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Misère canonical forms of partizan games

AARON N. SIEGEL

We show that partizan games admit canonical forms in misère play. The proof is a synthesis of Conway's simplest form theorems for normal-play partizan games and misère-play impartial games. As an immediate application, we show that there are precisely 256 games born by day 2, and obtain a bound on the number of games born by day 3.

1. Introduction

Disjunctive compounds of short combinatorial games have been studied for many years under a variety of assumptions. A structure theory for normal-play impartial games was established in the 1930s by the *Sprague–Grundy theorem* [Grundy 1939; Sprague 1935; 1937]. Every such game G is equivalent to a Nim-heap, and the size of this heap, known as the *nim value* of G, completely describes the behavior of G in disjunctive sums. The Sprague–Grundy theorem underpins virtually all subsequent work on impartial combinatorial games.

Decades later, Conway generalized the Sprague–Grundy theorem in two directions [Berlekamp et al. 2003; Conway 2001]. First, he showed that every partizan game G can be assigned a *value* that exactly captures its disjunctive behavior, and this value is represented by a unique simplest form for G. Conway's game values are partizan analogues of nim values, and his simplest form theorem directly generalizes the Sprague–Grundy theorem.

Conway also introduced a misère-play analogue of the Sprague–Grundy theorem. He showed that every impartial game G is represented by a unique *misère* simplest form [Conway 2001]. Unfortunately, in misère play such simplifications tend to be weak, and as a result the canonical theory of misère games is less useful in practice than its normal-play counterparts.

In each case — normal-play impartial, normal-play partizan, and misère-play impartial — the identification of simplest forms proved to be a key result, at once establishing a structure theory and opening the door to further investigations. In this paper, we prove an analogous simplest form theorem for the misère-play partizan case. The proof integrates techniques drawn from each of Conway's advances, together with a crucial lemma from [Mesdal and Ottaway 2007].

As an immediate application of this theorem, we also calculate the number of partizan misère games born on day 2. As it turns out, all 256 nonisomorphic games born by day 2 are already in simplest form, so there are no day-2 reductions at all! This sobering result is evidence that partizan misère games are yet more unruly than their impartial cousins.

2. Misère equivalence

We denote by $o(G)$ the misère outcome class of G:

$$o(G) = \begin{cases} \mathscr{L} & \text{if Left can win no matter who plays first,} \\ \mathscr{R} & \text{if Right can win no matter who plays first,} \\ \mathscr{P} & \text{if second player (the Previous player) can win,} \\ \mathscr{N} & \text{if first player (the Next player) can win,} \end{cases}$$

with misère play assumed in all cases. These four outcome classes are naturally partially ordered by *favorability to Left*:

$$\begin{tikzcd}[column sep=small] & \mathscr{L} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathscr{N} \arrow[dr, dash] & & \mathscr{P} \arrow[dl, dash] \\ & \mathscr{R} & \end{tikzcd}$$

This induces the usual partial order on all misère games.

$$G \geq H \quad \text{if and only if} \quad o(G+X) \geq o(H+X) \text{ for all games } X.$$

We also define equality in the usual manner.

$$G = H \quad \text{if and only if} \quad G \geq H \text{ and } H \geq G.$$

These definitions are exactly the same as the corresponding definitions for the normal-play theory: the only difference is in the meaning of $o(G)$. The normal-play theory was studied by Berlekamp, Conway and Guy [Berlekamp et al. 2003; Conway 2001] with great success. They showed that there is a simple recursive test for normal-play equality:

Fact 2.1. *In normal play*, $G \geq H$ *if and only if no* $G^R \leq H$ *and* $G \leq$ *no* H^L.

From this it follows, by a straightforward induction, that $G = 0$ for every $\mathscr{P}$-position G (still assuming normal play). These facts are crucial ingredients in the following key result.

Fact 2.2 (simplest form theorem [Berlekamp et al. 2003; Conway 2001]). *In normal play, suppose $G = H$, and assume that neither G nor H has any dominated or reversible options. Then for every H^L there is a G^L such that $G^L = H^L$, and vice versa; and likewise for Right options.*

Remarkably, the statement of the simplest form theorem is *exactly* the same in misère play, with *exactly* the same definitions of dominated and reversible options! However the misère analogue of Fact 2.1 is badly false. A recent result due to Mesdal and Ottaway shows just how decisively it fails:

Fact 2.3 [Mesdal and Ottaway 2007]. *Let G be any game. Then $G \neq 0$ in misère play, unless G is identically zero.*

The following proposition is the nearest misère analogue to Fact 2.1.

Proposition 2.4. *In misère play, $G \geq H$ if and only if the following two conditions hold:*

(i) *for all X with $o(H + X) \geq \mathscr{P}$, we have $o(G + X) \geq \mathscr{P}$; and*

(ii) *for all X with $o(H + X) \geq \mathscr{N}$, we have $o(G + X) \geq \mathscr{N}$.*

Note that $o(G) \geq \mathscr{P}$ is equivalent to "Left can win playing second on G," and $o(G) \geq \mathscr{N}$ is equivalent to "Left can win playing first on G."

Proof of Proposition 2.4. $\Rightarrow$ is immediate. For the converse, we must show that $o(G + X) \geq o(H + X)$, for all X. If $o(H + X) = \mathscr{R}$, then there is nothing to prove; if $o(H + X) = \mathscr{P}$ or $\mathscr{N}$, it is immediate from (i) or (ii), respectively. Finally, if $o(H + X) = \mathscr{L}$, then by (i) and (ii) we have $o(G + X) \geq$ both $\mathscr{P}$ and $\mathscr{N}$, whence $o(G + X) = \mathscr{L}$. □

3. Ends and adjoints

In normal play, $G + (-G)$ is a $\mathscr{P}$-position, and hence equal to 0, for every game G. This is emphatically false in misère play: by Fact 2.3, $G + (-G) \neq 0$ for *every* game G except 0 itself. Nonetheless, we can give an explicit example of a game G°, the *adjoint* of G, such that $G + G^\circ$ is a misère $\mathscr{P}$-position. Readers familiar with the impartial theory will recognize it as the partizan analogue of Conway's *mate* [Conway 2001].

Games of the form $\{G^L \mid \}$ and $\{ \mid G^R\}$, with options for just one player, will feature prominently in the succeeding analysis. Recall that in normal play, every such game is equal to an integer [Berlekamp et al. 2003]. In misère play, by contrast, such games emerge as a crucial and complicated pathology. We will sometimes use a dot to indicate "no moves", when it is useful to clarify the notation; for example, we may write $\{0 \mid \cdot\}$ in place of $\{0 \mid\}$.

We now focus exclusively on misère play, and we won't consider normal play again until Section 7. All further uses of the notation $o(G)$, and the relations $\geq$ and $=$, refer unambiguously to misère play.

Definition 3.1. G is a *Left (Right) end* if G has no Left (Right) option.

Definition 3.2. The *adjoint* of G, denoted G°, is given by

$$G^\circ = \begin{cases} * & \text{if } G = 0, \\ \{(G^R)^\circ \mid 0\} & \text{if } G \neq 0 \text{ and } G \text{ is a Left end}, \\ \{0 \mid (G^L)^\circ\} & \text{if } G \neq 0 \text{ and } G \text{ is a Right end}, \\ \{(G^R)^\circ \mid (G^L)^\circ\} & \text{otherwise.} \end{cases}$$

Note that G° always has at least one option for each player, and hence is never an end.

Proposition 3.3. *$G + G^\circ$ is a $\mathscr{P}$-position.*

Proof. By symmetry, it suffices to show that Left can win $G + G^\circ$ moving second. G° is not a Right end, so $G + G^\circ$ has at least one Right option. There are two cases:

- If Right moves to $G^R + G^\circ$ or $G + (G^L)^\circ$, Left makes the mirror image move on the other component, which wins by induction on G.
- If G is a Left end and Right moves to $G + 0$, then Left has no move, and so wins a priori. □

Theorem 3.4. *If $G \not\geq H$, then:*

(a) *there is some T such that $o(G+T) \leq \mathscr{P}$ but $o(H+T) \geq \mathscr{N}$; and*

(b) *there is some U such that $o(G+U) \leq \mathscr{N}$ but $o(H+U) \geq \mathscr{P}$.*

Proof. We know by Proposition 2.4 that one of (a) or (b) must hold, so it suffices to show that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a). The arguments are identical, so we will show that (a) $\Rightarrow$ (b).

Fix T so that $o(G+T) \leq \mathscr{P}$ and $o(H+T) \geq \mathscr{N}$, and put

$$U = \{(H^R)^\circ \mid T\}.$$

Now from $G+U$, Right has a winning move, to $G+T$. Therefore $o(G+U) \leq \mathscr{N}$. Likewise, consider $H+U$. It is certainly not a Right end, since Right has a move from U to T. Now if Right moves to $H^R + U$, Left has a winning response to $H^R + (H^R)^\circ$. If instead Right moves to $H+T$, then since $o(H+T) \geq \mathscr{N}$, Left wins *a priori*. Therefore $o(H+U) \geq \mathscr{P}$, as needed. □

The following lemma is a straightforward generalization of the Mesdal–Ottaway result (Fact 2.3), and it proves to be a crucial piece of the analysis.

Lemma 3.5. *If H is a Left end and G is not, then $G \not\geq H$.*

Proof. Put

$$T = \left\{(H^R)^\circ \;\middle\|\; \cdot \;\middle|\; (G^L)^\circ\right\}.$$

Consider $H+T$. If Right moves to H^R+T, then Left can respond to $H^R+(H^R)^\circ$, winning by Proposition 3.3; if Right moves to $H + \left\{\cdot \;\middle|\; (G^L)^\circ\right\}$, then Left wins outright, since he has no further move. Therefore $o(H+T) \geq \mathscr{P}$.

Now consider $G+T$. Right has a move to $G + \left\{\cdot \;\middle|\; (G^L)^\circ\right\}$. Left's only response is to $G^L + \left\{\cdot \;\middle|\; (G^L)^\circ\right\}$, which must exist since G is not a Left end. But Right may then respond to $G^L + (G^L)^\circ$, winning. Therefore $o(G+T) \leq \mathscr{N}$.

This shows that $o(G+T) \not\geq o(H+T)$, so in fact $G \not\geq H$. □

4. Dominated and reversible options

Definition 4.1. Let G be a game.

(a) A Left option G^L is said to be *dominated* if $G^{L'} \geq G^L$ for some other Left option $G^{L'}$.

(b) A Right option G^R is said to be *dominated* if $G^{R'} \leq G^R$ for some other Right option $G^{R'}$.

(c) A Left option G^L is said to be *reversible* if $G^{LR} \leq G$ for some Right option G^{LR}.

(d) A Right option G^R is said to be *reversible* if $G^{RL} \geq G$ for some Left option G^{RL}.

Lemma 4.2. *Suppose G^{L_1} is dominated by G^{L_2}, and let G' be the game obtained by eliminating G^{L_1} from G. Then $G = G'$.*

Proof. Since the Left options of G' are a subset of those of G, and since G' still has at least one Left option (namely, G^{L_2}), we trivially have $G' \leq G$. Thus it suffices to show that $G' \geq G$.

By Proposition 2.4, we must show that for all X,

(i) if $o(G+X) \geq \mathscr{P}$, then $o(G'+X) \geq \mathscr{P}$;

(ii) if $o(G+X) \geq \mathscr{N}$, then $o(G'+X) \geq \mathscr{N}$.

The proof is by induction on X.

(i) Suppose $o(G+X) \geq \mathscr{P}$. Then every $o(G^R+X) \geq \mathscr{N}$ and every $o(G+X^R) \geq \mathscr{N}$. Now G and G' have exactly the same Right options, so every $o((G')^R+X) \geq \mathscr{N}$. Also, by induction on X, we have

$$o(G'+X^R) = o(G+X^R) \geq \mathscr{N}$$

for all X^R. This shows that $o(G'+X) \geq \mathscr{P}$, except possibly in the case where $G'+X$ is a Right end. But since G and G' have the same Right options, this would imply that $G+X$ is a Right end, contradicting the assumption $o(G+X) \geq \mathscr{P}$.

(ii) Suppose $o(G+X) \geq \mathscr{N}$. We know that $G+X$ is not a Left end (since G is assumed to have a Left option), so either some $o(G+X^L) \geq \mathscr{P}$ or some $o(G^L+X) \geq \mathscr{P}$. If $o(G+X^L) \geq \mathscr{P}$, then by induction $o(G'+X^L) \geq \mathscr{P}$, so we are done. If $o(G^L+X) \geq \mathscr{P}$ and G^L is a Left option of G', then the conclusion is immediate. So the only remaining case is when $G^L = G^{L_1}$. But since $G^{L_2} \geq G^{L_1}$, we have

$$o(G^{L_2}+X) \geq o(G^{L_1}+X) \geq \mathscr{P}.$$

In all cases, $o(G'+X) \geq \mathscr{N}$. $\square$

Lemma 4.3. *Suppose G^{L_1} is reversible through $G^{L_1R_1}$, and let G' be the game obtained by bypassing G^{L_1}:*

$$G' = \left\{G^{L_1R_1L}, G^{L'} \;\middle|\; G^R\right\},$$

where $G^{L'}$ is understood to range over all Left options of G except G^{L_1}. Then $G = G'$.

Proof. By Proposition 2.4, we must show that for all X,

(i) $o(G+X) \geq \mathscr{P}$ if and only if $o(G'+X) \geq \mathscr{P}$;

(ii) $o(G+X) \geq \mathscr{N}$ if and only if $o(G'+X) \geq \mathscr{N}$.

The proof is by induction on X.

(i) If either $G+X$ or $G'+X$ is a Right end, then both must be, since G and G' have the same Right options. If both are Right ends, then necessarily $o(G+X) \leq \mathscr{N}$ and $o(G'+X) \leq \mathscr{N}$, so (i) is satisfied. So assume neither is a Right end; then by induction on X we have

$$\begin{aligned} o(G+X) \geq \mathscr{P} &\iff \text{every } o(G^R+X) \geq \mathscr{N} \text{ and every } o(G+X^R) \geq \mathscr{N} \\ &\iff \text{every } o((G')^R+X) \geq \mathscr{N} \text{ and every } o(G'+X^R) \geq \mathscr{N} \\ &\iff o(G'+X) \geq \mathscr{P}. \end{aligned}$$

(ii) First suppose $o(G+X) \geq \mathscr{N}$. We know that G is not a Left end (since it has G^{L_1} as a Left option), so either some $o(G+X^L) \geq \mathscr{P}$, or else some $o(G^L+X) \geq \mathscr{P}$. There are three subcases.

- If $o(G+X^L) \geq \mathscr{P}$, then $o(G'+X^L) \geq \mathscr{P}$, by induction on X.
- If $o(G^L+X) \geq \mathscr{P}$ for $G^L \neq G^{L_1}$, then G^L is also a Left option of G' and the conclusion is immediate.

- Suppose instead that $o(G^{L_1}+X) \geq \mathscr{P}$. Then in particular,
$$o(G^{L_1 R_1}+X) \geq \mathscr{N}.$$
So either $o(G^{L_1 R_1 L}+X) \geq \mathscr{P}$ or $o(G^{L_1 R_1}+X^L) \geq \mathscr{P}$. In the first case we are done, since $G^{L_1 R_1 L}$ is a Left option of G'. But in the second case, we have $G \geq G^{L_1 R_1}$ (by hypothesis) so
$$o(G+X^L) \geq o(G^{L_1 R_1}+X^L) \geq \mathscr{P},$$
and therefore $o(G'+X^L) \geq \mathscr{P}$, by induction on X.

Finally, suppose $o(G'+X) \geq \mathscr{N}$. Then again we have three subcases: either $o(G^L+X) \geq \mathscr{P}$, or $o(G'+X^L) \geq \mathscr{P}$, or $o(G^{L_1 R_1 L}+X) \geq \mathscr{P}$. The first two subcases are just the same as before, so suppose $o(G^{L_1 R_1 L}+X) \geq \mathscr{P}$. Then in particular, $o(G^{L_1 R_1}+X) \geq \mathscr{N}$. But $G \geq G^{L_1 R_1}$ so
$$o(G+X) \geq o(G^{L_1 R_1}+X) \geq \mathscr{N}. \qquad \square$$

5. The simplest form theorem

Theorem 5.1 (simplest form theorem). *Suppose $G = H$, and assume that neither G nor H has any dominated or reversible options. Then for every H^L there is a G^L such that $G^L = H^L$, and vice versa; and likewise for Right options.*

In order to prove Theorem 5.1, we generalize some machinery from Conway's proof [2001, Theorem 76] that impartial games admit misère simplest forms.

Definition 5.2. (a) G is *downlinked to H (by T)* if
$$o(G+T) \leq \mathscr{P} \quad \text{and} \quad o(H+T) \geq \mathscr{P}.$$
(b) G is *uplinked to H (by T)* if $o(G+T) \geq \mathscr{P}$ and $o(H+T) \leq \mathscr{P}$.

Theorem 5.3. *$G \geq H$ if and only if the following four conditions hold.*

(i) *G is downlinked to no H^L*;

(ii) *no G^R is downlinked to H*;

(iii) *if H is a Left end, then so is G*;

(iv) *if G is a Right end, then so is H.*

Proof. For $\Rightarrow$ (i), fix any game T. If $o(G+T) \leq \mathscr{P}$, then certainly $o(H+T) \leq \mathscr{P}$ as well. Therefore $o(H^L+T) \leq \mathscr{N}$, so T cannot downlink G to H^L. $\Rightarrow$ (ii) is similar, and $\Rightarrow$ (iii) and (iv) are just restatements of Lemma 3.5 (and its mirror image).

We now prove $\Leftarrow$. By Proposition 2.4, we must show that for all T,

- if $o(H+T) \geq \mathscr{P}$, then $o(G+T) \geq \mathscr{P}$; and

- if $o(H+T) \geq \mathscr{N}$, then $o(G+T) \geq \mathscr{N}$.

The proof proceeds by induction on T. First suppose (for contradiction) that $o(H+T) \geq \mathscr{P}$ but $o(G+T) \leq \mathscr{N}$. Then either

$$o(G+T^R) \leq \mathscr{P} \quad \text{or} \quad o(G^R+T) \leq \mathscr{P},$$

or else $G+T$ is a Right end. If $o(G+T^R) \leq \mathscr{P}$, then by induction on T we have $o(H+T^R) \leq \mathscr{P}$, contradicting the assumption that $o(H+T) \geq \mathscr{P}$. If $o(G^R+T) \leq \mathscr{P}$, then T downlinks G^R to H, contradicting (ii). Finally, if $G+T$ is a Right end, then in particular G is a Right end, so by (iv) H is a Right end. Therefore $H+T$ is a Right end, contradicting the assumption that $o(H+T) \geq \mathscr{P}$.

This shows that $o(H+T) \geq \mathscr{P}$ implies $o(G+T) \geq \mathscr{P}$. The proof that $o(H+T) \geq \mathscr{N}$ implies $o(G+T) \geq N$ is identical, with (i) and (iii) used in place of (ii) and (iv). □

Theorem 5.4. *G is downlinked to H if and only if no $G^L \geq H$ and $G \geq$ no H^R.*

Proof. Suppose T downlinks G to H, so that $o(G+T) \leq \mathscr{P}$ and $o(H+T) \geq \mathscr{P}$. Then necessarily $o(G^L+T) \leq \mathscr{N}$ and $o(H^R+T) \geq \mathscr{N}$, so T witnesses both $G^L \not\geq H$ and $G \not\geq H^R$.

Conversely, suppose that no $G^L \geq H$ and $G \geq$ no H^R. Then for each G_i^L, Theorem 3.4 yields an X_i such that

$$o(G_i^L+X_i) \leq \mathscr{P} \quad \text{and} \quad o(H+X_i) \geq \mathscr{N}.$$

Likewise, for each H_j^R, there is some Y_j such that

$$o(G+Y_j) \leq \mathscr{N} \quad \text{and} \quad o(H_j^R+Y_j) \geq \mathscr{P}.$$

Put

$$T = \begin{cases} * & \text{if } G = H = 0, \\ \{0 \mid (H^L)^\circ\} & \text{if } G = 0 \text{ and } H \text{ is a nonzero Right end,} \\ \{(G^R)^\circ \mid 0\} & \text{if } H = 0 \text{ and } G \text{ is a nonzero Left end,} \\ \{Y_j, (G^R)^\circ \mid X_i, (H^L)^\circ\} & \text{otherwise.} \end{cases}$$

We claim that G is downlinked to H by T. We will show that $o(G+T) \leq \mathscr{P}$; the proof that $o(H+T) \geq \mathscr{P}$ is identical.

We first show that $G+T$ has a Left option. If G has a Left option, this is automatic. If G or H has a Right option, then T necessarily has a Left option. This exhausts every case except when $G = 0$ and H is a Right end; but then Left's move to 0 is built into the definition of T.

Thus $G+T$ is not a Left end, and it therefore suffices to show that every Left option is losing. If Left moves to G_i^L+T, Right can respond to $G_i^L+X_i$, which wins by choice of X_i. If Left moves to $G + (G^R)^\circ$, Right can respond to $G^R + (G^R)^\circ$, which wins by Proposition 3.3. Left's move to $G + Y_j$ loses automatically, by choice of Y_j. The only remaining possibility is Left's additional move to 0 in the first two cases of the definition of T. But that move is only available when $G = 0$, so it ends the game immediately. □

Proof of Theorem 5.1. Fix H^L. Since $G \geq H$, Theorem 5.3 implies that G is not downlinked to H^L. By Theorem 5.4, either $G^L \geq H^L$ or $G \geq H^{LR}$. The latter would imply that $H \geq H^{LR}$, contradicting the assumption that H has no reversible options. So we must have $G^L \geq H^L$.

An identical argument, using the fact that $H \geq G$, shows that $H^{L'} \geq G^L$ for some $H^{L'}$. Therefore

$$H^{L'} \geq G^L \geq H^L.$$

Since H has no dominated options, we must have $H^{L'} = H^L$, so that

$$H^{L'} = G^L = H^L.$$

The same argument suffices for the remaining cases. □

6. Games born by day 2

There are four games born by day 1; and they are familiar from the normal-play theory:

$$0 = \{\cdot \mid \cdot\}, \quad * = \{0 \mid 0\}, \quad 1 = \{0 \mid \cdot\}, \quad \bar{1} = \{\cdot \mid 0\}.$$

Remarkably, they are pairwise incomparable.

Proposition 6.1. *The four games* 0, $*$, 1, *and* $\bar{1}$ *are pairwise incomparable.*

Proof. Theorem 5.3(iii) and (iv) immediately yield $*, 1 \not\geq 0, \bar{1}$ and $0, 1 \not\geq *, \bar{1}$. Since 0 is downlinked to 0 (by $*$), (i) and (ii) furthermore imply $0 \not\geq 1$ and $\bar{1} \not\geq 0$.

Now as a trivial consequence of Theorem 5.4, we have that $\bar{1}$ is downlinked to 0 and 0 is downlinked to 1. It therefore follows from Theorem 5.3(i) that $\bar{1} \not\geq 1, *$ and from (ii) that $\bar{1}, * \not\geq 1$. This exhausts all possibilities. □

Theorem 6.2. *There are* 256 *games born by day* 2.

Proof. There are 16 subsets of $\{0, *, 1, \bar{1}\}$. This gives 256 isomorphism types for games born by day 2, so it suffices to show that every (formal) game born by day 2 is already in simplest form.

So fix such a game G. By Proposition 6.1, G has no dominated options, so it suffices to show that G has no reversible options. Consider some G^{LR}. Since G is born by day 2, G^{LR} is born by day 0, so necessarily $G^{LR} = 0$. But G^{LR} is a Left end and G is not (since it has G^L as a Left option), so by Lemma 3.5 $G \not\geq G^{LR}$. Likewise, by symmetry, $G \not\leq G^{RL}$ for any G^{RL}. □

Let $\mathbb{P}$ denote the set of games born by day 2. We next describe the partial order structure of $\mathbb{P}$. Define

$$\mathbb{P}^+ = \{G \in \mathbb{P} : G \text{ is a nonzero Left end}\},$$
$$\mathbb{P}^- = \{G \in \mathbb{P} : G \text{ is a nonzero Right end}\},$$
$$\mathbb{P}^0 = \{G \in \mathbb{P} : G \text{ is not an end.}\}$$

Then we can write $\mathbb{P}$ as a disjoint union

$$\mathbb{P} = \mathbb{P}^+ \cup \mathbb{P}^- \cup \mathbb{P}^0 \cup \{0\}.$$

Now let $\mathbb{B}_4$ denote the complete Boolean lattice of dimension 4. Let $\mathbb{B}_4^+$ be the partial order obtained by removing the largest element from $\mathbb{B}_4$, and likewise delete the smallest element to obtain $\mathbb{B}_4^-$. We will show that

$$\mathbb{P}^+ \cong \mathbb{B}_4^+, \quad \mathbb{P}^- \cong \mathbb{B}_4^-, \quad \mathbb{P}^0 \cong \mathbb{B}_4^+ \times \mathbb{B}_4^-. \tag{†}$$

In order to characterize the structure of $\mathbb{P}$, it will then suffice to describe the interaction between components.

We first show that there are certain order relations $G \geq H$ that can be inferred from the top-level structure of G and H.

Definition 6.3. We say that G *simply exceeds* H, and write $G \geq_S H$, if:

(i) every H^L is equal to some G^L;

(ii) every G^R is equal to some H^R;

(iii) if H is a Left end, then so is G;

(iv) if G is a Right end, then so is H.

Lemma 6.4. *If $G \geq_S H$, then $G \geq H$.*

Proof. Certainly G is downlinked to no G^L, since whenever Right can win $G+T$ moving second, then he can necessarily win $G^L + T$ moving first. So by (i) G is downlinked to no H^L. Likewise, no H^R can be downlinked to H, so by (ii) no G^R is downlinked to H. The conclusion now follows from Theorem 5.3. □

Theorem 6.5. *Fix G and H satisfying Definition 6.3(iii) and (iv), and assume that G and H are both born by day 2. If $G \geq H$, then $G \geq_S H$.*

Proof. We show that every Left option of H is a Left option of G; the argument that every Right option of G is a Right option of H is identical.

So fix an H^L; by Theorem 5.3, G is not downlinked to H^L. By Theorem 5.4, either $G^L \geq H^L$ for some G^L, or else $G \geq H^{LR}$. Now since H has the Left option H^L, it is not a Left end, whence by assumption neither is G. Since H is born by day 2, we know that every $H^{LR} = 0$, so by Lemma 3.5 we cannot have $G \geq H^{LR}$. Therefore $G^L \geq H^L$ for some G^L.

But G^L and H^L are both born by day 1. By Proposition 6.1, this implies $G^L = H^L$. □

Now if G and H are in the same component of $\mathbb{P}$, then they necessarily satisfy Definition 6.3(iii) and (iv). Therefore, *on each component*, the relations $\geq$ and $\geq_S$ coincide. But this immediately establishes †. For example, for the isomorphism $\mathbb{P}^+ \to \mathbb{B}_4^+$, we can regard $\mathbb{B}_4$ as the powerset lattice of $\{0, *, 1, \bar{1}\}$; then each G maps to its set of Right options.

To complete the picture of $\mathbb{P}$, we must characterize the interaction between the four components. We are concerned specifically with the case where H is a Right end, but G is not; or where G is a Left end, but H is not (the converses are ruled out by Lemma 3.5).

Theorem 6.6. *The ordering of $\mathbb{P}$ is generated by its restrictions to $\mathbb{P}^+$, $\mathbb{P}^-$, and $\mathbb{P}^0$, together with the following four relations and their mirror images*:

$$\{\cdot \mid *, 1\} \geq 0, \quad \{* \mid *, 1\} \geq \{* \mid \cdot\}, \quad \{\bar{1} \mid *, 1\} \geq \{\bar{1} \mid \cdot\}, \quad \{*, \bar{1} \mid *, 1\} \geq \{*, \bar{1} \mid \cdot\}.$$

Proof. It is easy to verify each of the four stated relations. To prove the theorem, we must show that they collectively generate every relation $G \geq H$, for games $G, H \in \mathbb{P}$.

It suffices to prove the following facts (together with their mirror images, which follow by symmetry):

(i) If $G > 0$, then $G \in \mathbb{P}^+$ and $G \geq \{\cdot \mid *, 1\}$;

(ii) If $G \geq H$, $H \in \mathbb{P}^-$, and $G \in \mathbb{P}^0$, then one of the following three inequalities holds:

$$G \geq \{* \mid *, 1\} \geq \{* \mid \cdot\} \geq H,$$
$$G \geq \{\bar{1} \mid *, 1\} \geq \{\bar{1} \mid \cdot\} \geq H, \quad \text{or}$$
$$G \geq \{*, \bar{1} \mid *, 1\} \geq \{*, \bar{1} \mid \cdot\} \geq H.$$

(i) Suppose $G \geq 0$. By Theorem 5.3, G is necessarily a Left end, and furthermore no G^R is downlinked to 0. By Theorem 5.4, the Right options of G must therefore be a subset of $\{*, 1\}$. So either $G = 0$, or $G \geq_S \{\cdot \mid *, 1\}$.

(ii) Suppose $G \geq H$, H is a Right end, and G is not. Consider any G^R. By Theorem 5.3, G^R is not downlinked to H. Since H is a Right end, it is necessarily the case that $G^{RL} \geq H$, by Theorem 5.4. In particular, G^R is not a Left end. Therefore $G^R \in \{*, 1\}$. This shows that the Right options of G form a subset of $\{*, 1\}$.

Furthermore, G is born by day 2, so G^{RL} is born on day 0. Therefore $G^{RL} = 0$ and we have $H \leq 0$. By (i), this implies that $H \leq_S \{*, \bar{1} \mid \cdot\}$. This shows that the Left options of H form a subset of $\{*, \bar{1}\}$.

Finally, by Theorem 6.5, we know that $G \geq_S H$. So the Left options of G form a superset of those of H.

There are now three possibilities for the Left options of H: either $\{*\}$, $\{\bar{1}\}$, or $\{*, \bar{1}\}$. In each case we obtain one of the three inequalities described in (ii) (since the Right options of G form a subset of $\{*, 1\}$, and the Left options of G form a superset of those of H). $\square$

Antichains by day 2. Since $\mathbb{P}$ has such a clean structure, we can get a tight bound on the number of antichains. From a standard reference such as [OEIS], we find that $\mathbb{B}_4$ has 168 antichains. This shows that $\mathbb{P}^+$ (and hence $\mathbb{P}^-$ as well) has precisely 167, since there is a unique antichain containing the largest element of $\mathbb{B}_4$.

Consider $\mathbb{P}^0$. We have $\mathbb{B}_4 \times \mathbb{B}_4 \cong \mathbb{B}_8$, trivially. Again from a standard reference, we find that $\mathbb{B}_8$ has 56130437228687557907788 antichains. Since every antichain of $\mathbb{P}^0$ is an antichain of $\mathbb{B}_8$, this gives an upper bound for the number of antichains of $\mathbb{P}^0$.

Finally, $\{0\}$ trivially admits just two antichains, $\varnothing$ and $\{0\}$. Since every antichain on $\mathbb{P}$ restricts to an antichain on each component, this gives an upper bound of

$$M = 2 \times 167 \times 167 \times 56130437228687557907788$$

for the number of antichains of $\mathbb{P}$. Thus we obtain an upper bound of M^2 games born by day 3. This number is large, indeed (roughly 2^{183}), but it is much smaller than 2^{512}, the number of nonisomorphic game trees of height 3.

7. Relationships to other theories

In this section we consider how the partizan misère theory relates to other theories of combinatorial games. To avoid confusion, we now abandon the notation $o(G)$, and denote explicitly by $o^+(G)$ and $o^-(G)$ the normal-play and misère-play outcome classes of G. We also write

$$G \geq^+ H \quad \text{if and only if} \quad o^+(G+X) \geq o^+(H+X) \text{ for every game } X;$$

$$G \geq^- H \quad \text{if and only if} \quad o^-(G+X) \geq o^-(H+X) \text{ for every game } X.$$

$\geq^-$ is the relation that we have been calling $\geq$, and o^- is the same mapping as o; for this section only, we include the minus signs for clarity. $\geq^+$ is, of course, the usual Berlekamp–Conway–Guy inequality for normal-play partizan games. The following result shows that $\geq^+$ is a coarsening of $\geq^-$.

Theorem 7.1. *If $G \geq^- H$, then $G \geq^+ H$.*

Proof. We must show that Left can get the last move in $G - H$. Suppose Right plays to $G^R - H$ (the argument is the same if he plays to $G - H^L$). Since

$G \geq^- H$, Theorem 5.3 implies that G^R is not downlinked to H. By Theorem 5.4, either $G^{RL} \geq^- H$ or $G^R \geq^- H^R$. By induction, we may assume that either $G^{RL} \geq^+ H$ or $G^R \geq^+ H^R$. In the first case, Left wins by moving to $G^{RL} - H$; in the second, by moving to $G^R - H^R$. □

In addition to the usual equivalences $=^+$ and $=^-$, obtained by symmetrizing $\geq^+$ and $\geq^-$, we have two further equivalences when G and H are impartial.

$$G =^+_I H \quad \text{if and only if} \quad o^+(G + X) = o^+(H + X) \text{ for every impartial game } X;$$

$$G =^-_I H \quad \text{if and only if} \quad o^-(G + X) = o^-(H + X) \text{ for every impartial game } X.$$

It is a well-known fact that $=^+_I$ is just the restriction of $=^+$ to impartial games. It is worth pointing out that the analogous statement does *not* hold for misère games, as the following proposition demonstrates.

Proposition 7.2. *There exist impartial games G and H such that $G =^-_I H$ but $G \neq^- H$.*

Proof. It is well-known that $* + * =^-_I 0$ (see [Conway 2001]). However, $* + * \neq^- 0$, by Lemma 3.5. □

Therefore $=^-_I$ is a strict coarsening of $=^-$. This highlights an interesting difference between normal and misère play: there exist impartial games G and H that are distinct in partizan misère play, but that are not distinguishable by any impartial game. This behavior arises in other theories, as well; for example, there exist impartial loopy games G and H that are distinct (in normal play), but that are not distinguishable by any impartial loopy game [Siegel 2009]. Indeed, the coincidence of $=^+$ and $=^+_I$ appears to be an artifact of the special nature of short games in normal play: it is the exception rather than the rule.

8. Partizan misère quotients

Recently Thane Plambeck [2005] observed that, if $\mathscr{A}$ is any set of impartial games, then its misère-play structure can often be simplified by localizing the misère equivalence relation to $\mathscr{A}$. Plambeck showed that many important aspects of the theory can be generalized to the local setting, and the structure theory of such quotients has been explored in detail; see [Plambeck and Siegel 2008; Siegel 2007].

It is not our intention to replicate that analysis here, but merely to remark that a partizan generalization exists. The construction is exactly the same, but instead of a bipartite monoid $(\mathcal{Q}, \mathcal{P})$, we now have a *tetrapartite monoid* $(\mathcal{Q}, \Pi)$, where $\Pi : \mathcal{Q} \to \{\mathscr{L}, \mathscr{R}, \mathscr{P}, \mathscr{N}\}$ is the *outcome partition* for $\mathcal{Q}$.

Intriguingly, such monoids have an induced partial order structure, given by

$$x \geq y \quad \text{if and only if} \quad \Pi(xz) \geq \Pi(yz) \text{ for all } z \in \mathcal{Q}.$$

If $G \geq H$, then it is certainly true that $\Phi(G) \geq \Phi(H)$. However, the quotient may also gain new order-relations that are not present in the universe of games. We include one example to illustrate the rich possibilities. In the previous section we remarked that 1 and $\bar{1}$ are incomparable, and we have also seen that $1+\bar{1} \neq 0$. In $\mathcal{Q}(1, \bar{1})$, however, the expected inequalities hold:

$$\Phi(\bar{1}) > \Phi(0) > \Phi(1) \quad \text{and} \quad \Phi(\bar{1})\Phi(1) = \Phi(0);$$

and indeed we have $\mathcal{Q}(1, \bar{1}) \cong \mathbb{Z}$, equipped with the usual partial-order structure. We leave it to the reader to verify these assertions.

In the time since this manuscript was first prepared, Meghan Allen has obtained some exciting results on partizan misère games, including identifying several interesting quotients. These results represent the current state of the art on the subject.

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References

[Berlekamp et al. 2003] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning ways for your mathematical plays, II*, 2nd ed., A K Peters, Natick, MA, 2003.

[Conway 2001] J. H. Conway, *On numbers and games*, 2nd ed., A K Peters, Natick, MA, 2001.

[Grundy 1939] P. M. Grundy, "Mathematics and games", *Eureka* **2** (1939), 6–8. Reprinted in *Eureka*, **27**, 1974: 9–11.

[Mesdal and Ottaway 2007] G. A. Mesdal and P. Ottaway, "Simplification of partizan games in misère play", *Integers* **7** (2007), G06, 12.

[OEIS] N. J. A. Sloane, A000372 – Dedekind numbers or Dedekind's problem: number of monotone Boolean functions of n variables…, *Online encyclopedia of integer sequences*, http://oeis.org/A000372.

[Plambeck 2005] T. E. Plambeck, "Taming the wild in impartial combinatorial games", *Integers* **5**:1 (2005), G5, 36.

[Plambeck and Siegel 2008] T. E. Plambeck and A. N. Siegel, "Misère quotients for impartial games", *J. Combin. Theory Ser. A* **115**:4 (2008), 593–622.

[Siegel 2007] A. N. Siegel, "The structure and classification of misère quotients", preprint, 2007. arXiv math/0703070

[Siegel 2009] A. N. Siegel, "Coping with cycles", pp. x+575 in *Games of no chance 3*, vol. 56, edited by M. H. Albert and R. J. Nowakowski, Mathematical Sciences Research Institute Publications, Cambridge University Press, Cambridge, 2009.

[Sprague 1935] R. P. Sprague, "Über mathematische Kampfspiele", *Tôhoku Math. J.* **41** (1935), 438–444.

[Sprague 1937] R. P. Sprague, "Über zwei Abarten von Nim", *Tôhoku Math. J.* **43** (1937), 351–359.

aaron.n.siegel@gmail.com *San Francisco, CA 94188, United States*