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The structure and classification of misère quotients

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A *bipartite monoid* is a commutative monoid $\mathcal{Q}$ together with an identified subset $\mathcal{P} \subset \mathcal{Q}$. In this paper we study a class of bipartite monoids, known as *misère quotients*, that are naturally associated to impartial combinatorial games.

We introduce a structure theory for misère quotients with $|\mathcal{P}| = 2$, and give a complete classification of all such quotients up to isomorphism. One consequence is that if $|\mathcal{P}| = 2$ and $\mathcal{Q}$ is finite, then $|\mathcal{Q}| = 2^n + 2$ or $2^n + 4$.

We then develop computational techniques for enumerating misère quotients of small order, and apply them to count the number of nonisomorphic quotients of order at most 18. We also include a manual proof that there is exactly one quotient of order 8.

1. Introduction

An *impartial combinatorial game* Γ is a two-player game with no hidden information and no chance elements, in which both players have exactly the same moves available at all times. When Γ is played under the *misère play convention*, the player who makes the last move loses.

Thirty years ago, Conway [] showed that the misère-play combinatorics of such games are often frighteningly complicated. However, new techniques recently pioneered by Plambeck [2005] have reinvigorated the subject. At the core of these techniques is the *misère quotient*, a commutative monoid that encodes the additive structure of an impartial combinatorial game (or a set of such games). See [Siegel 2015] for a gentle introduction to misère quotients, and [Plambeck and Siegel 2008] for a more rigorous one; see [Plambeck 2009] for a survey of the theory.

The introduction of misère quotients opens up a fascinating new area of study: the investigation of their algebraic properties. Such investigations are intrinsically interesting, and also have the potential to reveal new insights into the misère-play structure of combinatorial games. In this paper, we introduce several new results that expose quite a bit of structure in misère quotients.

Bipartite monoids and misère quotients. We recall some basic facts about misère quotients from Plambeck and Siegel [2008]. A *bipartite monoid* is a commutative monoid $\mathcal{Q}$ together with an identified subset $\mathcal{P} \subset \mathcal{Q}$. Two elements $x, y \in \mathcal{Q}$ are *indistinguishable* if, for all $z \in \mathcal{Q}$, we have $xz \in \mathcal{P} \Leftrightarrow yz \in \mathcal{P}$. A bipartite monoid $(\mathcal{Q}, \mathcal{P})$ is *reduced* if the elements of $\mathcal{Q}$ are pairwise distinguishable. In [Plambeck and Siegel 2008] it is shown that every bipartite monoid has a unique reduced quotient (up to isomorphism), the *reduction* of $(\mathcal{Q}, \mathcal{P})$. We'll write r.b.m. as shorthand for reduced bipartite monoid.

$(\mathcal{Q}, \mathcal{P})$ is a *subbipartite monoid* of $(\mathcal{S}, \mathcal{R})$ if $\mathcal{Q}$ is a submonoid of $\mathcal{S}$ and $\mathcal{R} \cap \mathcal{Q} = \mathcal{P}$. In this case we write $(\mathcal{Q}, \mathcal{P}) < (\mathcal{S}, \mathcal{R})$.

For a closed set $\mathscr{A}$ of impartial combinatorial games, the *indistinguishability relation* on $\mathscr{A}$ is given by

$$G \equiv H \pmod{\mathscr{A}} \quad \text{if and only if} \quad o^-(G+X) = o^-(H+X) \quad \text{for all } X \in \mathscr{A},$$

where $o^-(G)$ denotes the misère-play outcome of G. We write $[G]$ for the equivalence class of G modulo $\equiv$. Then the *misère quotient* of $\mathscr{A}$ is the bipartite monoid $(\mathcal{Q}, \mathcal{P})$, where $\mathcal{Q}$ is the quotient monoid of $\mathscr{A}$ modulo $\equiv$, and $\mathcal{P}$ is the corresponding *$\mathscr{P}$-portion*:

$$\mathcal{Q} = \{[G] : G \in \mathscr{A}\}; \qquad \mathcal{P} = \{[G] : G \in \mathscr{A},\ o^-(G) = \mathscr{P}\}.$$

We denote the misère quotient of $\mathscr{A}$ by $\mathcal{Q}(\mathscr{A})$. It is necessarily reduced, and therefore isomorphic to the reduction of $\mathscr{A}$. The *quotient map* $\Phi : \mathscr{A} \to \mathcal{Q}$ is given by $\Phi(G) = [G]$.

If $\mathcal{Q}$ is a finite commutative monoid, then the intersection of all its ideals is called the *kernel* of $\mathcal{Q}$. The kernel $\mathcal{K}$ is itself an ideal; it is the smallest ideal of $\mathcal{Q}$, and is also the largest group that can be written as a homomorphic image of $\mathcal{Q}$. A misère quotient $(\mathcal{Q}, \mathcal{P})$ is said to be *normal* if $\mathcal{K} \cap \mathcal{P} = \{z\}$, where z is the group identity of $\mathcal{K}$.

The basic theory of misère quotients is introduced in [Plambeck 2005; Plambeck and Siegel 2008; Siegel 2015], and some familiarity with it is assumed throughout this paper.

Tame and restive quotients. The following facts were also established in [Plambeck 2005; Plambeck and Siegel 2008], and are discussed in [Siegel 2015] in this volume. When every element of $\mathscr{A}$ is tame (in the sense of [ONAG]), then its quotient is isomorphic to one of the *tame quotients* $\mathcal{T}_n$. Here $\mathcal{T}_0 = \{1\}$ (with empty $\mathscr{P}$-portion); $\mathcal{T}_1 = \{1, a\}$ (with the structure of $\mathbb{Z}_2$ and $\mathcal{P} = \{a\}$); and

$$\mathcal{T}_n \cong \langle a, b_1, b_2, \ldots, b_{n-1} \mid a^2 = 1, b_1^3 = b_1, b_2^3 = b_2,\ \ldots,\ b_{n-1}^3 = b_{n-1}, \\ b_1^2 = b_2^2 = \cdots = b_{n-1}^2 \rangle$$

for $n \geq 2$, with $\mathcal{P} = \{a, b_1^2\}$. (The condition that $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{T}_n$, for some n, can be taken as an equivalent definition of tame.) The structure of $\mathcal{T}_n$, for $n \geq 2$, is best described as:

- a *kernel* $\mathcal{K}_n$ isomorphic to $\mathbb{Z}_2^n$, with identity element $z = b_1^2$, and generators $az, b_1, b_2, \ldots, b_{n-1}$;
- a separate component $\{1, a\}$ isomorphic to $\mathbb{Z}_2$, which maps onto $\{z, az\}$ under multiplication by z.

Thus the cardinality of $\mathcal{T}_n$ ($n \geq 2$) is $2^n + 2$.

Related to the tame quotients is the sequence of *restive quotients* $\mathcal{R}_{2^n+4}$, for $n \geq 2$, obtained by adjoining a new element t to $\mathcal{T}_n$ with $tz = t^2 = z$ (so that t is nilpotent to $\mathcal{K}_n$). We have $\mathcal{R}_{2^n+4} = \mathcal{T}_n \cup \{t, at\}$, so that $|\mathcal{R}_{2^n+4}| = 2^n + 4$; and $\mathcal{P} = \{a, b^2\}$, so that $\mathcal{R}_{2^n+4}$ also has just two $\mathscr{P}$-positions.

Tame extensions. The first result of this paper is a complete classification of misère quotients whose $\mathscr{P}$-portion has cardinality 2. If $(\mathcal{Q}, \mathcal{P})$ is a misère quotient, then the *tame extension* $\mathcal{T}(\mathcal{Q}, \mathcal{P})$ is a certain extension of $(\mathcal{Q}, \mathcal{P})$ (to be defined in Section 3) that adds no new $\mathscr{P}$-positions. It is defined in such a way that

$$\mathcal{T}_3 = \mathcal{T}(\mathcal{T}_2), \quad \mathcal{T}_4 = \mathcal{T}(\mathcal{T}_3), \quad \mathcal{T}_5 = \mathcal{T}(\mathcal{T}_4), \ldots$$

If we replace the "base" $\mathcal{T}_2$ by $\mathcal{R}_8$, then we recover the family of restive quotients:

$$\mathcal{R}_{12} = \mathcal{T}(\mathcal{R}_8), \quad \mathcal{R}_{20} = \mathcal{T}(\mathcal{R}_{12}), \quad \mathcal{R}_{36} = \mathcal{T}(\mathcal{R}_{20}), \ldots$$

The main result is that *every* finite quotient with $|\mathcal{P}| = 2$ is isomorphic to a quotient in one of these two families. It will follow that every finite quotient with $|\mathcal{P}| = 2$ has order $2^n + 2$ or $2^n + 4$, for some $n \geq 2$. Furthermore, if $(\mathcal{Q}, \mathcal{P})$ is an *infinite* quotient with $|\mathcal{P}| = 2$, then $(\mathcal{Q}, \mathcal{P}) \cong$ either $\mathcal{T}_\infty$ or $\mathcal{R}_\infty$, the limits of the two families (in a sense to be defined in Section 2).

"Almost tame" octal games. Tame extensions also have a useful application to octal games. Fix a (finite-length) octal game Γ and an integer M, and consider the partial quotient $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}_M(\Gamma)$ and quotient map Φ_M. Assume that $(\mathcal{Q}, \mathcal{P})$ is normal and $\mathcal{Q}$ is finite, and let $\mathcal{K} \subset \mathcal{Q}$ be the kernel of $\mathcal{Q}$. In Section 3, we will show that if $\Phi_M(H_n) \in \mathcal{K}$ for sufficiently many heaps H_n, then $\mathcal{Q}_{M+1}(\Gamma)$ is either $(\mathcal{Q}, \mathcal{P})$ or $\mathcal{T}(\mathcal{Q}, \mathcal{P})$, and $\Phi_{M+1}(H_{M+1}) \in \mathcal{K}$. "Sufficiently many" will be in the sense of [Guy and Smith 1956]: there exists n_0 such that $n_0 \leq n < 2n_0 + d$, where d is the octal length of Γ.

This theorem can be iterated, with strong consequences. In particular, if we determine that $\Phi_M(H_n) \in \mathcal{K}$ for sufficiently many n, then we can conclude that $\mathcal{Q}(\Gamma)$ is one of

$$(\mathcal{Q}, \mathcal{P}), \ \mathcal{T}(\mathcal{Q}, \mathcal{P}), \ \mathcal{T}(\mathcal{T}(\mathcal{Q}, \mathcal{P})), \ldots,$$

n	2	4	6	8	10	12	14	16	18
Quotients of order n	1	0	1	1	1	6	9	50	211

Table 1. The number of misère quotients of order $n \leq 18$ (up to isomorphism).

or possibly the limit $\mathcal{T}^{\infty}(\mathcal{Q}, \mathcal{P})$ of this sequence. Furthermore, $\Phi(H_n) \in \mathcal{K}$ for all but finitely many n. Since $\mathcal{Q}(\Gamma)$ is normal, normal and misère play coincide on $\mathcal{K}$; so we conclude that misère play reduces to normal play unless all the heaps are small. In practice, this means that once we have computed $\mathcal{Q}_M(\Gamma)$, then we have completely characterized the "misère-play divergence" of Γ; and its misère-play solution now depends only on finding a normal-play solution.

An example is the game **0.414**, which we mentioned in [Plambeck and Siegel 2008]. Its normal-play solution is unknown, despite the computation of at least 2^{24} $\mathscr{G}$-values by Flammenkamp [2012]. However, it is easy enough to compute $\mathcal{Q}_{18}(\mathbf{0.414})$, and to verify using the above logic that $\Phi(H_n) \in \mathcal{K}$ for all $n > 18$. Thus we know $\mathcal{Q}(\mathbf{0.414}) \cong \mathcal{T}^k(\mathcal{Q}_{18})$, for some k, and we need invest no further worry in the misère play of **0.414**: we may sit back and await a normal-play solution.

Recall the misère-play strategy for NIM: play normal NIM unless your move would leave only heaps of size 1. In that case, play to leave an odd number of heaps of size 1. We can now state an analogous strategy for **0.414**: play normal **0.414** unless your move would leave only heaps of size ≤ 18. In that case, consult the fine structure of $\mathcal{Q}_{18}$. We can state this reduction with confidence, despite the fact that the fullnormal-play strategy for **0.414** remains unknown.

Quotients of small order. We'll prove in Section 6 that $\mathcal{R}_8$ is the only quotient of order 8 (up to isomorphism). The succeeding sections focus instead on developing computational techniques for classifying quotients of small order.

In Section 7, we show that an arbitrary r.b.m. $(\mathcal{Q}, \mathcal{P})$ is a misère quotient if and only if there exists a *valid transition table* for $(\mathcal{Q}, \mathcal{P})$ — a certain combinatorial structure superimposed on $(\mathcal{Q}, \mathcal{P})$. This yields a computational method for testing whether $(\mathcal{Q}, \mathcal{P})$ is a misère quotient, which is optimized and applied in Section 8. The fruits of this effort are summarized in Table 1.

Preliminaries. We recall a few more key facts from [Plambeck and Siegel 2008].

Definition 1.1. Let $(\mathcal{Q}, \mathcal{P})$ be a bipartite monoid and fix $x \in \mathcal{Q}$. The *meximal set of x in $(\mathcal{Q}, \mathcal{P})$*, denoted $\mathcal{M}_x$, is defined by

$$\mathcal{M}_x = \{y \in \mathcal{Q} : \text{there is no } z \in \mathcal{Q} \text{ such that } xz \in \mathcal{P} \text{ and } yz \in \mathcal{P}\}.$$

The following statement is slightly more general than the rule given in [Plambeck and Siegel 2008], but the proof is identical.

Fact 1.2 (generalized mex rule). *Let* $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$, *and let* $(\mathcal{Q}, \mathcal{P}) < (\mathcal{S}, \mathcal{R})$. *Fix* G *with* $\operatorname{opts}(G) \subset \mathscr{A}$ *and fix* $x \in \mathcal{S}$. *The following are equivalent.*

(a) $\mathcal{Q}(\mathscr{A} \cup \{G\}) \cong (\mathcal{S}, \mathcal{R})$ *and* $\Phi(G) = x$.

(b) $\mathcal{S}$ *is generated by* $\mathcal{Q} \cup \{x\}$, *and the following two conditions hold.*

(i) $\Phi'' G \subset \mathcal{M}_x$.

(ii) *For each* $Y \in \mathscr{A}$ *and* $n \geq 0$ *such that* $x^{n+1}\Phi(Y) \notin \mathcal{P}$, *we have either*: $x^{n+1}\Phi(Y') \in \mathcal{P}$ *for some option* Y' *of* Y; *or else* $x^n x' \Phi(Y) \in \mathcal{P}$ *for some* $x' \in \Phi'' G$.

In [Plambeck and Siegel 2008] we stated Fact 1.2 for the special case $\mathcal{S} = \mathcal{Q}$. This will often be the case of greatest interest, but we shall have several occasions to use the more general form.

2. Limits and one-stage extensions

In this section we show that every misère quotient is the limit of a sequence of finitely generated quotients. Furthermore, each term of this sequence is an extension of the previous term, in a way we now make precise.

Definition 2.1. Let $(\mathcal{Q}, \mathcal{P})$, $(\mathcal{Q}^+, \mathcal{P}^+)$ be reduced bipartite monoids. We say that $(\mathcal{Q}^+, \mathcal{P}^+)$ is an *extension* of $(\mathcal{Q}, \mathcal{P})$ if there is some submonoid $(\mathcal{S}, \mathcal{R}) < (\mathcal{Q}^+, \mathcal{P}^+)$ such that $(\mathcal{Q}, \mathcal{P})$ is (isomorphic to) the reduction of $(\mathcal{S}, \mathcal{R})$. If $\mathcal{Q}^+$ is generated by $\mathcal{S} \cup \{x\}$ for some single element $x \in \mathcal{Q}^+ \setminus \mathcal{S}$, then we say that $(\mathcal{Q}^+, \mathcal{P}^+)$ is a *one-stage extension* of $(\mathcal{Q}, \mathcal{P})$.

Note that Definition 2.1 does *not* require that $\mathcal{Q}$ be a submonoid of $\mathcal{Q}^+$: it is only required that $(\mathcal{Q}, \mathcal{P})$ be the *reduction* of a subbipartite monoid of $(\mathcal{Q}^+, \mathcal{P}^+)$.

Lemma 2.2. *Let* $(\mathcal{Q}, \mathcal{P})$ *be a finitely generated misère quotient. Then there is a sequence of misère quotients*

$$0 = (\mathcal{Q}_0, \mathcal{P}_0), (\mathcal{Q}_1, \mathcal{P}_1), \ldots, (\mathcal{Q}_n, \mathcal{P}_n) = (\mathcal{Q}, \mathcal{P})$$

such that each $(\mathcal{Q}_{i+1}, \mathcal{P}_{i+1})$ *is a one-stage extension of* $(\mathcal{Q}_i, \mathcal{P}_i)$.

Proof. Write $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ and choose a finite set $\mathscr{H} \subset \mathscr{A}$ so that $\Phi''\mathscr{H}$ generates $\mathcal{Q}$. (Here and throughout the sequel, $\Phi''\mathscr{H} = \{\Phi(H) : H \in \mathscr{H}\}$.) Since the hereditary closure of a finite set is finite, we may assume that $\mathscr{H}$ is hereditarily closed. Enumerate

$$\mathscr{H} = \{H_0, H_1, \ldots, H_m\}$$

so that the successive H_i's have nondecreasing birthdays, and put

$$(\mathcal{Q}_i, \mathcal{P}_i) = \mathcal{Q}(H_0, \ldots, H_i).$$

It is easily seen that either $(\mathcal{Q}_{i+1}, \mathcal{P}_{i+1}) = (\mathcal{Q}_i, \mathcal{P}_i)$, or else it is a one-stage extension of $(\mathcal{Q}_i, \mathcal{P}_i)$. A suitable reindexing gives the lemma. □

Now let $(\mathcal{Q}_n, \mathcal{P}_n)$ be a sequence of bipartite monoids, and suppose that for each n, there exists $(\mathcal{Q}_n^+, \mathcal{P}_n^+) < (\mathcal{Q}_{n+1}, \mathcal{P}_{n+1})$ such that $(\mathcal{Q}_n, \mathcal{P}_n)$ is the reduction of $(\mathcal{Q}_n^+, \mathcal{P}_n^+)$, with quotient map $\pi_n : \mathcal{Q}_n^+ \to \mathcal{Q}_n$. We call $(\mathcal{Q}_n, \mathcal{P}_n, \pi_n)$ a *partial inverse system*.

Let $\overleftarrow{\mathcal{Q}} = (\mathcal{Q}_n, \mathcal{P}_n, \pi_n)$ be a partial inverse system. It is convenient to regard the underlying sets of the $\mathcal{Q}_n$ as formally disjoint. A *thread of* $\overleftarrow{\mathcal{Q}}$ *starting at* n is a sequence $(x_n, x_{n+1}, x_{n+2}, \ldots)$, where $x_n \in \mathcal{Q}_n$ and for each $i > n$ we have $x_{i+1} \in \mathcal{Q}_i^+$ and $\pi_i(x_{i+1}) = x_i$. We say two threads $\overleftarrow{x}$ and $\overleftarrow{y}$ are equivalent, and write $\overleftarrow{x} \sim \overleftarrow{y}$, if one is a terminal segment of the other.

If $\overleftarrow{x} = (x_m, x_{m+1}, x_{m+2}, \ldots)$ and $\overleftarrow{y} = (y_n, y_{n+1}, y_{n+2}, \ldots)$ are threads, we can define their product as follows. Without loss of generality, assume that $m \le n$, and put

$$\overleftarrow{x} \cdot \overleftarrow{y} = (x_n y_n, x_{n+1} y_{n+1}, x_{n+2} y_{n+2}, \ldots).$$

It is easy to check that $\overleftarrow{x} \cdot \overleftarrow{y}$ is a thread and that the product respects the equivalence $\sim$. Further, $\overleftarrow{1} \cdot \overleftarrow{x} = \overleftarrow{x}$, where $\overleftarrow{1} = (1, 1, 1, \ldots)$ is a list of the identity elements of each $\mathcal{Q}_n$. Thus the threads modulo $\sim$ form a commutative monoid $\mathcal{Q}$. We can define a subset $\mathcal{P} \subset \mathcal{Q}$ by

$$\mathcal{P} = \{(x_n, x_{n+1}, x_{n+2}, \ldots) \in \mathcal{Q} : \text{some } x_i \in \mathcal{P}_i\}.$$

(Note that the condition that *some* $x_i \in \mathcal{P}_i$ is equivalent to *all* $x_i \in \mathcal{P}_i$, since each π_i is a bipartite monoid homomorphism.) This makes $(\mathcal{Q}, \mathcal{P})$ into a bipartite monoid, which we call the *partial inverse limit* of the system $\overleftarrow{\mathcal{Q}}$. We write $(\mathcal{Q}, \mathcal{P}) = \lim \overleftarrow{\mathcal{Q}} = \lim_n (\mathcal{Q}_n, \mathcal{P}_n)$.

The following lemma is an easy exercise.

Lemma 2.3. *If* $(\mathcal{Q}_n, \mathcal{P}_n)$ *is reduced for infinitely many values of* n, *then so is* $\lim_n (\mathcal{Q}_n, \mathcal{P}_n)$.

Theorem 2.4. *Suppose that* $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots$ *is a chain of closed sets of games. Then the quotients* $\mathcal{Q}(\mathscr{A}_n)$ *form a partial inverse system, and we have*

$$\mathcal{Q}\left(\bigcup_n \mathscr{A}_n\right) \cong \lim_n \mathcal{Q}(\mathscr{A}_n).$$

Proof. Let $\Phi_n : \mathscr{A}_n \to \mathcal{Q}(\mathscr{A}_n)$ be the quotient maps and put $\mathcal{Q}_n^+ = \Phi''_{n+1} \mathscr{A}_n$. Define $\pi_n : \mathcal{Q}_n^+ \to \mathcal{Q}(\mathscr{A}_n)$ by $\pi_n(\Phi_{n+1}(X)) = \Phi_n(X)$. Now if $X \equiv Y \pmod{\mathscr{A}_{n+1}}$, then necessarily $X \equiv Y \pmod{\mathscr{A}_n}$, so π_n is well-defined.

Now by Lemma 2.3, $\lim_n \mathcal{Q}(\mathscr{A}_n)$ is reduced. To complete the proof, it suffices to exhibit a surjective homomorphism $\Phi : \bigcup_n \mathscr{A}_n \to \lim_n \mathcal{Q}(\mathscr{A}_n)$. Let n be least so that $X \in \mathscr{A}_n$, and put

$$\Phi(X) = (\Phi_n(X), \Phi_{n+1}(X), \Phi_{n+2}(X), \ldots).$$

It is easily verified that Φ has the desired properties. □

An easy corollary of Theorem 2.4 will be central to the classification theory.

Corollary 2.5. *Suppose that* $(\mathcal{Q}, \mathcal{P})$ *is a misère quotient that is* not *finitely generated. Then there is some partial inverse system* $(\mathcal{Q}_n, \mathcal{P}_n, \pi_n)$ *of finitely generated misère quotients such that*:

(i) $(\mathcal{Q}_0, \mathcal{P}_0) = 0$;

(ii) *each* $(\mathcal{Q}_{n+1}, \mathcal{P}_{n+1})$ *is a one-stage extension of* $(\mathcal{Q}_n, \mathcal{P}_n)$; *and*

(iii) $(\mathcal{Q}, \mathcal{P}) = \lim_n (\mathcal{Q}_n, \mathcal{P}_n)$.

Proof. Write $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ with $\mathscr{A}$ closed. Enumerate $\mathscr{A} = \{H_0, H_1, H_2, \ldots\}$ so that the birthdays of the H_n are nondecreasing. (This can always be done, since there are only finitely many games of each fixed birthday.) Then for each n, we have $\mathrm{opts}(H_n) \subset \{H_0, \ldots, H_{n-1}\}$. Put

$$(\mathcal{Q}_n, \mathcal{P}_n) = \mathcal{Q}(H_0, \ldots, H_n),$$

with quotient map Φ_n. Let $\mathcal{Q}_n^+$ be the submonoid of $\mathcal{Q}_{n+1}$ generated by

$$\{\Phi_{n+1}(H_0), \ldots, \Phi_{n+1}(H_n)\}.$$

The map $\pi_n : \mathcal{Q}_n^+ \to \mathcal{Q}_n$ defined by

$$\pi_n(\Phi_{n+1}(H)) = \Phi_n(H)$$

is well-defined, since each $G \equiv G' \pmod{\mathscr{A}_{n+1}}$ implies $G \equiv G' \pmod{\mathscr{A}_n}$. Now (i) is immediate, since necessarily $H_0 = 0$, and (ii) follows easily (after reindexing to eliminate cases where $\mathcal{Q}_{n+1} = \mathcal{Q}_n$). Now by Lemma 2.3, we know that $\lim_n (\mathcal{Q}_n, \mathcal{P}_n)$ is reduced. To prove (iii), it therefore suffices (by uniqueness of reductions) to show that $\lim_n (\mathcal{Q}_n, \mathcal{P}_n)$ is a quotient of $\mathscr{A}$.

Let $\Phi_n : \mathrm{cl}(\{H_0, \ldots, H_n\}) \to \mathcal{Q}_n$ be the usual quotient map, and define $\Phi : \mathscr{A} \to \mathcal{Q}$ by

$$\Phi(H_n) = (\Phi_n(H_n), \Phi_{n+1}(H_n), \Phi_{n+2}(H_n), \ldots).$$

It is easily checked that Φ is a surjective homomorphism of bipartite monoids. □

3. Normal quotients and tame extensions

In this section we introduce a certain algebraic property known as *faithful normality*, and we study one-stage extensions of faithfully normal quotients. In particular, we show that certain one-stage extensions of faithfully normal quotients behave just like extensions in normal play. The vast majority of quotients encountered in practice are faithfully normal, so this work has useful applications to octal games.

Definition 3.1. Let $(\mathcal{Q}, \mathcal{P})$ be a misère quotient with kernel $\mathcal{K}$, and let $z \in \mathcal{K}$ be the kernel identity. We say that $(\mathcal{Q}, \mathcal{P})$ is *regular* if $|\mathcal{K} \cap \mathcal{P}| = 1$, and *normal* if $\mathcal{K} \cap \mathcal{P} = \{z\}$.

Definition 3.2. Let $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ and let $\Phi : \mathscr{A} \to \mathcal{Q}$ be the quotient map. Suppose that

$$\Phi(G) = \Phi(H) \Longrightarrow \mathscr{G}(G) = \mathscr{G}(H) \quad \text{for all } G, H \in \mathscr{A}.$$

Then we say that Φ is *faithful*. If in addition $(\mathcal{Q}, \mathcal{P})$ is normal, then we say that Φ is *faithfully normal*.

(Here and throughout, we write $\mathscr{G}(G)$ for the normal-play nim value of G.) Often we will abuse terminology and refer to the *quotient* as being faithful (or faithfully normal), rather than the quotient map. We recall the following fact from [Plambeck and Siegel 2008].

Fact 3.3. *Suppose $\Phi : \mathscr{A} \to \mathcal{Q}$ is faithfully normal. Then $\mathcal{K}$ is isomorphic to the normal quotient of $\mathscr{A}$.*

Roughly speaking, therefore, faithful normality asserts that normal and misère play coincide on $\mathcal{K}$. We have $\mathcal{K} \cong \mathbb{Z}_2^n$ for some n, and for each $i < 2^n$ there is a unique $z_i \in \mathcal{K}$ representing games of nim value i. For convenience, when $\mathcal{E} \subset \mathcal{K}$, we write $z_m = \operatorname{mex}(\mathcal{E})$ to mean $m = \operatorname{mex}\{i : z_i \in \mathcal{E}\}$.

Now fix a faithfully normal quotient $\mathcal{Q}(\mathscr{A})$ with kernel $\mathcal{K}$, and let $G \neq 0$ be a game such that $\operatorname{opts}(G) \subset \mathscr{A}$. Then $\mathcal{Q}(\mathscr{A} \cup \{G\})$ is necessarily a one-stage extension of $\mathcal{Q}(\mathscr{A})$. For the remainder of this section, we will focus on the special case where $\Phi''G \subset \mathcal{K}$. We will show that in this case, one-stage extensions behave *exactly* like normal-play extensions. In particular:

- Extensions by a *proper* subset of the kernel follow the mex rule. Formally, if $\Phi''G \subsetneq \mathcal{K}$, then $\mathcal{Q}(\mathscr{A} \cup \{G\}) \cong \mathcal{Q}(\mathscr{A})$ and $\Phi(G) = \operatorname{mex}(\Phi''G)$.
- Extensions by the *entire* kernel cause the kernel to grow (from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^{n+1}$). They behave like normal-play extensions whose nim values are new powers of 2. Formally, if $\Phi''G = \mathcal{K}$, then $\mathcal{Q}(\mathscr{A} \cup \{G\}) \cong \mathcal{T}(\mathcal{Q}(\mathscr{A}))$, where $\mathcal{T}(\mathcal{Q}(\mathscr{A}))$ is a certain "tame extension" of $\mathcal{Q}(\mathscr{A})$ generalizing the extension $\mathbb{Z}_2^n < \mathbb{Z}_2^{n+1}$.

We begin with the case $\Phi''G \subsetneq \mathcal{K}$.

Lemma 3.4. *Suppose* $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ *is faithfully normal with kernel* $\mathcal{K}$. *Let* G *be a game with* $\mathrm{opts}(G) \subset \mathscr{A}$ *and suppose* $\Phi''G \subsetneq \mathcal{K}$. *Then* $\mathcal{Q}(\mathscr{A} \cup \{G\}) \cong \mathcal{Q}(\mathscr{A})$ *and* $\Phi(G) = \mathrm{mex}(\Phi''G)$.

Proof. We verify conditions (i) and (ii) of the generalized mex rule, with $(\mathcal{S}, \mathcal{R}) = (\mathcal{Q}, \mathcal{P})$ and $x = \mathrm{mex}(\Phi''G)$. Note that $x = z_m$, where $m = \mathscr{G}(G)$.

For (i), normality implies that $\mathcal{K} \setminus \{z_m\} \subset \mathcal{M}_x$. Since $\Phi''G \subset \mathcal{K} \setminus \{z_m\}$, this suffices. For (ii), fix $Y \in \mathscr{A}$ and $n \geq 0$, and suppose $x^{n+1}\Phi(Y) \notin \mathcal{P}$. If n is odd, then since $x \in \mathcal{K}$ and the quotient is faithfully normal, we have $\mathscr{G}(Y) > 0$. Thus $\mathscr{G}(Y') = 0$ for some Y', whence $x^{n+1}\Phi(Y') \in \mathcal{P}$.

Conversely, suppose that n is even. Then $\mathscr{G}(Y) \neq m$. If $\mathscr{G}(Y) > m$, then $\mathscr{G}(Y') = m$ for some Y', whence $x^{n+1}\Phi(Y') \in \mathcal{P}$. Otherwise, let $i = \mathscr{G}(Y)$. Since $\Phi''G \subset \mathcal{K}$ and $z_m = \mathrm{mex}(\Phi''G)$, we necessarily have $z_i \in \Phi''G$. But $z_i\Phi(Y) = z$, so $x^n z_i \Phi(Y) = z \in \mathcal{P}$. □

Tame extensions. We now consider the case where $\Phi''G = \mathcal{K}$. Let $(\mathcal{Q}, \mathcal{P})$ be a bipartite monoid with kernel $\mathcal{K}$, and define

$$\overline{\mathcal{K}} = \{\bar{x} : x \in \mathcal{K}\},$$

where each $\bar{x}$ is taken to be a formal symbol.

Definition 3.5. The *first tame extension* $\mathcal{T}(\mathcal{Q}, \mathcal{P}) = (\mathcal{Q}^+, \mathcal{P}^+)$ is defined as follows. $\mathcal{Q}^+ = \mathcal{Q} \cup \overline{\mathcal{K}}$, $\mathcal{P}^+ = \mathcal{P}$, and multiplication is extended by:

$$x \cdot \bar{y} = \overline{xy}\ (x \in \mathcal{Q},\ y \in \mathcal{K}); \quad \bar{x} \cdot \bar{y} = xy\ (x, y \in \mathcal{K}).$$

The *n-th tame extension* $\mathcal{T}^n(\mathcal{Q}, \mathcal{P})$ is defined by

$$\mathcal{T}^0(\mathcal{Q}, \mathcal{P}) = (\mathcal{Q}, \mathcal{P}); \quad \mathcal{T}^{n+1}(\mathcal{Q}, \mathcal{P}) = \mathcal{T}(\mathcal{T}^n(\mathcal{Q}, \mathcal{P})).$$

Finally, we define

$$\mathcal{T}^\infty(\mathcal{Q}, \mathcal{P}) = \lim_n \mathcal{T}^n(\mathcal{Q}, \mathcal{P}).$$

Observe that the sequence of normal quotients

$$0, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \ldots, \mathbb{Z}_2^{\mathbb{N}}$$

can be written

$$\mathcal{T}^0(0), \mathcal{T}^1(0), \mathcal{T}^2(0), \mathcal{T}^3(0), \ldots, \mathcal{T}^\infty(0)$$

while the sequence of tame misère quotients

$$\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \ldots, \mathcal{T}_\infty$$

can be written

$$\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}^0(\mathcal{T}_2), \mathcal{T}^1(\mathcal{T}_2), \mathcal{T}^2(\mathcal{T}_2), \ldots, \mathcal{T}^\infty(\mathcal{T}_2).$$

Thus the normal quotients can be viewed as a tame sequence with base 0, and the tame misère quotients can be viewed as a tame sequence with base $\mathcal{T}_2$.

If $(\mathcal{Q}, \mathcal{P})$ is a misère quotient, then so is $\mathcal{T}(\mathcal{Q}, \mathcal{P})$, as the following lemma establishes (cf. Lemma 3.4). (Here we say that $(\mathcal{Q}, \mathcal{P})$ *is a misère quotient* if it is isomorphic to $\mathcal{Q}(\mathscr{A})$, for some set $\mathscr{A}$ of impartial games.)

Lemma 3.6. *Suppose* $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ *is faithfully normal with kernel* $\mathcal{K}$. *Let* G *be a game with* $\mathrm{opts}(G) \subset \mathscr{A}$ *and suppose* $\Phi'' G = \mathcal{K}$. *Then* $\mathcal{Q}(\mathscr{A} \cup \{G\}) \cong \mathcal{T}(\mathcal{Q}(\mathscr{A}))$ *and* $\Phi(G) = \bar{z}$.

Proof. Identical to the proof of Lemma 3.4. □

Corollary 3.7. *Suppose* $\mathcal{Q}(\mathscr{A})$ *is faithfully normal with kernel* $\mathcal{K}$. *Then for all* $n \in \mathbb{N} \cup \{\infty\}$, $\mathcal{T}^n(\mathcal{Q}(\mathscr{A}))$ *is a misère quotient.*

Proof. Let $G_0 = 0$, $\mathscr{A}_0 = \mathscr{A}$. Recursively choose G_{n+1} so that $\mathrm{opts}(G_{n+1}) \subset \mathscr{A}_n$ and $\Phi'' G_{n+1} = \ker \mathcal{Q}(\mathscr{A}_n)$. Put $\mathscr{A}_{n+1} = \mathrm{cl}(\mathscr{A}_n \cup \{G_{n+1}\})$.

By repeated application of Lemma 3.6, we have $\mathcal{Q}(\mathscr{A}_n) \cong \mathcal{T}^n(\mathcal{Q}(\mathscr{A}))$, and Theorem 2.4 therefore gives $\mathcal{Q}\big(\bigcup_n \mathscr{A}_n\big) = \mathcal{T}^\infty(\mathcal{Q}(\mathscr{A}))$. □

The quotients $\mathcal{R}_{2^n+4}$**.** If we start with a different base $(\mathcal{Q}, \mathcal{P})$, we obtain another sequence of quotients $\mathcal{T}^n(\mathcal{Q}, \mathcal{P})$. For example, if $(\mathcal{Q}, \mathcal{P}) = \mathcal{R}_8$, then for all $n \geq 2$, $\mathcal{T}^{n-2}(\mathcal{Q}, \mathcal{P})$ is a quotient of order $2^n + 4$, which we denote by $\mathcal{R}_{2^n+4}$. Likewise, we define $\mathcal{R}_\infty = \mathcal{T}^\infty(\mathcal{R}_8)$. Since $|\mathcal{P}| = 2$, all the $\mathcal{R}_n$'s have $\mathscr{P}$-portions of size 2. A major goal of this paper is to prove the following theorem.

Theorem 3.8. *Suppose* $(\mathcal{Q}, \mathcal{P})$ *is a misère quotient with* $|\mathcal{P}| = 2$. *Then either* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{T}_n$ *or* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{R}_n$, *for some* $n \in \mathbb{N} \cup \{\infty\}$.

Thus if $(\mathcal{Q}, \mathcal{P})$ is a misère quotient with $|\mathcal{P}| = 2$, it follows that either $|\mathcal{Q}| = \infty$, or $|\mathcal{Q}| = 2^n + 2$ or $2^n + 4$ for some $n \geq 2$. Furthermore, there is exactly one such quotient of each permissible finite order, and exactly two such infinite quotients.

"Almost tame" octal games. Lemmas 3.4 and 3.6 have useful implications for octal games, as summarized by the following theorem.

Theorem 3.9. *Let* Γ *be an octal game with last nonzero code digit* d. *Fix* n_0, *and suppose that* $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}_{2n_0+d-1}(\Gamma)$ *is faithfully normal with kernel* $\mathcal{K}$. *Suppose furthermore that*

$$\Phi(H_n) \in \mathcal{K} \text{ for all } n \text{ such that } n_0 \leq n < 2n_0 + d.$$

Then:

(i) $\mathcal{Q}(\Gamma)$ *is a faithfully normal quotient.*

(ii) $\mathcal{Q}(\Gamma) \cong \mathcal{T}^k(\mathcal{Q}, \mathcal{P})$ *for some* $k \in \mathbb{N} \cup \{\infty\}$.

(iii) $\Phi(H_n) \in \ker \mathcal{Q}(\Gamma)$, *for all* $n \geq n_0$.

Proof. We first show that (i)–(iii) hold for $\mathcal{Q}_n(\Gamma)$, for all n. By hypothesis we may assume that $n \geq 2n_0 + d$. Then a typical option of H_n is a position $H_a + H_b$, with $a + b \geq 2n_0$. Without loss of generality, we have $a \geq n_0$, so that $\Phi(H_a) \in \mathcal{K}$. Thus $\Phi(H_a)x \in \mathcal{K}$ for all x, and in particular $\Phi(H_a + H_b) \in \mathcal{K}$. This shows that $\Phi'' H_n \subset \mathcal{K}$, and Lemmas 3.4 and 3.6 immediately imply (i)–(iii).

If the partial quotients $\mathcal{Q}_n(\Gamma)$ eventually converge to some $\mathcal{T}^k(\mathcal{Q}, \mathcal{P})$, then $\mathcal{Q}(\Gamma) \cong \mathcal{T}^k(\mathcal{Q}, \mathcal{P})$. Otherwise $\mathcal{Q}(\Gamma) \cong \mathcal{T}^\infty(\mathcal{Q}, \mathcal{P})$; and in either case (i)–(iii) are immediate. □

Thus when the hypotheses of Theorem 3.9 are satisfied, we know that beyond heap n_0, the misère-play analysis of Γ is no harder than its normal-play analysis. It follows that we can stop computing partial quotients of Γ and revert to the much easier task of calculating nim values. We say that Γ is *tame relative to heap* n_0.

The hypotheses of Theorem 3.9 may seem rather restrictive, but there are several three-digit octal games that satisfy them; for example, **0.414**, **0.776**, and **4.76**. The misère-play solutions to these games now depend only on finding normal-play solutions, and we can regard them as "relatively solved."

The hexadecimal game **0.9092** is another interesting case. It is known to be arithmetic periodic in normal play (using methods due to Austin [1976] and Howse and Nowakowski [2004]). Furthermore, in misère play we can show that it is tame relative to heap 12 (using *MisereSolver*, say; see [Plambeck and Siegel 2007]). Now $\mathcal{Q}_{12}(\mathbf{0.9092}) \cong \mathcal{R}_8$, so by Theorem 3.9 (suitably generalized to hexadecimal games) we have $\mathcal{Q}(\mathbf{0.9092}) \cong \mathcal{T}^k(\mathcal{R}_8)$ for some k. Since the $\mathscr{G}$-values of **0.9092** are unbounded, k is necessarily ∞. Therefore $\mathcal{Q}(\mathbf{0.9092})$ is exactly $\mathcal{R}_\infty$.

4. One-stage extensions of $\mathcal{T}_n$

We next focus our attention on proving Theorem 3.8. The crux of the proof is an analysis of one-stage extensions of $\mathcal{T}_n$ and $\mathcal{R}_{2^n+4}$. This analysis also yields a useful structure theory for these quotients. In particular, we will prove the following two theorems.

Theorem 4.1. *If* $(\mathcal{Q}, \mathcal{P})$ *is a one-stage extension of* $\mathcal{T}_n$ *and* $|\mathcal{P}| = 2$, *then either* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{T}_n$, *or* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{T}_{n+1}$, *or else* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{R}_{2^n+4}$.

Theorem 4.2. *If* $(\mathcal{Q}, \mathcal{P})$ *is a one-stage extension of* $\mathcal{R}_{2^n+4}$ *and* $|\mathcal{P}| = 2$, *then either* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{R}_{2^n+4}$, *or else* $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{R}_{2^{n+1}+4}$.

In this section we focus on Theorem 4.1, and we prove Theorem 4.2 in the following section.

Throughout the discussion there will be the implicit assumption that all quotients encountered are faithful. This is a slightly suspicious assumption, since it is unknown whether there exists an unfaithful quotient. However, since the argument proceeds "ground-up" by one-stage extensions, we are safe: a careful check of the proofs reveals that every extension under consideration preserves faithfulness. Therefore, if there exists an unfaithful quotient, it must necessarily satisfy $|\mathcal{P}| > 2$, and so will not interfere with the present argument. We will not be too careful about stating and restating this assumption of faithfulness, but in all cases the checks are routine.

The structure of $\mathcal{T}_n$. For the remainder of this section, fix a set of games $\mathscr{A}$, and suppose that $\mathcal{Q}(\mathscr{A}) \cong \mathcal{T}_n$, where $n \geq 2$. The structure of $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ is described as follows. $\mathcal{Q} = \mathcal{K} \cup \{1, a\}$, where $\mathcal{K} \cong \mathbb{Z}_2^n$ and $a^2 = 1$. We write $\mathcal{K} = \{z_0, z_1, \ldots, z_{2^n-1}\}$, where z_0 is the identity, $z_1 = az_0$, and z_i corresponds to nim value i.

Now fix a game $G \not\cong 0$ with $\operatorname{opts}(G) \subset \mathscr{A}$, and write $m = \mathscr{G}(G)$, $\mathscr{B} = \operatorname{cl}(\mathscr{A} \cup \{G\})$, and $(\mathcal{Q}^+, \mathcal{P}^+) = \mathcal{Q}(\mathscr{B})$.

Definition 4.3. Let $\mathcal{E} \subset \mathcal{Q}$. We say that $\mathcal{E}$ is *complemented* if $\mathcal{E} \cap \{a, z\} \neq \varnothing$ and $\mathcal{E} \cap \{1, az\} \neq \varnothing$.

Lemma 4.4. *If $\Phi''G$ is complemented, then $2n \cdot G$ is a $\mathscr{P}$-position for all $n \geq 1$.*

Proof. Write the copies of G in pairs, as $n \cdot (G + G)$. Second player follows the mirror-image strategy on each pair *until* her move would remove the last copy of G. If that is the case, then the position must be

$$G + G' + Y, \qquad \text{with } Y \in \mathscr{A},$$

and since second player has been following the mirror-image strategy, we necessarily have $\mathscr{G}(Y) = 0$.

Case 1: $\Phi(G' + Y) \in \mathcal{K}$. Then second player moves to $G' + G' + Y$. Now

$$\mathscr{G}(G' + G' + Y) = \mathscr{G}(G' + G') + \mathscr{G}(Y) = 0 + 0.$$

Since also $\Phi(G' + G' + Y) \in \mathcal{K}$, faithful normality implies that

$$\Phi(G' + G' + Y) = z \in \mathcal{P}.$$

Case 2: $\Phi(G' + Y) = 1$. Then second player chooses an $H \in \operatorname{opts}(G)$ with $\Phi(H) \in \{a, z\}$, as guaranteed by complementarity, and we have

$$\Phi(H + G' + Y) = \Phi(H) \cdot \Phi(G' + Y) = \Phi(H) \cdot 1 \in \mathcal{P}.$$

Case 3: $\Phi(G'+Y) = a$. Then second player chooses H with $\Phi(H) \in \{1, az\}$, to the same effect.

Since $\mathcal{Q} = \mathcal{K} \cup \{1, a\}$, this exhausts all possibilities for $\Phi(G'+Y)$. □

Lemma 4.5. *Assume that* $|\mathcal{P}^+| \geq 2$, $\Phi''G$ *is complemented*, *and* $m = 0$ *or* 1. *Fix* $Y \in \mathscr{A}$ *with* $\Phi(Y) \in \mathcal{K}$ *and* $\mathscr{G}(Y) \neq \mathscr{G}(G)$. *Then* $G+Y$ *is an* $\mathscr{N}$*-position.*

Proof. The $m = 0$ and 1 cases are similar, so suppose $m = 0$. By Lemma 4.4, $G+G$ is a $\mathscr{P}$-position, so that $\Phi(G+G) \in \{a, z\}$. But $4 \cdot G$ is also a $\mathscr{P}$-position (again by Lemma 4.4), and $a^2 = 1$, so necessarily $\Phi(G+G) \neq a$. Therefore $\Phi(G+G) = z$. Moreover, since $\Phi(Y) \in \mathcal{K}$, we necessarily have $\Phi(Y+Y) = z$, so that $G+G \equiv Y+Y \pmod{\mathscr{B}}$.

Now consider $G+G+G$. A typical option is $G'+G+G \equiv G'+Y+Y \pmod{\mathscr{B}}$; but $\mathscr{G}(G') \neq 0$, so

$$\Phi(G'+Y+Y) = \Phi(G')z \notin \mathcal{P}.$$

Therefore $G+G+G$ is also a $\mathscr{P}$-position. Assume (for contradiction) that $G+Y$ is also a $\mathscr{P}$-position. Then either $G+Y \equiv * \pmod{\mathscr{B}}$ or $G+Y \equiv Y+Y \pmod{\mathscr{B}}$. But $G+G+Y+Y$ is also a $\mathscr{P}$-position, since it is equivalent to $4 \cdot Y$, so necessarily $G+Y \equiv Y+Y \pmod{\mathscr{B}}$. But now

$$G+G+G \equiv G+Y+Y \equiv Y+Y+Y \pmod{\mathscr{B}},$$

a contradiction, since $Y+Y+Y$ is an $\mathscr{N}$-position. □

Definition 4.6. Fix $\mathcal{E} \subset \mathcal{Q}$. The *discriminant* $\Delta = \Delta(\mathcal{E})$ is given by

$$\Delta = \mathcal{E} \cap \{1, a, z, az\}.$$

We say that $\mathcal{E}$ is *restive* if $\Delta = \{1, z\}$ or $\{a, az\}$, *restless* if $\Delta = \{a, z\}$ or $\{1, az\}$, and *tame* otherwise. We say that $\mathcal{E}$ is *wild* if it is restive or restless.

Lemma 4.7. *Assume that* $\Phi''G$ *is tame. If* $m < 2^n$, *then* $\mathcal{Q}(\mathscr{B}) \cong \mathcal{T}_n$; *if* $m = 2^n$, *then* $\mathcal{Q}(\mathscr{B}) \cong \mathcal{T}_{n+1}$. *In either case*, *we have*

$$\Phi(G) = \begin{cases} 1 & \text{if } \Delta = \{a\}, \\ a & \text{if } \Delta = \{1\}, \\ z_m & \text{otherwise}, \end{cases}$$

where $\Delta = \Delta(\Phi''G)$.

Proof. Let $\mathcal{E} = \Phi''G$. In each of the three possibilities for Δ, it is easily seen that $\mathcal{E}$ satisfies condition (i) of the generalized mex rule. We now verify condition (ii).

Case 1: $\Delta = \{a\}$. With $x = 1$, condition (ii) is equivalent to: for every $\mathscr{N}$-position $Y \in \mathscr{A}$, either $\Phi(Y') \in \mathcal{P}$ for some Y', or else $x'\Phi(Y) \in \mathcal{P}$ for some $x' \in \mathcal{E}$. But if $Y \neq 0$, then the first of these two conditions is satisfied a priori; while if $Y = 0$, then $x' = a$ suffices for the second.

Case 2: $\Delta = \{1\}$. We must verify (ii) with $x = a$. Fix $Y \in \mathscr{A}$ and $n \geq 0$ and suppose $a^{n+1}\Phi(Y) \notin \mathcal{P}$. If n is odd, then Y is an $\mathscr{N}$-position, so either $Y = 0$ or some Y' is a $\mathscr{P}$-position. If $Y = 0$, then we have $a^n \cdot 1 \cdot \Phi(Y) = a \in \mathcal{P}$; if Y' is a $\mathscr{P}$-position, then $a^{n+1}\Phi(Y') \in \mathcal{P}$. Finally, if n is even, then $Y + *$ is an $\mathscr{N}$-position. So either Y is a $\mathscr{P}$-position, in which case $a^n \cdot 1 \cdot \Phi(Y) = \Phi(Y) \in \mathcal{P}$; or else $Y' + *$ is a $\mathscr{P}$-position, in which case $a^{n+1}\Phi(Y') = a\Phi(Y') \in \mathcal{P}$.

Case 3: $\Delta \neq \{a\}, \{1\}$. Fix $Y \in \mathscr{A}$ and $n \geq 0$ and suppose $x^{n+1}\Phi(Y) \notin \mathcal{P}$. If n is odd, then $x^{n+1} = z$, so $\Phi(Y) \neq 1, z$. Therefore $\mathscr{G}(Y) \neq 0$, and Y has some option Y' with $\mathscr{G}(Y') = 0$. Therefore $x^{n+1}\Phi(Y') = z \in \mathcal{P}$.

If n is even, then $x^{n+1} = z_k$, so $\mathscr{G}(Y) \neq k$. If $\mathscr{G}(Y) > k$, then there is some option Y' with $\mathscr{G}(Y') = m$; hence $x^{n+1}\Phi(Y') = z \in \mathcal{P}$. So suppose $\mathscr{G}(Y) < m$. Then there is some option G' of G with $\mathscr{G}(G') = \mathscr{G}(Y)$. There are three subcases.

Subcase 3a: $n > 0$ or $\Phi(G') \in \mathcal{K}$ or $\Phi(Y) \in \mathcal{K}$. Then we have immediately that $x^n\Phi(G')\Phi(Y) = z \in \mathcal{P}$.

Subcase 3b: $n = 0$ and $\Phi(G') = \Phi(Y) = 1$. Then $1 \in \Delta$. Now $\Delta \neq \{1\}$ (since we are in Case 3), and furthermore $\Delta \neq \{1, az\}$ (since $\Phi''G$ is tame). So either $a \in \Phi''G$ or $z \in \Phi''G$. But if $x' = a$ or z, then $x'\Phi(Y) \in \mathcal{P}$, as needed.

Subcase 3c: $n = 0$ and $\Phi(G') = \Phi(Y) = a$. Then $a \in \Delta$. Now $\Delta \neq \{a\}$ (since we are in Case 3), and furthermore $\Delta \neq \{a, z\}$ (since $\Phi''G$ is tame). So either $1 \in \Phi''G$ or $az \in \Phi''G$. But if $x' = 1$ or az, then $x'\Phi(Y) \in \mathcal{P}$, as needed. □

Lemma 4.8. *Assume that $\Phi''G$ is restless. Then $|\mathcal{P}^+| \geq 3$.*

Proof. As in Lemma 4.7, we write $\mathcal{E} = \Phi''(G)$ and $\Delta = \Delta(\mathcal{E})$.

Case 1: $\Delta = \{1, az\}$. Then $\{a, z\} \cap \mathcal{E} = \varnothing$, so G is a $\mathscr{P}$-position. Furthermore, if G' is an option with $\Phi(G') = 1$ (resp. az), then $\Phi(G')z \in \mathcal{P}$ (resp. $\Phi(G')az \in \mathcal{P}$). This shows that $G + *2_2$ (resp. $G + *2_3$) is an $\mathscr{N}$-position. Therefore $G \not\equiv *2_2$ (mod $\mathscr{B}$) and $G \not\equiv *$ (mod $\mathscr{B}$); since G is a $\mathscr{P}$-position, this implies $|\mathcal{P}^+| \geq 3$.

Case 2: $\Delta = \{a, z\}$. Then $\{1, az\} \cap \mathcal{E} = \varnothing$, so $\{a, z\} \cap a\mathcal{E} = \varnothing$, and hence $G + *$ is a $\mathscr{P}$-position. Just as in Case 1, we see that $G + * \not\equiv *$ (mod $\mathscr{B}$) and $G + * \not\equiv *2_2$ (mod $\mathscr{B}$), so again $|\mathcal{P}^+| \geq 3$. □

Lemma 4.9. *Assume that $\Phi''G$ is restive and $|\mathcal{P}^+| = 2$. Then $\mathcal{Q}(\mathscr{B}) \cong \mathcal{R}_{2^n+4}$ and*

$$\Phi(G) = \begin{cases} t & \text{if } \Delta = \{a, az\}, \\ at & \text{if } \Delta = \{1, z\}. \end{cases}$$

(*Here t denotes the same element as in the discussion of $\mathcal{R}_{2^n+4}$ in Section 1*, *and $\Delta = \Delta(\Phi''G)$ as in Lemma 4.7.*)

Proof. As before, let $\mathcal{E} = \Phi''(G)$. The argument is similar in both cases, so suppose that $\Delta = \{a, az\}$. Now in $\mathcal{R}_{2^n+4}$ it is easy to compute $\mathcal{M}_t = \mathcal{Q} \setminus \{1, t, z\}$. Since $\mathcal{E} \cap \{1, z\} = \varnothing$, condition (i) of the generalized mex rule is therefore trivially satisfied.

For (ii), fix $Y \in \mathscr{A}$ and $n \geq 0$ and suppose that $t^{n+1}\Phi(Y) \notin \mathcal{P}$. There are three cases.

Case 1: $n > 0$. Then $t^{n+1} = z$, so necessarily $\mathscr{G}(Y) > 0$. Therefore $t^{n+1}\Phi(Y') \in \mathcal{P}$, where Y' is any option with $\mathscr{G}(Y') = 0$.

Case 2: $n = 0$ and $\Phi(Y) \notin \mathcal{K}$. If $\Phi(Y) = 1$, then we have $a\Phi(Y) \in \mathcal{P}$; if $\Phi(Y) = a$, then $az\Phi(Y) \in \mathcal{P}$. Since $a, az \in \mathcal{E}$, this suffices.

Case 3: $n = 0$ and $\Phi(Y) \in \mathcal{K}$. Then $\mathscr{G}(Y) \neq 0$. If $\Phi(Y') = z$ for some Y', then we are done, since $t\Phi(Y') \in \mathcal{P}$, so assume $\Phi(Y') \neq z$ for all Y'.

Now since G is restive, it is complemented, so by Lemma 4.5 $G + Y$ is an $\mathscr{N}$-position. Consider a typical $G + Y'$. By assumption, $\Phi(Y') \neq z$. If $\Phi(Y') = 1$, then $G' + Y'$ is a $\mathscr{P}$-position, where $\Phi(G') = a$. If $\Phi(Y') = a$ or az, then $G' + Y'$ is a $\mathscr{P}$-position, where $\Phi(G') = az$. If $\mathscr{G}(Y') \geq 2$, then by Lemma 4.5 $G + Y'$ is *a priori* an $\mathscr{N}$-position. (We automatically have $m = 0$ or 1, since G is restive; cf. [ONAG].) So in all cases, $G + Y'$ is an $\mathscr{N}$-position.

But $G + Y$ is an $\mathscr{N}$-position, so we must have $G' + Y$ a $\mathscr{P}$-position, for some G'. Then $x'\Phi(Y) \in \mathcal{P}$, where $x' = \Phi(G')$, completing the proof. ☐

Proof of Theorem 4.1. Immediate from the preceding lemmas. ☐

5. One-stage extensions of $\mathcal{R}_{2^n+4}$

In this section we generalize much of the machinery of Section 4. Recall (from Section 1) that $\mathcal{R}_{2^n+4} = \mathcal{T}_n \cup \{t, at\}$, where $t^2 = tz = z$, and $\mathcal{P} = \{a, z\}$.

For the rest of this section, assume that $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ is faithful, with $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{R}_{2^n+4}$. Fix G with $\text{opts}(G) \subset \mathscr{A}$, and write $\mathscr{B} = \text{cl}(\mathscr{A} \cup \{G\})$, $(\mathcal{Q}^+, \mathcal{P}^+) = \mathcal{Q}(\mathscr{B})$, $\mathcal{E} = \Phi''G$, and $m = \mathscr{G}(G)$. Assume throughout this section that $|\mathcal{P}^+| = 2$.

We can very quickly reduce to the case where $\mathcal{E}$ is complemented (in the sense of Definition 4.3: $\mathcal{E} \cap \{a, z\} \neq \varnothing$ and $\mathcal{E} \cap \{1, az\} \neq \varnothing$).

Lemma 5.1. *Assume that $\Phi''G$ is* not *complemented. Then $\mathcal{Q}(\mathscr{B}) \cong \mathcal{Q}(\mathscr{A})$.*

Proof. Case 1: $\{a, z\} \cap \mathcal{E} = \varnothing$. Since $\mathcal{P} = \{a, z\}$, this immediately implies that G is a $\mathscr{P}$-position, so since $|\mathcal{P}^+| = |\mathcal{P}| = 2$, we must have $G \equiv Y \pmod{\mathscr{B}}$ for some $Y \in \mathscr{A}$. Therefore $\mathcal{Q}(\mathscr{B}) \cong \mathcal{Q}(\mathscr{A})$.

Case 2: $\{1, az\} \cap \mathcal{E} = \varnothing$. If G is a $\mathscr{P}$-position, then the argument is just as in Case 1. Otherwise, consider $G + *$. Since $\{1, az\} \cap \mathcal{E} = \varnothing$, we have $\mathcal{P} \cap a\mathcal{E} = \varnothing$, so every $G' + *$ is an $\mathscr{N}$-position. Since $G + 0$ is also an $\mathscr{N}$-position, we conclude that $G + *$ is a $\mathscr{P}$-position.

But this implies $G + * \equiv Y \pmod{\mathscr{B}}$ for some $Y \in \mathscr{A}$, whence $G \equiv Y + * \pmod{\mathscr{B}}$, and again we have $\mathcal{Q}(\mathscr{B}) \cong \mathcal{Q}(\mathscr{A})$. □

We now consider the case when $\mathcal{E}$ is complemented. The key fact about complementarity is the following (cf. Lemma 4.4).

Lemma 5.2. *If $\Phi''G$ is complemented, then $2n \cdot G$ is a $\mathscr{P}$-position for all $n \geq 1$.*

Proof. Identical to the proof of Lemma 4.4. □

Lemma 5.3. *Assume that $\Phi''G$ is complemented and $m = 0$ or 1, and fix $Y \in \mathscr{A}$ with $\mathscr{G}(Y) \geq 2$. Then $G + Y$ is an $\mathscr{N}$-position.*

Proof. Identical to the proof of Lemma 4.5. □

Lemma 5.4. *Assume that $m \geq 2$, and fix $Y \in \mathscr{A}$ with $\Phi(Y) \in \{t, at\}$. Then $G + Y$ is an $\mathscr{N}$-position.*

Proof. Since $m \geq 2$ and $\mathscr{G}(Y) \leq 1$, there must exist an option G' such that $\mathscr{G}(G') = \mathscr{G}(Y)$. Then $\mathscr{G}(G' + Y) = 0$, so that $\Phi(G' + Y) \in \{1, t, z\}$. In all cases, $\Phi(G' + Y)z \in \mathcal{P}$. Fix $Z \in \mathscr{A}$ with $\Phi(Z) = z$; then $G + Y + Z$ is an $\mathscr{N}$-position and $Z + Z$ a $\mathscr{P}$-position, so $G + Y \not\equiv Z \pmod{\mathscr{B}}$.

Next choose an option G' with $\mathscr{G}(G') = \mathscr{G}(Y) \oplus 1$. Then $\mathscr{G}(G' + Y) = 1$, so $\Phi(G' + Y) \in \{a, at, az\}$. In all cases, $\Phi(G' + Y)az \in \mathcal{P}$. Fix $W \in \mathscr{A}$ with $\Phi(W) = az$; then $G + Y + W$ is an $\mathscr{N}$-position and $* + W$ a $\mathscr{P}$-position, so $G + Y \not\equiv * \pmod{\mathscr{B}}$.

Thus $G + Y \not\equiv Z \pmod{\mathscr{B}}$, and $G + Y \not\equiv * \pmod{\mathscr{B}}$. But $|\mathcal{P}^+| = 2$, so these are the only two classes of $\mathscr{P}$-position in $\mathscr{B}$, and it follows that $G + Y$ is an $\mathscr{N}$-position. □

Lemma 5.5. *Assume that $\Phi''G$ is complemented and $m \geq 2$, and fix $Y \in \mathscr{A}$. Then $G + Y$ is a $\mathscr{P}$-position if and only if $\Phi(Y) = z_m$.*

Proof. If $\Phi(Y) = z_i$, for some $i < m$, then $G' + Y$ (with $\mathscr{G}(G') = i$) is *a priori* a $\mathscr{P}$-position, so $G + Y$ is an $\mathscr{N}$-position. If $\Phi(Y) = z_i$ for some $i > m$, then $\mathscr{G}(Y') = m$ for some Y', so $\Phi(Y') = z_m$. By induction on the birthday of Y, we have that $G + Y'$ is a $\mathscr{P}$-position, so again $G + Y$ is an $\mathscr{N}$-position.

If $\Phi(Y) = z_m$, then by induction every $G + Y'$ is an $\mathscr{N}$-position. Likewise, for every G' we have $\mathscr{G}(G' + Y) \neq 0$ and $\Phi(G' + Y) \in \mathcal{K}$, so every $G' + Y$ is also an $\mathscr{N}$-position. Therefore $G + Y$ is a $\mathscr{P}$-position.

This leaves only the cases $\Phi(Y) \in \{1, a, t, at\}$. But if $\Phi(Y) = 1$ (resp. a), then $G' + Y$ is a $\mathscr{P}$-position, where $\Phi(G') \in \{a, z\}$ (resp. $\{1, az\}$), as guaranteed by complementarity. Therefore $G + Y$ is an $\mathscr{N}$-position. Conversely, if $\Phi(Y) \in \{t, at\}$, then Lemma 5.4 guarantees that $G + Y$ is an $\mathscr{N}$-position. □

We now proceed with the proof of Theorem 4.2. There are two fundamental cases, each stated as a separate lemma: $m \geq 2$, and $m \in \{0, 1\}$.

Lemma 5.6. *Assume that $\Phi'' G$ is complemented and $m \geq 2$. Then $\mathcal{Q}(\mathscr{B}) \cong \mathcal{R}_{2^n+4}$ or $\mathcal{R}_{2^{n+1}+4}$.*

Proof. This is much like Lemma 3.4. It suffices to verify conditions (i) and (ii) in the generalized mex rule. Since $m \geq 2$, we have $\mathcal{M}_{z_m} = \mathcal{Q} \setminus \{z_m\}$. Since $\mathscr{G}(G) = m$, this suffices for (i). For (ii), fix $Y \in \mathscr{A}$ and $n \geq 0$, and suppose $z_m^{n+1}\Phi(Y) \notin \mathcal{P}$.

If n is odd, then $\mathscr{G}(Y) > 0$, so $z_m^{n+1}\Phi(Y') \in \mathcal{P}$, where $\mathscr{G}(Y') = 0$.

If n is even, then $\mathscr{G}(Y) \neq m$. If $\mathscr{G}(Y) > m$, then $z_m^{n+1}\Phi(Y') \in \mathcal{P}$, where $\mathscr{G}(Y') = m$. If $2 \leq \mathscr{G}(Y) < m$, then let $i = \mathscr{G}(Y)$. In this case z_i is the unique element of $\mathcal{Q}$ with $\mathscr{G}$-value i, so necessarily $z_i \in \mathcal{E}$. Since $z_i \cdot z_i \in \mathcal{P}$, this suffices.

If $\mathscr{G}(Y) = 0$, then we have $\Phi(Y) \in \{1, t, z\}$. If $\Phi(Y) = z$, then $x'\Phi(Y) \in \mathcal{P}$ for any $x' \in \mathcal{E} \cap \{1, t, z\}$. (This intersection must be nonempty, since $1, t, z$ are the only elements of $\mathcal{Q}$ corresponding to $\mathscr{G}$-value 0.) If $\Phi(Y) = 1$, then since $\mathcal{E}$ is complemented, we have $\mathcal{E} \cap \{a, z\} \neq \varnothing$; and either choice suffices. This leaves only the case $\Phi(Y) = t$. If $x't \in \mathcal{P}$ for some $x' \in \mathcal{E}$, then we are done. Otherwise, $G' + Y$ is an $\mathscr{N}$-position for every G'. But by Lemma 5.4 (and the assumption $|\mathcal{P}^+| = 2$), we know that $G + Y$ is an $\mathscr{N}$-position, so some $G + Y'$ must be a $\mathscr{P}$-position. By Lemma 5.5, we have specifically $\Phi(Y') = z_m$, whence $z_m^{n+1}\Phi(Y') = z \in \mathcal{P}$, as needed.

Finally, if $\mathscr{G}(Y) = 1$, then $\Phi(Y) \in \{a, at, az\}$, and the proof proceeds just as in the $\mathscr{G}(Y) = 0$ case. ☐

Lemma 5.7. *Assume that $\Phi'' G$ is complemented and $m = 0$ (resp. 1). Then $\mathcal{Q}(\mathscr{B}) \cong \mathcal{R}_{2^n+4}$, and $\Phi(G) = t$ (resp. at).*

Proof. The two cases are essentially identical, so assume $m = 0$. As always, we use the generalized mex rule. Note that

$$\mathcal{M}_t = \mathcal{Q} \setminus \{1, t, z\} = \{x : \mathscr{G}(x) \neq 0\},$$

and since $\mathscr{G}(G) = 0$, this suffices for (i). For (ii), fix $Y \in \mathscr{A}$ and $n \geq 0$, and suppose $t^{n+1}\Phi(Y) \notin \mathcal{P}$. There are four cases.

Case 1: $n \geq 1$. Then $t^{n+1} = z$, so $z\Phi(Y) \notin \mathcal{P}$. Thus $\Phi(Y) \neq 1, t, z$, so $\mathscr{G}(Y) \neq 0$. We conclude that $t^{n+1}\Phi(Y') \in \mathcal{P}$, where Y' is any option with $\mathscr{G}(Y') = 0$.

Case 2: $n = 0$ and $\mathscr{G}(Y) = 0$. Then $\Phi(Y) \in \{1, t, z\}$, and since $t\Phi(Y) \notin \mathcal{P}$, necessarily $\Phi(Y) = 1$. But since $\mathcal{E}$ is complemented, $\mathcal{E} \cap \{a, z\} \neq \varnothing$, so $x'\Phi(Y) \in \mathcal{P}$, where $x' = a$ or z.

Case 3: $n = 0$ and $\mathscr{G}(Y) = 1$. Since $\mathcal{E}$ is complemented, we have $\mathcal{E} \cap \{1, az\} \neq \varnothing$. Since $m = 0$, we know that $1 \notin \mathcal{E}$, so necessarily $az \in \mathcal{E}$. Since $\mathscr{G}(Y) = 1$, we always have $az\Phi(Y) \in \mathcal{P}$, so this suffices.

Case 4: $n = 0$ and $\mathscr{G}(Y) \geq 2$. If $t \in \Phi''Y$ or $z \in \Phi''Y$, then $t\Phi(Y') = z$ and there is nothing to prove. Otherwise, put $i = \mathscr{G}(Y)$; to complete the proof, it suffices to show that $z_i \in \mathcal{E}$, because $z_i\Phi(Y) = z \in \mathcal{P}$. So consider $G + Y$. We first show that every $G + Y'$ is an $\mathscr{N}$-position. If $\mathscr{G}(Y') = 0$, then $\Phi(Y') = 1$ (since we are assuming $t, z \notin \Phi''Y$). Since G is complemented and $\mathscr{G}(G) = 0$, we necessarily have $a \in \Phi''G$, so $a\Phi(Y') \in \mathcal{P}$ and hence $G + Y'$ is an $\mathscr{N}$-position. If $\mathscr{G}(Y') = 1$, then since G is complemented and $\mathscr{G}(G) = 0$, we necessarily have $az \in \Phi''G$, so $az\Phi(Y') \in \mathcal{P}$ and again $G + Y'$ is an $\mathscr{N}$-position. Finally, if $\mathscr{G}(Y') \geq 2$, then the desired conclusion follows from Lemma 5.3.

This shows that every $G + Y'$ is an $\mathscr{N}$-position. But by Lemma 5.3, $G + Y$ itself is an $\mathscr{N}$-position. Therefore some $G' + Y$ is necessarily a $\mathscr{P}$-position. Since $\Phi(Y) = z_i$, we conclude that $\Phi(G') = z_i$ as well, completing the proof. □

6. Uniqueness of $\mathcal{R}_8$

The following theorem emerges readily from previous work.

Theorem 6.1. *$\mathcal{R}_8$ is the only misère quotient of order* 8 (*up to isomorphism*).

Proof. Let $(\mathcal{Q}, \mathcal{P})$ be a misère quotient of order 8. By Lemma 2.2, $(\mathcal{Q}, \mathcal{P})$ must arise as a one-stage extension of $\mathcal{T}_2$ (since $|\mathcal{T}_2|$ has order 6, and all nontrivial quotients must have even order, as shown in [Plambeck and Siegel 2008]). So there is some closed set $\mathscr{A}$, and some G with $\text{opts}(G) \subset \mathscr{A}$, such that

$$\mathcal{Q}(\mathscr{A}) \cong \mathcal{T}_2 \quad \text{and} \quad \mathcal{Q}(\mathscr{A} \cup \{G\}) \cong (\mathcal{Q}, \mathcal{P}).$$

Let $\Phi : \text{cl}(\mathscr{A} \cup \{G\}) \to \mathcal{Q}$ be the quotient map, and write

$$a = \Phi(*), \quad b = \Phi(*2), \quad t = \Phi(G).$$

Since $\mathcal{Q}(\mathscr{A}) \cong \mathcal{T}_2$ and $\mathcal{Q}(\mathscr{A} \cup \{G\}) \ncong \mathcal{T}_2$, t is not in the submonoid generated by a, b. Thus neither is at (since $a^2 = 1$), and it follows immediately that

$$\mathcal{Q} = \mathcal{T}_2 \cup \{t, at\}.$$

Now put $\mathcal{E} = \Phi''G$. $\mathcal{E}$ cannot be tame, since then Lemma 4.7 would imply that $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{T}_2$ or $\mathcal{T}_3$, neither of which has order 8.

If $\mathcal{E}$ is restive, then either $\{1, z\} \subset \mathcal{E}$ or $\{a, az\} \subset \mathcal{E}$, and it follows that G and $G + *$ are both $\mathscr{N}$-positions (since $\Phi''G$ contains either z or a directly, and $\Phi''(G + *)$ contains either $1 \cdot a$ or $az \cdot a$, equal to a or z respectively). Therefore $t, at \notin \mathcal{P}$, so $|\mathcal{P}| = 2$. By Theorem 4.1, we have $(\mathcal{Q}, \mathcal{P}) \cong \mathcal{R}_8$.

We complete the proof by assuming $\mathcal{E}$ to be restless and obtaining a contradiction. There are two cases.

Case 1: $\Delta = \{1, az\}$. Then $a, z \notin \mathcal{E}$, so G is a $\mathscr{P}$-position. Therefore $*2 + G$ is an $\mathscr{N}$-position; and since $\Phi(*2 + G) = bt$, we have $bt \notin \mathcal{P}$, so that $bt \in \{1, b, ab, az, at\}$. To obtain a contradiction, we show that $*2 + G$ is distinguishable from some representative of each of these possibilities.

The table below summarizes. The first column of each row lists one possibility for bt, along with an inequality $x \neq y$ that rules out this possibility. In each case, x is known to be in $\mathcal{P}$, and the second column exhibits an $\mathscr{N}$-position Y that witnesses $y \notin \mathcal{P}$. The winning move Y' is shown in the third column; the notation $\Phi^{-1}(x)$ is used to represent a typical option of G with $\Phi(G) = x$.

Distinction(s)	Typical $\mathscr{N}$-position	Winning Move
$1 \neq bt \Leftarrow a \neq abt$ $az \neq bt \Leftarrow z \neq abt$ $at \neq bt \Leftarrow t \neq abt$	$* + *2 + G$	$* + * + G$
$b \neq bt \Leftarrow z \neq zt$	$*2 + *2 + G$	$*2 + *2 + \Phi^{-1}(1)$
$ab \neq bt \Leftarrow z \neq azt$	$* + *2 + *2 + G$	$* + *2 + *2 + \Phi^{-1}(az)$

Case 2: $\Delta = \{a, z\}$. This is similar. Clearly G is an $\mathscr{N}$-position, so since $1, az \notin \mathcal{E}$, we have that $* + G$ is a $\mathscr{P}$-position. As before, this implies that $*2 + G$ is an $\mathscr{N}$-position. The following table parallels the table from Case 1.

Distinction(s)	Typical $\mathscr{N}$-position	Winning Move
$1 \neq bt \Leftarrow a \neq abt$ $az \neq bt \Leftarrow z \neq abt$ $t \neq bt \Leftarrow at \neq abt$	$* + *2 + G$	$* + G$
$b \neq bt \Leftarrow z \neq zt$	$*2 + *2 + G$	$*2 + *2 + \Phi^{-1}(z)$
$ab \neq bt \Leftarrow z \neq azt$	$* + *2 + *2 + G$	$* + *2 + *2 + \Phi^{-1}(a)$

This exhausts all possibilities and completes the proof. □

Theorem 6.1 can be extended: for example, $\mathcal{T}_3$ is the unique misère quotient of order 10. But the proof of Theorem 6.1 gives us pause. The uniqueness of $\mathcal{R}_8$ takes shape through a somewhat subtle combinatorial analysis. To prove the uniqueness of $\mathcal{T}_3$ by hand, we would need to sharpen the restless cases of Theorem 6.1, and then show that every one-stage extension of $\mathcal{R}_8$ has order ≥ 12. This appears to be quite a lot of work, so we now refocus our efforts on automating this sort of analysis.

7. Valid transition tables

Transition algebras were introduced in [Plambeck and Siegel 2008], and there they proved to be useful in the study of mex functions. We now abstract out some of their structure.

Definition 7.1. Let $\mathcal{Q}$ be a commutative monoid. A *transition table on* $\mathcal{Q}$ is a subset $T \subset \mathcal{Q} \times Pow(\mathcal{Q})$.

Note that if $\mathscr{A}$ is a closed set of games, then $T(\mathscr{A})$ is a transition table on $\mathcal{Q}(\mathscr{A})$.

Definition 7.2. Let T be a transition table on a bipartite monoid $(\mathcal{Q}, \mathcal{P})$. T is said to be *valid* if and only if the following four conditions hold.

(i) (parity) For each $(x, \mathcal{E}) \in T$, we have

$$x \in \mathcal{P} \iff \mathcal{E} \neq \varnothing \text{ and } \mathcal{E} \cap \mathcal{P} = \varnothing.$$

(ii) (completeness) For each $x \in \mathcal{Q}$, there is some set $\mathcal{E}$ such that $(x, \mathcal{E}) \in T$.

(iii) (closure) If $(x, \mathcal{E}), (y, \mathcal{F}) \in T$, then $(xy, x\mathcal{F} \cup y\mathcal{E}) \in T$.

(iv) (well-foundedness) There exists a map $R : \mathcal{Q} \to \mathbb{N}$ (a *rank function* for $\mathcal{Q}$) with the following property. $R(1) = 0$, and for each $x \in \mathcal{Q}$, there is some $(x, \mathcal{E}) \in T$ such that $R(y) < R(x)$ for all $y \in \mathcal{E}$.

We note that condition (iv) implies (ii), but nonetheless we include (ii) for clarity. Note also that condition (iii) implies a monoid structure.

Definition 7.3. A transition table T is a *transition algebra* if it is closed (in the sense of Definition 7.2(iii)).

We will use the terms "valid transition table" and "valid transition algebra" interchangeably. The main result is the following.

Theorem 7.4. *Let $(\mathcal{Q}, \mathcal{P})$ be a r.b.m. with $1 \notin \mathcal{P}$. The following are equivalent.*

(i) *There exists a closed set of games $\mathscr{A}$ with $\mathcal{Q}(\mathscr{A}) = (\mathcal{Q}, \mathcal{P})$.*

(ii) *There exists a valid transition table T on $(\mathcal{Q}, \mathcal{P})$.*

Proof. (i) $\Rightarrow$ (ii): Put $T = T(\mathscr{A})$. It is straightforward to check that T is valid. A suitable rank function is given by $R(x) = \min\{\text{birthday}(G) : \Phi(G) = x\}$.

(ii) $\Rightarrow$ (i): First define, for each $x \in \mathcal{Q}$, a game H_x as follows. The definition is by induction on $R(x)$. Let $(x, \mathcal{E}) \in T$ be such that $R(y) < R(x)$ for each $y \in \mathcal{E}$, and put

$$H_x = \{H_y : y \in \mathcal{E}\}.$$

Now define a game H_t for each $t \in T$:

$$H_{(x,\mathcal{E})} = \{H_y : y \in \mathcal{E}\}.$$

Let

$$\mathscr{A} = \mathrm{cl}(\{H_t : t \in T\}).$$

We claim that $\mathcal{Q}(\mathscr{A}) = (\mathcal{Q}, \mathcal{P})$.

Since $(\mathcal{Q}, \mathcal{P})$ is a r.b.m., it suffices (by [Plambeck and Siegel 2008, Proposition 4.7]) to exhibit a surjective homomorphism $\Phi : \mathscr{A} \to \mathcal{Q}$. Regarding $\mathscr{A}$ as a free commutative monoid on the generators H_t, we define Φ as a monoid homomorphism by

$$\Phi(H_{(x,\mathcal{E})}) = x.$$

By completeness (condition (ii) in the definition of validity), Φ is surjective. To complete the proof, we need to show that, for all $G \in \mathscr{A}$,

$$\Phi(G) \in \mathcal{P} \Longleftrightarrow G \neq 0 \quad \text{and} \quad \Phi(G') \notin \mathcal{P} \quad \text{for any option } G'.$$

So fix $G = H_{t_1} + \cdots + H_{t_k}$, and write $t_i = (x_i, \mathcal{E}_i)$. Write $x = x_1 x_2 \cdots x_k$, and denote by x/x_i the product $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_k$. Put

$$\mathcal{E} = \bigcup_{1 \le i \le k} \frac{x}{x_i} \mathcal{E}_i,$$

and let $t = (x, \mathcal{E})$. By closure (condition (iii) in the definition of validity), $t \in T$. By parity (condition (i)), we have

$$x \in \mathcal{P} \Longleftrightarrow \mathcal{E} \neq \varnothing \text{ and } \mathcal{E} \cap \mathcal{P} = \varnothing.$$

But clearly $\Phi(G) = x$, and $\mathcal{E} = \Phi'' G$. This suffices except for the case when $\mathcal{E} = \varnothing$; but then G has no options, so $\Phi(G) = 1$. Since we assumed that $1 \notin \mathcal{P}$, this completes the proof. □

Theorem 7.4 yields an algorithm for counting the number of misère quotients of order n: for each r.b.m. of order n, iterate over all transition tables and check whether any are valid. This is an atrociously poor algorithm, however; even if one could effectively enumerate the r.b.m.'s of order n, each one admits 2^{n2^n} transition tables! Theorem 7.4 is still important, however, since it reduces the search for misère quotients to a finite problem.

8. Enumerating quotients of small order

We now show how the techniques of the previous section can be made (reasonably) efficient. We first show that every misère quotient can be represented by a certain restricted type of transition algebra.

Definition 8.1. Let $(\mathcal{Q}, \mathcal{P})$ be a bipartite monoid. Fix $x_1, \ldots, x_k \in \mathcal{Q}$, and for $0 \le i \le k$ let $\mathcal{S}_i$ be the submonoid of $\mathcal{Q}$ generated by $x_1, \ldots, x_i$. We say that $x_1, \ldots, x_k$ is a *construction sequence* for $(\mathcal{Q}, \mathcal{P})$ if:

(i) $\mathcal{S}_k = \mathcal{Q}$;

(ii) for each i, $x_i \notin \mathcal{S}_{i-1}$;

(iii) for each $i < k$, the reduction of $(\mathcal{S}_i, \mathcal{P} \cap \mathcal{S}_i)$ is a misère quotient.

Definition 8.2. Let $(\mathcal{Q}, \mathcal{P})$ be a bipartite monoid. A transition algebra T on $(\mathcal{Q}, \mathcal{P})$ is said to be a *minimex algebra* if there exists a construction sequence $x_1, \ldots, x_k \in \mathcal{Q}$ that generates T in the following sense. Write $\mathcal{E}_i = \mathcal{M}_{x_i} \cap \mathcal{S}_{i-1}$, where the $\mathcal{S}_i$'s are as in the previous definition. Then T is generated by

$$(x_1, \mathcal{E}_1), \ldots, (x_k, \mathcal{E}_k).$$

We say that T is the minimex algebra *constructed by* $x_1, \ldots, x_k$.

Lemma 8.3. *Suppose T is a transition algebra on a finite r.b.m. $(\mathcal{Q}, \mathcal{P})$. Fix generators $x_1, \ldots, x_k \in \mathcal{Q}$ and suppose that, for each i, there is an $\mathcal{E}_i \subset \mathcal{S}_{i-1}$ such that $(x_i, \mathcal{E}_i) \in T$. Then T admits a rank function.*

Proof. Define a map $R^* : \mathcal{Q} \to \mathbb{N}^k$ as follows. For each $x \in \mathcal{Q}$, write

$$x = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

choosing the *lexicographically least* expression on the generators $x_1, \ldots, x_k$. Put $R^*(x) = (n_1, \ldots, n_k)$.

Now order the elements of $\mathbb{N}^k$ lexicographically. We claim that R^* is a "rank function" under this ordering. For if $R^*(x) = (n_1, \ldots, n_k)$, then let

$$(x, \mathcal{E}) = (x_1, \mathcal{E}_1)^{n_1} (x_2, \mathcal{E}_2)^{n_2} \cdots (x_k, \mathcal{E}_k)^{n_k}.$$

By the assumptions on the $\mathcal{E}_i$, we know that $R^*(y) < R^*(x_i)$ for each $y \in \mathcal{E}_i$. Therefore $R^*(y) < R^*(x)$ for each $y \in \mathcal{E}$.

Finally, R^* can be converted into a suitable rank function $R : \mathcal{Q} \to \mathbb{N}$ by enumerating the finite range of R^*. □

Theorem 8.4. *Let $(\mathcal{Q}, \mathcal{P})$ be a finitely generated r.b.m. with $1 \notin \mathcal{P}$. The following are equivalent.*

(i) *There exists a closed set of games $\mathscr{A}$ with $\mathcal{Q}(\mathscr{A}) = (\mathcal{Q}, \mathcal{P})$.*

(ii) *There exists a valid minimex algebra on $(\mathcal{Q}, \mathcal{P})$.*

Proof. (ii) ⇒ (i) is immediate from Theorem 7.4, since every minimex algebra is automatically a valid transition table. So we must prove (i) ⇒ (ii).

Since $\mathcal{Q}$ is finitely generated, we may assume that $\mathscr{A}$ is also finitely generated (passing, if necessary, to a suitable finitely generated subset of $\mathscr{A}$, and noting that the closure of a finitely generated set is finitely generated). Choose generators $H_1, \ldots, H_l$ for $\mathscr{A}$ such that $\text{opts}(H_i) \subset \langle H_1, \ldots, H_{i-1} \rangle$ for each i. (Here $\langle H_1, \ldots, H_{i-1} \rangle$ denotes the submonoid of $\mathscr{A}$ generated by $H_1, \ldots, H_{i-1}$.)

Put $y_i = \Phi(H_i)$ and consider the sequence $y_1, \dots, y_l \in \mathcal{Q}$. Define a subsequence $y_{j_1}, \dots, y_{j_k}$ inductively: let j_i be the *least* index such that

$$y_{j_i} \notin \mathcal{S}_{i-1} = \langle y_{j_1}, \dots, y_{j_{i-1}} \rangle,$$

and stop when the subsequence $y_{j_1}, \dots, y_{j_k}$ generates $\mathcal{Q}$. To avoid excessive use of nested subscripts, put $x_i = y_{j_i}$.

We claim that $x_1, \dots, x_k$ is a construction sequence. Conditions (i) and (ii) are immediate from the inductive definition, and for (iii) note that

$$(\mathcal{S}_i, \mathcal{P} \cap \mathcal{S}_i) \text{ reduces to } \mathcal{Q}(H_1, H_2, H_3, \dots, H_{j_i}).$$

Next let $\mathcal{E}_i = \Phi'' H_{j_i}$ and let U be the submonoid of $T(\mathscr{A})$ generated by $(x_i, \mathcal{E}_i)$. We claim that U is valid. Conditions (i) and (iii) (in the definition of "valid") are immediate, since U is a submonoid of a valid transition table; and condition (ii) follows because the x_i's generate $\mathcal{Q}$. Finally, the choice of x_i's guarantees that $\mathcal{E}_i \subset \mathcal{S}_{i-1}$, so (iv) is a consequence of Lemma 8.3.

Finally, let $\mathcal{E}'_i = \mathcal{M}_{x_i} \cap \mathcal{S}_{i-1}$. Let U' be generated by $(x_i, \mathcal{E}'_i)$. To complete the proof, we show that U' is valid; then U' will satisfy all the requirements of a minimex algebra. Conditions (ii), (iii) and (iv) follow as before. It remains to prove (i). Now for each i, we know that $\mathcal{E}_i \subset \mathcal{S}_{i-1}$. Since U is valid, we have furthermore that $\mathcal{E}_i \subset \mathcal{M}_{x_i}$. Therefore $\mathcal{E}_i \subset \mathcal{E}'_i$. It follows that, whenever $(x, \mathcal{E}') \in U'$, then there is some $\mathcal{E} \subset \mathcal{E}'$ with $(x, \mathcal{E}) \in U$.

To conclude, fix any $(x, \mathcal{E}') \in U'$. If $x \in \mathcal{P}$, then $\mathcal{E} \cap \mathcal{P} = \varnothing$ because each $\mathcal{E}'_i \subset \mathcal{M}_{x_i}$. If $x \notin \mathcal{P}$, then choose $\mathcal{E} \subset \mathcal{E}'$ with $(x, \mathcal{E}) \in U$. Since U is valid, we know that $\mathcal{E} \cap \mathcal{P} \neq \varnothing$. Therefore $\mathcal{E}' \cap \mathcal{P} \neq \varnothing$. This proves (i), showing that U' is a minimex algebra. □

We now describe the algorithm for enumerating quotients of order n. Define a *construction scheme* to be a tuple $(\mathcal{Q}, \mathcal{P}, x_1, \dots, x_k)$, such that $(\mathcal{Q}, \mathcal{P})$ is a bipartite monoid and $x_1, \dots, x_k$ is a construction sequence for $\mathcal{Q}$. A *simple extension* of $(\mathcal{Q}, \mathcal{P}, x_1, \dots, x_k)$ is a construction scheme $(\mathcal{Q}^+, \mathcal{P}^+, x_1, \dots, x_{k+1})$ such that $\mathcal{Q} \subset \mathcal{Q}^+$ and $\mathcal{P}^+ \cap \mathcal{Q} = \mathcal{P}$.

It is worth emphasizing a subtle, but crucial, technicality in the definition of construction scheme. No restrictions are placed on the b.m. $(\mathcal{Q}, \mathcal{P})$. However, it is required that every *proper* initial segment $(\mathcal{S}_i, \mathcal{P} \cap \mathcal{S}_i)$ reduce to a genuine misère quotient. Therefore, simple extensions are meaningful only in the special case where $(\mathcal{Q}, \mathcal{P})$ is indeed a misère quotient.

By the above theorems, $(\mathcal{Q}, \mathcal{P})$ is a misère quotient if and only if there is a construction scheme $(\mathcal{Q}, \mathcal{P}, x_1, \dots, x_k)$ such that the minimex algebra constructed by $x_1, \dots, x_k$ is valid. To find all misère quotients of order n, we can therefore enumerate all construction schemes of order n and check which ones generate valid minimex algebras.

	$\mathcal{Q}$	$\mathcal{P}$	G
$\mathcal{S}_{12}$	$\langle a, b, c \mid a^2 = 1, b^4 = b^2, b^2c = b^3, c^2 = 1\rangle$	$\{a, b^2, ac\}$	$*2_{\#}1$
$\mathcal{S}'_{12}$	$\langle a, b, c \mid a^2 = 1, b^3 = b, c^2 = 1\rangle$	$\{a, b^2, c\}$	$*2_{\#}321$
$\mathcal{R}_{12}$	$\langle a, b, c, d \mid a^2 = 1, b^3 = b, b^2c = c, c^2 = b^2, bd = b, cd = c, d^2 = b^2\rangle$	$\{a, b^2\}$	$*2_{\#\#}54321$
	$\langle a, b, c \mid a^2 = 1, b^4 = b^2, b^2c = b^3, c^2 = b^2\rangle$	$\{a, b^2, c\}$	$*H2_{\#\#}$
	$\langle a, b, c, d \mid a^2 = 1, b^3 = b, bc = b, c^2 = b^2, bd = ab, d^2 = b^2\rangle$	$\{a, b^2, d\}$	$*H_{\#}$
	$\langle a, b, c, d \mid a^2 = 1, b^4 = b^2, b^2c = ab^3, c^2 = abc\rangle$	$\{a, b^2, c\}$	$*HK2_{\#\#}0$

Table 2. The six misère quotients of order 12. $H = *2_{\#\#}321$, $K = *2_{\#\#}2_{\#}$.

$*\,2_{\#}0$
$*\,G2_{\#}32$
$*\,H_{\#G}320$
$*\,(G2_{\#})(G2_{\#2}2_{\#})$
$*\,(G2_{\#})(G2_{\#2}2_{\#}1)$
$*\,(G2_{\#})(G2_{\#3}2_{\#}1)$
$*\,(G2_{\#2}2_{\#}1)(G2_{\#3}2_{\#}1)$
$*\,2_{\#\#}4_{2}54320$
$*\,K_2K_1KG2_{\#}321$

Table 3. Nine games that generate nonisomorphic quotients of order 14. $G = *2_{\#}320$, $H = *2_{\#\#}321$, $K = *2_{\#\#}2_{\#}32$.

This method is made useful by a crucial optimization. Built into the definition of construction sequence is the assumption that *each proper initial segment reduces to a known misère quotient*. We can therefore use the following strategy. First, recursively compute *all* misère quotients of order $< n$. Now start with the trivial construction scheme $(\{1\}, \varnothing)$. Given a construction scheme $\Sigma = (\mathcal{Q}, \mathcal{P}, x_1, \ldots, x_k)$, consider every possible simple extension $\Sigma^+ = (\mathcal{Q}^+, \mathcal{P}^+, x_1, \ldots, x_{k+1})$ such that $|\mathcal{Q}^+| \le n$. The key is that if $|\mathcal{Q}^+| < n$, then $(\mathcal{Q}^+, \mathcal{P}^+)$ *must* reduce to a known quotient. If it does not, then we can discard Σ^+ from further consideration.

We have therefore reduced the search space to small simple extensions of known quotients. Since a simple extension is just a monoid extension by a single generator, there are relatively few possibilities, and the algorithm is tractable. It is summarized as Algorithm 1 on the next page.

```
Recursively compute all quotients of size < n
𝒳 ← ∅
Put the trivial construction scheme ({1}, ∅) into 𝒳
for all Σ = (𝒬, 𝒫, x_1, ..., x_k) in 𝒳 do
    𝒴 ← the set of all simple extensions of Σ of order ≤ n
    for all (𝒬^+, 𝒫^+, x_1, ..., x_{k+1}) in 𝒴 do
        if |𝒬^+| = n then
            T ← the minimex algebra on (𝒬^+, 𝒫^+)
              constructed by x_1, ..., x_{k+1}
            if (𝒬^+, 𝒫^+) is reduced and T is valid then
                Output (𝒬^+, 𝒫^+)                ▷ It's a misère quotient
            end if
        else                                      ▷ |𝒬^+| ≤ n − 2
            (𝒮, ℛ) ← the reduction of (𝒬^+, 𝒫^+)
            if (𝒮, ℛ) is a misère quotient then
                Put (𝒬^+, 𝒫^+, x_1, ..., x_{k+1}) into 𝒳
            end if
        end if
    end for
end for
```

Algorithm 1. Classification algorithm.

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