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Toppling Conjectures

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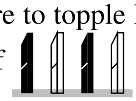
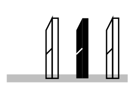
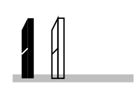
Positions of the game of TOPPLING DOMINOES exhibit many familiar combinatorial game theory values, often arranged in unusual and striking patterns. We show that for any given dyadic rational x, there is a *unique* TOPPLING DOMINOES position G equal to x, and that G is necessarily a palindrome. We also exhibit positions of value $+_x$ for each $x > 0$. We show that for each integer $m \geq 0$, there are exactly m distinct LR-TOPPLING DOMINOES positions of value $*m$ (modulo a trivial symmetry). Lastly, every infinitesimal TOPPLING DOMINOES position has atomic weight 0, 1 or -1.

1. Introduction

TOPPLING DOMINOES, introduced by Albert, Nowakowski and Wolfe [1], is a combinatorial game played with a row of dominoes, such as the one pictured in Figure 1. Here each domino is colored blue or red (black or white, respectively, when color printing is unavailable). On his turn, Left selects any bLue (black) domino and topples it either east or west (his choice). This removes the toppled domino from the game, together with all other dominoes in the chosen direction. Likewise, Right's options are to topple Red (white) dominoes east or west. For example, the Left options of  are

$A =$ , $B =$, $C =$ , $D =$ .

Here A and B result from toppling the westmost domino respectively west or

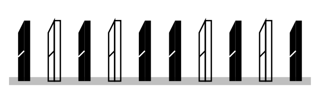


Figure 1. A typical TOPPLING DOMINOES position.

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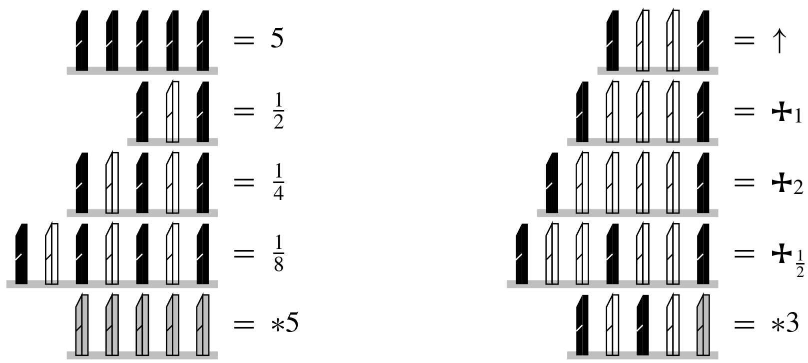


Figure 2. Some familiar combinatorial game values that arise in TOPPLING DOMINOES.

east, while C and D result from toppling the eastern black domino respectively west or east.

Positions in TOPPLING DOMINOES might also include grEen (gray) dominoes, which Either player may topple. Positions consisting entirely of black and white dominoes, such as the example above, are sometimes called LR-TOPPLING DOMINOES positions.

The reader might wish to verify some of the sample values in Figure 2. (Exercise: what is the value of Figure 1?)

The principal results of this paper are the following four theorems.

- For each number x, there is exactly one TOPPLING DOMINOES position G equal to x. Furthermore, G is necessarily a palindrome. The proof involves constructing a new LR-TOPPLING DOMINOES position $f(G)$ with value $1{:}G$ and applying this construction recursively.
- For each number $x > 0$, there exists a TOPPLING DOMINOES position G equal to $⧾_x$. It is not known whether G is unique.
- For each integer $m \geq 0$, there are exactly m distinct LR-TOPPLING DOMINOES positions of value $*m$ (modulo east/west symmetry).
- Every infinitesimal TOPPLING DOMINOES position has atomic weight 0, 1 or -1.

In Section 2, we introduce some basic facts and strategies for TOPPLING DOMINOES. The three main results are proved in Sections 3, 4 and 5, as Theorems 16, 21 and 29, respectively. Finally, in Section 6 we note that every infinitesimal TOPPLING DOMINOES position necessarily has atomic weight 0, 1 or -1.

We adopt the following notational conventions:

L	a single Left (black) domino
R	a single Right (white) domino
E	a single domino for Either (gray)
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	an arbitrary string of dominoes

Familiarity with combinatorial game theory is assumed; the necessary background may be found in a standard reference such as [1] or [2].

2. Preliminaries

The following easy result is very useful.

Lemma 1. *A* TOPPLING DOMINOES *position's outcome class is completely determined by its end dominoes*:

- *The empty string has value* 0 (*and is the unique position of value* 0);
- $L > 0$, *and* $L\boldsymbol{x}L > 0$ *for any string* $\boldsymbol{x}$;
- $R > 0$, *and* $R\boldsymbol{x}R > 0$ *for any string* $\boldsymbol{x}$;
- $L\boldsymbol{x}R \ngtrless 0$ *and* $R\boldsymbol{x}L \ngtrless 0$ *for any string* $\boldsymbol{x}$.

Proof. From $L\boldsymbol{x}L$, any opening move for Right annihilates at most one of the two endpoints, and hence reverts to 0. Finally, from $L\boldsymbol{x}R$ (or $R\boldsymbol{x}L$) both players have moves to 0. □

The next results shed light on some simple winning strategies.

Lemma 2. *For any string* $\boldsymbol{x}$, *every Left move from* $L\boldsymbol{x}L$ *to some* $L\boldsymbol{y}R$ *or* $L\boldsymbol{y}E$ *is reversible through* 0.

Proof. In particular, $L\boldsymbol{x}L > 0$, and so Left's move to $L\boldsymbol{y}R$ or $L\boldsymbol{y}E$ reverses through Right's response toppling the remaining dominoes. □

Lemma 3 (sandwich lemma). *Let G be a* TOPPLING DOMINOES *position*, *and suppose G^L is obtained from G by toppling east* (*resp. west*). *Suppose that G^{LR} is obtained from G^L by toppling in the same direction. Then $G^{LR} \mathrel{|\!\!\rhd} G$.*

Proof. Assume (by symmetry) both moves toppled east. Write $\boldsymbol{x} = G^{LR}$, so that $G^L = \boldsymbol{x}\boldsymbol{y}$ for some $\boldsymbol{y}$ and $G = \boldsymbol{x}\boldsymbol{y}\boldsymbol{z}$ for some $\boldsymbol{z}$. Note that $\boldsymbol{x} = G^{LR}$ is available directly as a Right option of G, so that $G^{LR} = G^R \mathrel{|\!\!\rhd} G$. □

The importance of Lemma 3 is that in any difference $G - H = 0$, if Left moves to any $G^L - H$, then Right's winning response cannot be to topple in the same direction in G.

Lemma 4. *For any* $\boldsymbol{x}$, $L\boldsymbol{x} > \boldsymbol{x}$.

Proof. In $L\boldsymbol{x}-\boldsymbol{x}$ Left can win by moving to $\boldsymbol{x}-\boldsymbol{x}$. When Right moves first, Left plays the corresponding move in the other component leaving either $\boldsymbol{y}-\boldsymbol{y}=0$ or $L\boldsymbol{y}-\boldsymbol{y}$ which, by induction, is positive. □

Corollary 5. *Within any block of adjacent blue dominoes, the two moves that topple an end domino away from the block dominate all other moves within the block.*

3. Uniqueness of numbers

In this section, we show that for every number x there is a unique TOPPLING DOMINOES position $G = x$. The *ordinal sum* defined in [2; 3] plays a central role in this analysis. We recall the definition here.

Definition 6. Let G and H be combinatorial games. The *ordinal sum* of G and H, denoted $G{:}H$, is defined recusively by

$$G{:}H = \left\{G^L,\, G{:}H^L \;\middle|\; G^R,\, G{:}H^R\right\}.$$

Intuitively, we may think of the ordinal sum as a modified disjunctive sum in which *any move on G annihilates the entire component H*.

The special case $1{:}G = \left\{0,\, 1{:}G^L \;\middle|\; 1{:}G^R\right\}$ has particular importance in our analysis, owing to the following classical theorem.

Fact 7 (Berlekamp–Conway–Guy). *Let $x > 0$ be a number. Then $x = 1{:}y$ for some simpler number y.*

Corollary 8. *The set of numbers is generated from* $\{0\}$ *by the transformations* $x \mapsto -x$, $x \mapsto 1{:}x$.

With these facts in hand, the outline of our argument becomes clear. We will associate to each TOPPLING DOMINOES position G a new position $f(G)$ of value $1{:}G$. Since there is a trivial transformation $G \mapsto -G$ (replace blue dominoes with red ones and vice versa), we immediately obtain all numbers. We then show that *every* positive number G is necessarily of the form $f(H)$ for some H, after which a simple inductive argument establishes uniqueness.

For completeness, we include a proof of Fact 7 here. The theorem was originally proved in the context of BLUE-RED HACKENBUSH strings, where it was used to show that every such string encodes a unique number [2].

Proof of Fact 7. It is easily seen that $1{:}n = n+1$ and $1{:}-n = 2^{-n}$, both of which satisfy the simplicity conclusion, so assume $x = \left\{x^L \;\middle|\; x^R\right\}$ in simplest form, with $x^R > x^L > 0$. By induction, $x^L = 1{:}y^L$ and $x^R = 1{:}y^R$ for some y^L, y^R. Put $y = \left\{y^L \;\middle|\; y^R\right\}$. Then

$$1{:}y = \left\{0,\, 1{:}y^L \;\middle|\; 1{:}y^R\right\} = \left\{0,\, x^L \;\middle|\; x^R\right\} = x,$$

since 0 is dominated by x^L. By induction we may assume that y^L and y^R are simpler than x^L and x^R, respectively, whence y is simpler than x. □

With this machinery in hand, we now introduce our transformation f.

Definition 9. Let f be the morphism of strings such that, for a TOPPLING DOMINOES string $\boldsymbol{x}$, $f(\boldsymbol{x})$ is obtained by inserting one additional L in every maximal substring of Ls in $\boldsymbol{x}$, including empty substrings. For example, $f(RRLLRL) = LRLRLLLRLL$.

Equivalently, f can be described by the following transformation, where $\wedge$ denotes the "start of string" marker for $\boldsymbol{x}$:

$$\begin{array}{lll} \wedge \to \wedge L & \qquad & R \to RL \\ L \to L & \qquad & E \to EL \end{array}$$

Theorem 10. *For any* TOPPLING DOMINOES *position* G, *we have* $f(G) = 1{:}G$.

Proof. Let $H = \{0, f(G^L) \mid f(G^R)\}$. It suffices to show that $f(G) = H$, for then the theorem follows by induction.

Step 1: We first show that every Left (resp. Right) option of H is a Left (resp. Right) option of $f(G)$. Consider a typical G^R. We may write $G = \boldsymbol{x} R \boldsymbol{y}$ (or $G = \boldsymbol{x} E \boldsymbol{y}$) with $G^R = \boldsymbol{x}$ or $\boldsymbol{y}$. By definition of f, we have $f(G) = f(\boldsymbol{x})Rf(\boldsymbol{y})$ (or $f(\boldsymbol{x})Ef(\boldsymbol{y})$), so $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ are necessarily Right options of $f(G)$.

Likewise, for any G^L we may write $G = \boldsymbol{x} L \boldsymbol{y}$ (or $G = \boldsymbol{x} E \boldsymbol{y}$) with $G^L = \boldsymbol{x}$ or $\boldsymbol{y}$. If $G = \boldsymbol{x} E \boldsymbol{y}$, then the argument is identical to the G^R case. If $G = \boldsymbol{x} L \boldsymbol{y}$, then $f(G) = f(\boldsymbol{x}) f(\boldsymbol{y})$, so $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ are necessarily Left options of $f(G)$ (note that both $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ have blue dominoes on both ends).

Finally, $f(G)$ has a move to 0, since it necessarily has a blue domino on the west end.

Step 2: To complete the proof, we show that for every option of T of $f(G)$, *either* T is an option of H, *or* T reverses through 0.

First of all, inspection of Definition 9 reveals that *every* Right option of $f(G)$ is obtained as in Step 1, since every red (resp. green) domino in $f(G)$ arises as the image of a red (resp. green) domino in G.

However, there are additional Left options $T = f(G)^L$ that are not of the form 0 or $f(G^L)$. In particular, they are those that expose a red or green domino, but these moves reverse out through Right's moves to 0, since $0 < f(G)$. □

Corollary 11. *For every number* x, *there is a* TOPPLING DOMINOES *position* $G = x$.

Proof. Theorem 10 yields the mapping $G \mapsto 1{:}G$, so this result follows from Corollary 8. □

We now prove uniqueness. A few more lemmas are needed.

Lemma 12. *Let G be an* LR-TOPPLING DOMINOES *position with $G > 0$. If G does not contain RR as a substring, then there exists a* TOPPLING DOMINOES *position H such that $G = f(H)$.*

Proof. H is obtained by removing a single L from every maximal substring of Ls in G. □

Lemma 13. $1{:}H \geq 1{:}K$ *if and only if $H \geq K$.*

Proof. When playing $1{:}H - 1{:}K$, a move on the base 1 taking a component to 0 can never be winning unless the other component is already 0. Hence, winning play on $1{:}H - 1{:}K$ is tantamount to play on $H - K$ until the last two moves. □

Lemma 14. *If a* TOPPLING DOMINOES *position G has consecutive non-blue dominoes, then G is not a number.*

Proof. Suppose (for contradiction) that $G = L\boldsymbol{x}AB\boldsymbol{y}L$ is a number where A and B are non-blue dominoes. Consider Right's move to $G^R = L\boldsymbol{x}A$. Since $G^R \mathrel{|\!\!\triangleright} G$, Left must have a winning move on $G^R - G$. Since $-G$ is a number and G^R is not, Left must have a winning move on the G^R component. By Lemma 3, this move must be to topple *west* on $L\boldsymbol{x}A$, to a position of the form $\boldsymbol{z}A - G$. But then Right has a response to $-G < 0$, a contradiction. □

Lemma 15. *Let G be a* TOPPLING DOMINOES *position. If G is a number and $G > 0$, then $G \cong f(H)$ for some H.*

Proof. It follows from the lemma that G is in the image of f. □

Theorem 16. *Let G and G' be* TOPPLING DOMINOES *positions, and suppose that $G = G' = x$, a number. Then $G \cong G'$.*

Proof. For $x = 0$ this is just Lemma 1. We proceed by induction on x.

First suppose $x > 0$. By the previous lemma, we have $G = f(H)$ and $G' = f(H')$ for some H, H', so by Theorem 10, $G = 1{:}H$ and $G' = 1{:}H'$. Now we know that $x = 1{:}y$ for some number y simpler than x. By Lemma 13, we have $H = H' = y$, so by induction $H \cong H'$. Therefore $G \cong G'$.

The case $x < 0$ is symmetric. □

Corollary 17. *All* TOPPLING DOMINOES *positions that are numbers are palindromes.*

Proof. If G' is the reversal of G, then certainly $G' = G$, so by Theorem 16 we have $G' \cong G$. □

4. Tinies

The main result of this section is the construction of $⧾_x$ and $⧿_x$. We present the main theorem here, but defer its proof until the end of the section.

Theorem 18. *Let $\boldsymbol{x}$ be the* TOPPLING DOMINOES *position of value x, where $x \geq 0$ is a number. The position $RL\boldsymbol{x}LR$ has value $⧿_x$.*

The proof of the theorem naturally entails showing that the games $\boldsymbol{x}LR + \{0 \mid -x\}$ and $RL\boldsymbol{x}LR + ⧾_x$ are second player wins. In the arguments, Left sometimes moves twice in the TOPPLING DOMINOES component to some $\boldsymbol{x}^{LL}$, and we will need to show this is bad for Left:

Lemma 19. *Let $\boldsymbol{x}$ be a* TOPPLING DOMINOES *number (possibly negative). Let $\boldsymbol{x}'$ be a position reachable from $\boldsymbol{x}$ by one or more Left moves. Then $\boldsymbol{x}' < \boldsymbol{x}$.*

Proof. Recall that $\boldsymbol{x}$ consists only of L and R. Furthermore, if $\boldsymbol{x} > 0$, Lemma 14 tells us $\boldsymbol{x}$ never has consecutive R's and begins and ends in an L.

As usual, consider the difference game $\boldsymbol{x} - \boldsymbol{x}'$, and consider the case when $\boldsymbol{x} > 0$. If $\boldsymbol{x}'$ begins and/or ends in R, Left wins: Left's primary plan is to first move $-\boldsymbol{x}'$ to 0, and then move $\boldsymbol{x}$ to 0. Right cannot thwart this plan except possibly by making a first move which eliminates the only L end of $-\boldsymbol{x}'$, but Left has a good response to this opening by the next case.

In case $\boldsymbol{x}'$ begins and ends in L, then $\boldsymbol{x}'$ is in the image of f (Definition 9), and we can rewrite $\boldsymbol{x} = 1{:}\boldsymbol{y}$ and $\boldsymbol{x}' = 1{:}\boldsymbol{y}'$. Furthermore, since $\boldsymbol{x}'$ begins and ends in an L, observe that $\boldsymbol{y}'$ is reachable from $\boldsymbol{x}'$ by Left moves. So, by induction and Lemma 13, $\boldsymbol{y}' < \boldsymbol{y}$ and we have $\boldsymbol{x}' < \boldsymbol{x}$.

Finally, if $\boldsymbol{x} < 0$, $\boldsymbol{x}'$ must begin and end in R, and we can rewrite $\boldsymbol{x} - \boldsymbol{x}'$ as $(-1{:}\boldsymbol{y}) - (-1{:}\boldsymbol{y}')$ and use induction as above. □

Lemma 20. *If $\boldsymbol{x}$ is a* TOPPLING DOMINOES *number $x > 0$, then*

$$\boldsymbol{x}LR = \{x \mid 0\}.$$

Proof. Consider play on

$$\boldsymbol{x}LR + \{0 \mid -x\}.$$

If the first player moves in one component, the second player almost always moves in the other component; the only exception is when Right topples west in the first component wherein Left topples the rightmost LR, leaving $\boldsymbol{x}^R + \{0 \mid -x\}$. Right's move to $\boldsymbol{x}^R - x$ loses by Lemma 19 while when Right topples, Left can move to $\boldsymbol{x}^{RR} + 0 > 0$ since $\boldsymbol{x}^{RR}$ ends in Ls.

When the second player does respond to the first player's move in the other component, typically this leaves either $0 + 0$ or $\boldsymbol{x} - x$ to win. However, when Right is the first player and topples east, Left leaves games of the form $\boldsymbol{x}^R + 0$ where $\boldsymbol{x}^R > 0$ (since $\boldsymbol{x}$ never has consecutive Rs). When Left makes an initial move toppling a domino east, Right leaves $\boldsymbol{x}^L - x < 0$ by Lemma 19. When Left makes an initial move toppling west, Right again leaves $\boldsymbol{x}^R LR - x$ or $R - x$. The latter is clearly negative, while in the former Left's only hope is to topple east, but that leaves $\boldsymbol{x}^{RR} - x < 0$ by Lemma 19. □

Theorem 21. *If $\boldsymbol{x}$ is a positive* TOPPLING DOMINOES *number $x > 0$, then*

$$RL\boldsymbol{x}LR = \text{-}_x = \{x \mid 0 \| 0\}.$$

Proof. Consider play on

$$RL\boldsymbol{x}LR + \text{+}_x = RL\boldsymbol{x}LR + \{0 \| 0 \mid -x\}.$$

All moves outside those in $\boldsymbol{x}$ have natural winning responses. When Right moves in $\boldsymbol{x}$ or topples an end R only, Left immediately responds to $0 + \text{+}_x > 0$. The remaining case is Left's move to some $\boldsymbol{x}^L LR + \text{+}_x$ whence Right can play to

$$\boldsymbol{x}^L LR + \{0 \mid -x\}.$$

If Left now plays the second component to 0, Right can move the first component to 0. If, on the other hand, Left plays in the first component, Right replies in the second component leaving either $\boldsymbol{x}^{LL} - x$ which Right wins by Lemma 19, or $R - x$ which Right wins trivially. □

Conjecture 22. *$RL\boldsymbol{x}LR$ is the* unique TOPPLING DOMINOES *position of value -_x, where $\boldsymbol{x} > 0$ is a number.*

This has been confirmed for TOPPLING DOMINOES strings of length up to 15.

5. Nimbers

In the sequel, we use $\boldsymbol{x}^a$ to mean the block of dominoes represented by $\boldsymbol{x}$ repeated a times.

Unlike dyadic rationals, nimbers have more than one representation; in LR-TOPPLING DOMINOES, $*m$ has m representations, or $2m$ counting reversals. The shortest representation is given by the following lemma.

Lemma 23. *For any non-negative integer m, $(LR)^m = *m$.*

Proof. The right followers of the form $(LR)^k L$ are positive and therefore dominated, while the ones of the form $(LR)^k$ are, by induction, $*k$. □

One position recurs as a Right follower in the arguments to come, and so we highlight it as a losing move for Right:

Lemma 24. *Let $\boldsymbol{x}$ be a* TOPPLING DOMINOES *position. Then $L\boldsymbol{x}L + *m \mathrel{|\!\!\triangleright} 0$.*

Proof. The sum $L\boldsymbol{x}L + *m$ is a positive game added to a game equal to or fuzzy with 0. □

The main theorem of the section is that the following lemma describes the *only* forms of $*m$, and the proof of this fact will take the bulk of this section.

Lemma 25. *The position* $(LR)^a(RL)^m(LR)^a = *m$, *for* $0 \le a < m$.

Proof. The proof is routine. First, let $G = (LR)^a(RL)^m(LR)^a$ with $a < m$. We show that $G + *m = 0$. By the symmetry of G we need only consider Right going first.

If Right plays in G, all moves, except one, which topple east leave both ends L and lose by Lemma 24. The exception is the move to $(LR)^a + *m$ whence Left responds to $(LR)^a + *a = 0$.

When Right topples west, Left mirrors moves in the initial block $(LR)^a$ by moving symmetrically in the final block. Right's moves in the final block leave $(LR)^{a'} + *m$ for $a' < m$ and lose. Lastly, if Right topples west from the middle block $(RL)^m$, Right can remove the latter block leaving a position of the form $(LR)^{m'}L + *m = 1/2^{m'} + *m > 0$.

If Right plays in the second component of $G + *m$ to some $G + *m'$, there are two cases. If $m' \le a$, Left plays to $(LR)^{m'} + *m' = 0$. If $m' > a$, Left plays to $(LR)^a(RL)^m + *m'$. From here, each of Right's plays is easily seen as losing, for Right cannot leave a position of the form $LxL + *k$, yet moves to $(LR)^a + *m' = *a + *m'$ and to $(RL)^m + *m' = *m + *m'$ also lose. □

For the remainder of this section, let $G = *m$ be an LR-TOPPLING DOMINOES position. By Lemma 1, G cannot have Ls at both ends and so by symmetry we may assume $G = L\boldsymbol{x}R$.

Lemma 26. $G = *m$ *contains no substring of the form* $LL(RL)^nL$.

Proof. Say $G = \boldsymbol{x}LL(RL)^nL\boldsymbol{y}$. From $G + *m$ Left can move to $\boldsymbol{x}LL(RL)^n + *m$. By Lemma 3 if Right has a winning move in $\boldsymbol{x}LL(RL)^n$, it must be to topple a domino west. In these cases, then Left either has a move to or is left with a position of the form $L(RL)^j + *m$ for some j, which is $1/2^j + *m > 0$. If Right moves on the $*m$ then Left moves to $L(RL)^n + *k > 0$. □

The last lemma, and the symmetric one with L and R interchanged, is used to prove:

Lemma 27. $G = *m$ *is of the form* $(LR)^{a_1}(RL)^{a_2}(LR)^{a_3} \cdots (LR)^{a_n}$ (*or its reverse*) *for some* $n \ge 1$ *odd and* $a_1, \dots, a_n \ge 0$.

Proof. By the previous lemma, G has no two instances of RR which aren't separated by an LL, and in particular no RRR (and the same holds with the roles of L and R reversed). Yet G cannot begin $(LR)^aLL\dots$ (nor end $\dots RR(LR)^a$) for Left's move to $(LR)^aL = 1/2^a$ is too strong. So if there are any pairs of consecutive like dominoes, these must alternate between RR and LL, beginning with RR and ending with LL. The form claimed in the lemma is the only possibility remaining. □

Lemma 28. *$G = *m$ is of the form $(LR)^a(RL)^m(LR)^a$ (or its reverse) for $a < m$.*

Proof. The previous lemma has G of one of the forms

$$(LR)^a \quad \text{or} \quad (LR)^a(RL)^b(LR)^c\boldsymbol{x}$$

for some string $\boldsymbol{x}$ either empty or starting in R. The first form is fine; in the second we must show that $\boldsymbol{x} = 0$, that $a = c$ and that $b > a$.

Since Left has moves from G to $0, *, \dots, *(a-1)$, and Right has a move to $*a$, we have $m > a$. Moreover, Left has a move to $(LR)^a(RL)^b$, from which Right must respond to either $(LR)^a$ or $(RL)^b$ (since Right will lose if she leaves a double-Left-ended string). Since $m > a$, Right's move to $(LR)^a + *m$ is losing, so necessarily $(RL)^b + *m <= 0$. Therefore $b = m > a$.

Next consider c. We can't have $c > a$ since then Left would have a move from G to $(LR)^a(RL)^m(LR)^a$, which we know is equal to $*m$. Suppose instead $c < a$. Then Right can move from G to $(LR)^c(RL)^m(LR)^c\boldsymbol{x}$. If $\boldsymbol{x} = 0$ this wins immediately. Otherwise, consider Left's responses. Toppling to the west must lose, since all such moves are available to Left directly from $G + *m$. If Left topples east from within $\boldsymbol{x}$, then Right can revert to $(LR)^c(RL)^m(LR)^c$ which is equal to $*m$. Finally, if Left topples east annihilating $\boldsymbol{x}$, then Left's move must a priori be losing, since it's also available from $(LR)^c(RL)^m(LR)^c + *m$, which is equal to 0. Therefore Right's move from G to $(LR)^c(RL)^m(LR)^c\boldsymbol{x} + *m$ was a winning move, a contradiction.

This shows that $c = a$. Finally, this implies $\boldsymbol{x} = 0$, since otherwise Right could win by simply annihilating $\boldsymbol{x}$. □

Theorem 29. *The* LR-TOPPLING DOMINOES *position $G = *m$ if and only if it is of the form $(LR)^a(RL)^m(LR)^a$ for $0 \le a < m$.*

Proof. The last lemma combined with Lemma 25 proves the theorem. □

6. Atomic weights

In [1, p. 202], it is proved that any TOPPLING DOMINOES position with two gray ends is an infinitesimal with atomic weight 0, 1, or -1. In this section we generalize the result to include all TOPPLING DOMINOES infinitesimals, including those that end in L or R.

Lemma 30. *If G is an infinitesimal* TOPPLING DOMINOES *position beginning in L, then it must be of the form $(LR)^n$ or $(LR)^nE\boldsymbol{x}$ or $(LR)^nR\boldsymbol{x}$.*

Proof. If G is not of one of the above forms, fix G and rewrite it as $(LR)^m L\boldsymbol{y}$ where m is maximal. The string $\boldsymbol{y}$ cannot begin in R, for then G would either be one of the above forms or m would not be maximal. So $\boldsymbol{y}$ is either empty, in which case G equals some $(LR)^m L = 1/2^n$, or Left can move to $(LR)^m L$, contradicting that G is infinitesimal. □

Lemma 31. *If G is an infinitesimal* TOPPLING DOMINOES *position, then the first player has a move to a nimber. Furthermore, if the first player moves to any non-nimber, the second player can respond to a nimber.*

Proof. The proof is clear from the forms of G given by the last lemma. We first observe that the first player can move to a nimber. If G begins with an L, Left can move to 0, while Right can move to some $(LR)^n = {*n}$. If G begins with an E, either player can move to 0.

We now show that if the first player moves to a non-nimber, the second player can respond to a nimber. Without loss of generality, the first player topples east. When G begins with an E, the second player has a move to 0, and when G begins with an L, Left moving second has a move to 0. In the remaining case, G begins in L, and Left moved first toppling east. If Left's move was to some initial $(LR)^m$, then that already equals ${*m}$. If, on the other hand, the move is to some $(LR)^n E\boldsymbol{x}'$ or $(LR)^n R\boldsymbol{x}'$, then Right can move to $(LR)^n = {*n}$. □

Theorem 32. *All infinitesimal* TOPPLING DOMINOES *positions have atomic weight* -1, 0, *or* 1.

Proof. The proof is identical to that found in [1, pp. 202–203], and so we only provide a brief sketch here. Let G be an infinitesimal. The last lemma shows that G is of the form

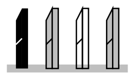
$$\Big\{\mathbf{0}, \{?\mid \mathbf{0}\}, \ldots \;\Big|\; \mathbf{0}, \{\mathbf{0}\mid ?\}, \ldots\Big\}$$

where the $\mathbf{0}$s are nimbers, and therefore have atomic weight 0. If each ? were all-small, then the rule for recursively computing atomic weights given by [1, p. 199] or [2] would yield at atomic weight of -1, 0, or 1. But if the ? are not all-small, we can replace each positive number stop in G with ⤊, and each negative number with ⤋, arriving at a new game G'. Observe that $G + H$ has the same outcome as $G' + H$ for $H \in \{⇑☆, ↑☆, ☆, ↓☆, ⇓☆\}$, and so G has the same atomic weight as G' in $\{-1, 0, 1\}$. □

7. Conjectures

We end with a few conjectures. This first one would prove that green (gray) dominoes can introduce new values:

Conjecture 33. *The value*

 $= \{1 \mid 0,\ \{0 \mid -1\}\}$

does not appear in LR-TOPPLING DOMINOES.

Conjecture 34. *For all* TOPPLING DOMINOES *positions* $\boldsymbol{x}$ *and* $\boldsymbol{y}$, $\boldsymbol{x}L\boldsymbol{y} > \boldsymbol{x}\boldsymbol{y}$. (*Proving* $\boldsymbol{x}L\boldsymbol{y} \geq \boldsymbol{x}\boldsymbol{y}$ *is trivial.*)

Conjecture 35. *In* LR-TOPPLING DOMINOES, *if G is a palindrome then G's value appears uniquely.*

Next, here are a few conjectures that generalize the construction of $+_x$. In what follows, $a \geq b \geq c \geq d$ are numbers (as are x, x^L, and x^R), and are the toppling dominoes positions given by the construction of Section 3.

Theorem 36. *If* $x = \{x^L \mid x^R\}$ *is a number in canonical form*, x *is given uniquely by the* TOPPLING DOMINOES *position* $x^L L R x^R$.

Conjecture 37. *The game* $\{a \mid b\}$ *is given* (*uniquely*) *by the* TOPPLING DOMINOES *position* $aLRb$.

Conjecture 38. *The game* $\{a \parallel b \mid c\}$ *is given* (*uniquely*) *by the* TOPPLING DOMINOES *position* $aLRcRLb$.

Conjecture 39. *The game* $\{a \mid b \parallel c \mid d\}$ *is given* (*uniquely*) *by the* TOPPLING DOMINOES *position* $bRLaLRdRLc$.

Conjecture 40. *Some games* do not *occur in* TOPPLING DOMINOES. (*By way of comparison*, HACKENBUSH *contains no hot games.*)

References

[1] M. Albert, R. J. Nowakowski, and D. Wolfe. *Lessons in play: An introduction to combinatorial game theory*. A K Peters, Wellesley, MA, 2007.

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