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The Rat game and the Mouse game

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We define three new take-away games, the Rat game, the Mouse game and the Fat Rat game. Three winning strategies are given for the Rat game and outlined for the Mouse and Fat Rat games. The efficiencies of the strategies are determined. Whereas the winning strategies of nontrivial take-away games are based on irrational numbers, our games are based on rational numbers. Another motivation stems from a problem in combinatorial number theory.

1. Description of the game

The Rat game is played on 3 piles of tokens by 2 players who play alternately. Positions in the game are denoted throughout in the form (x, y, z), with $0 \le x \le y \le z$, and moves in the form $(x, y, z) \to (u, v, w)$, where of course also $0 \le u \le v \le w$ (see below). The player first unable to move — because the position is $(0, 0, 0)$ — loses; the opponent wins. There are 3 types of moves:

(I) Take any positive number of tokens from up to 2 piles.

(II) Take $\ell > 0$ from the x pile, $k > 0$ from the y pile, and an arbitrary positive number from the z pile, subject to the constraint $|k - \ell| < a$, where

$$a = \begin{cases} 1 & \text{if } y - x \not\equiv 0 \pmod 7, \\ 2 & \text{if } y - x \equiv 0 \pmod 7. \end{cases}$$

(III) Take $\ell > 0$ from the x pile, $k > 0$ from the z pile, and an arbitrary positive number from the y pile, subject to the constraint $|k - \ell| < b$, where $b = 3$ if $w = u$; otherwise,

$$b = \begin{cases} 5 & \text{if } w - u \not\equiv 4 \pmod 7, \\ 6 & \text{if } w - u \equiv 4 \pmod 7. \end{cases}$$

In a move of type (II) we permit the permutation $x \to v,\ y \to w,\ z \to u$ (so $\ell = x - v$, $k = y - w$), in addition to $x \to u,\ y \to v,\ z \to w$ ($\ell = x - u$, $k = y - v$). No other permutations are allowed for (II), and none (except $x \to u$, $y \to v$, $z \to w$) for (III). For (I), any rearrangement is possible. When we write $(x, y, z) \to (u, v, w)$, we always mean $x \to u,\ y \to v,\ z \to w$.

Note that in (II), the congruence conditions depend only on 2 of the piles moved *from*: the smallest and the intermediate; whereas in (III) they depend only on 2 of the piles moved *to*: the smallest and the largest. The case $w = u$ in (III) is an initial condition, to accommodate the end position $(0, 0, 0)$.

Examples. Given the position $p_1 = (1, 2, 4)$. If player I takes one of the piles in its entirety, player II wins with a type (I) move to $\Phi := (0, 0, 0)$. If player I moves $p_1 \to (1, 2, t)$, $t \in \{1, 2, 3\}$, player II wins with a type (III) move to Φ. If player I moves $p_1 \to (1, 1, 4)$, player II wins with a type (II) move to Φ. It's straightforward to see that if player I makes a move of type (I), or (II) or (III), then player II can win by moving to Φ. Consider now the position $p_2 = (3, 6, 10)$. Then player I can make a type (III) move to p_1. Indeed, $(10-4)-(3-1) = 4 < 5 = b$.

*What's the motivation for inventing and analyzing this game? Why are the move rules complicated? What's the connection to rats? Where's the Mouse game? How about the **Fat** Rat game?*

2. Two characterizations of the *P*-positions

Let $T \subsetneqq \mathbb{Z}_{\geq 0}$. Define the mex operator by $\text{mex}(T) = \min(\mathbb{Z}_{\geq 0} \setminus T) =$ smallest nonnegative integer not in T. Recall that the set of P-positions of a game is the set of positions for which the *P*revious (second) player can win, and the set of N-positions is the set of positions for which the *N*ext (first) player can win [WW; Albert et al. 2007]. We begin with a recursive characterization of the P-positions of the Rat game.

Theorem 1. *The P-positions of the Rat game are given by*

$$R = \bigcup_{n=0}^{\infty} \{(a_n, b_n, c_n)\},$$

where $(a_0, b_0, c_0) = (0, 0, 0)$, *and for* $n \geq 1$,

$$a_n = \text{mex}\{a_i, b_i, c_i : 0 \leq i < n\},$$
$$b_n = a_n + \lfloor (7n-2)/4 \rfloor,$$
$$c_n = b_n + \lfloor (7n-3)/2 \rfloor.$$

The first few triples of R are displayed in Table 1.

We now turn to an explicit characterization of the P-positions.

Define an infinite set of triples $S_n := (A_n, B_n, C_n)$ as follows. $S_0 = (0, 0, 0)$, and for $n \in \mathbb{Z}_{\geq 1}$,

$$A_n = \lfloor 7n/4 \rfloor, \quad B_n = \lfloor 7n/2 \rfloor - 1, \quad C_n = 7n - 3. \tag{1}$$

Put $S = \bigcup_{n=0}^{\infty} S_n$.

n	a_n	b_n	c_n
0	0	0	0
1	1	2	4
2	3	6	11
3	5	9	18
4	7	13	25
5	8	16	32
6	10	20	39
7	12	23	46
8	14	27	53
9	15	30	60
10	17	34	67
11	19	37	74
12	21	41	81
13	22	44	88
14	24	48	95
15	26	51	102

Table 1. The first few P-positions of the Rat game.

Theorem 2. *The collection S constitutes the set of P-positions of the Rat game, so $S = R$.*

In Sections 4–6 we prove Theorems 1 and 2. Various extensions are given in Sections 7–8, efficiencies of the winning strategies are discussed in Section 9, and we wrap up in Section 10.

3. Preliminaries

For proving Theorem 2, which is proved first, we begin by collecting a few properties of the set S.

Lemma 1. *Let $p, q \in \mathbb{Z}_{\geq 1}$, with $p > q$, $s \in \mathbb{Z}$. Then*:

(i) *For every $t \in \mathbb{Z}$, the q values $\lfloor (pn+s)/q \rfloor$, $n \in \{t+1, \ldots, t+q\}$ are distinct* (mod p).

(ii) *For every $k \in \mathbb{Z}$, $\lfloor (p(n+kq)+s)/q \rfloor = \lfloor (pn+s)/q \rfloor + kp$.*

Proof. (i) Let $n_1, n_2 \in \{t+1, \ldots, t+q\}$, $n_1 \neq n_2$, say $n_2 > n_1$. Then

$$0 < (p/q) - 1 < \lfloor (pn_2+s)/q \rfloor - \lfloor (pn_1+s)/q \rfloor < (p(q-1)/q) + 1 < p,$$

as claimed.

(ii) Obvious. □

It follows that for $n \in \{1, \dots, q\}$, $\lfloor (pn+s)/q \rfloor$ contains distinct residues $r_1 < \cdots < r_q \pmod{p}$; for $n \in \{q+1, \dots, 2q\}$ it contains $p+r_1, \dots, p+r_q$; for $n \in \{kq+1, \dots, (k+1)q\}$ it contains $kp+r_1, \dots, kp+r_q$.

Let

$$A = \bigcup_{i=1}^{\infty} A_i, \quad B = \bigcup_{i=1}^{\infty} B_i, \quad C = \bigcup_{i=1}^{\infty} C_i.$$

Lemma 2. (i) *Each of the sequences A_i, B_i, C_i is strictly increasing.*

(ii) *The sets A, B, C partition $\mathbb{Z}_{\geq 1}$.*

Proof. (i) Follows directly from the definition of the 3 sequences.

(ii) Note that $\left(\bigcup_{n=1}^{4} A_n\right) \cup \left(\bigcup_{n=1}^{2} B_n\right) \cup C_1 = \{1, \dots, 7\}$. The result now follows from Lemma 1(i). □

For $n \in \mathbb{Z}_{\geq 0}$, let

$$d_n = B_n - A_n, \quad \delta_n = C_n - B_n, \quad \Delta_n = C_n - A_n.$$

Lemma 3. *For $n \in \mathbb{Z}_{\geq 1}$,*

$$d_n = \lfloor (7n-2)/4 \rfloor, \quad \delta_n = \lfloor (7n-3)/2 \rfloor, \quad \Delta_n = d_n + \delta_n.$$

Proof. The assertion for d_n is seen to hold for $n = 1, 2, 3, 4$. Therefore it holds for all $n \in \mathbb{Z}_{\geq 1}$ by Lemma 1(i). Similarly, the assertion about δ_n is seen to hold for $n = 1, 2$, therefore it holds for all $n \in \mathbb{Z}_{\geq 1}$. Finally,

$$\Delta_n = C_n - A_n = (C_n - B_n) + (B_n - A_n) = \delta_n + d_n. \qquad \square$$

Table 2 depicts the differences d_n, δ_n, Δ_n together with the P-positions. It also illustrates the next few lemmas.

Lemma 4. *For $n \in \mathbb{Z}_{\geq 1}$,*

$$d_n = \begin{cases} A_n & \text{if } n \equiv 1, \text{ or } 2 \pmod 4, \\ A_n - 1 & \text{if } n \equiv 0, \text{ or } 3 \pmod 4, \end{cases} \qquad \delta_n = \begin{cases} B_n - 1 & \text{if } n \text{ is even}, \\ B_n & \text{if } n \text{ is odd}. \end{cases}$$

Proof. The first one follows from Lemma 3 and an easy computation. Similarly for the second. □

Lemma 5. (i) *Each of the sequences d_n, δ_n, Δ_n is strictly increasing.*

(ii) *For $n \in \mathbb{Z}_{\geq 1}$, $d_n < \delta_n < \Delta_n$.*

(iii) *For $n \in \mathbb{Z}_{\geq 1}$, the sequences d_n and δ_n are disjoint. In fact,*

$$d_n \equiv \{1, 3, 4, 6\} \pmod 7, \quad \delta_n \equiv \{2, 5\} \pmod 7,$$

n	A_n	d_n	B_n	δ_n	C_n	Δ_n
0	0	0	0	0	0	0
1	1	1	2	2	4	3
2	3	3	6	5	11	8
3	5	4	9	9	18	13
4	7	6	13	12	25	18
5	8	8	16	16	32	24
6	10	10	20	19	39	29
7	12	11	23	23	46	34
8	14	13	27	26	53	39
9	15	15	30	30	60	45
10	17	17	34	33	67	50
11	19	18	37	37	74	55
12	21	20	41	40	81	60
13	22	22	44	44	88	66
14	24	24	48	47	95	71
15	26	25	51	51	102	76

Table 2. *P*-positions of the Rat game with their differences.

and each of the residues (*mod* 7) *of* d_n *and* δ_n *are assumed for infinitely many* n. *Also*,

$$d_n > \delta_0 = 0, \quad \delta_n > d_0 = 0, \quad \textit{and} \quad \Delta_n \equiv \{3, 1, 6, 4\} \pmod 7.$$

(iv) $\bigcup_{n=1}^{\infty}(d_n \cup \delta_n) = \mathbb{Z}_{\geq 1} \setminus \bigcup_{i=1}^{\infty}\{7i\}$.

Proof. (i) and (ii) follow from Lemma 3.

(iii) By inspection, this holds for $n = 1, 2, 3, 4$. It follows in general from Lemma 1(i).

(iv) From (iii) we see that $\left(\bigcup_{n=1}^{4} d_n\right) \cup \left(\bigcup_{n=1}^{2} \delta_n\right) = \{1, 2, 3, 4, 5, 6\}$. The result now follows from Lemma 1(ii) by induction on k. □

Lemma 6. *For* $n \in \mathbb{Z}_{\geq 1}$, $d_{n+1} - d_n \in \{1, 2\}$, $\delta_{n+1} - \delta_n \in \{3, 4\}$,

$$\Delta_{n+1} - \Delta_n = \begin{cases} 5 & \textit{if } \Delta_n \equiv 1,\ 3 \textit{ or } 6 \pmod 7, \\ 6 & \textit{if } \Delta_n \equiv 4 \pmod 7, \end{cases}$$

and $\Delta_1 - \Delta_0 = 3$.

Proof. Table 2 shows that the statements hold for $n = 1, 2, 3, 4$. Their general truth then follows from Lemma 1. □

4. Proof of Theorem 2

It suffices to show:

(A) *Every move from* $(A_n, B_n, C_n) \in S$ *results in a position outside* S.

(B) *For every position* $(x, y, z) \notin S$, *there exists a move resulting in a position in* S.

(A) Since A_n, B_n, C_n are each strictly increasing, a move of type (I) from $(A_n, B_n, C_n) \in S$, $n \in \mathbb{Z}_{\geq 1}$, clearly results in a position not in S. Suppose that there is a move $S \to S$ of type (II) or (III). Such a move can have one of the following two forms:

(i) $(A_n, B_n, C_n) \to (A_i, B_i, C_i)$, or

(ii) $A_n \to B_i,\ B_n \to C_i,\ C_n \to A_i$.

In both cases, $i < n$. These moves have to satisfy the following conditions. Either:

(i1) $|(B_n - B_i) - (A_n - A_i)| < a$. Now $|(B_n - B_i) - (A_n - A_i)| = |d_n - d_i| \geq 1$ by Lemma 5(i). By Lemma 5(iv), $y - x = d_n \not\equiv 0 \pmod 7$. Hence by the rule of a move of type (II), $a = 1$, a contradiction. Or:

(i2) $|(C_n - C_i) - (A_n - A_i)| < b$. Now

$$|(C_n - C_i) - (A_n - A_i)| = |\Delta_n - \Delta_i| \geq |\Delta_n - \Delta_{n-1}| = b,$$

a contradiction to a move of type (III).

(ii) The constraint is $|(B_n - C_i) - (A_n - B_i)| < a$. Now

$$|(B_n - C_i) - (A_n - B_i)| = |d_n - \delta_i| \geq 1$$

by Lemma 5(i). As in case (i1) we have $a = 1$, a contradiction.

(B) Let $(x, y, z) \notin S$ with $0 < x \leq y \leq z$. (If $x = 0$ there is a type (I) move to $(0, 0, 0)$.) Throughout we use the notation:

$$d = y - x, \qquad D = z - x.$$

Since the sets A, B, C partition $\mathbb{Z}_{\geq 1}$, there exists $n \in \mathbb{Z}_{\geq 1}$, such that either

(i) $x = C_n$, or

(ii) $x = B_n$, or

(iii) $x = A_n$.

Note that since $x > 0$, we have $n > 0$, so by Lemma 3,

$$0 < A_n < B_n < C_n.$$

(i) $x = C_n$. Then do a type (I) move, $y \to A_n$, $z \to B_n$.

(ii) $x = B_n$. If $z \geq C_n$, make a type (I) move, $z \to C_n$, $y \to A_n$. So we may assume $z < C_n$. We have $A_n < B_n = x \leq y \leq z < C_n$. Then make a move of type (III): $(x, y, z) \to (A_i, B_i, C_i)$, where i is the largest index such that $\Delta_i \leq D$. This move is legal:

(a) $\Delta_i \leq D = z - x = z - B_n < z - A_n < C_n - A_n = \Delta_n$. Hence $i < n$.

(b) $x = B_n > A_n > A_i$; $y \geq x = B_n > B_i$ since $i < n$;

$$z = x + D \geq A_n + \Delta_i > A_i + \Delta_i = C_i.$$

(c) The move has to satisfy $|(z - C_i) - (x - A_i)| < b$. Indeed,

$$|(z - C_i) - (x - A_i)| = |D - \Delta_i| = D - \Delta_i < \Delta_{i+1} - \Delta_i = b,$$

by Lemma 5(iii) and Lemma 6.

(iii) $x = A_n$. If $y \geq B_n$ and $z \geq C_n$ (at least one of these inequalities is necessarily strict), then make a move of type (I), $y \to B_n$, $z \to C_n$. Below we consider the remaining 3 subcases:

(α) $y \geq B_n$, $z < C_n$,

(β) $y < B_n$, $z \geq C_n$,

(γ) $y < B_n$, $z < C_n$.

(α) $y \geq B_n$, $z < C_n$. Then $A_n = x < B_n \leq y \leq z < C_n$. We make a move of type (III) as in case (ii) above: $(x, y, z) \to (A_i, B_i, C_i)$, where i is the largest index such that $\Delta_i \leq D$. This move is legal:

(a) $\Delta_i \leq D = z - x = z - A_n < C_n - A_n = \Delta_n$. Hence $i < n$.
(b) $x = A_n > A_i$; $y \geq B_n > B_i$; $z = x + D \geq x + \Delta_i = A_n + \Delta_i > A_i + \Delta_i = C_i$.
(c) $|(z - C_i) - (x - A_i)| = |D - \Delta_i| < b$, as in case (ii)(c) above.

(β) $y < B_n$, $z \geq C_n$. We have $A_n = x \leq y < B_n < C_n \leq z$.

(β1) We first consider the case $d \not\equiv 0 \pmod 7$. Then $d \in \{d_i, \delta_i\}$ for some $i \in \mathbb{Z}_{\geq 0}$ by Lemma 5(iv). Since $d = y - A_n < B_n - A_n = d_n < \delta_n$, we have $i < n$.

(β11) Assume $d = d_i$. Then move $(x, y, z) \to (A_i, B_i, C_i)$. This is indeed a move of type (II):

(a) $x = A_n > A_i$; $y = A_n + d = A_n + d_i > A_i + d_i = B_i$; $z \geq C_n > C_i$.
(b) $|(y - B_i) - (x - A_i)| = |d - d_i| = 0 < a$.

(β12) Assume $d = \delta_i$. Then move $x \to B_i$, $y \to C_i$, $z \to A_i$. It's a move of type (II):

(a) Clearly $z \geq C_n > A_i$. It remains to show that $x > B_i$ and $y > C_i$. By Lemma 4, $A_n \geq d_n = B_n - A_n > y - x = d = \delta_i \geq B_i - 1$. Thus $A_n \geq B_i$. By Lemma 2 we then actually have $A_n > B_i$. Therefore also $y = x + d = A_n + \delta_i > B_i + \delta_i = C_i$.

(b) $|(y - C_i) - (x - B_i)| = |d - \delta_i| = 0 < 1 = a$.

(β2) We now consider the case $d \equiv 0 \pmod 7$. Then Lemma 5(iii) implies $d - 1 = d_i$ for some $i \in \mathbb{Z}_{\geq 0}$. Then move $(x, y, z) \to (A_i, B_i, C_i)$. This is a move of type (II):

(a) $d_i = d - 1 = y - x - 1 < B_n - A_n - 1 = d_n - 1 < d_n$. Hence $i < n$.

(b) $x = A_n > A_i$; $y = A_n + d = A_n + d_i + 1 > A_i + d_i = B_i$; $z \geq C_n > C_i$.

(c) $|(y - B_i) - (x - A_i)| = |d - d_i| = 1 < 2 = a$.

(γ) $y < B_n$, $z < C_n$.

(γ1) We first assume $d \not\equiv 0 \pmod 7$. By Lemma 5 there exists $i \in \mathbb{Z}_{\geq 0}$ such that $d \in \{d_i, \delta_i\}$. Now $d = y - A_n < B_n - A_n = d_n < \delta_n$. Hence $i < n$.

(γ11) We consider first the case $d = d_i$, and try a move of type (II): $(x, y, z) \to (A_i, B_i, C_i)$. Note that $x = A_n > A_i$ and

$$y = x + d = A_n + d_i > A_i + d_i = B_i.$$

If $z \geq C_i$, this is a legitimate move of type (II) (or one of type (I) if $z = C_i$). Indeed, $|(y - B_i) - (x - A_i)| = |d - d_i| = 0 < 1 = a$. So suppose that $z < C_i$. We then do a move of type (III). Move $(x, y, z) \to (A_j, B_j, C_j)$, where j is the largest index such that $\Delta_j \leq D$. This move is legal:

(a) $\Delta_j \leq D = z - A_n < C_n - A_n = \Delta_n$. Hence $j < n$.

(b) $x = A_n > A_j$. Since $z < C_i$ we have

$$\Delta_j \leq D = z - x < C_i - A_n < C_i - A_i = \Delta_i.$$

Hence $j < i$. We showed above that $y > B_i$. Hence $y > B_j$. Also $z = x + D = A_n + D \geq A_n + \Delta_j > A_j + \Delta_j = C_j$.

(c) $|(z - C_j) - (x - A_j)| = |D - \Delta_j| = D - \Delta_j < \Delta_{j+1} - \Delta_j = b$.

(γ12) We now deal with the case $d = \delta_i$. Then move $x \to B_i$, $y \to C_i$, $z \to A_i$. Recall from (γ1) that $i < n$. It's a move of type (II):

(a) Clearly $z \geq y \geq x = A_n > A_i$. It remains to show that $x > B_i$ and $y > C_i$. As in case (β12), Lemma 4 implies $x = A_n \geq d_n > d = \delta_i \geq B_i - 1$. Thus $A_n \geq B_i$. By Lemma 2 we have actually $A_n > B_i$. Therefore $y = x + d = A_n + \delta_i > B_i + \delta_i = C_i$.

(b) $|(y - C_i) - (x - B_i)| = |d - \delta_i| = 0 < 1 = a$.

($\gamma 2$) We now take up the remaining case, $d \equiv 0 \pmod 7$. Then Lemma 5(iii) implies $d-1=d_i$ for some $i \in \mathbb{Z}_{\geq 0}$. Then try a move of type (II) $(x, y, z) \to (A_i, B_i, C_i)$. We have:

(a) $d_i = d-1 = y-A_n-1 < B_n-A_n-1 = d_n-1 < d_n$. Hence $i<n$.

(b) $x = A_n > A_i$; $y = A_n+d = A_n+d_i+1 > A_i+d_i = B_i$. If $z \geq C_i$ we indeed made a legitimate move of type (II) (or one of type (I) if $z = C_i$).

(c) $|(y-B_i)-(x-A_i)| = |d-d_i| = 1 < 2 = a$.

So assume $z < C_i$. We then do a move of type (III):

$$(x, y, z) \to (A_j, B_j, C_j),$$

where j is the largest index such that $\Delta_j \leq D$. The legality of this move is established in precisely the same way as for the type (III) move of case ($\gamma 11$) above. □

Remark. Regarding case (B)(iii) of the proof, i.e., $x = A_n$, a move of type (III), as in case (ii), is not always possible. Example: $(x, y, z) = (17, 28, 66)$. Then $D = z-x = 49$, so Table 1 shows that a (II)(b) move would have to be to $(15, 30, 60)$. This, however, is impossible, since $y = 28 < 30$.

5. Proof of Theorem 1

For proving Theorem 1, it evidently suffices to prove the following result.

Theorem 3. *For all $n \in \mathbb{Z}_{\geq 0}$, $a_n = A_n, b_n = B_n, c_n = C_n$. In other words, the set of triples $R = \bigcup_{n=0}^{\infty}\{(a_n, b_n, c_n)\}$, defined recursively, constitutes the set of P-positions of the Rat game.*

Proof. We note that $(a_0, b_0, c_0) = (A_0, B_0, C_0) = (0, 0, 0)$. Suppose we have shown already that $(a_n, b_n, c_n) = (A_n, B_n, C_n)$ for all $n < N$ $(N \geq 1)$. Recall from Lemma 2(ii) that the sets A, B, C partition $\mathbb{Z}_{\geq 1}$, and clearly $A_n < B_n < C_n$ for all $n \in \mathbb{Z}_{\geq 1}$. Therefore $A_N = \text{mex}\{A_i, B_i, C_i : 0 \leq i < N\}$. Otherwise A_N would never be attained in the complementary sets A, B, C. Thus $A_N = a_N$. Now $B_n - A_n = \lfloor (7n-2)/4 \rfloor$, and $C_n - B_n = \lfloor (7n-3)/2 \rfloor$ for all $n \in \mathbb{Z}_{\geq 1}$ (Lemma 3), the same as in the recursive definition of the triples (a_n, b_n, c_n). Hence also $B_N = b_N$, and $C_N = c_N$. □

6. A numeration systems for the Rat game

Let α be a rational or irrational number satisfying $1 < \alpha < 2$. Denote its *simple continued fraction* expansion by $\alpha = [1, a_1, \ldots]$, $a_i \in \mathbb{Z}_{\geq 1}$ for all i. This expansion is unique if α is irrational. If $\alpha = [1, a_1, \ldots, a_n]$ is rational, there are 2 expansions, since for $a_n > 1$, $a_n = (a_n - 1) + \frac{1}{1}$. In the latter case we assume, for our purposes here, that $\alpha = [1, a_1, \dots, a_{2n}]$. The *convergents* $p_k/q_k = [1, a_1, \dots, a_k]$ of α ($k \le 2n$ if α is rational) are defined recursively in the form

$$p_{-1} = 1, \quad p_0 = 1, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \ge 1),$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \ge 1).$$

For properties of simple continued fractions see [Hardy and Wright 2008, Chapter 10] or [Fraenkel 1982, Section 4]. It is well-known (see the previous reference) that every positive integer N has a unique representation in the form

$$N = \sum_{i=0}^{m} s_i p_i, \quad 0 \le s_i \le a_{i+1},\ s_{i+1} = a_{i+2} \Longrightarrow s_i = 0 \quad (i \ge 0),$$

and also in the form

$$N = \sum_{i=0}^{m} t_i q_i, \quad 0 \le t_0 < a_1,\ 0 \le t_i \le a_{i+1},\ t_i = a_{i+1} \Longrightarrow t_{i-1} = 0 \quad (i \ge 1).$$

Remarks.

- The *p-representation* of any positive integer is its representation $\sum_{i=0}^{m} s_i p_i$ in the p-numeration system. Analogously for the *q-representation*.
- If $\alpha = [1, a_1, \dots, a_{2n}]$ is rational, we may assume that there is an arbitrarily large partial quotient a_{2n+1}, so the digits s_{2n} and t_{2n} can be arbitrarily large. This permits to represent every positive number N in the numeration systems with only finitely many p_i, q_i.
- Notice that $7/4 = [1, 1, 3]$, $p_0 = 1$, $p_1 = 2$, $p_2 = 7$; $q_0 = 1$, $q_1 = 1$, $q_2 = 4$. Further, $0 \le s_0 \le 1$, $0 \le s_1 \le 3$, $s_2 \ge 0$. Since $q_1 = 1$, we have $t_0 = 0$ for the q-representation of every $N \in \mathbb{Z}_{\ge 1}$. The q- and p-numeration systems for this example are portrayed in Table 3.

Theorem 4. *The set A is identical to the set of numbers whose p-representation ends in an even number of* 0*s. The set B is identical to the set of numbers ending in* 10 *or* 30, *and the set C is identical to the set of numbers ending in* 20.

Proof. Every term in the set A must end in an even number of 0s in the numeration system, as was shown in [Fraenkel 1982, Section 4]. (There the results were proved for the continued fraction of an irrational number, but the same holds for rational numbers.) Therefore every term in B and C must end in an odd number of 0s. Now 4 is the smallest positive number in C, it has representation 20, and every subsequent number in C is larger than its predecessor by 7. Hence all numbers in C have representations of the form $t20$, $t \in \mathbb{Z}_{\ge 0}$. Since A, B, C are complementary, the representations of all numbers in B must end in 10 or 30. $\square$

p-numeration			q-numeration			
7	2	1	4	1	1	n
		1		1	0	1
	1	0		2	0	2
	1	1		3	0	3
	2	0	1	0	0	4
	2	1	1	1	0	5
	3	0	1	2	0	6
1	0	0	1	3	0	7
1	0	1	2	0	0	8
1	1	0	2	1	0	9
1	1	1	2	2	0	10
1	2	0	2	3	0	11
1	2	1	3	0	0	12
1	3	0	3	1	0	13
2	0	0	3	2	0	14
2	0	1	3	3	0	15
2	1	0	4	0	0	16

Table 3. The q- and p-numeration systems for the Rat game.

Theorem 5. *Let $n \in \mathbb{Z}_{\geq 1}$, and let its digits in the q-numeration system be $t_2t_1t_0$. Then*

$$t_1 p_1 + t_2 p_2 = \begin{cases} A_n & \text{if } t_1 = 0, \\ A_n + 1 & \text{if } t_1 > 0. \end{cases}$$

Proof. The proof is similar to one given in [Fraenkel 1982, Section 4], and is therefore omitted. □

Remark. Theorem 5 states that to compute the p-representation of A_n, it suffices to compute the q-representation of n, and then *interpret* it in the p-system (i.e., replace q_i by p_i).

7. The Mouse game

The Mouse game is played on 2 piles of tokens by 2 players who play alternately. Analogously to the Rat game, positions are denoted in the form (x, y), with $0 \leq x \leq y$, and moves in the form $(x, y) \to (u, v)$, where of course also $0 \leq u \leq v$. The player first unable to move — because the position is $(0, 0)$ — loses; the opponent wins. There are 2 types of moves:

(I) Take any positive number of tokens from a single pile.

(II) Take $\ell > 0$ from one of the piles, $k > 0$ from the other, subject to the constraint $|k - \ell| < a$, where

$$a = \begin{cases} 1 & \text{if } y - x \not\equiv 0 \pmod 3, \\ 2 & \text{if } y - x \equiv 0 \pmod 3. \end{cases}$$

We then have:

Theorem 6. *The P-positions of the Mouse game are given by* $(0, 0)$, *and for* $n \geq 1$, *we have*

$$A_n = \text{mex}\{A_i, B_i : 0 \leq i < n\},$$
$$B_n = A_n + \lfloor (3n - 1)/2 \rfloor.$$

The following is an explicit description of the P-positions.

Theorem 7. *The P-positions of the Mouse game are given by* $(0, 0)$, *and for* $n \geq 1$, $A_n = \lfloor 3n/2 \rfloor$, $B_n = 3n - 1$.

We omit the proofs, since they are analogous to and simplified versions of those of Theorems 1 and 2. The first few P-positions (A_n, B_n) together with their differences $d_n = B_n - A_n$ are shown in Table 4.

We leave it to the reader to characterize the P-positions of the Mouse game in terms of an appropriate numeration system.

n	A_n	d_n	B_n
0	0	0	0
1	1	1	2
2	3	2	5
3	4	4	8
4	6	5	11
5	7	7	14
6	9	8	17
7	10	10	20
8	12	11	23
9	13	13	26
10	15	14	29
11	16	16	32
12	18	17	35
13	19	19	38
14	21	20	41
15	22	22	44

Table 4. P-positions of the Mouse game with their differences d_n.

8. The Fat Rat game

The *Fat Rat game* is the case of the Rat game played on an arbitrary number of $m \in \mathbb{Z}_{\geq 2}$ piles. The games for $m \in \{2, 3\}$ were analyzed in the previous sections. The P-positions for $m = 4$ are given in Table 5. This follows the general rule of

$$A_n^k = \lfloor (2^m - 1)n/2^{m-k} \rfloor - 2^{k-1} + 1, \quad k = 1, \ldots, m, \quad n \geq 1.$$

It is not hard to see that for $m = 4$ the differences $d_n^i := A_n^{i+1} - A_n^i$ are

$$d_n^1 = \left\lfloor \frac{15n - 4}{8} \right\rfloor, \quad d_n^2 = \left\lfloor \frac{15n - 6}{4} \right\rfloor, \quad d_n^3 = \left\lfloor \frac{15n - 7}{2} \right\rfloor,$$

and

$$\bigcup_{n=1}^{\infty} \left\{ d_n^1, d_n^2, d_n^3, \{15n\} \right\} = \mathbb{Z}_{\geq 1}.$$

We note that the numeration system for the case $m = 4$ is based on the continued fraction $15/8 = [1, 1, 7]$, and for the Fat Rat game,

$$\frac{2^m - 1}{2^{m-1}} = [1, 1, 2^{m-1} - 1].$$

n	A_n^1	d_n^1	A_n^2	d_n^2	A_n^3	d_n^3	A_n^4	Δ_n
1	1	1	2	2	4	4	8	7
2	3	3	6	6	12	11	23	20
3	5	5	10	9	19	19	38	33
4	7	7	14	13	27	26	53	46
5	9	8	17	17	34	34	68	59
6	11	10	21	21	42	41	83	72
7	13	12	25	24	49	49	98	85
8	15	14	29	28	57	56	113	98
9	16	16	32	32	64	64	128	112
10	18	18	36	36	72	71	143	125
11	20	20	40	39	79	79	158	138
12	22	22	44	43	87	86	173	151
13	24	23	47	47	94	94	188	164
14	26	25	51	51	102	101	203	177
15	28	27	55	54	109	109	218	190
16	30	29	59	58	117	116	233	203

Table 5. P-positions with their differences.

So we have the P-positions. But what are the game rules? Even for the case $m = 4$, there are a priori various possibilities to be checked out. It appears that 4 types of moves are required. Perhaps the case $m = 4$ will point to the game rules for general m. It appears that the transition from 3 to 4 piles is a stumbling block in a number of games. This seems to be the case, for example, for the 3-pile Tribonacci game, based on the *Tribonacci word* [Duchêne and Rigo 2008]. The P-positions of the 3-pile *Raleigh* game [Fraenkel 2007] are, for all $n \in \mathbb{Z}_{\geq 0}$,

$$A_n = \lfloor \lfloor n\varphi \rfloor \varphi \rfloor, \quad B_n = \lfloor n\varphi^2 \rfloor, \quad C_n = \lfloor \lfloor n\varphi^2 \rfloor \varphi \rfloor,$$

where $\varphi = (1 + \sqrt{5})/2$ (golden section). A natural generalization to $m > 3$ piles may also be nontrivial.

Sometimes already the transition from 2 to 3 piles looks difficult. Wythoff's game [1906; Coxeter 1953; Fraenkel 1982; Landman 2002; Duchêne and Gravier 2009] is played on 2 piles. A natural generalization to $m > 2$ piles was suggested in [Fraenkel 1996], and 2 conjectures about the asymptotic structure of their P-positions were given. Their latest form appears in [Guy and Nowakowski 2009]. Some progress on the conjectures was achieved. See [Fraenkel and Krieger 2004; X. Sun and Zeilberger 2004; X. Sun 2005]. For example, the case $m = 3$ was solved, but it is considerably more complicated than $m = 2$.

9. Complexity

We gave three winning strategies for the Rat game: recursive (Theorem 1), algebraic (Theorem 2), and arithmetic (Theorem 4). Given an arbitrary game position $(x, y) \in \mathbb{Z}^2$ of input size $O(\log x + \log y)$, what's the computational complexity of deciding whether or not (x, y) is a P-position? We indicate briefly that all three strategies are efficient.

Theorem 8. *All three winning strategies for the Rat game are polynomial-time.*

Proof. A sequence of positive integers $\{a_n\}_{n \geq 0}$ is *approximately linear*, if there exist constants α, u_1, $u_2 \in \mathbb{R}$ such that $u_1 \leq a_n - n\alpha \leq u_2$ for all $n \in \mathbb{Z}_{\geq 1}$. In [Fraenkel and Peled 2015] it is shown that for approximately linear sequences, the mex function can be computed in polynomial time. Moreover, the sequences $\{a_n\}$ and $\{b_n\}$ are both approximately linear if and only if their difference $\{b_n - a_n\}$ is (Theorem 4 there). Now by Theorem 1, $b_n - a_n = \lfloor (7n - 2)/4 \rfloor$ which is clearly approximately linear, and so is $c_n - b_n = \lfloor (7n - 3)/2 \rfloor$. This implies that the recursive strategy is polynomial. For the algebraic strategy this follows from the discussion following Theorem 2 in [Fraenkel 1982], and for the arithmetic one it follows from the end of that paper. □

It is not hard to see that the same result holds for the Mouse game and the Fat Rat game.

10. Epilogue

In addition to the quest for the analysis of multipile take-away games, there is another motivation for inventing and analyzing the Rat game. The analysis of most games on piles of tokens is associated with irrational numbers. Thus, the generalized Wythoff game $W(a)$ depends on $\alpha = (2 - a + \sqrt{a^2+4})/2$ and $\beta = \alpha + a$, where a is an integer parameter [Fraenkel 1982]. Also games on more piles often depend on irrational numbers, such as the multipile Wythoff game. This is also the case for the Raleigh game.

Here we were interested in investigating whether there is a game whose strategy depends on distinct *rational* numbers. Since the sequences of P-positions of a game split $\mathbb{Z}_{\geq 1}$, this leads naturally to a question that has been solved for the integers, solved for the irrationals, but is wide open for the rationals! This fact may explain, in part, why the move rules for the Rat game are more complicated than those where the strategy depends on irrationals.

Since the game we constructed here depends on rational numbers, it is appropriately dubbed the *Rat* game. The *Mouse* game is a small Rat game with only 2 piles. This is as far as etymology is concerned.

The *m-Fat Rat* game is played on an arbitrary finite number of piles. Let $0 < \alpha_1 \leq \cdots \leq \alpha_m$, $\gamma_1, \ldots, \gamma_m$ be reals, and suppose that the $m \geq 2$ *Beatty sequences*

$$\lfloor n\alpha_i + \gamma_i \rfloor, \quad i = 1, \ldots, m, \tag{2}$$

split $\mathbb{Z}_{\geq 1}$. If the *moduli* α_i are all integers (so $\lfloor n\alpha_i + \gamma_i \rfloor = n\alpha_i + \lfloor \gamma_i \rfloor$) and $m \geq 2$, then $\alpha_m = \alpha_{m-1}$. A short "proof from the Book", due to Mirsky, D. Newman, Davenport and Rado, involving a generating function and a primitive (complex) root of unity proves this. See [Erdős 1952]. First elementary proofs were given in [Berger, Felzenbaum and Fraenkel 1986a; Simpson 1986; Lewis 1996]. The essence of the elementary proof was expounded in [Zeilberger 2001]. For finite splittings of the integers with irrational moduli there is the well-known result that if α, β are positive irrationals satisfying

$$\alpha^{-1} + \beta^{-1} = 1, \tag{3}$$

then the sequences $\lfloor n\alpha \rfloor$ and $\lfloor n\beta \rfloor$ $(n = 1, 2, \ldots)$ split $\mathbb{Z}_{\geq 1}$. For a "proof from the Book", see [Fraenkel 1982]. Thus also $\lfloor 2n\alpha \rfloor$, $\lfloor (2n-1)\alpha \rfloor$, $\lfloor n\beta \rfloor$, where $n = 1, 2, \ldots$, is a splitting. This is a simple example of the *composition* of the integer splitting $2n$, $2n-1$ with an irrational splitting. In fact, the composition of one or two integer moduli splittings with one or both of any irrational moduli α, β respectively satisfying (3) is also a splitting.

So far we have seen that any splitting of the positive integers by m sequences that are multiples of moduli, contains at least two sequences with identical moduli

($m \geq 2$ for integer moduli, $m \geq 3$ for irrational moduli). But a counterexample for the case when the α_k are rational was constructed in [Fraenkel 1973]:

$$\left\lfloor \frac{2^m - 1}{2^{m-k}} n \right\rfloor - 2^{k-1} + 1, \quad k = 1, \ldots, m, \quad n \geq 1,$$

which splits the positive integers for every $m \in \mathbb{Z}_{\geq 2}$. In fact, the following was conjectured there:

Conjecture 1. If $0 < \alpha_1 < \cdots < \alpha_m$ are any real numbers and $m \geq 3$, then the system (2) splits $\mathbb{Z}_{\geq 1}$ if and only if

$$\alpha_k = \frac{2^m - 1}{2^{m-k}} \quad \text{for } k = 1, \ldots, m.$$

In other words, the only disjoint covering system with *distinct* moduli is conjectured to have this form. Some special cases were proved in [Fraenkel 1973]. See also [Erdős and Graham 1980, Section 1] and [Berger, Felzenbaum and Fraenkel 1986b]. The most substantial progress towards settling the conjecture was made by Graham [1973], who showed that if the moduli are irrational and $m \geq 3$, then again $\alpha_i = \alpha_j$ for some $i \neq j$. He did this by proving that any finite irrational splitting of the integers by at least 3 sequences is a composition as stated above, so the result follows from the integer case. However, the remaining rational case appears to be rather stubborn.

Morikawa [1982/83; 1985b] (and in a number of additional papers) investigated splittings of $\mathbb{Z}_{\geq 1}$ by rational Beatty sequences and proved the conjecture for $m = 3$. This case was proved independently in [Tijdeman 1996]. Morikawa [1985a] gave necessary and sufficient conditions for two rational Beatty sequences to be disjoint. Simpson [2004] simplified this proof and dubbed it "Japanese remainder theorem" in honor of Morikawa. He also gave there a generating function method for the splitting of $\mathbb{Z}_{\geq 1}$ by rational Beatty sequences, similar to that of Mirsky, D. Newman, Davenport and Rado for expressing the splitting of $\mathbb{Z}_{\geq 1}$ by integer Beatty sequences. Simpson [1991] proved that the conjecture holds if $\alpha_1 \leq 3/2$. Altman, Gaujal and Hordijk [Altman et al. 2000] proved it for $m = 4$, using the notion of balanced words. Using this method, Tijdeman [2000a; 2000b] established it for $m = 5$ and 6. Using the same method, Barát and Varjú [2003] extended it to $m = 7$. The conjecture was generalized by Graham and O'Bryant [2005] to exact t-fold coverings, and proved several special cases of it using Fourier methods. For further developments see [Vuillon 2003; Paquin and Vuillon 2007].

It is of some interest to note that the conjecture has applications in the theory of scheduling and just-in-time sequencing. See, e.g., [Kubiak 2003; Brauner and Jost 2008; Brauner and Crama 2004]. These applications, in turn, precipitated proofs of special cases of the conjecture.

But despite all these really nice results, the conjecture is still open. Ron Graham summed it all up in a slight rephrasing of Piet Hein's saying:

A problem worthy of attack,
proves its worth by fighting back.

Summarizing, the 3-fold motivation for this work: (i) a games approach might help to settle this conjecture, and (ii) find a take-away game whose winning strategy depends on *rational* numbers, and (iii) find another analyzable nonNim take-away game played on more than 2 piles.

We remark finally that it may be of interest to characterize infinite disjoint covering systems with rational moduli. For the case of integer moduli (arithmetic sequences), see [Barát and Varjú 2005; Z.-W. Sun 2005].

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