b$, and $c$ are parametrized by (2-3), and the function

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma - t)\sinh(\gamma + t)}, \quad |t| < \gamma, \tag{4-37}$$

is the Laplace transform of a discrete measure supported on the integers:

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma - t)\sinh(\gamma + t)} = 2\sum_{l=-\infty}^{\infty} e^{2tl - 2\gamma|l|}. \tag{4-38}$$

Then

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1,\dots,l_n=-\infty}^{\infty} \Delta(l)^2 \prod_{i=1}^{n} e^{2tl_i - 2\gamma|l_i|}. \tag{4-39}$$

This is again the partition function of a DOPE, and we introduce the discrete monic polynomials $P_n(l) = l^n + \cdots$ via the orthogonality condition

$$\sum_{l=-\infty}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}, \quad w(l) = e^{2tl - 2\gamma|l|}. \tag{4-40}$$

Then it follows from (4-39) that

$$\tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k. \tag{4-41}$$

***Critical line between the antiferroelectric and disordered phases.*** When the parameters $a, b$, and $c$ are such that $a + b = c$, (so $\Delta = -1$ in (2-1)), the Izergin–Korepin formula is not directly applicable, and we must consider a limiting case of the orthogonal polynomial formula (4-41). On the critical line

$$\frac{a}{c} + \frac{b}{c} = 1, \tag{4-42}$$

we fix a point,

$$\frac{a}{c} = \frac{1-\alpha}{2}, \quad \frac{b}{c} = \frac{1+\alpha}{2}, \qquad -1 < \alpha < 1, \tag{4-43}$$

and consider the partition function

$$Z_n = Z_n\left(\frac{1-\alpha}{2}, \frac{1-\alpha}{2}, \frac{1+\alpha}{2}, \frac{1+\alpha}{2}, 1, 1\right). \tag{4-44}$$

This corresponds to taking a limit of the Izergin–Korepin formula in the antiferroelectric phase as $t, \gamma \to 0$, and $t/\gamma = \alpha$. Introduce the rescaled variable

$$x = -2tl + 2\gamma l, \tag{4-45}$$

and the rescaled limiting weight,

$$w_\alpha(x) = \lim_{t,\gamma\to+0,\ \frac{t}{\gamma}\to\alpha} e^{2tl - 2\gamma|l|} = \begin{cases} e^{-x}, & x \ge 0, \\ e^{rx}, & x < 0, \end{cases} \tag{4-46}$$

where

$$r = \frac{1+\alpha}{1-\alpha} > 0. \tag{4-47}$$

Consider monic orthogonal polynomials $P_j(x;\alpha)$ satisfying the orthogonality condition,

$$\int_{\mathbb{R}} P_j(x;\alpha) P_k(x;\alpha) w_\alpha(x)\, dx = h_{k,\alpha}\delta_{jk}, \tag{4-48}$$

which can be obtained from the polynomials (4-40) by taking the appropriate scaling limit as $t, \gamma \to 0$, and $t/\gamma = \alpha$. Similar to (4-36), we obtain

$$Z_n = Z_n\Big(\frac{\alpha-1}{2}, \frac{\alpha-1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+1}{2}, 1, 1\Big) = \Big(\frac{1+\alpha}{2}\Big)^{n^2} \prod_{k=0}^{n-1} \frac{h_{k,\alpha}}{(k!)^2}. \tag{4-49}$$

## 5. Large $n$ asymptotics of $Z_n$

The asymptotic evaluation of $Z_n$ in the different regions of the phase diagram thus reduces to asymptotic evaluation of different systems of orthogonal polynomials. In general, this may be done by formulating the orthogonal polynomials as the solution to a $2 \times 2$ matrix valued Riemann–Hilbert problem as in [Fokas et al. 1992]. One may then perform the steepest descent analysis of [Deift and Zhou 1993]. In the case that the weight of orthogonality is a continuous one on $\mathbb{R}$, this analysis was performed for weights of the form $\exp(-nV(x))$ for a very general class of analytic potential functions $V(x)$ in [Deift et al. 1999]. The analysis was adapted to the case that the orthogonality is with respect to a discrete measure in [Baik et al. 2007; Bleher and Liechty 2011]. The steepest descent analysis yields the following results in the different regions of the phase diagram.

***Disordered phase.***

**Theorem 5.1** [Bleher and Fokin 2006]. *Let the weights $a$, $b$, and $c$, in the six-vertex model with DWBC be parametrized as in* (2-4). *Then, as $n \to \infty$, the partition function $Z_n$ has the asymptotic expansion*

$$Z_n = Cn^\kappa F^{n^2}(1 + O(n^{-\varepsilon})), \quad \varepsilon > 0, \tag{5-1}$$

*where*

$$F = \frac{\pi ab}{2\gamma \cos \frac{\pi t}{2\gamma}}, \quad \kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}, \tag{5-2}$$

*and $C > 0$ is a constant.*

This proves the conjecture of Zinn-Justin, and it gives the exact value of the exponent $\kappa$. Let us remark that the presence of the power-like factor $n^\kappa$ in the asymptotic expansion of $Z_n$ in (5-1) is rather unusual from the point of view of random matrix models. Also, in the one-cut case the usual large $n$ asymptotics of $\log Z_n$ in a noncritical random matrix model is the so called "topological expansion", which gives $(\log Z_n)/n^2$ as an asymptotic series in powers of $1/n^2$ (see, e.g., [Ercolani and McLaughlin 2003; Bleher and Its 2005]). In this case the asymptotic expansion of $\log Z_n$ includes the term $\kappa \log n$.

It is noteworthy that, as shown in [Bogoliubov et al. 2002], asymptotic formula (5-1) remains valid on the borderline between the disordered and antiferroelectric phases. In this case $\kappa = \frac{1}{12}$, which corresponds to $\gamma = 0$. In [Bleher and Bothner 2012] the constant $C$ in the asymptotic expansion (5-1) is calculated on the borderline between the disordered and antiferroelectric phases, up to a universal constant factor, which is still unknown. Also, it is shown in [Bleher and Bothner 2012] that the error term $O(n^{-\varepsilon})$ in (5-1) can be replaced by $O(n^{-1})$. The calculations of [Bleher and Bothner 2012] can be extended to the whole disordered region, where they give an explicit dependence of the constant $C$ on the parameter $t$ in parametrization (2-4), and improve the error term in (5-1) to $O(n^{-1})$.

***Ferroelectric phase.*** We have obtained the large $n$ asymptotics of $Z_n$ in the ferroelectric phase, $\Delta > 1$ [Bleher and Liechty 2009a], and also on the critical line between the ferroelectric and disordered phases, $\Delta = 1$ [2009b]. In the ferroelectric phase we use parametrization (2-2) for $a, b$ and $c$. The large $n$ asymptotics of $Z_n$ in the ferroelectric phase is given by the following theorem.

**Theorem 5.2** [Bleher and Liechty 2009a]. *Let the weights $a$, $b$, and $c$ in the six-vertex model with DWBC be parametrized as in* (2-2) *with $t > \gamma > 0$. For any $\varepsilon > 0$, as $n \to \infty$,*

$$Z_n = C G^n F^{n^2}\big(1 + O(e^{-n^{1-\varepsilon}})\big), \tag{5-3}$$

*where $C = 1 - e^{-4\gamma}$, $G = e^{\gamma - t}$ and $F = b$.*

On the critical line between the ferroelectric and disordered phases we use the parametrization $b = a + 1$, $c = 1$. The main result here is the following asymptotic formula for $Z_n$.

**Theorem 5.3** [Bleher and Liechty 2009b]. *As $n \to \infty$,*

$$Z_n = C n^{\kappa} G^{\sqrt{n}} F^{n^2}[1 + O(n^{-1/2})], \tag{5-4}$$

*where $C > 0$,*

$$\kappa = \tfrac{1}{4}, \quad G = \exp\left[-\zeta\left(\tfrac{3}{2}\right)\sqrt{\frac{a}{\pi}}\right], \tag{5-5}$$

*and*

$$F = b. \tag{5-6}$$

Notice that in both Theorems 5.2 and 5.3, the limiting free energy $F$ is the weight $b$. The ground state in this phase is unique and is achieved when there is exactly one $c$-type vertex in each row and column, and the rest of the vertices are of type $b$. That is, the diagonal consists of type 5 vertices while above the diagonal all vertices are type 3 and below all vertices are type 4. The weight of the ground state is $b^{n^2}(c/b)^n$, and thus the free energy in the ferroelectric phase is completely determined by the ground state. This is a reflection of the fact that local fluctuations from the ground state can take place only in a thin neighborhood of the diagonal. The conservation laws (1-3) forbid local fluctuations away from the diagonal.

***Antiferroelectric phase.*** The large $n$ asymptotics in the antiferroelectric phase were obtained nonrigorously in [Zinn-Justin 2000], and rigorously, using the Riemann–Hilbert method, in [Bleher and Liechty 2010]. They are given in the following theorem. In this theorem $\vartheta_1$ and $\vartheta_4$ are the Jacobi theta functions with elliptic nome $q = e^{-\pi^2/2\gamma}$ (see, e.g., [Whittaker and Watson 1996]), and the phase $\omega$ is given as

$$\omega = \frac{\pi}{2}\left(1 + \frac{t}{\gamma}\right). \tag{5-7}$$

**Theorem 5.4** [Bleher and Liechty 2010]. *Let the weights $a$, $b$, and $c$ in the six-vertex model with DWBC be parametrized as in* (2-2). *As $n \to \infty$,*

$$Z_n = C\vartheta_4(n\omega)F^{n^2}(1 + O(n^{-1})), \tag{5-8}$$

*where $C > 0$ is a constant, and*

$$F = \frac{\pi ab\vartheta_1'(0)}{2\gamma\vartheta_1(\omega)}. \tag{5-9}$$

In contrast to the disordered phase, note the lack of a power like term. In contrast to the ferroelectric phase, notice that the free energy depends transcendentally on the weight of the ground state configuration. Only in the limit as $\gamma \to \infty$, which can be regarded as the low temperature limit, does the weight of the ground state become dominant. For a discussion of this limit, see [Zinn-Justin 2000].

## 6. The Riemann–Hilbert approach

All the above asymptotic results are obtained in the Riemann–Hilbert approach, but the concrete asymptotic analysis of the Riemann–Hilbert problem is quite different in the different phase regions. Let us discuss it.

***Disordered phase region.*** To apply the Riemann–Hilbert approach, we introduce a rescaled weight as

$$w_n(x) = w\left(\frac{nx}{\gamma}\right). \tag{6-1}$$

It can be written as

$$w_n(x) = e^{-nV_n(x)}, \tag{6-2}$$

where

$$V_n(x) = -\zeta x - \frac{1}{n} \ln \frac{\sinh\left(n\left(\frac{\pi}{2\gamma} - 1\right)x\right)}{\sinh \frac{n\pi x}{2\gamma}}, \quad \zeta = \frac{t}{\gamma}. \tag{6-3}$$

The external potential $V_n(x)$ is real analytic for any finite $n$, but it has logarithmic singularities on the imaginary axis, which accumulate to the origin as $n \to \infty$. In fact, the limiting external potential,

$$\lim_{n\to\infty} V_n(x) = V(x) = -\zeta x + |x|, \tag{6-4}$$

is not analytic at $x = 0$. The Riemann–Hilbert approach developed in [Bleher and Fokin 2006] is based on an opening of lenses whose boundary approaches the origin as $n \to \infty$. This turns out to be possible due to the fact that the density of the equilibrium measure $\rho_n(x)$ for the external potential $V_n(x)$ diverges logarithmically at the origin as $n \to \infty$, and as a result, the jump matrix on the boundary of the lenses converges to the unit matrix (for details, see [ibid.]). The calculation of subdominant asymptotic terms in the partition function as $n \to \infty$ is the central difficult part of the work [ibid.], and it is done by an asymptotic analysis of the solution to the Riemann–Hilbert problem near the turning points and near the origin.

***Ferroelectric phase region.*** In the ferroelectric region, the measure of orthogonality is a discrete one on $\mathbb{N}$. To apply the Riemann–Hilbert approach to discrete orthogonal polynomials, we need to rescale both the weight and the lattice that supports the measure so that the mesh of the lattice goes to zero as $n \to \infty$. Introduce the rescaled lattice and weight

$$L_n = \left(\frac{2t}{n}\right)\mathbb{N}, \qquad w_n(x) = e^{-nx(1-\zeta)}(1 - e^{-4nx}) = e^{-nV_n(x)}, \tag{6-5}$$

where

$$V_n(x) = x(1-\zeta) - \frac{1}{n}\log(1 - e^{-2nx\zeta}), \quad 0 < \zeta = \frac{\gamma}{t} < 1. \tag{6-6}$$

Then the orthogonality condition (4-20) can be written as

$$\sum_{x\in L_n} P_j\left(\frac{nx}{2t}\right) P_k\left(\frac{nx}{2t}\right) w_n(x) = h_k \delta_{jk}. \tag{6-7}$$

Notice that, as $n \to \infty$, $V_n(x)$ has the limit

$$\lim_{n\to\infty} V_n(x) = x(1-\zeta), \tag{6-8}$$

which would indicate that, in the large $n$ limit, the polynomials (4-20) behave as polynomials orthogonal on $\mathbb{N}$ with a simple exponential weight. These polynomials are a special case of the classical *Meixner polynomials*, and there are exact formulae for their recurrence coefficients (see, e.g., [Koekoek et al. 2010]). The monic Meixner polynomials which concern us are defined from the orthogonality condition

$$\sum_{l=1}^{\infty} Q_j(l)Q_k(l)q^l = h_k^{\mathrm{Q}}\delta_{jk}, \quad q = e^{2\gamma-2t}, \tag{6-9}$$

and the normalizing constants are given exactly as

$$h_k^{\mathrm{Q}} = \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}}. \tag{6-10}$$

Up to the constant factor, Theorem 5.2 can therefore be proven by showing that $h_k$ and $h_k^{\mathrm{Q}}$ are asymptotically close as $k \to \infty$. More precisely, it is shown in [Bleher and Liechty 2009a] that as $k \to \infty$, for any $\varepsilon > 0$,

$$h_k = h_k^{\mathrm{Q}}\big(1 + O(e^{-k^{1-\varepsilon}})\big). \tag{6-11}$$

***Antiferroelectric region.*** In the antiferroelectric region, the orthogonal polynomials are with respect to a discrete weight, and we rescale the weight in (4-40) and the integer lattice as

$$L_n = \left(\frac{2\gamma}{n}\right)\mathbb{Z}, \quad w_n(x) = e^{-nV(x)}, \quad V(x) = |x| - \zeta x, \;\; \zeta = \frac{t}{\gamma} < 1, \tag{6-12}$$

so that the orthogonality condition (4-40) can be written as

$$\sum_{x\in L_n} P_j\left(\frac{nx}{2\gamma}\right)P_k\left(\frac{nx}{2\gamma}\right)w_n(x) = h_k\delta_{jk}. \tag{6-13}$$

The mesh of the lattice $L_n$ is $2\gamma/n$, which places an upper constraint on the equilibrium measure, which is the limiting distribution of zeroes of the orthogonal polynomials. This upper constraint is realized. The equilibrium measure, which has density $\rho(x)$, is supported on a single interval $[\alpha, \beta]$, but within that interval is an interval $[\alpha', \beta']$ on which $\rho(x) \equiv 1/2\gamma$. This interval is called the *saturated region*, and it separates the single band of support $[\alpha, \beta]$ into the two bands of analyticity $[\alpha, \alpha']$ and $[\beta', \beta]$. Thus in effect we have a "two-cut" situation, which is the source of the quasiperiodic factor $\vartheta_4(n\omega)$ in Theorem 5.4.

In principle, a problem could come from the fact that the potential $V(x)$ is not analytic at the origin. However, it turns out that this point of nonanalyticity is always in the saturated region and therefore does not present a problem in the steepest descent analysis.

As previously noted, there is no power-like term in the asymptotic formula for $Z_n$ in the antiferroelectric phase. The Riemann–Hilbert approach to orthogonal polynomials generally gives an expansion of the normalizing constants $h_n$ in inverse powers of $n$. In the two-cut case, the coefficients in this expansion may be quasiperiodic functions of $n$. For the orthogonal polynomials (4-40) it is a tedious calculation involving the Jacobi theta functions to show that the term of order $n^{-1}$ vanishes in the expansion of $h_n$, which then implies the absence of the power-like term in $Z_n$.

## References

[Allison and Reshetikhin 2005] D. Allison and N. Reshetikhin, “Numerical study of the 6-vertex model with domain wall boundary conditions”, *Ann. Inst. Fourier* (*Grenoble*) **55**:6 (2005), 1847–1869.

[Baik et al. 2007] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin, and P. D. Miller, *Discrete orthogonal polynomials: asymptotics and applications*, Annals of Mathematics Studies **164**, Princeton University Press, 2007.

[Batchelor et al. 1995] M. T. Batchelor, R. J. Baxter, M. J. O’Rourke, and C. M. Yung, “Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions”, *J. Phys. A* **28**:10 (1995), 2759–2770.

[Baxter 1989] R. J. Baxter, *Exactly solved models in statistical mechanics*, reprint of the 1982 ed., Academic Press, London, 1989.

[Bleher and Bothner 2012] P. Bleher and T. Bothner, “Exact solution of the six-vertex model with domain wall boundary conditions: critical line between disordered and antiferroelectric phases”, *Random Matrices Theory Appl.* **1**:4 (2012), 1250012, 43.

[Bleher and Fokin 2006] P. M. Bleher and V. V. Fokin, “Exact solution of the six-vertex model with domain wall boundary conditions: disordered phase”, *Comm. Math. Phys.* **268**:1 (2006), 223–284.

[Bleher and Its 2005] P. M. Bleher and A. R. Its, “Asymptotics of the partition function of a random matrix model”, *Ann. Inst. Fourier* (*Grenoble*) **55**:6 (2005), 1943–2000.

[Bleher and Liechty 2009a] P. Bleher and K. Liechty, “Exact solution of the six-vertex model with domain wall boundary conditions: ferroelectric phase”, *Comm. Math. Phys.* **286**:2 (2009), 777–801.

[Bleher and Liechty 2009b] P. Bleher and K. Liechty, “Exact solution of the six-vertex model with domain wall boundary conditions: critical line between ferroelectric and disordered phases”, *J. Stat. Phys.* **134**:3 (2009), 463–485.

[Bleher and Liechty 2010] P. Bleher and K. Liechty, “Exact solution of the six-vertex model with domain wall boundary conditions: antiferroelectric phase”, *Comm. Pure Appl. Math.* **63**:6 (2010), 779–829.

[Bleher and Liechty 2011] P. Bleher and K. Liechty, "Uniform asymptotics for discrete orthogonal polynomials with respect to varying exponential weights on a regular infinite lattice", *Int. Math. Res. Not.* **2011** (2011), 342–386.

[Bogoliubov et al. 2002] N. M. Bogoliubov, A. V. Kitaev, and M. B. Zvonarev, "Boundary polarization in the six-vertex model", *Phys. Rev. E* (3) **65**:2 (2002), 026126, 4.

[Colomo and Pronko 2003] F. Colomo and A. G. Pronko, "On some representations of the six vertex model partition function", *Phys. Lett. A* **315**:3-4 (2003), 231–236.

[Colomo and Pronko 2004] F. Colomo and A. G. Pronko, "On the partition function of the six-vertex model with domain wall boundary conditions", *J. Phys. A* **37**:6 (2004), 1987–2002.

[Colomo and Pronko 2005] F. Colomo and A. G. Pronko, "Square ice, alternating sign matrices, and classical orthogonal polynomials", *J. Stat. Mech. Theory Exp.* 1 (2005), 005, 33 pp.

[Colomo and Pronko 2006] F. Colomo and A. G. Pronko, "The role of orthogonal polynomials in the six-vertex model and its combinatorial applications", *J. Phys. A* **39**:28 (2006), 9015–9033.

[Deift and Zhou 1993] P. Deift and X. Zhou, "A steepest descent method for oscillatory Riemann–Hilbert problems: asymptotics for the MKdV equation", *Ann. of Math.* (2) **137**:2 (1993), 295–368.

[Deift et al. 1999] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, "Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory", *Comm. Pure Appl. Math.* **52**:11 (1999), 1335–1425.

[Ercolani and McLaughlin 2003] N. M. Ercolani and K. D. T.-R. McLaughlin, "Asymptotics of the partition function for random matrices via Riemann–Hilbert techniques and applications to graphical enumeration", *Int. Math. Res. Not.* **2003** (2003), 755–820.

[Ferrari and Spohn 2006] P. L. Ferrari and H. Spohn, "Domino tilings and the six-vertex model at its free-fermion point", *J. Phys. A* **39**:33 (2006), 10297–10306.

[Fokas et al. 1992] A. S. Fokas, A. R. It·s, and A. V. Kitaev, "The isomonodromy approach to matrix models in 2D quantum gravity", *Comm. Math. Phys.* **147**:2 (1992), 395–430.

[Izergin 1987] A. G. Izergin, "Partition function of a six-vertex model in a finite volume", *Dokl. Akad. Nauk SSSR* **297**:2 (1987), 331–333. In Russian; translated in *Soviet Phys. Dokl.* **32**:11 (1987), 878–879.

[Izergin et al. 1992] A. G. Izergin, D. A. Coker, and V. E. Korepin, "Determinant formula for the six-vertex model", *J. Phys. A* **25**:16 (1992), 4315–4334.

[Koekoek et al. 2010] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their $q$-analogues*, Springer, Berlin, 2010.

[Korepin 1982] V. E. Korepin, "Calculation of norms of Bethe wave functions", *Comm. Math. Phys.* **86**:3 (1982), 391–418.

[Korepin and Zinn-Justin 2000] V. Korepin and P. Zinn-Justin, "Thermodynamic limit of the six-vertex model with domain wall boundary conditions", *J. Phys. A* **33**:40 (2000), 7053–7066.

[Kuperberg 1996] G. Kuperberg, "Another proof of the alternating-sign matrix conjecture", *Internat. Math. Res. Notices* **3** (1996), 139–150.

[Lieb 1967a] E. H. Lieb, "Exact solution of the problem of the entropy of two-dimensional ice", *Phys. Rev. Lett.* **18** (Apr 1967), 692–694.

[Lieb 1967b] E. H. Lieb, "Exact solution of the $F$ model of an antiferroelectric", *Phys. Rev. Lett.* **18** (1967), 1046–1048.

[Lieb 1967c] E. H. Lieb, "Exact solution of the two-dimensional Slater KDP Model of a ferroelectric", *Phys. Rev. Lett.* **19** (Jul 1967), 108–110.

[Lieb 1967d] E. H. Lieb, "Residual entropy of square ice", *Phys. Rev.* **162** (Oct 1967), 162–172.

[Lieb and Wu 1972] E. H. Lieb and F. Y. Wu, "Two dimensional ferroelectric models", pp. 331–490 in *Phase transitions and critical phenomena*, vol. 1, edited by C. Domb and M. Green, Academic Press, 1972.

[Sogo 1993] K. Sogo, "Toda molecule equation and quotient-difference method", *J. Phys. Soc. Japan* **62**:4 (1993), 1081–1084.

[Sutherland 1967] B. Sutherland, "Exact solution of a two-dimensional model for hydrogen-bonded crystals", *Phys. Rev. Lett.* **19** (Jul 1967), 103–104.

[Szegő 1975] G. Szegő, *Orthogonal polynomials*, 4th ed., Ame. Math. Soc., Colloquium Publications **23**, American Mathematical Society, Providence, R.I., 1975.

[Whittaker and Watson 1996] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, reprint of the 4th (1927) edition ed., Cambridge University Press, 1996.

[Wu and Lin 1975] F. Y. Wu and K. Y. Lin, "Staggered ice-rule vertex model: the Pfaffian solution", *Phys. Rev. B* **12** (Jul 1975), 419–428.

[Zinn-Justin 2000] P. Zinn-Justin, "Six-vertex model with domain wall boundary conditions and one-matrix model", *Phys. Rev. E* (3) **62**:3, part A (2000), 3411–3418.

bleher@math.iupui.edu *Department of Mathematical Sciences, Indiana University–Purdue University, 402 North Blackford Street, Indianapolis, IN 46202, United States*

kliechty@depaul.edu *Department of Mathematical Sciences, DePaul University, Chicago, IL 60614, United States*