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A ruler regularity in hexadecimal games

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An important problem in the theory of impartial games is to determine the regularities of their nim-sequences. Subtraction games have periodic nim-sequences and those of octal games are conjectured to be periodic, but the possible regularities of the nim-sequence of a hexadecimal game are unknown. Periodic and arithmetic periodic nim-sequences have been discovered but other patterns also exist. We present an infinite set of hexadecimal games, based on the game **0.2048**, that exhibit a regularity — ruler regularity — not yet reported or codified.

1. Introduction

A taking-and-breaking game [Albert et al. 2007; Berlekamp et al. 2001] is an impartial combinatorial game, played with heaps of beans on a table. A move for either player consists of choosing a heap, removing a certain number of beans from the heap, and then possibly splitting the remainder into several heaps; the winner is the player making the last move. For example, both Grundy's Game (choose a heap and split it into two unequal heaps) and Couples-Are-Forever (choose a heap with at least three beans and split it into two) are taking-and-breaking games with very simple rules, however neither has been solved.

We present an overview of the required theory of *impartial* games. The reader can consult the references above for a more in-depth grounding in the theory of, and for more details about, subtraction and octal games.

The *followers* of a game are all those positions which can be reached in one move. The *minimum excluded value* of a set S is the least nonnegative integer which is not included in S and is denoted $\operatorname{mex}(S)$. The *nim-value* of an impartial game G, denoted by $\mathcal{G}(G)$, is given by $\mathcal{G}(G) = \operatorname{mex}\{\mathcal{G}(H) | H \text{ is a follower of } G\}$. The values in the set $\{\mathcal{G}(H) | H \text{ is a follower of } G\}$ are called *excluded values* for $\mathcal{G}(G)$. An impartial game G is a previous player win (i.e., the next player has no good move) if and only if $\mathcal{G}(G) = 0$. The *exclusive or* (XOR), or *nim-sum*, of two nonnegative integers a, b, written $a \oplus b$, consists of adding their binary

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representations with no carrying; for example, $3 \oplus 6 = (11 \oplus 110)_{\text{base } 2} = 101_{\text{base } 2} = 5$. The *disjunctive sum* of games G and H, written $G + H$, is the game where a player chooses either of the two and plays in that game; the nim-value of a disjunctive sum satisfies $\mathcal{G}(G+H) = \mathcal{G}(G) \oplus \mathcal{G}(H)$. Taking-and-breaking games are examples of disjunctive games — choose one heap and play in it. To know how to play these game well, it suffices to know what the nim-values are for individual heaps. For a given game G, let $\mathcal{G}(i)$ be the nim-value of G played with a heap of size i. We define the nim-sequence for a taking-and-breaking game to be the sequence $\mathcal{G}(0), \mathcal{G}(1), \mathcal{G}(2), \ldots$. A nim sequence is *periodic* if there exist N and p such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq N$. It is *arithmetic-periodic* if there exist N, p and s such that $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ for all $n \geq N$, where s is called the *saltus*.

In a hexadecimal game, after removing beans from a heap the remainder can be split into at most three heaps. The rules for a hexadecimal game are described by a hexadecimal code $\mathbf{0.}d_1 d_2 \ldots d_u$ where $0 \leq d_i \leq 15$. This is an extension of the octal code used in [Berlekamp et al. 2001]. We use the letters A, B, C, D, E, F for the numbers 10 through 15 respectively. If $d_i = 0$ then a player cannot take i beans away from a heap. If $d_i = a_3 2^3 + a_2 2^2 + a_1 2^1 + a_0 2^0$ where a_i is 0 or 1, a player can remove i beans from the heap provided he leaves the remainder in exactly j heaps for some j with $a_j = 1$. If $0 \leq d_i \leq 7$ for all i then this is called an *octal* game. This restriction allows a heap to be split into no more than 2 heaps. A subtraction game has $d_i = 0$ or 3; that is, a player can remove beans but cannot split the heap.

The nim-sequence of a subtraction game with a finite subtraction set is periodic [Albert et al. 2007; Berlekamp et al. 2001]. In [Althöfer and Bültermann 1995], it is shown that the sequence of nim-values of games with small subtraction sets can have long periods. It is conjectured that all octal games also have periodic nim-sequence in unsolved problem 2 of [Guy and Nowakowski 2002]. This appears to be hard. For example, in the game **0.106** Flammenkamp computed the period length as 328226140474 and preperiod length as 465384263797. See [Flammenkamp 2012] for this and other searches for long periods, also see [Berlekamp et al. 2001; Caines et al. 1999; Gangolli and Plambeck 1989]. It is interesting to note that while octal games cannot have arithmetic-periodic nim-sequences [Austin 1976], if a single pass move is allowed then not only do arithmetic-period nim-sequences occur, but also nim-sequences composed of a periodic subsequence and an arithmetic-periodic subsequence occur called *sapp* sequences [Horrocks and Nowakowski 2003].

For hexadecimal games, there is an even richer selection of behaviours [Howse and Nowakowski 2004]):

- *Periodic*: **0.B**, period 2 with no preperiod.

- *Arithmetic-periodic*: **0.137F** has nim-sequence 0, 1, 1, 2, 2, 3, 3, ..., where $\mathcal{G}(2m-1) = \mathcal{G}(2m) = m$ and $\mathcal{G}(2m+1) = \mathcal{G}(2m) + 1 = m + 1$ for $m \geq 1$. In this case, the saltus is 1 and the period length is 2.
- *Sapp* regularity: **0.205200C** has the nim-sequence consisting of
  (a) periodic subsequences $\mathcal{G}(40k + 19) = 6$ and $\mathcal{G}(40k + 39) = 14$, and
  (b) arithmetic-periodic subsequences
  $$\mathcal{G}(40k + j) = \mathcal{G}(40(k-1) + j) + 16, \quad j \neq 19, 39,\ k \geq 0,$$
  with preperiod length of 4, period length of 40.
- In [Howse and Nowakowski 2004], it is noted that **0.123456789** exhibits another type of regularity. Starting with $n = 0$, the first fifteen nim-values are 0, 1, 0, 2, 2, 2, 1, 1, 3, 2, 4, 4, 5, 5, 6, 4, and thereafter
  $$\mathcal{G}(2m-1) = \mathcal{G}(2m) = m - 1,$$
  except $\mathcal{G}(2^k + 6) = 2^k - 1$. The nim-sequence is essentially arithmetic-periodic, but with an infinite number of exceptional values that occur in a geometric fashion.

In this paper, we report another new regularity: *ruler-regularity*. This concept is taken from the markings of a foot ruler. (See [Berlekamp et al. 2003, page 470, Figure 7], *"The $\mathcal{G}$-values for the* RULER *game"*, except for us the ruler needs to be rotated by 45 degrees and then flipped!)

Definition 1. A sequence is *ruler regular* if there are positive integers N, p, s, r and a finite set of integers K such that $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ for all $n \geq N$ except
$$n = r((q+1)2^{m+1} + 2^m + 1) + k, \quad k \in K, q, m \geq 0,$$
when
$$\mathcal{G}(r((q+1)2^{m+1} + 2^m + 1) + k) = \mathcal{G}(r(q2^{m+1} + 2^m + 1) + k) + 2^{m+3}.$$

This is illustrated in Figure 1 for **0.2048**, where $r = 13$. New "lines" start at $(m, q) = (1, 0), (2, 0), \ldots$, and are indicated in the top graph. A ruler-regular nim-sequence appears arithmetic-periodic regardless of how far we extend the sequence, but before the sequence goes far enough to ensure arithmetic-periodicity (approximately $3e + 8ps^3$, where e is the size of the largest heap which is not in the period; see [Howse and Nowakowski 2004, Theorem 4]), a new term arises which essentially doubles the length of the apparent period. In this paper we show that **0.20...048**, where there are an odd number of 0s in the hexadecimal code, are ruler-regular. These are not the only such games, for instance **0.21317809532**, **0.31711188** (Figure 2, top), **0.321432132900903213**,

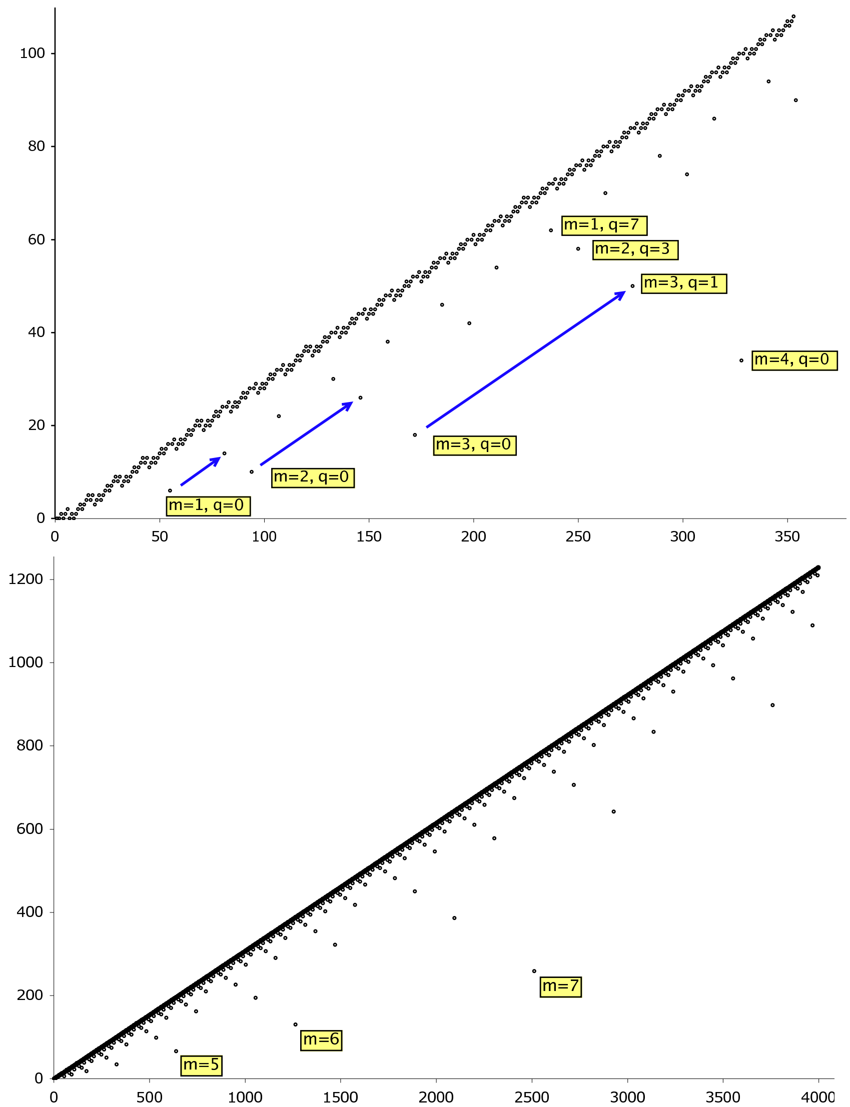
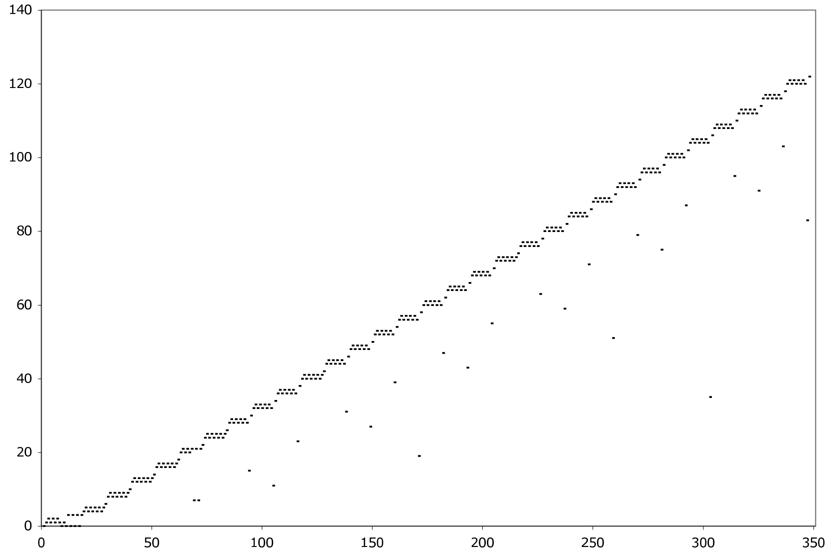


Figure 1. The first 350 values (top) and the first 4000 (bottom) of the $\mathcal{G}$-sequence of **0.2048**.

and **0.404008** (Figure 2, bottom) all appear to be ruler-regular with $|K| = 1$. (Actually, **0.404008** isn't precisely ruler-regular; the sharp-eyed reader will have noticed the same behaviour as in **0.123456789** happening along the first diagonal underneath the main diagonal, in tandem with the point starting the new "line".) The game **0.9138B835B** has $|K| = 3$; see Figure 3.

All of the previously known regularities — periodic, arithmetic-periodic and sapp — have the property that only a finite number of nim-values have to be



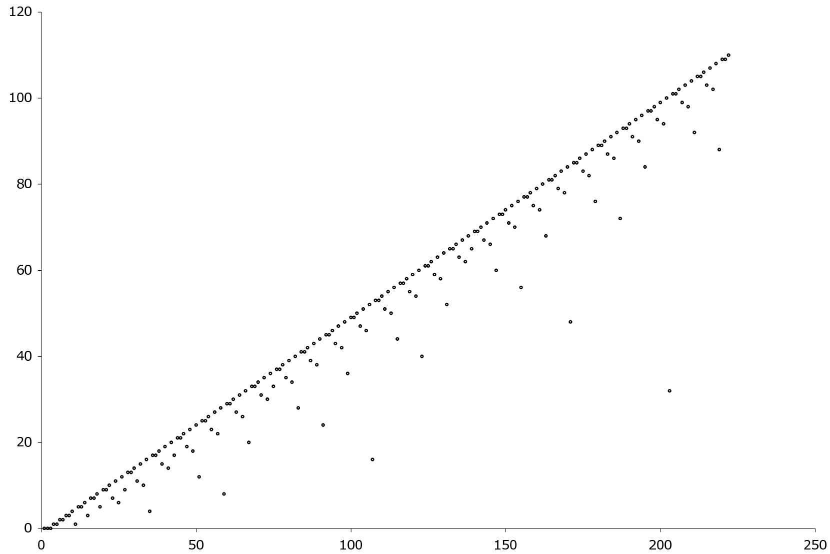


Figure 2. Top: **0.31711188**. Bottom: **0.404008**.

calculated to identify the type of regularity. However, as yet we have no similar mechanism to check for ruler-regularity.

2. The hexadecimal game 0.2048

In the hexadecimal game **0.2048**, the legal moves are to:

- subtract 1 from a heap of size ≥ 2;

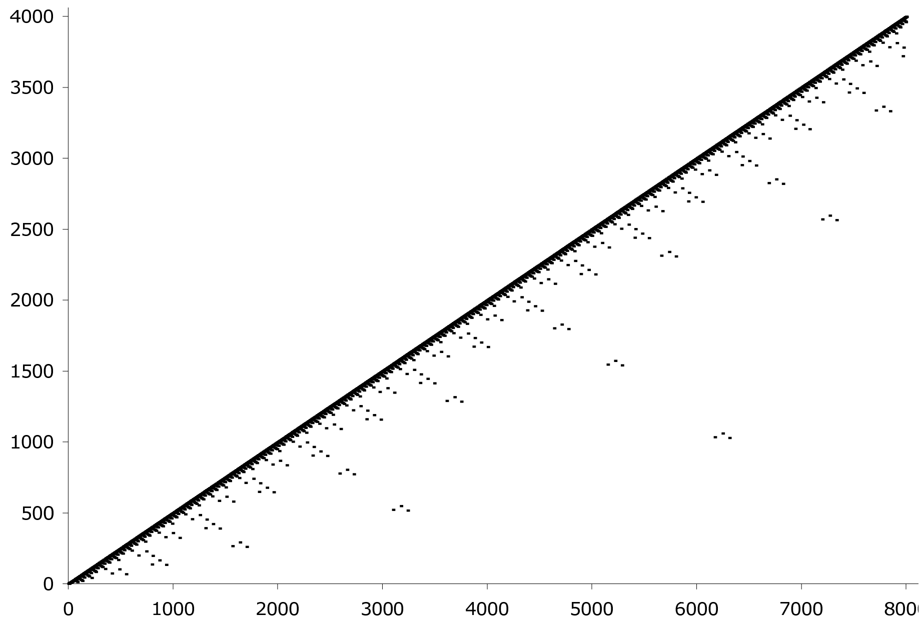


Figure 3. 0.9138B835B.

- subtract 3 from a heap and split the remainder into exactly two heaps;
- subtract 4 from a heap and split the remainder into exactly three heaps.

Here are the first 21 nim-values for this game; they can be computed fairly easily by hand.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathcal{G}(n)$	0	0	1	0	1	2	0	1	0	1	2	3	2	3	4	5	4	5	3	4	5

Theorem 2. *For the game* **0.2048**:

(a) *If* $k \geq 0$ *and* $j \in \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 18\}$, *then* $\mathcal{G}(13k + j) = 4k + \mathcal{G}(j)$ *unless* $j = 2$ *and* k *is of the form*

$$(q + 1)2^{m+1} + 2^m + 1, \quad (q, m \geq 0).$$

(b) *For* $q, m \geq 0$, $\mathcal{G}\big(13((q + 1)2^{m+1} + 2^m + 1) + 2\big) = 2^{m+3}q + 2^{m+2} + 2.$

Part (b) of the theorem describes the exceptional values which form the ruler pattern. Equivalently stated, if

$$k = (q + 1)2^{m+1} + 2^m + 1$$

then

$$\mathcal{G}(13k + 2) = 4(k - 2^{m+1} - 1) + 2.$$

m	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	Differences
0	(54,6)	(80,14)	(106,22)	(132,30)	(158,38)	(13·2,8)
1	(93,10)	(145,26)	(197,42)	(249,58)	(301,74)	(13·4,16)
2	(171,18)	(275,50)	(379,82)	(483,114)	(587,146)	(13·8,32)
3	(327,34)	(535,98)	(743,162)	(951,226)	(1159,290)	(13·16,64)
4	(639,66)	(1055,194)	(1471,322)	(1887,450)	(2303,578)	(13·32,128)
5	(1263,130)	(2095,386)	(2927,642)	(3759,898)	(4591,1154)	(13·64,256)
6	(2511,258)	(4175,770)	(5839,1282)	(7503,1794)	(9167,2306)	(13·128,512)

Table 1. The "lines" $m = 0, 1, \ldots, 6$.

These exception values form lines, each of slope $\frac{4}{13}$, indexed by $m \geq 0$ starting at $(39 \times 2^m + 15, 2^{m+2} + 2)$, and the other points on the "line" are given by $(39 \times 2^m + 15, 2^{m+2} + 2) + 13(2^{m+1}, 2^{m+3})$. This is illustrated in Table 1. The annotations "$m =$" in Figure 1 indicate the start of the m-th line. In the top half of the figure, the four points $m = 4$, $q = 0$ through $m = 1$, $q = 7$ lie on a "rule" of negative slope. For a given $k \geq 0$, starting with the point in the bottom right position of this line, the coordinates of the k-th new rule are given by Theorem 2(b) with

$$(m, q) = (k, 0),\ (k-1, 1),\ (k-2, 3),\ \ldots,\ (k-i, 2^i - 1),\ \ldots,\ (0, 2^k - 1).$$

The difference between consecutive points is $(-13 \cdot 2^{m-1}, 2^{m+1})$ which gives a slope of $-\frac{4}{13}$. The other set of interesting points are the ones given by $(m, 0)$, $m = 1, 2, \ldots$. These also fall on a line and this has slope $\frac{4}{3 \cdot 13}$.

Before we embark on the proof of Theorem 2, let us take a moment to discuss the statement of the theorem and make a few simple observations that will be of use later on. First, the set $J = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 18\}$ is a modified set of remainders (mod 13) chosen so that for $j \in J$ we have $0 \leq \mathcal{G}(j) \leq 3$. These are the values of $\mathcal{G}(j)$ for $j \in J$:

j	1	2	3	4	6	7	8	9	10	11	12	13	18
$\mathcal{G}(j)$	0	1	0	1	0	1	0	1	2	3	2	3	3

Our proof of Theorem 2 is by induction, where we divide into cases based on j, the residue modulo 13. For each $j \in J$, we must show that $\mathcal{G}(13k + j)$ is as described by the theorem. In order to show that $\mathcal{G}(n) = g$, it is necessary to (i) prove that for all $g' < g$ there is a move from a heap of size n to a position with nim-value g', and (ii) prove that there is no move from a heap of size n to a position with nim-value g. Note that (a) holds trivially with $k = 0$, so in all that follows, let $k > 0$ and $j \in J$, and suppose inductively that the statement of Theorem 2 holds for heaps smaller than $13k + j$. We begin with a lemma:

Lemma 3. *For all* $1 \leq a \leq k$ *there is a move from a heap of size* $13k + j$ *to a position with nim-value* $4(k-a)+2$ *unless* $k = (q+1)2^{m+1} + 2^m + 1$, $j = 2$ *and* $a = 2^{m+1} + 1$, *in which case there is no move from a heap of size* $13k + j$ *to a position with nim-value* $4(k-a)+2$.

Proof. We first consider the nonexceptional cases where $j \neq 2$ or $a \neq 2^{m+1} + 1$ or $k \neq (q+1)2^{m+1} + 2^m + 1$, and show that there is a move from a heap of size $13k + j$ to a position with nim-value $4(k-a)+2$. We can subtract 4 and split the remainder into three heaps of size r, r and $13k + j - 4 - 2r$. The nim-value of this position is simply $\mathcal{G}(13k + j - 4 - 2r)$, so it certainly suffices to find $r \geq 1$ such that $\mathcal{G}(13k + j - 4 - 2r) = 4(k-a)+2$. In other words, it suffices to find $n \leq 13k + j - 6$ such that $13k + j - n \equiv 0 \pmod 2$ and $\mathcal{G}(n) = 4(k-a)+2$. For $a = k$, either $n = 5$ or $n = 10$ will satisfy these conditions unless $13k + j = 14$, but for this case we simply subtract 3 and divide into heaps of 10 and 1. For $a < k$ we consider two cases:

Case 1: $\boldsymbol{j + a \equiv 0 \pmod 2}$**.** Take $n = 13(k-a) + 10$. Then by induction, $\mathcal{G}(n) = 4(k-a)+2$, and also $13k + j - n \equiv j + a \equiv 0 \pmod 2$, so we are done if $13(k-a) + 10 \leq 13k + j - 6$, which is equivalent to $16 \leq 13a + j$. This is certainly the case if $a > 1$, so we need only to consider $a = 1$ in which case the inequality becomes $j \geq 3$. This in turn is true unless $j = 1$ or $j = 2$, but $j \neq 2$ because we are considering the case $j + a \equiv 0 \pmod 2$. Thus, it remains to show that there is a move from a heap of size $13k + 1$ to a position with nim-value $4(k-1)+2$. We can accomplish this by subtracting 3 and splitting the remainder into two heaps of size 1 and $13k - 3$; by induction $\mathcal{G}(1) \oplus \mathcal{G}(13k-3) = 0 \oplus \mathcal{G}(13(k-1)+10) = 4(k-1)+2$.

Case 2: $\boldsymbol{j + a \equiv 1 \pmod 2}$**.** Let m be the number of trailing zeros in the binary representation of $k - a$. Theorem 2(b) states that $4(k-a)+2$ appears as an exceptional value $\mathcal{G}(n)$ for $n = 13(k - a + 2^{m+1} + 1) + 2$. If either $a > 2^{m+1} + 1$ or $a = 2^{m+1} + 1$ and $j > 2$ then $n = 13(k - a + 2^{m+1} + 1) + 2 < 13k + j$, so by induction we can assume that $\mathcal{G}(n) = 4(k-a)+2$. Then there is some q such that $4(k-a)+2 = 2^{m+3}q + 2^{m+2} + 2$, and so $13k + j - n \equiv j + a - 1 \equiv 0 \pmod 2$, so we're done if $n \leq 13k + j - 6$, that is, $8 \leq 13(a - 2^{m+1} - 1) + j$. This in turn is true if either $a > 2^{m+1} + 1$ or $a = 2^{m+1} + 1$ and $j \geq 8$; note that these conditions on (a, j) subsume the previous ones. Thus, we may assume from this point forward that $a \leq 2^{m+1} + 1$ and either $a < 2^{m+1} + 1$ or $j < 8$. In particular, for $j \geq 8$ we may assume that $a < 2^{m+1} + 1$. Note that by definition of m, if $d < 2^m$ then the 1s in the binary representations of $k - a$ and d do not overlap, so $d \oplus (k-a) = d + (k-a)$, which we rewrite as

$$k - a = d \oplus (k - a + d). \qquad (1)$$

We proceed by considering three cases.

Case 2.1: $a \equiv 0 \pmod 2$, $j \equiv 1 \pmod 2$. Since $a \geq 1$ we in fact have $a \geq 2$. Let $d = (a-2)/2$. Then $d \leq (2^{m+1}-1)/2 < 2^m$ and (1) becomes

$$k - a = d \oplus (k - 2 - d). \tag{2}$$

Our strategy is now to subtract 4 and split the remaining $13k + j - 4$ into exactly three heaps of sizes $13(k-2-d)+j_1$, $13d+j_2$ and j_3 where $j_1+j_2+j_3=22+j$, $\mathcal{G}(j_1) \oplus \mathcal{G}(j_2) \oplus \mathcal{G}(j_3) = 2$, $j_3 \in J \cup \{5\}$ and $j_1, j_2 \in J$. Using induction, (2) and the fact that $0 \leq \mathcal{G}(j_i) \leq 3$, the nim-value of this position is

$$\begin{aligned}(4(k-2-d)+\mathcal{G}(j_1)) \oplus (4d+\mathcal{G}(j_2)) \oplus \mathcal{G}(j_3)\\ = 4(d \oplus (k-2-d)) + \mathcal{G}(j_1) \oplus \mathcal{G}(j_2) \oplus \mathcal{G}(j_3) = 4(k-a)+2.\end{aligned}$$

Since $j \equiv 1 \pmod 2$, we only need to consider $j \in \{1, 3, 7, 9, 11, 13\}$. We can verify that the j_i in the following table satisfy the given conditions for each choice of j:

$22+j$	23	25	29	31	33	35
j_1	9	10	12	13	7	8
j_2	9	10	12	13	8	9
j_3	5	5	5	5	18	18

Case 2.2a: $a \equiv 1 \pmod 2$, $j \equiv 0 \pmod 2$, $a < 2^{m+1}+1$, $j \geq 8$ *or* $j = 2$. Let $d = (a-1)/2 < 2^{m+1}/2 = 2^m$. Then (1) becomes

$$k - a = d \oplus (k - 1 - d). \tag{3}$$

The conditions on j imply that $j \in \{2, 8, 10, 12, 18\}$ and hence

$$(j+4)/2 \in \{3, 6, 7, 8, 11\} \subset J.$$

We can therefore subtract 4 and split the remaining $13k + j - 4$ into three heaps of size $13(k-1-d) + (j+4)/2$, $13d + (j+4)/2$ and 5. Using induction and (3), we again find that the nim-value of this position is $4(k-a)+2$.

Case 2.2b: $a \equiv 1 \pmod 2$, $a \leq 2^{m+1}+1$, $j = 4, 6$. We dispense with the case $a = 1$ by observing that we can subtract 4 from $13k+4$ and split the remaining $13k$ into heaps of size 5, 4 and $13(k-1)+4$; we can subtract 3 from $13k+6$ and split the remaining $13k+3$ into heaps of size 6 and $13(k-1)+10$. By induction, the nim-value of each resulting position is $4(k-1)+2$. Now assume that $a \geq 3$, and let $d = (a-3)/2 \leq (2^{m+1}-2)/2 < 2^m$. Then (1) becomes

$$k - a = d \oplus (k - 3 - d). \tag{4}$$

Our strategy is to again subtract 4 and split the remaining $13k+j-4$ into exactly three heaps of size $13(k-3-d)+j_1$, $13d+j_2$ and j_3 where $j_1+j_2+j_3=35+j$, $\mathcal{G}(j_1)\oplus\mathcal{G}(j_2)\oplus\mathcal{G}(j_3)=2$, $j_3\in J\cup\{5\}$ and $j_1, j_2\in J$. Using induction, (4) and the fact that $0\le\mathcal{G}(j_i)\le 3$, we again find that the nim-value of this position is $4(k-a)+2$. We can verify that for $j=4$ we can take $(j_1, j_2, j_3)=(10, 11, 18)$ and for $j=6$ we can take $(j_1, j_2, j_3)=(11, 12, 18)$.

The only remaining values of j and a to consider are $j=2$ and $a=2^{m+1}+1$; in all other cases we have shown that there is a move from a heap of size $13k+j$ to a position with nim-value $4(k-a)+2$. Recall that m is the number of trailing zeros in the binary representation of $k-a$, thus $k-a$ is of the form $q\cdot 2^{m+1}+2^m$ for some $q\ge 0$, that is $k=(k-a)+a=(q+1)\cdot 2^{m+1}+2^m+1$. This is exactly the case excepted by the statement of the lemma, so all that remains to be shown is that in this case there is no move from a heap of size $13k+j$ to a position with nim-value $4(k-a)+2$. First, observe that subtracting 1 leaves a heap which by induction has nim-value $4k$. Next, suppose we subtract 4 and divide the remaining $13k-2$ into exactly 3 heaps. We can express the size of these heaps as $13k_1+j_1$, $13k_2+j_2$, and $13k_3+j_3$, where for $i=1, 2, 3$ either $j_i\in J$ or $j_i=5$ and $k_i=0$. By induction, the nim-value of the resulting position is:

$$(4k_1+\mathcal{G}(j_1))\oplus(4k_2+\mathcal{G}(j_2))\oplus(4k_3+\mathcal{G}(j_3)) \\ =4(k_1\oplus k_2\oplus k_3)+\mathcal{G}(j_1)\oplus\mathcal{G}(j_2)\oplus\mathcal{G}(j_3).$$

If this is equal to $4(k-a)+2$ then the following three equations must hold:

$$(13k_1+j_1)+(13k_2+j_2)+(13k_3+j_3)=13k-2, \tag{5}$$

$$k_1\oplus k_2\oplus k_3=k-a, \tag{6}$$

$$\mathcal{G}(j_1)\oplus\mathcal{G}(j_2)\oplus\mathcal{G}(j_3)=2. \tag{7}$$

From (5) we have $j_1+j_2+j_3\equiv -2 \pmod{13}$. On the other hand, since $j_i\le 18$ we have $j_1+j_2+j_3\le 54$, thus $j_1+j_2+j_3\in\{11, 24, 37, 50\}$. There are no choices of $j_1, j_2, j_3\in J$ which sum to 50. If $j_1+j_2+j_3=37$ then they can't all be ≤ 13, so one of them is 18 and the other two sum to 19, but we can verify by hand that for all such choices we have $\mathcal{G}(j_1)\oplus\mathcal{G}(j_2)\oplus\mathcal{G}(j_3)=0$. If $j_1+j_2+j_3=24$ then from (5) $k_1+k_2+k_3=k-2$. But $a=2^{m+1}+1$ is odd and $k_1\oplus k_2\oplus k_3=k-a$, so in this case $k_1+k_2+k_3$ and $k_1\oplus k_2\oplus k_3$ would have opposite parity which is impossible. Finally, if $j_1+j_2+j_3=11$ then each $j_i\le 9$, so in order to satisfy (7) one of them must be 5, say j_3, hence $k_3=0$. We are then left with two equations:

$$k_1+k_2=k-1, \tag{8}$$

$$k_1\oplus k_2=k-a. \tag{9}$$

Now $k_1 + k_2 = k_1 \oplus k_2 + 2(k_1 \& k_2)$, where & denotes bitwise Boolean AND. Equations (8) and (9) therefore give us $2(k_1 \& k_2) = k_1 + k_2 - k_1 \oplus k_2 = a - 1 = 2^{m+1}$ which implies that $k_1 \& k_2 = 2^m$ and so k_1 and k_2 both have a 1 in the 2^m place of their binary representations. But $k_1 \oplus k_2 = k - a = q \cdot 2^{m+1} + 2^m$ so that exactly one of k_1 and k_2 have a 1 in the 2^m place of their binary representations, a contradiction. Hence there is no way to subtract 4 and divide the remaining $13k-2$ into three heaps such that the resulting position has nim-value $4(k-a)+2$. Finally, suppose we subtract 3 and divide the remaining $13k-1$ into exactly 2 heaps of size $13k_1 + j_1$ and $13k_2 + j_2$ where again either $j_i \in J$ or $j_i = 5$ and $k_i = 0$. By induction, the nim-value of the resulting position is $4(k_1 \oplus k_2) + \mathcal{G}(j_1) \oplus \mathcal{G}(j_2)$. We therefore have the following three equations, corresponding to Equations (5)–(7):

$$(13k_1 + j_1) + (13k_2 + j_2) = 13k - 1, \tag{10}$$

$$k_1 \oplus k_2 = k - a, \tag{11}$$

$$\mathcal{G}(j_1) \oplus \mathcal{G}(j_2) = 2. \tag{12}$$

From (10) we have $j_1 + j_2 \equiv -1 \pmod{13}$, but $j_i \le 18$ so that $j_1 + j_2 \le 36$ giving $j_1 + j_2 \in \{12, 25\}$. Hence from (10), $k_1 + k_2 = k - 1$ or $k_1 + k_2 = k - 2$. But we saw previously that $k_1 + k_2 = k - 1$ is inconsistent with (11) (which is the same as (9)), and $k - 2$, $k - a$ have opposite parity. Hence there is no way to subtract 3 and divide the remaining $13k - 1$ into two heaps such that the resulting position has nim-value $4(k - a) + 2$, and this concludes the proof of Lemma 3. □

Lemma 4. *For all $a \ge 1$, there is a move from a heap of size $13k + j$ to positions with nim-values $4(k - a) + 1$, $4(k - a)$ and $4(k - a) - 1$.*

Proof. As in the proof of the previous lemma, we can move to a position with a given nim-value if a heap of size n has the desired value where $n \le 13k + j - 6$ and $13k + j - n \equiv 0 \pmod 2$. Since $j \ge 1$, this will certainly be the case if $n \le 13(k-1)+8$ and $13k+j-n$ is even. Hence, we obtain nim-value $4(k-a)+1$ with either $n = 13(k-a)+7$ or $n = 13(k-a)+4$ (one or the other will always do since they have opposite parity), we obtain nim-value $4(k - a)$ with either $n = 13(k-a)+6$ or $n = 13(k-a)+3$, and we obtain nim-value $4(k-a)-1$ with either $n = 13(k-a-1)+11$ or $n = 13(k-a-1)+18$. □

We now know that from a heap of size $13k + j$, we can move to a position with nim-value g for any $g \le 4k - 2$, with one exception given by Lemma 3. This in fact proves part (b) of Theorem 2, as it shows that from a heap of size

$$13((q+1) \cdot 2^{m+1} + 2^m + 1) + 2,$$

we cannot move to a position with value $2^{m+3}q+2^{m+2}+2$, but we can move to a position with value g for all $g<2^{m+3}q+2^{m+2}+2$, so

$$\mathcal{G}(13((q+1)\cdot 2^{m+1}+2^m+1)+2)=2^{m+3}q+2^{m+2}+2.$$

To finish the proof of Theorem 2, we must show that for all nonexceptional heaps $13k+j$ we can move to a position with value g for $4k-1\le g<4k+\mathcal{G}(j)$, but we cannot move to a position with value $4k+\mathcal{G}(j)$. Table 2 accomplishes the former, giving explicit moves for each $j\in J$.

It remains to show that there is no move from a heap of size $13k+j$ to a position with nim-value $4k+\mathcal{G}(j)$. First, subtracting 1 leaves $13k+j-1$. By induction, if $j-1\in$J, then $\mathcal{G}(13k+j-1)=4k+\mathcal{G}(j-1)$, but $\mathcal{G}(j)\neq\mathcal{G}(j-1)$ for $j,\ j-1\in J$; otherwise $j+12\in J$ and

$$\mathcal{G}(13k+j-1)=\mathcal{G}(13(k-1)+j+12)=4(k-1)+\mathcal{G}(j+12)<4k+\ G(j).$$

This leaves us with two cases to consider: moves that split $13k+j-4$ into exactly three heaps, and moves that split $13k+j-3$ into exactly two heaps. Following the approach in the proof of Lemma 3, splitting $13k+j-4$ into heaps of size $13k_i+j_i$ with either $j_i\in J$ or $j_i=5$ and $k_i=0$ gives us three equations, for which we must show there is no solution:

$$(13k_1+j_1)+(13k_2+j_2)+(13k_3+j_3)=13k+j-4, \tag{13}$$

$$k_1\oplus k_2\oplus k_3=k, \tag{14}$$

$$\mathcal{G}(j_1)\oplus\mathcal{G}(j_2)\oplus\mathcal{G}(j_3)=\mathcal{G}(j). \tag{15}$$

Equation (14) tells us that $k_1+k_2+k_3\ge k_1\oplus k_2\oplus k_3=k$, so we can rewrite (13) as:

$$j-4-j_1-j_2-j_3=13(k_1+k_2+k_3-k)\ge 0. \tag{16}$$

For each j there are limited choices for $j_1,\ j_2,\ j_3$ satisfying both

$$j_1+j_2+j_3\equiv j-4 \pmod{13} \quad\text{and}\quad j_1+j_2+j_3\le j-4,$$

and we can verify by hand that none of them also satisfy (15). The same approach can be used for moves which split $13k+j-3$ into two heaps; in this case we verify by hand that there are no $j_1,\ j_2$ satisfying $j_1+j_2\equiv j-3 \pmod{13}$, $j_1+j_2\le j-3$, and $\mathcal{G}(j_1)\oplus\mathcal{G}(j_2)=\mathcal{G}(j)$. This completes the proof of Theorem 2.

3. The games 0.200048, 0.20000048, 0.2000000048, . . .

Theorem 5. *Let G be the game* $0.2(0)^{2n+1}48$ (*i.e.*, $2n+1$ *zeroes between the* 2 *and* 4) *and let* $A=6(n+2)+1$.

j	Move	Nim-value
1	$(13k+1)-1=(13(k-1)+13)$	$4(k-1)+3=4k-1$
2	$(13k+2)-3=(1)+(13(k-1)+11)$ $(13k+2)-1=(13k+1)$	$4(k-1)+3=4k-1$ $4k$
3	$(13k+3)-4=(3)+(5)+(13(k-1)+4)$	$4(k-1)+3=4k-1$
4	$(13k+4)-4=(1)+(1)+(13(k-1)+11)$ $(13k+4)-1=(13k+3)$	$4(k-1)+3=4k-1$ $4k$
6	$(13k+6)-1=(13(k-1)+18)$	$4(k-1)+3=4k-1$
7	$(13k+7)-3=(6)+(13(k-1)+11)$ $(13k+7)-1=(13k+6)$	$4(k-1)+3=4k-1$ $4k$
8	$(13k+8)-4=(2)+(2)+(13(k-1)+13)$	$4(k-1)+3=4k-1$
9	$(13k+9)-3=(8)+(13(k-1)+11)$ $(13k+9)-1=(13k+8)$	$4(k-1)+3=4k-1$ $4k$
10	$(13k+10)-4=(3)+(3)+(13(k-1)+13)$ $(13k+10)-3=(1)+(13k+6)$ $(13k+10)-1=(13k+9)$	$4(k-1)+3=4k-1$ $4k$ $4k+1$
11	$(13k+11)-3=(3)+(13(k-1)+18)$ $(13k+11)-4=(2)+(2)+(13k+3)$ $(13k+11)-3=(1)+(13k+7)$ $(13k+11)-1=(13k+10)$	$4k-1$ $4k$ $4k+1$ $4(k-1)+3=4k+2$
12	$(13k+12)-4=(4)+(4)+(13(k-1)+13)$ $(13k+12)-4=(1)+(1)+(13k+6)$ $(13k+12)-4=(2)+(2)+(13k+4)$	$4(k-1)+3=4k-1$ $4k$ $4k+1$
13	$(13k+13)-4=(1)+(3)+(13(k-1)+18)$ $(13k+13)-4=(3)+(3)+(13k+3)$ $(13k+13)-3=(3)+(13k+7)$ $(13k+13)-1=(13k+12)$	$4(k-1)+3=4k-1$ $4k$ $4k+1$ $4k+2$
18	$(13k+18)-4=(7)+(7)+(13(k-1)+13)$ $(13k+18)-4=(3)+(3)+(13k+8)$ $(13k+18)-4=(5)+(5)+(13k+4)$ $(13k+18)-4=(1)+(1)+(13k+12)$	$4(k-1)+3=4k-1$ $4k$ $4k+1$ $4k+2$

Table 2. Moving to a position with nim-value g, $4k-1 \le g < 4k+\mathcal{G}(j)$.

(a) *If* $k > 0$ *then* $\mathcal{G}(Ak + j) = 4k + \mathcal{G}(j)$ *unless* $j = 2$ *and* k *is of the form* $(q+1) \cdot 2^{m+1} + 2^m + 1$, $q, m \geq 0$.

(b) *For* $q, m \geq 0$, $\mathcal{G}(A((q+1) \cdot 2^{m+1} + 2^m + 1) + 2) = 2^{m+3}q + 2^{m+2} + 2$.

Proof. It is not too difficult to prove that the generic game $\mathbf{0.2(0)}^{2n+1}\mathbf{48}$ has initial nim-sequence 0, $(0, 1)^{n+2}$, 2, $(0, 1)^{n+2}$, $(2, 3)^{n+2}$. The proof is now a repetition of that for **0.2048**, with fewer special cases to be considered. We leave this to the reader. □

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