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Deformation of expressions for elements of an algebra

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1. Introduction

We introduce a notion of deformation of expressions for elements of an algebra. Deformation quantization [Bayen et al. 1978a; 1978b] deforms the commutative world into a noncommutative world. In contrast, our formalism involves the deformation of expressions of elements of algebras from one commutative world into another.

2. Definition of $*$-functions and intertwiners

Let $\mathbb{C}[w]$ be the space of polynomials in one variable w. For a complex parameter τ, we define a new product on this space:

$$f *_\tau g = \sum_{k\geq 0} \frac{\tau^k}{2^k k!} \partial_w^k f \partial_w^k g \quad \big(= f e^{\frac{\tau}{2}\overleftarrow{\partial_w}\overrightarrow{\partial_w}} g\big). \tag{2-1}$$

We see easily that $*_\tau$ makes $\mathbb{C}[w]$ into a commutative and associative algebra, which we denote by $(\mathbb{C}[w], *_\tau)$. If $\tau{=}0$, then $(\mathbb{C}[w], *_0)$ is the usual polynomial algebra, and $\tau \in \mathbb{C}$ is called a deformation parameter. What is deformed is not the algebraic structure, but the expression of elements.

Intertwiners and infinitesimal intertwiners. It is not hard to verify that the mapping

$$e^{\frac{\tau}{4}\partial_w^2} : (\mathbb{C}[w], *_0) \to (\mathbb{C}[w], *_\tau) \tag{2-2}$$

gives an algebra isomorphism: $e^{\frac{\tau}{4}\partial_w^2}$ has the inverse $e^{-\frac{\tau}{4}\partial_w^2}$ and we have

$$e^{\frac{\tau}{4}\partial_w^2}(f *_0 g) {=} (e^{\frac{\tau}{4}\partial_w^2} f) *_\tau (e^{\frac{\tau}{4}\partial_w^2} g).$$

The isomorphism $I_0^\tau {=} e^{\frac{\tau}{4}\partial_w^2}$ is called the *intertwiner*. Defining $I_\tau^{\tau'} = I_0^{\tau'}(I_0^\tau)^{-1}$

gives the intertwiner from $(\mathbb{C}[w], *_\tau)$ onto $(\mathbb{C}[w], *_{\tau'})$. Its differential

$$dI_\tau = I_\tau^{\tau+d\tau} = \frac{d}{d\tau'} I_\tau^{\tau'}\Big|_{\tau'=\tau} = \tfrac{1}{4}\partial_w^2$$

is called the *infinitesimal intertwiner*.

Defining $w_{*\tau}^n$ by $I_0^\tau w^n$ we get

$$w_{*\tau}^n = P_n(w,\tau) = \sum_{k\le[n/2]} \frac{n!}{4^k k!\,(n-2k)!}\tau^k w^{n-2k}. \tag{2-3}$$

Let $\mathrm{Hol}(\mathbb{C})$ be the space of all entire functions on $\mathbb{C}$ with the topology of uniform convergence on each compact domain. $\mathrm{Hol}(\mathbb{C})$ is known to be a Fréchet space defined by a countable family of seminorms. It is easy to see that the product $*_\tau$ extends naturally to $f, g \in \mathrm{Hol}(\mathbb{C})$ if either f or g is a polynomial. By the inductive limit topology $\mathbb{C}[w]$ is a complete topological algebra with uncountable basis of neighborhoods of 0. We easily see the following:

Theorem 2.1. *For a polynomial $p(w)$, the multiplication $p(w)*_\tau$ is a continuous linear mapping of* $\mathrm{Hol}(\mathbb{C})$ *into itself. By polynomial approximations, the associativity $f*_\tau(g*_\tau h) = (f*_\tau g)*_\tau h$ holds if two of f, g, h are polynomials.* $\mathrm{Hol}(\mathbb{C})$ *is a topological* $\mathbb{C}[w]$ *bimodule.*

Star-exponential functions and τ-expressions. We now study the deformation of the exponential function e^{aw}. Although the ordinary exponential function e^{aw} is not a polynomial, the intertwiner I_0^τ given by (2-2) extends to give

$$I_0^\tau(e^{2aw}) = e^{2aw+a^2\tau} = e^{a^2\tau}e^{2aw}, \quad \tau\in\mathbb{C}. \tag{2-4}$$

Using Taylor expansion, we get

$$e^{2aw}*_\tau e^{2bw} = e^{2(a+b)w+2ab\tau}, \quad e^{2aw}*_\tau f(w) = e^{2aw} f(w+a\tau), \tag{2-5}$$

for every $f\in\mathrm{Hol}(\mathbb{C})$. We have also associativity,

$$e^{2aw}*_\tau(e^{2bw}*_\tau f(w)) = (e^{2aw}*_\tau e^{2bw})*_\tau f(w),$$

for every $f\in\mathrm{Hol}(\mathbb{C})$. Computation via intertwiners gives

$$I_\tau^{\tau'}(e^{\frac14 s^2\tau}e^{sw}) = e^{\frac14 s^2\tau'}e^{sw}.$$

We denote by e_*^{sw} the family $\{e^{\frac14 s^2\tau}e^{sw}; \tau\in\mathbb{C}\}$ and call this the $*$-*exponential function*.

Associated with polynomials and exponential functions $f(w)$, we construct a family of functions

$$\{f_\tau(w); \tau\in\mathbb{C}\}, \quad f_\tau = I_0^\tau(f(w)), \tag{2-6}$$

which is denoted by $f_*(w)$. We view $f_*(w)$ as an *element* of the abstract algebra. Given f we refer to the object (2-6) as a $*$-*function*. Using the notation $:\cdot:_\tau$ we write

$$:f_*(w):_\tau = f_\tau(w).$$

We view $:f_*:_\tau$ as the τ-*expression* of f_*. Then we have $:e_*^{sw}:_\tau = e^{\frac{1}{4}s^2\tau}\, e^{sw}$. We call the right-hand side the τ-*expression* of e_*^{sw}. The product formula (2-1) gives the exponential law

$$:e_*^{sw}:_\tau *_\tau :e_*^{tw}:_\tau = :e_*^{(s+t)w}:_\tau, \quad \text{for all } \tau \in \mathbb{C}. \tag{2-7}$$

Note that $:e_*^{tw}:_\tau$ is the solution for every τ of the differential equation $dg(t)/dt = w*_\tau g(t)$ with the initial condition $g(0) = 1$. It is easy to see the exponential law $e_*^{tw}e^s = e_*^{tw+s}$ holds for the ordinary exponential function e^s. The formula $:e_*^{tw}:_\tau = \sum_n t^n/n!:w_*^n:_\tau$ also holds.

For every $f \in \mathrm{Hol}(\mathbb{C})$, the formula (2-5) gives

$$:e_*^{2sw}:_\tau *_\tau f(w) = e^{2sw+s^2\tau} f(w + s\tau). \tag{2-8}$$

Using this, we have several basic properties of $*$-exponential functions:

Proposition 2.1. *Associativity holds in every τ-expression*:

$$e_*^{rw}*(e_*^{sw}*f) = e_*^{(r+s)w}*f = e_*^{rw}*(f*e_*^{sw}).$$

If $f(w) \in \mathrm{Hol}(\mathbb{C})$ satisfies $:e_*^{isw}:_\tau *_\tau f(w) = 0$, then $f(w) = 0$.

As $:e_*^{2niw}:_\tau = e^{-n^2\tau}e^{2niw}$, if $\mathrm{Re}\,\tau>0$, then $:e_*^{2niw}:_\tau$ tends to 0 very quickly. Using this we have:

Proposition 2.2. *If a power series $\sum_{n=0}^\infty a_n z^n$ has a positive radius of convergence, then $:e_*^{\ell iw}:_\tau *_\tau \sum_{n=0}^\infty a_n :e_*^{niw}:_\tau$ is an entire function of w for every $\ell \in \mathbb{Z}$.*

On the other hand we note the following:

Proposition 2.3. *If $\ell \geq 3$ and $\tau \neq 0$, the radius of convergence of the power series $\sum_{n=0}^\infty t^n/n!:w_*^{n\ell}:_\tau$ in t is 0. That is, $e_*^{tw_*^\ell}$ cannot be defined as a power series for $\ell \geq 3$.*

Applications to generating functions. As is well known, exponential functions contribute to construct generating functions. We now show how $*$-exponential functions relates to generating functions.

The generating function of the *Hermite polynomials* is given by

$$e^{\sqrt{2}tx-\frac{1}{2}t^2} = \sum_{n=0}^\infty H_n(x)\frac{t^n}{n!}.$$

This is the Taylor expansion formula of $:e_*^{\sqrt{2}tw}:_{-1}$.

Noting that $e_*^{taw} = \sum (t^n/n!)(aw)_*^n$ and setting

$$e_*^{\sqrt{2}tw} = \sum_{n\geq 0} (\sqrt{2}w)_*^n \frac{t^n}{n!},$$

we see $H_n(w) = :(\sqrt{2}w)_*^n:_{-1}$. Hence it is easy to see that $H_n(w)$ is a polynomial of degree n. For every $\tau \in \mathbb{C}$, we define the $*$-Hermite polynomials $H_n(w,*)$ by

$$e_*^{\sqrt{2}tw} = \sum_{n\geq 0}^{\infty} H_n(w,*) \frac{t^n}{n!}, \tag{2-9}$$

where

$$H_n(w,\tau) = :H_n(w,*):_\tau \quad \text{and} \quad H_n(w,-1) = H_n(w).$$

Since $\frac{d}{dt} e_*^{\sqrt{2}tw} = \sqrt{2}w * e_*^{\sqrt{2}tw}$, we have

$$\frac{\tau}{\sqrt{2}} H_n'(w,\tau) + \sqrt{2}w H_n(w,\tau) = H_{n+1}(w,\tau),$$

where $H_n'(w,\tau) = \partial H(w,\tau)/\partial w$.

The exponential law yields

$$\sum_{k+\ell=n} \frac{n!}{k!\ell!} H_k(w,*) * H_\ell(w,*) = H_n(w,*).$$

On the other hand, differentiating both sides of (2-9) with respect to w gives $\sqrt{2}nH_{n-1}(w,*) = H_n'(w,*)$. Differentiate again and use the equality above to get

$$\tau H_n''(w,\tau) + 2wH_n'(w,\tau) - 2nH_n(w,\tau) = 0.$$

By setting

$$\sqrt{2}tw + \frac{\tau}{2}t^2 = \frac{\tau}{2}\left(t + \frac{\sqrt{2}}{\tau}w\right)^2 - \frac{1}{\tau}w^2,$$

the Hermite polynomial $H_n(w,*)$ is obtained via the formula

$$H_n(w,\tau) = \frac{d^n}{dt^n} e^{\frac{\tau}{2}(t+\frac{\sqrt{2}}{\tau}w)^2}\Big|_{t=0} e^{-\frac{1}{\tau}w^2} = e^{-\frac{1}{\tau}w^2} \left(\frac{\tau}{\sqrt{2}}\right)^n \frac{d^n}{dw^n} e^{\frac{1}{\tau}w^2}.$$

The orthogonality of $\{H_n(w,\tau)\}_n$ is shown under the condition $\operatorname{Re}\tau < 0$ as follows:

$$\int_{\mathbb{R}} e^{\frac{1}{\tau}w^2} H_n(w,\tau) H_m(w,\tau) dw = \int_{\mathbb{R}} \left(\frac{\tau}{\sqrt{2}}\right)^n \frac{d^n}{dw^n} e^{\frac{1}{\tau}w^2} H_m(w,\tau) dw.$$

If $n \neq m$, one may suppose $n > m$ without loss of generality. Hence this vanishes by the integration by parts n times. For the case $n = m$, we set

$$:e_*^{\sqrt{2}tw}:_\tau = e^{\frac{\tau}{2}t^2+\sqrt{2}tw} = \sum_{n=0}^{\infty} H_n(w,\tau)\frac{t^n}{n!}.$$

Hence we see that

$$\frac{1}{n!}H_n(w,\tau) = \sum_{p=0}^{[n/2]} \frac{\sqrt{2}^n \tau^p}{p!\,(n-2p)!\,4^p} w^{n-2p}, \quad \frac{d^n}{dw^n}H_n(w,\tau) = \sqrt{2}^n n!.$$

It follows that

$$\int_{\mathbb{R}} e^{\frac{1}{\tau}w^2} H_n(w,\tau)H_n(w,\tau)dw = n!\,(-\tau)^n \int_{\mathbb{R}} e^{\frac{1}{\tau}w^2}\,dw = n!\,(-\tau)^n\sqrt{-\tau}\sqrt{\pi}.$$

The generating function of the *Bessel functions* $J_n(z)$ is known to be

$$e^{iz\sin s} = \sum_{n=-\infty}^{\infty} J_n(z)e^{ins}.$$

Keeping this in mind, we define $*$-Bessel functions by

$$e_*^{\frac{1}{2}(e^{is}-e^{-is})aw} = \sum_{n=-\infty}^{\infty} J_n(aw,*)e^{ins}, \quad :J_n(aw,*):_\tau = J_n(aw,\tau), \quad a\in\mathbb{C}.$$

Replacing s by $s+\pi/2$ gives

$$e_*^{\frac{i}{2}(e^{is}+e^{-is})aw} = \sum_{n=-\infty}^{\infty} i^n J_n(aw,*)e^{ins},$$

and basic symmetric properties hold. First we see $J_n(aw,*) = (-1)^n J_{-n}(aw,*)$. Replacing w by $-w$ in the first equality gives $J_n(-aw,*) = J_{-n}(aw,*)$. Since

$$\begin{aligned}:e_*^{\frac{1}{2}(e^{is}-e^{-is})aw}:_\tau &= e^{\frac{1}{16}a^2\tau(e^{is}-e^{-is})^2} e^{\frac{1}{2}(e^{is}-e^{-is})aw}\\ &= e^{\frac{1}{8}a^2\tau} e^{-\frac{1}{16}a^2\tau(e^{2is}+e^{-2is})} e^{\frac{1}{2}(e^{is}-e^{-is})aw},\end{aligned}$$

$J_n(aw,\tau)$ and $J_n(aw)$ are related by

$$\sum_{n=-\infty}^{\infty} J_n(aw,\tau)e^{ins} = e^{\frac{1}{8}a^2\tau} e^{-\frac{1}{16}a^2\tau(e^{2is}+e^{-2is})} \sum_{n=-\infty}^{\infty} J_n(aw)e^{ins}.$$

Setting $s=0$, we see in particular $1 = \sum_{n=-\infty}^{\infty} J_n(aw,\tau) = \sum_{n=-\infty}^{\infty} J_n(aw)$. The exponential law of left-hand side of the defining equality gives that

$$\begin{aligned}e_*^{\frac{1}{2}(e^{is}-e^{-is})aw} * e_*^{\frac{1}{2}(e^{is}-e^{-is})bw} &= e_*^{\frac{1}{2}(e^{is}-e^{-is})(a+b)w}\\ &= \sum_n J_n(aw+bw,*)e^{nis}\end{aligned}$$

and

$$J_n(aw+bw,*)=\sum_{m=-\infty}^{\infty} J_m(aw,*)*J_{n-m}(bw).$$

If $a^2+b^2=1$, then

$$\begin{aligned} e_*^{\frac{1}{2}(e^{is}-e^{-is})aw} * e_*^{\frac{i}{2}(e^{is}+e^{-is})bw} &= e_*^{\frac{1}{2}((a+ib)e^{is}-(a-ib)e^{-is})w} \\ &= \sum_n J_n(w,*)(a+ib)^n e^{nis} \end{aligned}$$

and

$$\sum_{k=-\infty}^{\infty} J_k(aw,*)e^{iks} * \sum_{\ell=-\infty}^{\infty} i^\ell J_\ell(bw,*)e^{i\ell s} = \sum_n J_n(w,*)(a+ib)^n e^{nis}.$$

The generating function of the *Legendre polynomials* $P_n(z)$ is

$$\frac{1}{\sqrt{1-2tz+t^2}}=\sum_{n=0}^{\infty} P_n(z)t^n, \quad \text{for small } |t|.$$

It is known that $P_n(z)=\dfrac{1}{2^n n!}\dfrac{d^n}{dz^n}(z^2-1)^n$. Hence

$$\frac{1}{\sqrt{1-2t(z+a)+t^2}}=\sum_n \frac{1}{2^n n!}\frac{d^n}{da^n}((z+a)^2-1)^n t^n$$

is viewed as the Taylor expansion of the left-hand side. Using the Laplace transform, we rewrite the left-hand side, and we see

$$\frac{1}{\sqrt{1-2t(z+a)+t^2}}=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}}e^{-s(1-2t(z+a)+t^2)}\,ds=\sum_{n=0}^{\infty} P_n(z+a)t^n.$$

This implies also that

$$\frac{d^n}{dt^n}\Big|_{t=0}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}}e^{-s(1-2t(z+a)+t^2)}\,ds=\frac{1}{2^n}\frac{d^n}{da^n}((z+a)^2-1)^n. \quad (2\text{-}10)$$

Replacing the exponential function in the integrand by the $*$-exponential function, we define $*$-Legendre polynomial by

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}}e_*^{-s(1-2t(w+a)+t^2)}\,ds=\sum_{n=0}^{\infty} P_n(w+a,*)t^n.$$

As $:e_*^{-s(1-2t(w+a)+t^2)}:_\tau = e^{\tau s^2t^2}e^{-s(1-2t(w+a)+t^2)}$, we assume that $\operatorname{Re}\tau<0$ so that the integral converges. Setting $P_n(w+a,\tau)={:}P_n(w+a,*){:}_\tau$, we have

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}}e^{\tau s^2t^2}e^{-s(1-2t(w+a)+t^2)}\,ds=\sum_{n=0}^{\infty} P_n(w+a,\tau)t^n.$$

As the variable z is used formally in (2-10), the same formula as in (2-10) holds for $*$-exponential functions; i.e.,

$$\frac{d^n}{dt^n}\Big|_{t=0}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}} e_*^{-s(1-2t(w+a)+t^2)}\,ds = \frac{1}{2^n}\frac{d^n}{da^n}((w+a)_*^2-1)_*^n.$$

By means of this trick we see that

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{s}} e_*^{-s(1-2t(w+a)+t^2)}\,ds &= \sum_{n=0}^{\infty} P_n(w+a,*)t^n \\ &= \sum_{n=0}^{\infty}\frac{1}{2^n n!}\frac{d^n}{dz^n}((w+a)_*^2-1)_*^n t^n.\end{aligned}$$

Generating functions for Bernoulli numbers, Euler numbers and Laguerre polynomials will be mentioned in later sections, for there are some other problems for the treatment.

Jacobi's theta functions, and imaginary transformations. For arbitrary $a \in \mathbb{C}$, consider the $*$-exponential function $e_*^{t(w+a)}$. Since $:e_*^{t(w+a)}:_\tau = e^{\frac{\tau}{4}t^2}e^{t(w+a)}$, we see, by supposing Re $\tau < 0$, that this is rapidly decreasing on $\mathbb{R}$. Hence both

$$\int_{-\infty}^{\infty} :e_*^{t(w+a)}:_\tau\,dt \quad \text{and} \quad \sum_{n=-\infty}^{\infty} :e_*^{n(w+a)}:_\tau$$

converge absolutely on every compact domain in w to give entire functions of w.

In this section, we treat first a special case $\theta(w,*) = \sum_n e_*^{2inw}$ under the condition Re $\tau > 0$. If we set $q = e^{-\tau}$, the τ-expression $\theta(w,\tau) = :\theta(w,*):_\tau$ is given by $\theta(w,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}e^{2niw}$. This is Jacobi's elliptic θ-function $\theta_3(w,\tau)$. Furthermore, Jacobi's elliptic theta functions $\theta_i, i = 1 \sim 4$ are τ-expressions of bilateral geometric series of $*$-exponential functions as follows (cf. [Andrews et al. 1999]):

$$\begin{aligned}\theta_1(w,*) &= \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n e_*^{(2n+1)iw}, &\quad \theta_2(w,*) &= \sum_{n=-\infty}^{\infty} e_*^{(2n+1)iw}, \\ \theta_3(w,*) &= \sum_{n=-\infty}^{\infty} e_*^{2niw}, & \theta_4(w,*) &= \sum_{n=-\infty}^{\infty}(-1)^n e_*^{2niw}.\end{aligned} \tag{2-11}$$

This was fact mentioned in [Omori 2007], but no further investigation of this fact has been made.

The exponential law $e_*^{aw+s} = e_*^{aw}e^s$ for $s \in \mathbb{C}$ gives that the $\theta_i(w,*)$ are 2π-periodic. (Precisely, $\theta_1(w,*)$ and $\theta_2(w,*)$ are alternating π-periodic, while

$\theta_3(w,*)$ and $\theta_4(w,*)$ are π-periodic.) Furthermore the exponential law (2-7) gives the trivial identities

$$e_*^{2iw}*\theta_i(w,*)=\theta_i(w,*),\qquad (i=2,3),$$
$$e_*^{2iw}*\theta_i(w,*)=-\theta_i(w,*),\quad (i=1,4).$$

For every τ such that Re $\tau>0$, τ-expressions of these are given by using $:e_*^{2iw}:_\tau=e^{-\tau}e^{2iw}$ and (2-8) as follows:

$$\begin{aligned} e^{2iw-\tau}\theta_i(w+i\tau,\tau)&=\theta_i(w,\tau), && (i=2,3),\\ e^{2iw-\tau}\theta_i(w+i\tau,\tau)&=-\theta_i(w,\tau), && (i=1,4).\end{aligned}\tag{2-12}$$

$\theta_i(w;*)$ is a parallel section defined on the open right half-plane, but the expression parameter τ turns out to give the quasi-periodicity with the exponential factor $e^{2iw-\tau}$.

Noting that $(e_*^{2iw}-1)*\theta_3(w,*)=0$ in the computation of $*_\tau$-product, we have

Proposition 2.4. *If* $f\in \mathrm{Hol}(\mathbb{C})$ *satisfies* $f(w+\pi)=f(w)$ *and*

$$:(e_*^{2iw}-1):_\tau *_\tau f=0,$$

then $f=c{:}\theta_3(w,*){:}_\tau,\quad c\in\mathbb{C}$.

Proof. By periodicity, the Fourier expansion theorem gives

$$f(w)=\sum a_n e^{2inw},$$

but by the formula of $*$-exponential functions, this is rewritten as

$$f(w)=:\sum c_n e_*^{2inw}:_\tau.$$

This gives the result, for the second identity implies $c_{n+1}=c_n$. ☐

Two different inverses of an element and $*$-delta functions. The convergence of bilateral geometric series for a $*$-exponential functions is responsible for somewhat strange features. If Re $\tau>0$, the τ-expressions of $\sum_{n=0}^{\infty}e_*^{2niw}$ and $-\sum_{n=-\infty}^{-1}e_*^{2niw}$ both converge in $\mathrm{Hol}(\mathbb{C})$ to give inverses of the element $:(1-e_*^{2iw}):_\tau$, and $\theta_3(w,\tau)$ is the difference of these inverses. We denote these inverses by using short notations:

$$(1-e_*^{2iw})_{*+}^{-1}=\sum_{n=0}^{\infty}e_*^{2niw},\quad (1-e_*^{2iw})_{*-}^{-1}=-\sum_{n=1}^{\infty}e_*^{-2niw},$$
$$(1-e_*^{-2iw})_{*+}^{-1}=\sum_{n=0}^{\infty}e_*^{-2niw}$$

Apparently, this breaks associativity:

$$\big((1-e_*^{2iw})_{*+}^{-1}*_\tau(1-e_*^{2iw})\big)*_\tau(1-e_*^{2iw})_{*-}^{-1} \neq (1-e_*^{2iw})_{*+}^{-1}*_\tau\big((1-e_*^{2iw})*_\tau(1-e_*^{2iw})_{*-}^{-1}\big).$$

Similarly, $\theta_4(w,*)$ is the difference of two inverses of $1+e_*^{2iw}$:

$$(1+e_*^{2iw})_{*+}^{-1}=\sum_{n=0}^{\infty}(-1)^n e_*^{2niw},\quad (1+e_*^{2iw})_{*-}^{-1}=-\sum_{n=1}^{\infty}(-1)^n e_*^{-2niw}.$$

Note also that

$$2e_*^{iw}*\sum_{n\geq 0}(-1)^n e_*^{2inw}\quad\text{and}\quad 2e_*^{-iw}*\sum_{n\geq 0}(-1)^n e_*^{-2inw}$$

are both $*$-inverses of $\frac{1}{2}(e_*^{iw}+e_*^{-iw})$. We denote them by

$$(\cos_* w)_{*+}^{-1}\quad\text{and}\quad(\cos_* w)_{*-}^{-1}.$$

Then, we see that

$$2i\theta_1(w,*)=(\cos_* w)_{*+}^{-1}-(\cos_* w)_{*-}^{-1}.$$

Every $\theta_i(w,*)$ is written as a difference of two different inverses.

Next, we note the similar phenomenon as above for the generator of the algebra:

Proposition 2.5. *If* Re $\tau>0$, *then for every* $a\in\mathbb{C}$, *the integrals*

$$i\int_{-\infty}^{0}{:}e_*^{it(a+w)}{:}_\tau\,dt,\ -i\int_{0}^{\infty}{:}e_*^{it(a+w)}{:}_\tau\,dt$$

converge in $\mathrm{Hol}(\mathbb{C})$ *to give inverses of* $a+w$.

Denote these inverses by

$$:(a+w)_{*+}^{-1}:_\tau\ = i\int_{-\infty}^{0}{:}e_*^{it(a+w)}{:}_\tau\,dt,$$
$$:(a+w)_{*-}^{-1}:_\tau\ = -i\int_{0}^{\infty}{:}e_*^{it(a+w)}{:}_\tau\,dt,\quad \mathrm{Re}\ \tau>0.$$

The difference of these two inverses is given by

$$(a+w)_{*+}^{-1}-(a+w)_{*-}^{-1}=i\int_{-\infty}^{\infty}e_*^{it(a+w)}\,dt,\quad \mathrm{Re}\ \tau>0. \tag{2-13}$$

The right-hand side may be viewed as a δ-function in the world of $*$-functions. We set

$$\delta_*(a+w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e_*^{it(a+w)}\,dt,\quad \mathrm{Re}\ \tau>0, \tag{2-14}$$

and call (2-14) the $*$-δ function. We see easily that $(a+w)*\delta_*(a+w)=0$. Note that $(a+w)^{-1}_{*+}+ci\delta_*(a+w)$ gives the inverse of $a+w$ for any constant c.

In the ordinary calculus, $\int_{-\infty}^{\infty} e^{it(a+x)}\,dt = 2\pi\delta(a+x)$ is not a function but a distribution. By contrast, in the world of $*$-functions, the τ-expression $:\delta_*(a+w):_\tau$ of $\delta_*(a+w)$ is an entire function:

$$
\begin{aligned}
:\delta_*(a+w):_\tau &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4}t^2\tau} e^{it(a+w)}\,dt \\
&= \frac{1}{\sqrt{\pi\tau}} e^{-\frac{1}{\tau}(a+w)^2}, \quad \operatorname{Re}\tau>0,\ a\in\mathbb{C}. \qquad (2\text{-}15)
\end{aligned}
$$

Jacobi's imaginary transformations. By (2-15), we see that the series

$$
\begin{aligned}
\tilde\theta_1(w,*) &= \sum_n (-1)^n \delta_*\Big(w+\frac{\pi}{2}+\pi n\Big), & \tilde\theta_2(w,*) &= \sum_n (-1)^n \delta_*(w+\pi n),\\
\tilde\theta_3(w,*) &= \sum_n \delta_*(w+\pi n), & \tilde\theta_4(w,*) &= \sum_n \delta_*\Big(w+\frac{\pi}{2}+\pi n\Big),
\end{aligned}
$$

converge in the τ-expression for Re $\tau>0$. These may be viewed as π-periodic or π-alternating periodic $*$-delta functions on $\mathbb{R}$. As $e^{2\pi i n}=1$, we have the identities

$$
\begin{aligned}
e_*^{2iw}*\tilde\theta_i(w,*) &= \tilde\theta_i(w,*) \qquad (i=2,3),\\
e_*^{2iw}*\tilde\theta_i(w,*) &= -\tilde\theta_i(w,*) \quad (i=1,4).
\end{aligned}
$$

By a slight modification of Proposition 2.4, we have $\theta_i(w,*)=\alpha_i\tilde\theta_i(w,*)$, $\alpha_i\in\mathbb{C}$. Note that α_i does not depend on the expression parameter τ. Taking the τ-expressions of both sides at $\tau=\pi$ and setting $w=0$, we have $\alpha_i=\frac{1}{2}$.

Proposition 2.6. *We have $\theta_i(w,*)=\frac{1}{2}\tilde\theta_i(w,*)$ for $i=1\sim 4$. The Jacobi imaginary transformation is given by taking the τ-expression of these identities.*

This may be proved directly as follows. Since $f(t)=\sum_n e_*^{2(n+t)iw}$ is a periodic function of period 1, the Fourier expansion formula gives

$$
f(t)=\sum_m \int_0^1 f(s)e^{-2\pi i m s}\,ds e^{2\pi i m t}
$$

and

$$
\theta_3(w,*)=f(0)=\sum_m \int_0^1 \Big(\sum_n e_*^{2(n+s)iw}\Big)e^{-2\pi i m s}\,ds.
$$

Since $e^{-2\pi i m s}=e^{-2\pi i m(s+n))}$, we have

$$f(0)=\sum_m\int_0^1\sum_n e_*^{2(n+s)iw}e^{-2(n+s)i\pi m}\,ds$$
$$=\sum_m\int_{-\infty}^{\infty}e_*^{2si(w+\pi m)}\,ds=\frac{1}{2}\sum_m\delta_*(w+\pi m).$$

Hence (2-15) gives

$$\theta_3(w,\tau)=\frac{2\pi}{2}{:}\sum_n\delta_*(w+\pi n){:}_\tau=\sqrt{\frac{\pi}{\tau}}\sum_n e^{-\frac{1}{\tau}(w+\pi n)^2}$$
$$=\sqrt{\frac{\pi}{\tau}}e^{-\frac{1}{\tau}w^2}\sum_n e^{-\pi^2n^2\tau^{-1}-2\pi n\tau^{-1}w}$$
$$=\sqrt{\frac{\pi}{\tau}}e^{-\frac{1}{\tau}w^2}\theta_3\Big(\frac{\pi w}{i\tau},\frac{\pi^2}{\tau}\Big). \tag{2-16}$$

This is a remarkable explicit relation between two different expressions (viewpoints).

In particular, Jacobi's theta relation is obtained by setting $w=0$ in (2-16):

$$\theta_3(0,\tau)=\sqrt{\frac{\pi}{\tau}}\theta_3\Big(0,\frac{\pi^2}{\tau}\Big). \tag{2-17}$$

This will be used to obtain the functional identities of the $*$-zeta function in a forthcoming paper.

Calculus of inverses. We first note that the method of constant variation creates many inverses of a single element. By the product formula $(a+w)*$, $a\in\mathbb{C}$, is viewed as a linear operator of $\mathrm{Hol}(\mathbb{C})$ into itself. If $\tau\neq 0$,

$$(a+w)*_\tau f(w)=0$$

gives a differential equation $(a+w)f(w)+\frac{\tau}{2}\partial_w f(w)=0$. Solving this, we have

$$(a+w)*_\tau Ce^{-\frac{1}{\tau}(a+w)^2}=Ce^{-\frac{1}{\tau}(a+w)^2}*_\tau(a+w)=0.$$

The method of constant variation gives a function $g_a(w)$ such that

$$(a+w)*_\tau g_a(w)=g_a(w)*_\tau(a+w)=1.$$

Thus, we have

$$g_a(w)=\frac{2}{\tau}\int_0^1 e^{\frac{1}{\tau}((a+wt)^2-(a+w)^2)}w\,dt+Ce^{-\frac{1}{\tau}(a+w)^2},\quad \tau\neq 0. \tag{2-18}$$

This breaks associativity:

$$\big(e^{-\frac{1}{\tau}(a+w)^2}*_\tau(a+w)\big)*_\tau g_a(w)\neq e^{-\frac{1}{\tau}(a+w)^2}*_\tau((a+w)*_\tau g_a(w)).$$

If $b+w$ has also two different $*$-inverses, then by providing $a\neq b$, the four elements

$$\frac{1}{b-a}\big((a+w)^{-1}_{*\pm}-(b+w)^{-1}_{*\pm}\big)$$

(with independent $\pm$ signs) give respectively $*$-inverses of $(a+w)*(b+w)$. Thus, we define $*$-inverses with independent $\pm$-sign by

$$(a+w)^{-1}_{*\pm}*(b+w)^{-1}_{\pm}=\frac{1}{b-a}\big((a+w)^{-1}_{*\pm}-(b+w)^{-1}_{*\pm}\big). \tag{2-19}$$

Direct computation of the $*$-product shows that, for any $a,b\in\mathbb{C},\ a\neq b$,

$$\big((a+w)^{-1}_{*+}-(a+w)^{-1}_{*}\big)*\big((b+w)^{-1}_{*+}-(b+w)^{-1}_{*-}\big)=0.$$

Half-series algebra. It is well known that if a formal power series satisfies $\sum_{n=0}^{\infty}a_nz^n=0$, then $a_n=0$. This is proved by setting $z=0$ to get $a_0=0$, and then taking $\partial_z|_{z=0}$ to get $a_1=0$ and so on. Hence this method cannot be applied to formal power series $\sum_{n=0}^{\infty}a_ne_*^{niw}$.

We suppose Re $\tau>0$ throughout this subsection. A formal power series $z^{\ell}\sum_{n=0}^{\infty}a_nz^n$, $\ell\in\mathbb{Z}$, is called *convergent* if $\sum_{n=0}^{\infty}a_nz^n$ has a positive radius of convergence. Proposition 2.2 shows that for a convergent power series of this form,

$$f(w)={:}e_*^{\ell iw}*\sum_{n=0}^{\infty}a_ne_*^{niw}{:}_{\tau}$$

is an entire function of w. Hence if $f(w)=0$, then Proposition 2.1 gives

$$\sum_{n=0}^{\infty}a_ne_*^{niw}{:}_{\tau}=0,$$

and $a_0=0$ by taking $w\to i\infty$. Thus the repeated use of Proposition 2.1 shows all the a_n vanish.

The product of two convergent power series is a convergent power series. If $z^{\ell}\sum_{n=0}^{\infty}a_nz^n$ $(\ell\in\mathbb{Z})$ is a convergent power series, its inverse $\big(z^{\ell}\sum_{n=0}^{\infty}a_nz^n\big)^{-1}$ obtained by the method of indeterminate constants is also a convergent power series. We denote by $\mathfrak{H}_+$ the space of power series ${:}e_*^{\ell iw}*\sum_{n=0}^{\infty}a_ne_*^{niw}{:}_{\tau}$ made by convergent power series $z^{\ell}\sum_{n=0}^{\infty}a_nz^n$. We call $\mathfrak{H}_+$ the half-series algebra. Its fundamental property is this:

Theorem 2.2. $(\mathfrak{H}_+,*_{\tau})$ *is a topological field of periodic entire functions of* w *of period* 2π.

Proof. The proof is completed by showing the uniqueness of the inverse. It is reduced to showing that

$$\sum_{n=0}^{\infty} a_n e_*^{niw}{:}_\tau *_\tau \sum_{k=0}^{\infty} b_k e_*^{kiw}{:}_\tau = 0,$$

and $a_0 \neq 0$ gives $\sum_{k=0}^{\infty} b_k e_*^{kiw}{:}_\tau = 0$. The repeated use of Proposition 2.1 gives all $b_n = 0$. □

Euler numbers. Recall the generating function of Euler numbers,

$$\frac{2}{e^z + e^{-z}} = \frac{e^z}{1+e^{2z}} + \frac{e^{-z}}{1+e^{-2z}} = \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} z^{2n}, \quad |z| < \pi.$$

The left-hand side is a convergent power series obtained by the method of indeterminate constants. Hence Proposition 2.2 gives

$$e_*^{e_*^{iw}} * \Big(1 + \sum_{k=0}^{\infty} 2^k e_*^{kiw} \frac{1}{k!}\Big)^{-1} + e_*^{-e_*^{iw}} * \Big(1 + \sum_{k=0}^{\infty} (-2)^k e_*^{kiw} \frac{1}{k!}\Big)^{-1} = \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} e_*^{2niw}, \quad \text{(2-20)}$$

where

$$e_*^{\pm e_*^{iw}} = \sum_{\ell=0}^{\infty} \frac{(\pm 1)^\ell}{\ell!} e_*^{\ell iw}.$$

On the other hand, by using the formal power series of $(iw)_*^n$, we can compute the inverses

$$\Big(1 + \sum_{k=0}^{\infty} \frac{(2iw)_*^k}{k!}\Big)^{-1}, \quad \Big(1 + \sum_{k=0}^{\infty} \frac{(-2iw)_*^k}{k!}\Big)^{-1}$$

by the method of indeterminate constants. Hence we have also

$$e_*^{iw} * \Big(1 + \sum_{k=0}^{\infty} (2iw)_*^k \frac{1}{k!}\Big)^{-1} + e_*^{-iw} * \Big(1 + \sum_{k=0}^{\infty} (-2iw)_*^k \frac{1}{k!}\Big)^{-1} = \sum_{n=0}^{\infty} E_{2n} \frac{1}{(2n)!} (iw)_*^{2n}. \quad \text{(2-21)}$$

It is clear that the replacement $(iw)_*^k$ by e_*^{kiw} gives (2-20). It is interesting to compare the left-hand side with $e_*^{iw}(1+e_*^{2iw})_{*+}^{-1} + e_*^{-iw} * (1+e_*^{-2iw})_{*+}^{-1}$. We are naturally led to the following:

Conjecture. By using another expression parameter τ' such that Re $\tau'>0$ and $\mathrm{Re}(\tau-\tau')>0$, the τ'-expression of $e_*^{iw}(1+e_*^{2iw})_{*+}^{-1}+e_*^{-iw}{*}(1+e_*^{-2iw})_{*+}^{-1}$ is an entire function of w. Denote this by

$$:e_*^{iw}(1+e_*^{2iw})_{*+}^{-1}+e_*^{-iw}{*}(1+e_*^{-2iw})_{*+}^{-1}:_{\tau'}=\sum_{n=0}^{\infty}a_{2n}(\tau,\tau'):(iw)_*^{2n}:_{\tau'},$$

and regard the right side as a τ'-expression of the $*$-function $\sum_n a_{2n}(\tau,\tau')(iw)_*^{2n}$. Then, the replacement $(iw)_*^{2n}$ by e_*^{2inw} gives

$$:\sum_n a_n(\tau,\tau')e_*^{niw}:_{\tau'}=:\sum_{n=0}^{\infty}E_{2n}\frac{1}{(2n)!}e_*^{2niw}:_{\tau-\tau'}.$$

Bernoulli numbers. Recall here the generating function of Bernoulli numbers:

$$z\Big(\frac{1}{2}+\frac{1}{e^z-1}\Big)=\frac{z}{2}\Big(\frac{1}{e^z-1}-\frac{1}{e^{-z}-1}\Big)=\sum_{n=0}^{\infty}B_{2n}\frac{1}{(2n)!}z^{2n}.$$

One computes $\dfrac{z}{e^z-1}$ and $\dfrac{-z}{e^{-z}-1}$ by the method of indeterminate constants as

$$\Big(\sum_n\frac{z^n}{(n+1)!}\Big)^{-1}=\sum B_{2n}\frac{1}{(2n)!}z^{2n}-\frac{1}{2}z,$$
$$\Big(\sum_n\frac{(-z)^n}{(n+1)!}\Big)^{-1}=\sum B_{2n}\frac{1}{(2n)!}z^{2n}+\frac{1}{2}z,$$

as in (2-20), the right-hand side is a convergent power series. Hence we have

$$\frac{1}{2}\Big(\sum_n\frac{e_*^{niw}}{(n+1)!}\Big)^{-1}+\frac{1}{2}\Big(\sum_n\frac{-e_*^{niw}}{(n+1)!}\Big)^{-1}=\sum_{n=0}^{\infty}B_{2n}\frac{1}{(2n)!}e_*^{2niw}. \tag{2-22}$$

On the other hand, we have for every τ' a formal power series

$$:\frac{1}{2}\Big(\sum_n\frac{(iw)_*^n}{(n+1)!}\Big)^{-1}+\frac{1}{2}\Big(\sum_n\frac{(-iw)_*^n}{(n+1)!}\Big)^{-1}:_{\tau'}=\sum_{k=0}^{\infty}B_{2k}:\frac{(iw)_*^{2k}}{(2k)!}:_{\tau'},$$

where both sides are computed as formal power series of (iw). It is clear that the replacement $(iw)_*^{2k}$ by e_*^{2kiw} in the right-hand side gives

$$\sum_{k=0}^{\infty}B_{2k}\frac{(e_*^{2kiw})}{(2k)!}.$$

Hence, we have the same conjecture for

$$:\tfrac{1}{2}iw{*}\big((e_*^{iw}-1)_{*+}^{-1}-(e_*^{-iw}-1)_{*+}^{-1}\big):_{\tau-\tau'}.$$

3. Srar-functions made by tempered distributions

Throughout this section, we assume Re $\tau > 0$. Note that

$$:\delta_*(x-w):_\tau = \frac{1}{\sqrt{\pi\tau}}\, e^{-\frac{1}{\tau}(x-w)^2}$$

is rapidly decreasing. Suppose $f(x)$ has $e^{|x|^\alpha}$ growth on $\mathbb{R}$ with $0<\alpha<2$. Then the integral $\int f(x){:}\delta_*(x-w){:}_\tau dx$ is well-defined to give an entire function with respect to w.

The next theorem is a key tool in extending the class of $*$-functions via the Fourier transform:

Theorem 3.1. *For every tempered distribution $f(x)$, the τ-expression of*

$$\int_{-\infty}^{\infty} f(x)\delta_*(x-w)\,dx$$

is an entire function of w whenever Re $\tau>0$. *In particular*,

$$\delta_*(a-w) = \int_{-\infty}^{\infty} \delta(x-a)\delta_*(x-w)\,dx.$$

Although the product $\delta_*(x-w)*\delta_*(x-w)$ diverges, the next one is important:

$$\delta_*(x-w)*\delta_*(x'-w) = \delta(x-x')\delta_*(x'-w) \tag{3-1}$$

in the sense of distributions. This is proved directly as follows:

$$\begin{aligned}\delta_*(x-w)*\delta_*(x'-w) &= \Big(\frac{1}{2\pi}\Big)^2 \iint e_*^{it(x-w)}*e_*^{is(x-w)}\,dt\,ds\\ &= \Big(\frac{1}{2\pi}\Big)^2 \iint e^{itx+isx'}e_*^{-i(t+s)w}\,dt\,ds\\ &= \Big(\frac{1}{2\pi}\Big)^2 \iint e^{is(x'-x)}e_*^{i\sigma(x-w)}\,ds\,d\sigma\\ &= \delta(x'-x)\delta_*(x-w).\end{aligned}$$

For every tempered distribution $f(x)$, we define a $*$-function $f_*(w)$ by

$$f_*(w) = \int_{-\infty}^{\infty} f(w)\delta_*(x-w)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \check{f}(t)e_*^{-itw}\,dt. \tag{3-2}$$

where $\check{f}(t)$ is the inverse Fourier transform of $f(x)$. Since $f(x)$ is a tempered distribution, one may write

$$\int f(x){:}\delta_*(x-w){:}_\tau\,dx = \frac{1}{2\pi}\iint f(x)e^{itx}{:}e_*^{-itw}{:}_\tau\,dt\,dx$$

under the existence of a rapidly decreasing function ${:}e_*^{-itw}{:}_\tau$ in the integrand.

By the definition of the Fourier transform of tempered distributions, one may exchange the order of integration. Letting $\check{f}(t)$ be the inverse Fourier transform of $f(x)$, we have

$$:\int_{\mathbb{R}} f(x)\delta_*(x-w)\,dx:_\tau = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \check{f}(t):e_*^{-itw}:_\tau\,dt = :f_*(w):_\tau. \tag{3-3}$$

If another $*$-function is given by $g_*(w) = \int g(x)\delta_*(x-w)\,dx$, we define their product by

$$\begin{aligned} f_*(w)*g_*(w) &= \int_{-\infty}^{\infty} f(w)g(w)\delta_*(x-w)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int \Big(\frac{1}{\sqrt{2\pi}}\int \check{f}(t-\sigma)\check{g}(\sigma)d\sigma\Big)e_*^{-itw}\,dt, \end{aligned} \tag{3-4}$$

if $f(x)g(x)$ is defined as a tempered distribution or the convolution product $\check{f}\bullet\check{g}(t) = (1/\sqrt{2\pi})\int \check{f}(t-\sigma)\check{g}(\sigma)d\sigma$ is defined as a tempered distribution. Hence (3-4) may be viewed as an integral representation of the intertwiner $I_0^\tau(f(x)) = f_*(w)$. If $f(x)$ is a slowly increasing function (a function with a value at each point $x \in \mathbb{R}$ and a tempered distribution), applying (3-4) to the case $g_*(w) = \delta_*(a-w)$ gives

$$f_*(w) * \delta_*(a-w) = \int f(x)\delta(a-x)\delta_*(x-w)\,dx = f(a)\delta_*(a-w). \tag{3-5}$$

Applications. The function $1/(a-w)$, for $a \notin \mathbb{R}$, is slowly increasing. Recalling our assumption that Re $\tau > 0$, it is not hard to verify that

$$\int \frac{1}{a-x}\delta_*(x-w)\,dx = \begin{cases} (a-w)_{*+}^{-1} & \text{if Im } a < 0, \\ (a-w)_{*-}^{-1} & \text{if Im } a > 0. \end{cases}$$

Define

$$Y_*(w) = \lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{\infty} \delta_*(x-w)\,dx, \quad Y_*(-w) = \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{-\varepsilon} \delta_*(x-w)\,dx.$$

It is clear that

$$\partial_w \lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{\infty} \delta_*(x-w)\,dx = -\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{\infty} \partial_x\delta_*(x-w)\,dx = \delta_*(-w) = \delta_*(w).$$

Using (3-1) we have

$$Y_*(w) + Y_*(-w) = \int_{\mathbb{R}} \delta_*(x-w)\,dx = 1$$

and $Y_*(w)*Y_*(w) = Y_*(w),\ Y_*(w)*Y_*(-w) = 0.$

Defining

$$\mathrm{sgn}_*(w) = Y_*(w) - Y_*(-w),$$

we have $\mathrm{sgn}_*(w) * \mathrm{sgn}_*(w) = Y_*(w) + Y_*(-w) = 1,\ \mathrm{sgn}_*(w) + \mathrm{sgn}_*(-w) = 0.$

Since $\delta_*(z-w)$ is holomorphic in z, the Cauchy integral theorem implies that every contour integral vanishes; but we see easily that, for every simple closed curve C,

$$\frac{1}{2\pi i}\int_C \frac{1}{z}\delta_*(z-w)\,dz = \delta_*(w), \quad \mathrm{Re}\,\tau > 0.$$

Note that v.p. $1/x$ and Pf. x^{-m}, for $m \in \mathbb{N}$, are tempered distributions rather than functions; but their Fourier transform may be viewed as slowly increasing functions. Hence we see that

$$\text{v.p.}\int_{\mathbb{R}} \frac{1}{x}\delta_*(x-w)\,dx = \frac{-i}{\sqrt{2\pi}}\int_{\mathbb{R}}\sqrt{\frac{\pi}{2}}\ \mathrm{sgn}(t)e_*^{-itw}\,dt = \frac{1}{2}(w_{*+}^{-1} + w_{*-}^{-1}),$$

$$\begin{aligned}\text{Pf.}\int_{\mathbb{R}} x^{-m}\delta_*(x-w)\,dx &= \frac{-i}{2}\int_{\mathbb{R}}\frac{1}{(m-1)!}(-it)^{m-1}\,\mathrm{sgn}(t)e_*^{-itw}\,dt\\ &= (-1)^{m-1}\tfrac{1}{2}(w_{*+}^{-m} + w_{*-}^{-m}).\end{aligned}$$

Periodical distributions. A tempered distribution $f(x)$ is said to be 2π-periodic if it satisfies $f(x+2\pi) = f(x)$. For every distribution $f(x)$ with compact support, the infinite sum $\sum_n f(x+2\pi n)$ is a 2π-periodic tempered distribution. The fundamental relation between 2π-periodic tempered distributions and Fourier series is

$$\sum_n \delta_*(a + 2\pi n + w) = \sum_n e_*^{in(a+w)}. \tag{3-6}$$

A continuous function $f(x)$ on $[-\pi,\pi]$ extends to a (not continuous) 2π-periodic function $\tilde{f}_\pi(x)$ to give a 2π-periodic tempered distribution, where

$$\tilde{f}_\pi(x) = \frac{1}{2\pi}\sum_n\Big(\int_{-\pi}^{\pi} f(s)e^{-ins}\,ds\Big)e^{inx} = \sum_n a_n e^{inx}.$$

Hence

$$\tilde{f}_{\pi*}(w) = \int_{\mathbb{R}} \tilde{f}_\pi(x)\delta_*(x-w)\,dx = \sum a_n e_*^{inw}. \tag{3-7}$$

4. Star-exponential function of w_*^2

As we have seen, the $*$-exponential function $e_*^{sh_*(w)}$ is well-behaved if the order of $h(w)$ is less than 2. In this section, we treat the $*$-exponential function of quadratic form w_*^2. As e^{-tx^2} is a slowly increasing function of x for Re $t \geq 0$, the integral $\int_{\mathbb{R}} e^{-tx^2}\delta_*(x-w)\,dx$ defines a semigroup $e_*^{-tw_*^2}$ under the expression parameter $\mathrm{Re}\,\tau > 0$. Noting that $:w_*^2:_\tau = w^2 + \tau/2$ in the τ-expression, we now define the star-exponential function of w_*^2 by the real analytic solution of the evolution equation

$$\frac{d}{dt} f_t = :w_*^2:_\tau *_\tau f_t, \quad f_0 = 1, \tag{4-1}$$

or explicitly

$$\frac{d}{dt} f_t = \frac{\tau^2}{4} f_t'' + \tau w f_t' + \Big(w^2 + \frac{\tau}{2}\Big) f_t, \quad f_0 = 1.$$

To solve this, we set $:f_t:_\tau = g(t)e^{h(t)w^2}$ by taking the uniqueness of real analytic solution in mind. Then, we have a system of ordinary differential equations:

$$\begin{cases} dh(t)/dt = (1 + \tau h(t))^2, & h(0) = 0, \\ dg(t)/dt = \frac{1}{2}(\tau^2 h(t) + \tau)g(t), & g(0) = 1. \end{cases}$$

The solution $:e_*^{tw_*^2}:_\tau$ is given by

$$:e_*^{tw_*^2}:_\tau = \frac{1}{\sqrt{1-\tau t}}\, e^{\frac{t}{1-\tau t} w^2}, \quad \text{whenever } \tau \neq 1,\ t\tau \neq 1 \text{ (double-valued)}. \tag{4-2}$$

It is rather surprising that the solution has a branching singular point, and hence this does not form a complex one parameter group whenever $\tau \neq 0$ is fixed. Moreover, the solution is double-valued with respect to the variable t. This solution is obtained also via the intertwiner $I_0^\tau e^{tw^2}$ of (4-4). Here that there is no restriction on τ: one obtains $e_*^{tw_*^2}$ for every τ.

Generating function of Laguerre polynomials. $L_n^{(\alpha)}(x)$ is given as follows:

$$\frac{1}{(1-t)^{\alpha+1}} e^{-\frac{t}{1-t}x} = \sum_{n\geq 0} L_n^{(\alpha)}(x) t^n \quad (|t| < 1).$$

If $\alpha = -\frac{1}{2}$, this is the $\tau = -1$ expression of $e_*^{-tw_*^2}$, that is,

$$:e_*^{-tw_*^2}:_{-1} = \frac{1}{(1-t)^{\frac{1}{2}}} e^{-\frac{t}{1-t}w^2} = \sum_{n\geq 0} L_n^{(-1/2)}(w^2) t^n.$$

Using these, we define the $*$-Laguerre polynomials $L_n(w^2, \tau) = :L_n(w^2, *):_\tau$:

$$\begin{aligned} e_*^{tw_*^2} &= \sum_n L_n^{(-1/2)}(w^2, *) \frac{1}{n!} t^n, \\ L_n^{(-1/2)}(w^2, \tau) &= \frac{d^n}{dt^n}\Big|_{t=0} \frac{1}{(1-t\tau)^{\frac{1}{2}}} e^{\frac{1}{\tau(1-t\tau)}w^2} e^{-\frac{1}{\tau}w^2}. \end{aligned} \tag{4-3}$$

As $t=0$ is a regular point, this is well defined, and the exponential law gives

$$L_n^{(-1/2)}(w^2,*)=\sum_{k+\ell=n}L_k^{(-1/2)}(w^2,*)*L_\ell^{(-1/2)}(w^2,*).$$

Setting $x=w^2$, we have $\frac{d}{dt}\frac{x^{\alpha-1}}{(1-t\tau)^\alpha}e^{\frac{1}{\tau(1-t\tau)}}=\frac{d}{dx}\frac{1}{\tau}\frac{x^\alpha}{(1-t\tau)^{\alpha+1}}e^{\frac{1}{\tau(1-t\tau)}}$. Using this, we see that

$$\frac{d^n}{dt^n}\bigg|_{t=0}\frac{1}{(1-t\tau)^{\frac{1}{2}}}e^{\frac{1}{\tau(1-t\tau)}w^2}e^{-\frac{1}{\tau}w^2}=\left(\tau^{-n}\frac{d^n}{dx^n}\left(x^{\frac{1}{2}+n}e^{\frac{1}{\tau}x}\right)\right)x^{-\frac{1}{2}}e^{-\frac{1}{\tau}x}.$$

It follows that setting $x=w^2$

$$L_n^{(-1/2)}(x,\tau)=\frac{1}{n!}\left(\tau^{-n}\frac{d^n}{dx^n}\left(x^{\frac{1}{2}+n}e^{\frac{1}{\tau}x}\right)\right)x^{-\frac{1}{2}}e^{-\frac{1}{\tau}x}.$$

As in the case of Hermite polynomials, this formula is used to obtain the orthogonality of $\{L_n^{(-1/2)}(x,\tau)\}_n$ by restricting x to the real axis and supposing $\operatorname{Re}\tau<0$. Namely, we want to show

$$\int_{\mathbb{R}}x^{\frac{1}{2}}e^{\frac{1}{\tau}x}L_n^{(-1/2)}(x,\tau)L_m^{(-1/2)}(x,\tau)\,dx=\delta_{n,m}.$$

First note that $L_n(x,\tau)$ is a polynomial of degree n, and

$$\int_{\mathbb{R}}x^{\frac{1}{2}}e^{\frac{1}{\tau}x}L_n^{(-1/2)}(x,\tau)L_m^{(-1/2)}(x,\tau)\,dx =\int_{\mathbb{R}}\frac{1}{\tau^n}\frac{1}{n!}\left(\frac{d^n}{dx^n}x^{\frac{1}{2}+n}e^{\frac{1}{\tau}x}\right)L_m^{(-1/2)}(x,\tau)\,dx.$$

If $n\neq m$, one may suppose $n>m$. Hence this vanishes by integration by parts n times.

For the case $n=m$, recalling that $L_n^{(-1/2)}(x,\tau)$ is a polynomial of degree n, and applying d^n/dx^n to both sides of the second equality in (4-3), we have

$$\begin{aligned}\frac{d^n}{dx^n}L_n^{(-1/2)}(x,\tau)&=\frac{1}{n!}\frac{d^n}{dt^n}\bigg|_{t=0}\frac{d^n}{dx^n}\frac{1}{(1-t\tau)^{\frac{1}{2}}}e^{\frac{t}{1-t\tau}x}\\&=\frac{1}{n!}\frac{d^n}{dt^n}\bigg|_{t=0}\frac{t^n}{(1-t\tau)^{\frac{1}{2}+n}}e^{\frac{t}{1-t\tau}x}.\end{aligned}$$

But the last term does not contain x, for it must be of degree 0. Hence

$$\frac{d^n}{dx^n}L_n^{(-1/2)}(x,\tau)=\frac{1}{n!}\frac{d^n}{dt^n}\bigg|_{t=0}\frac{t^n}{(1-t\tau)^{\frac{1}{2}+n}}=1.$$

In spite of double-valued nature of $e_*^{tw_*^2}$, one can treat $:e_*^{tw_*^2}:_\tau$ as a continuous function on any continuous curve C that does not hit singular points. In particular, one can treat the integral

$$\int_C :e_*^{tw_*^2}:_\tau\, dt$$

without ambiguity. The uniqueness of real analytic solutions gives the exponential law $e_*^{sw_*^2} * e_*^{tw_*^2} = e_*^{(s+t)w_*^2}$:

$$\frac{1}{\sqrt{1-\tau s}}\, e^{\frac{s}{1-\tau s}w^2} *_\tau \frac{1}{\sqrt{1-\tau t}}\, e^{\frac{t}{1-\tau t}w^2} = \frac{1}{\sqrt{1-\tau(s+t)}}\, e^{\frac{s+t}{1-\tau(s+t)}w^2}.$$

Indeed this holds through calculations such as

$$\sqrt{a}\sqrt{b} = \sqrt{ab}, \quad \sqrt{a}/\sqrt{a} = \sqrt{1} = \pm 1.$$

Similarly, we have the exponential law $e^s * e_*^{tw_*^2} = e_*^{s+tw_*^2}$ with an ordinary scalar exponential function e^s.

Intertwiners are 2-to-2 mappings. Recall that the intertwiner $I_\tau^{\tau'}$ is defined by $e^{\frac{1}{4}(\tau'-\tau)\partial_w^2}$. For the case of exponential functions of quadratic forms, this is treated by solving the evolution equation

$$\frac{d}{dt} f_t(w) = \partial_w^2 f(w), \quad f_0(w) = ce^{aw^2}.$$

Setting $f_t = g(t)e^{q(t)w^2}$, this equation is changed into

$$\begin{cases} dq(t)/dt = 4q(t)^2 & q(0) = a, \\ dg(t)/dt = 2g(t)q(t) & g(0) = c. \end{cases}$$

Solving this we get $g(t)e^{q(t)w^2} = \dfrac{c}{\sqrt{1-4ta}} e^{\frac{a}{1-4ta}w^2}$. Plugging in $t = \frac{1}{4}(\tau'-\tau)$, we obtain

$$I_\tau^{\tau'}(ce^{aw^2}) = \frac{c}{\sqrt{1-(\tau'-\tau)a}} e^{\frac{a}{1-(\tau'-\tau)a}w^2}.$$

To reveal its double-valued nature, we rewrite this equality as

$$I_\tau^{\tau'}\left(\frac{c}{\sqrt{1-\tau t}} e^{\frac{t}{1-\tau t}w^2}\right) = \frac{c}{\sqrt{1-\tau' t}} e^{\frac{t}{1-\tau' t}w^2}. \tag{4-4}$$

Since the branching singular point of the double-valued parallel section of the source space moves by the intertwiners, $I_\tau^{\tau'}$ must be viewed as a 2-to-2 mapping.

To describe (4-4) more clearly, we take two sheets with slit from τ^{-1} to ∞, and denote points by $(t;+)_\tau$ or $(t;-)_\tau$. $I_\tau^{\tau'}$ has the property that

$$I_\tau^{\tau'}((t;\pm)_\tau) = (t;\pm)_{\tau'}$$

as a set-to-set mapping, and one may define this locally as a 1-to-1 mapping. Note that

$$I^{\tau}_{\tau''}I^{\tau''}_{\tau'}I^{\tau'}_{\tau}((t,\pm)_{\tau})=(t,\pm)_{\tau},$$

but this is neither the identity nor -1. This depends on t discontinuously.

On the other hand, we want to retain the feature of being a complex one-parameter group. For that purpose, we have to set

$$:e_*^{0w_*^2}:_\tau = 1$$

as the multiplicative unit for every expression. The problem is caused by another sheet, for we have to distinguish 1 and -1.

It is important to recognize that there is no effective theory to understand such a vague system. This is something like an "air pocket" of the theory of point-set topology. As it will be seen in the next section, this system forms an object which may be viewed as a "double covering group" of $\mathbb{C}$. This is absurd since $\mathbb{C}$ is simply connected!

5. Extended notions for group-like objects

Recall that $:e_*^{tw_*^2}:_\tau$ does not form a group. However, using various expression parameters $\tau\neq 0$, $e_*^{tw_*^2}$ behaves like a group. To handle the group-like nature of the one-parameter family of $*$-exponential function $e_*^{tw_*^2}$, we introduce the notion of a *blurred covering group* of a topological group by using the notion of local groups. Thus, $\{e_*^{tw_*^2}; t\in\mathbb{C}\}$ is viewed as a blurred covering group of the abelian group $\{e^{tw^2}; t\in\mathbb{C}\}$. We need such a strange notion to understand the strange behavior of $*$-exponential functions for quadratic forms of several variables.

In spite of Lie's third theorem, which asserts that every finite-dimensional Lie algebra is the Lie algebra of a Lie group, we see in this section that the notion of local Lie groups is much wider than that of Lie groups, since it has to treat singular points.

A topological local group with unit. Recall that $:e_*^{tw_*^2}:_\tau$ is defined for t in $\mathbb{C}\setminus\{1/\tau\}$. Abstracting the property of an open connected neighborhood D of the identity e of a topological group leads to this:

Definition 5.1. A topological space D is called a topological local group with identity e if the following conditions are satisfied:

(a) For every $g\in D$, there are neighborhoods U of g and V of e such that both gh and hg are defined continuously for every $g\in U,\ h\in V$.

(b) g^{-1} is defined on an open dense subset of D and it is continuous.

(c) Associativity holds whenever both sides are defined.

A blurred covering group of a topological group. Let G be a locally simply arcwise connected topological group and let $\{\mathbb{O}_\alpha; \alpha \in I\}$ be an open covering of G. It may be helpful to keep in mind the correspondence

$$G \leftrightarrow \mathbb{C}, \quad \alpha \leftrightarrow \frac{1}{\tau}, \quad \mathbb{O}_\alpha \leftrightarrow \mathbb{C}\backslash\left\{\frac{1}{\tau}\right\}, \quad \Gamma \leftrightarrow \mathbb{Z}_2, \quad \widetilde{G} \leftrightarrow \{e_*^{tw_*^2}\}$$

in order to understand the following abstract conditions:

(a) For every $\alpha \in I$, $\mathbb{O}_\alpha$ contains the identity e. We call $\mathbb{O}_\alpha$ an *abstract expression space*, and α an expression parameter.

(b) For every $\alpha \in I$, $\mathbb{O}_\alpha$ is open, dense and connected, but it may not be simply connected.

(c) For every $\alpha, \beta \in I$, there is a homeomorphism $\phi_\alpha^\beta : \mathbb{O}_\alpha \to \mathbb{O}_\beta$.

(d) For every $g, h \in G$, there is $\alpha \in I$ and continuous path $g(t)$, $h(t) \in G$, $t \in [0, 1]$, such that

$$g(0) = h(0) = e, \quad g(1) = g, h(1) = h \quad \text{and} \quad g(t), h(t), g(t)h(t)$$

are in $\mathbb{O}_\alpha$ for every $t \in [0, 1]$.

An open covering $\{\mathbb{O}_\alpha; \alpha \in I\}$ is called a *natural covering* of G if it satisfies (a)–(d). Condition (c) shows that there is an abstract topological space X homeomorphic to every $\mathbb{O}_\alpha$. We consider a connected covering space $\pi : \widetilde{X} \to X$. This is same to say we consider a connected covering $\pi_\alpha : \widetilde{\mathbb{O}}_\alpha \to \mathbb{O}_\alpha$ for each α. It is easy see that $\pi_\alpha^{-1}(e)$ is a group given as a quotient group of the fundamental group of $\mathbb{O}_\alpha$. As G is locally simply connected, $\pi_\alpha^{-1}(e)$ forms a discrete group, and ϕ_α^β lifts to an isomorphism $\tilde{\phi}_\alpha^\beta : \pi_\alpha^{-1}(e) \to \pi_\beta^{-1}(e)$. We denote $\pi_\alpha^{-1}(e) = \Gamma_\alpha$, and the isomorphism class is denoted by Γ.

Choose $\tilde{e}_\alpha \in \pi_\alpha^{-1}(e)$ and call $\tilde{e}_\alpha$ a tentative identity. For any continuous path $g(t)$ in $\mathbb{O}_\alpha$ such that $g(0) = g(1) = e$, the continuous chasing among the set $\pi^{-1}(g(t))$ starting at $\tilde{e}_\alpha$ gives a group element $\gamma \in \Gamma_\alpha$.

By a standard argument, it is easy to make $\widetilde{\mathbb{O}}_\alpha$ a local group such that π_α is a homomorphism: We define first that $\tilde{e}_\alpha \tilde{e}_\alpha = \tilde{e}_\alpha$. For paths $g(t)$, $h(t)$, $g(t)h(t)$ such that they are in $\mathbb{O}_\alpha$ for every $t \in [0, 1]$ and $g(0) = h(0) = e$, we define the product by a continuous chasing among the set-to-set mapping

$$\pi_\alpha^{-1}(g(t))\pi_\alpha^{-1}(h(t)) = \pi_\alpha^{-1}(g(t)h(t)).$$

We set $\mathbb{O}_{\alpha\beta} = \mathbb{O}_\alpha \cap \mathbb{O}_\beta$ and $\mathbb{O}_{\alpha\beta\gamma} = \mathbb{O}_\alpha \cap \mathbb{O}_\beta \cap \mathbb{O}_\gamma$, for simplicity.

As G is locally simply connected, the full inverse $\pi_\alpha^{-1}V$ of a simply connected neighborhood $V \subset \mathbb{O}_\alpha$ of the identity $e \in G$ is the disjoint union $\coprod_\lambda \widetilde{V}_\lambda$, each member $\widetilde{X}_\lambda$ of which is homeomorphic to V.

Moreover $\pi_\alpha^{-1}\mathbb{O}_{\alpha\beta}$ is also a local group for every β.

Isomorphisms modulo Γ. For every α, β, we define the notion of "isomorphism" I_α^β of local groups, which corresponds to the notion of intertwiners in the previous section:

$$\begin{array}{ccccc}
\widetilde{\mathbb{O}}_\alpha \supset \pi_\alpha^{-1}\mathbb{O}_{\alpha\beta} & \xrightarrow{I_\alpha^\beta} & \pi_\beta^{-1}\mathbb{O}_{\beta\alpha} \subset \widetilde{\mathbb{O}}_\beta \\
\downarrow \pi_\alpha & & \downarrow \pi_\beta \\
\mathbb{O}_\alpha \supset \quad \mathbb{O}_{\alpha\beta} & = & \mathbb{O}_{\beta\alpha} \quad \subset \mathbb{O}_\beta
\end{array}$$

such that $I_\beta^\alpha = (I_\alpha^\beta)^{-1}$, but the cocycle condition $I_\alpha^\beta I_\beta^\gamma I_\gamma^\alpha = 1$ is not required for $\mathbb{O}_{\alpha\beta\gamma}$.

Since the correspondence I_α^β does not make sense as a point-set mapping, we should be careful about the definition.

Note that I_α^β is a collection of 1-to-1 mapping $I_\alpha^\beta(g) : \pi_\alpha^{-1}(g) \to \pi_\beta^{-1}(g)$ for every $g \in \mathbb{O}_{\alpha\beta} = \mathbb{O}_{\beta\alpha}$, which may not be continuous in g.

For each g there is a neighborhood V_g of the identity e such that $V_g g \subset \mathbb{O}_{\alpha\beta}$ and the local trivialization $\pi_\alpha^{-1}(V_g g) = V_g g \times \pi_\alpha^{-1}(g)$. Thus $I_\alpha^\beta(g)$ extends to the correspondence

$$\widetilde{I}_\alpha^\beta(h, g) : \pi_\alpha^{-1}(hg) \to \pi_\beta^{-1}(hg), \quad h \in V_g,$$

which commutes with the local deck transformations.

Definition 5.2. The collection $I_\alpha^\beta = \{I_\alpha^\beta(g); g \in \mathbb{O}_{\alpha\beta}\}$ is called an isomorphism modulo Γ, if $I_\beta^\alpha(hg)\widetilde{I}_\alpha^\beta(h, g)$ is in the group Γ for every $g \in \mathbb{O}_{\alpha\beta}$ and $h \in V_g$. (It follows the continuity of $I_\alpha^\beta(hg)$ with respect to h.)

The condition given by this definition means roughly that $I_\alpha^\beta(g)$ has discontinuity in g only in the group Γ.

$\widetilde{G} = \{\widetilde{\mathbb{O}}_\alpha, \pi_\alpha, I_\alpha^\beta; \alpha, \beta \in I\}$ is called a *blurred covering group* of G if each $\widetilde{\mathbb{O}}_\alpha$ is a covering local group of $\mathbb{O}_\alpha$, where $\{\mathbb{O}_\alpha; \alpha \in I\}$ is a natural open covering of a locally simply arcwise connected topological group G and I_α^β are isomorphisms modulo Γ.

Because of the failure of the cocycle condition, this object does neither form a covering group, nor a topological point set. However, this object looks like a covering group.

For g, let $I_g=\{\alpha\in I; \mathbb{O}_\alpha\ni g\}$ be the set of expression parameters involving g. For every $\alpha\in I(g,h,gh)=I_g\cap I_h\cap I_{gh}$, we easily see that $\pi_\alpha^{-1}(g)\pi_\alpha^{-1}(h)=\pi_\alpha^{-1}(gh)$. In general, this is viewed as set-to-set correspondence, but if g or h is in a small neighborhood of the identity, we can make these correspondence a genuine point-set mapping. Hence, we have the notion of indefinite small action or *infinitesimal left/right action* of small elements to the object. This corresponds to the infinitesimal action w_*^2* or $*w_*^2$ in the previous section.

Next, we choose an element $\tilde{e}_\alpha\in\pi_\alpha^{-1}(e)$, and call it a local identity. We call $\pi_\alpha^{-1}(e)$ the *set of local identities* of $\widetilde{G}$. The failure of the cocycle condition gives that $\mathfrak{M}_\alpha\tilde{e}_\alpha$ may not be a single point set, but forms a discrete abelian group. Hence an identity of our object is always a *local identity*.

Since G is locally simply connected, there is an open simply connected neighborhood V_β of e contained in $\mathbb{O}_\beta$. Hence, there is the unique lift $\widetilde{V}_\beta$ through $\tilde{e}_\beta$. Setting $\widetilde{V}_{\beta\gamma}=\widetilde{V}_\beta\cap\widetilde{V}_\gamma$ and so on, we see easily that $I_\beta^\gamma(\widetilde{V}_{\beta\gamma})=\widetilde{V}_{\gamma\beta}$.

The $\{\tilde{g}_\alpha\in\widetilde{\mathbb{O}}_\alpha; \alpha\in I\}$ may be viewed as an element of $\widetilde{G}$ if $I_\alpha^\beta\tilde{g}_\alpha=\tilde{g}_\beta$, but this is not a single point set by the same reason. In spite of this, one can distinguish individual points within a small local area.

The $*$-exponential function $e_*^{zw_*^2}$ may be viewed as a blurred covering group of $\mathbb{C}$ by treating this as a family $\{:e_*^{zw_*^2}:_\tau; \tau\}$, where the feature of complex one parameter group is retained.

Several remarks for the equation $(w_*^2-a^2)*f=0$. If f satisfies $w_*^2*f=a^2f$, then $e^{ta^2}f$ is the real analytic solution of the evolution equation

$$\frac{d}{dt}f_t(w)=w_*^2*f_t(w),$$

with the initial value f. Hence, one may write $e_*^{tw_*^2}*f=e^{ta^2}f$ by defining the $*$-product by this way. Next one gives a justification:

Proposition 5.1. *If* Re $\tau>0$, *then* $:e_*^{tw_*^2}*\delta_*(w+\alpha):_\tau$ *is holomorphic in* $t\in\mathbb{C}$. *That is*,

$$\big\{:e_*^{tw_*^2}:_\tau; t\in\mathbb{C}\big\}$$

acts on $\delta_*(w+\alpha):_\tau$ *as a genuine one parameter group. That is*,

$$:e_*^{tw_*^2}:_\tau*_\tau:\delta_*(w+\alpha):_\tau=e^{t\alpha^2}:\delta_*(w+\alpha):_\tau.$$

(See (3-3).)

Proof. Since $f(w)*_\tau e^{aw}=f(w+a\tau/2)e^{aw}$, we see

$$:e_*^{tw_*^2}:_\tau*_\tau:e_*^{i\sigma(w+\alpha)}:_\tau=\frac{1}{\sqrt{1-t\tau}}e^{\frac{t}{1-t\tau}w^2+\frac{i\sigma}{1-t\tau}w+i\sigma\alpha-\frac{\tau}{4(1-t\tau)}\sigma^2}.$$

If Re $\tau > 0$ and $t \neq \tau^{-1}$, the integral

$$\int_{\mathbb{R}} {:}e_*^{tw_*^2}{:}_\tau *_\tau {:}e_*^{i\sigma(w+\alpha)}{:}_\tau \, d\sigma$$
$$= \frac{1}{\sqrt{1-t\tau}} e^{\frac{t}{1-t\tau}w^2 - \frac{1}{\tau(1-t\tau)}(w+\alpha(1-t\tau))^2} \int_{\mathbb{R}} e^{-\frac{\tau}{4(1-t\tau)}(\sigma - \frac{2i}{\tau}(w+\alpha(1-t\tau))^2} d\sigma$$

converges. A calculation similar to that leading to (2-15) gives

$$\int_{\mathbb{R}} \exp\Big(-\frac{\tau}{4(1-t\tau)}\Big(\sigma - \frac{2i}{\tau}(w+\alpha(1-t\tau))^2\Big)\Big) d\sigma = \frac{2\sqrt{\pi(1-t\tau)}}{\sqrt{\tau}}.$$

Note that $t = \tau^{-1}$ is a removable singularity in this integral. Hence,

$$\begin{aligned} {:}e_*^{tw_*^2}{:}_\tau *_\tau {:}\delta_*(w+\alpha){:}_\tau &= \frac{1}{\sqrt{\pi\tau}} e^{\alpha^2 t} e^{-\frac{1}{\tau}(w+\alpha)^2} \\ &= e^{t\alpha^2} {:}\delta_*(w+\alpha){:}_\tau = e^{t(-\alpha)^2} {:}\delta_*(-\alpha-w){:}_\tau. \qquad \square \end{aligned}$$

Note. This exemplified the fact that even though the family $\{e_*^{tw_*^2}, t \in \mathbb{C}\}$ does not form a group, it can act as a genuine one-parameter group on some restricted family. This gives also an example that the formula $e_*^{tw_*^2} = \int e^{tx^2} \delta_*(x-w)\, dx$ does not extend for $t \in \mathbb{C}$.

The equation $(\alpha^2 - w_*^2) * f = 0$ can be solved by the Fourier transform. Namely, by setting $f = f_\alpha(w) = \int \hat{f}_\alpha(t) e_*^{itw}\, dt$, the equation is changed into

$$\int \hat{f}_\alpha(t)(\alpha^2 - w_*^2) * e_*^{itw}\, dt = \int \hat{f}_\alpha(t)\Big(\alpha^2 + \frac{d^2}{dt^2} e_*^{itw}\Big) dt = 0.$$

Integration by parts gives

$$\hat{f}_\alpha(t) = ae^{i\alpha t} + be^{-i\alpha t}, \quad a, b \in \mathbb{C}.$$

Hence we have

$$f_\alpha(w) = \int (ae^{i\alpha t} + be^{-i\alpha t}) e_*^{itw}\, dt, \quad a, b \in \mathbb{C}.$$

If Re $\tau > 0$, the right-hand side makes sense for any $\alpha \in \mathbb{C}$. This is equivalent to giving the solution as

$$f_\alpha(w) = a\delta_*(w+\alpha) + b\delta_*(w-\alpha).$$

By (2-15), the τ-expression of $f_\alpha(w)$ is

$${:}f_\alpha(w){:}_\tau = \frac{1}{\sqrt{\pi\tau}}\big(ae^{-\frac{1}{\tau}(w+\alpha)^2} + be^{-\frac{1}{\tau}(w-\alpha)^2}\big).$$

Thus, the equation $(\alpha^2 - w_*^2)*f = 0$ is solved uniquely by the boundary data $f_\alpha(0)$ and $f'_\alpha(0)$. Let $\Phi_\alpha(w,\tau)$, $\Psi_\alpha(w,\tau)$ be the solutions of $(\alpha^2 - w_*^2)*f = 0$ such that

$$\Phi_\alpha(0,\tau) = 1,\ \ \Phi'_\alpha(0,\tau) = 0, \quad \Psi_\alpha(0,\tau) = 0,\ \ \Psi'_\alpha(0,\tau) = 1.$$

As these are linear combinations of $*$-delta functions, Proposition 5.1 shows that $e_*^{zw_*^2} * \Phi_\alpha(w,*)$, $e_*^{zw_*^2} * \Psi_\alpha(w,*)$ are defined without singularity. This shows that the singular point of the differential equation $df_t/dt = (w^2 + \tau/2)*_\tau f_t$ depends on the initial functions. If $f_0 = 1$, the solution $:e_*^{tw_*^2}:_\tau$ has a singular point at $t = \tau^{-1}$, but if $f_0 = \Phi_\nu(w,\tau)$ or $\Psi_\nu(w,\tau)$, there is no singular point.

On the other hand, the integral along a closed path $\int_{C^2} e_*^{z(\nu + w_*^2)}\,dz$ satisfies

$$(\nu + w_*^2) * \int_{C^2} e_*^{z(\nu + w_*^2)}\,dz = 0,$$

where C^2 is the path turning around the same circle C twice avoiding singular points, so the integrand is closed on that path. As $\int_{C^2} e_*^{z(\nu + w_*^2)}\,dz$ is a function of w^2, we see that $\int_{C^2} :e_*^{z(\nu + w_*^2)}:_\tau\,dz = \alpha \Phi_\nu(w,\tau)$, the constant α being given by the value at $w^2 = 0$. Hence, we have

$$\int_{C^2} :e_*^{z(\nu + w_*^2)}:_\tau\,dz = \int_{C^2} \frac{e^{z\nu}}{\sqrt{1 - z\tau}}\,dz\ \Phi_\nu(w,\tau). \tag{5-1}$$

Computing the Laurent expansion of $\dfrac{e^{(\tau^{-1} + s^2)\nu}}{s\sqrt{-\tau}}$ at $s = 0$ and setting $z = s^2$ we see that

$$\int_{C^2} \frac{e^{z\nu}}{\sqrt{1 - z\tau}}\,dz = 0,$$

the secondary residue a_{-2} does not appear in the Laurent series. Hence, we have the following extraordinary property:

Proposition 5.2. $\displaystyle\int_{C^2} e_*^{z(\nu + w_*^2)}\,dz = 0$ *for any closed path* C^2.

Besides integrals along closed paths, the integral along a noncompact path Γ,

$$\int_\Gamma :e_*^{z(\nu + w_*^2)}:_\tau\,dz = \int_\Gamma \frac{e^{z\nu}}{\sqrt{1 - z\tau}} e^{\frac{z}{1 - z\tau} w^2}\,dz,$$

converges if Γ is suitably chosen under Re $\nu > 0$. By the continuity of $(\nu + w_*^2)*$, the integral must satisfy

$$(\nu + w_*^2) * \int_\Gamma e_*^{z(\nu + w_*^2)}\,dz = \int_\Gamma \frac{d}{dz} e_*^{z(\nu + w_*^2)}\,dz = 0.$$

This integral has a remarkable feature: it is given as the difference of two inverses of $\nu + w_*^2$. Specifically, let $\Gamma_\pm$ be different paths from $-\infty$ to 0 such that $\Gamma = \Gamma_+ \setminus \Gamma_-$. Then,

$$\int_{\Gamma_+}^0 e_*^{z(\nu+w_*^2)}\,dz - \int_{\Gamma_-}^0 e_*^{z(\nu+w_*^2)}\,dz$$

is nontrivial and satisfies the equation $(\nu + w_*^2)*f = 0$.

Residues and Laurent series. Note that $:e_*^{zw_*^2}:_\tau$ has a branching singular point at $z = \tau^{-1}$. Let D be a small disk with center at τ^{-1}. Let s be the complex coordinate of the double covering space $\widetilde{D}_*$ of $D\backslash\{1/\tau\}$ such that $z = s^2 + \tau^{-1}$. We view $:e_*^{zw_*^2}:_\tau$ as a single-valued holomorphic function of s on the double covering space $\widetilde{D}_*$. The residue at $s = 0$ is defined as the coefficient a_{-1} of $1/s$ in the Laurent series expansion at the isolated singular point $s = 0$. We extend the term *residue* to mean 0 at a regular point.

Using (4-2), we see that the 1-form

$$\begin{aligned} :e_*^{(\tau^{-1}+s^2)w_*^2}:_\tau\,ds &= \frac{ds}{s}\,e^{-\frac{w^2}{\tau^2 s^2}}\,\frac{1}{\sqrt{-\tau}}\,e^{-\frac{1}{\tau}w^2} \\ &= \frac{1}{\sqrt{-\tau}}\,e^{-\frac{1}{\tau}w^2}\left(\frac{1}{s} - \frac{w^2}{\tau^2 s^3} + \frac{w^4}{2!\,\tau^4 s^5} - \cdots\right)ds \end{aligned} \tag{5-2}$$

has terms only of negative odd degrees in s. The 1-form $:e_*^{(\tau^{-1}+s^2)w_*^2}:_\tau\,ds$ may be written as

$$:e_*^{zw_*^2}:_\tau\,\frac{dz}{2\sqrt{z-\tau^{-1}}},$$

by setting a suitable slit. The Cauchy's integral theorem gives that the residue is given by

$$\begin{aligned} \mathrm{Res}_{z=\tau^{-1}}(:e_*^{zw_*^2}:_\tau) &= \frac{1}{2\pi i}\int_{\widetilde{C}} :e_*^{(\tau^{-1}+s^2)w_*^2}:_\tau\,ds \\ &= \frac{1}{\sqrt{-\tau}}\,e^{-\frac{1}{\tau}w^2}\,\frac{1}{2\pi i}\int_{\widetilde{C}} \frac{1}{s}\,e^{-\frac{1}{s^2\tau^2}w^2}\,ds \\ &= \frac{1}{\sqrt{-\tau}}\,e^{-\frac{1}{\tau}w^2}, \end{aligned} \tag{5-3}$$

where $\widetilde{C}$ corresponds to C^2, the path turning around the same circle $C = \partial D$ twice so that the integrand is closed. As there are only two singular points, $s = 0$ and $s = \infty$, one does not need to take the radius of C small, but one may set $|s| = 1$. It is very suggestive to compare the residue formula with the (2-15). If $\mathrm{Re}\,\tau > 0$, then

$$\frac{1}{\sqrt{-\tau}}\,e^{-\frac{1}{\tau}w^2} = \sqrt{-\pi}:\delta_*(w):_\tau.$$

Note also that the integral obtaining the residue may be replaced as follows by taking the $\pm$ sheet and the slit in mind:

$$\begin{aligned}\mathrm{Res}_{z=\tau^{-1}}(:e_*^{z(\nu+w_*^2)}:_\tau) &= \frac{1}{2\pi i}\int_{C^2} :e_*^{z(\nu+w_*^2)}:_\tau \frac{dz}{2\sqrt{z-\tau^{-1}}}\\ &= \frac{1}{2\pi i}\int_{C} :e_*^{zw_*^2}:_\tau \frac{e^{z\nu}dz}{\sqrt{z-\tau^{-1}}},\end{aligned} \tag{5-4}$$

where C^2 means the union C_+ and C_- of C viewed as a curve in $\pm$-sheets. Note that the $\pm$-sign changes on $\pm$ sheets. The existence of the slit keeps the integrand single-valued, and dz is treated as $-dz$ in the negative sheet. Hence $dz/\sqrt{z-\tau^{-1}}$ does not change sign on the opposite sheet.

Discontinuity of Laurent coefficients. Recall that

$$:e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}:_\tau = e^{\tau^{-1}\nu}\frac{1}{\sqrt{-\tau}}e^{-\frac{1}{\tau}w^2}\frac{1}{s}e^{\nu s^2-\frac{1}{s^2}\frac{w^2}{\tau^2}}. \tag{5-5}$$

We can write the Laurent series for $\frac{1}{s}e^{\nu s^2-\frac{1}{\tau^2 s^2}w^2}$ as

$$\cdots+\frac{c_{-(2k+1)}(\nu,\tau,w)}{s^{2k+1}}+\cdots+\frac{c_{-1}(\nu,\tau)}{s}+c_1(\nu,\tau,w)s+c_3(\nu,\tau,w)s^3+\cdots,$$

without terms of even degree. We have $c_{2k+1}(\nu,\tau,w)=0$ at $\nu=0$ for $k\geq 0$, by (5-2). Hence the Laurent series of $:e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}:_\tau$ is given by

$$\sum_{k\in\mathbb{Z}} a_{2k-1}(\nu,\tau,w)s^{2k-1} = e^{\tau^{-1}\nu}\frac{1}{\sqrt{-\tau}}e^{-\frac{1}{\tau}w^2}\sum_k c_{2k-1}(\nu,\tau,w)s^{2k-1},$$

$$a_{2k-1}(\nu,\tau,w)=\mathrm{Res}_{s=0}\big(:s^{-2k}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}:_\tau\big),$$

$$a_{-1}(\nu,\tau,w)=\frac{e^{\frac{\nu}{\tau}}}{\sqrt{-\tau}}e^{-\frac{1}{\tau}w^2}\sum_k\frac{(-\nu)^k}{k!\,k!}\Big(\frac{w}{\tau}\Big)^{2k}. \tag{5-6}$$

Note that every $a_{2k-1}(\nu,\tau,w)$ can be written in the form

$$a_{2k-1}(\nu,\tau,w)=e^{\tau^{-1}\nu}\frac{1}{\sqrt{-\tau}}e^{-\frac{1}{\tau}w^2}p_{2k-1}(\tau^{-1},w^2),$$

for some polynomial $p_{2k-1}(\tau^{-1},w^2)$. The following is easy to see.

Proposition 5.3. *$a_{2k-1}(0,\tau,w)=0$ for $2k-1\geq 0$, and $a_{2k-1}(\nu,\tau,0)=0$ for $2k-1\leq -2$. Hence $a_{2k-1}(0,\tau,0)=0$ except for $k=0$: $a_{-1}(0,\tau,0)=1/\sqrt{-\tau}$.*

A strange fact arises by writing these as integrals:

$$
\begin{aligned}
a_{2k-1} =& \frac{1}{2\pi i}\int_{\widetilde{C}} {:}s^{-2k} e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau\, ds \\
=& \frac{1}{\sqrt{-\tau}} e^{-\frac{1}{\tau}w^2}\, e^{\frac{\nu}{\tau}} \frac{1}{2\pi i}\int_{\widetilde{C}} \frac{1}{s^{2k+1}}\, e^{\nu s^2 - \frac{1}{\tau^2 s^2}w^2}\, ds,
\end{aligned}
$$

where $\widetilde{C}$ is any simple closed curve in the covering space $\mathbb{C}\backslash\{\tau^{-1}\}$ turning positively around τ^{-1}. By Cauchy's theorem, it does not depend on $\widetilde{C}$, hence it may be infinitesimally small. Integration by parts gives

$$
\begin{aligned}
{:}(\nu + w_*^2){:}_\tau {*_\tau} a_{2k-1} &= {:}\frac{1}{2\pi i}\int_{\widetilde{C}} \frac{1}{2}\, s^{-2k-1} \frac{d}{ds}\, e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}\, ds{:}_\tau \\
&= \left(k+\tfrac{1}{2}\right)\frac{1}{2\pi i}\int_{\widetilde{C}} s^{-2k-2}{:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau\, ds \\
&= \left(k+\tfrac{1}{2}\right) a_{2k+1}. \qquad (5\text{-}7)
\end{aligned}
$$

(If $\nu = 0$, (5-2) shows that $a_{2k+1} = 0$ for $k \geq 0$.)

There is a strange phenomenon as follows.

Proposition 5.4. *Although* (5-7) *implies* ${:}(\nu+w_*^2){:}_\tau {*_\tau} a_{2k-1}(\nu,\tau) \neq 0$, *we have*

$$
{:}e_*^{t(\nu+w_*^2)}{:}_\tau {*_\tau} a_{2k-1}(\nu,\tau) = 0
$$

for any $t \neq 0$, *and this is not continuous at* $t = 0$. *Hence*, *differentiating by* t *at* $t = 0$ *is prohibited.*

Proof. Using the formula (5-3) and the exponential law, we have

$$
\begin{aligned}
&{:}e_*^{t(\nu+w_*^2)}{:}_\tau {*_\tau} \frac{1}{2\pi i}\int_{\widetilde{C}} s^{-2k}{:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}\, ds \\
&\qquad\qquad = \frac{1}{2\pi i}\int_{\widetilde{C}} s^{-2k}{:}e_*^{(t+\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau\, ds.
\end{aligned}
$$

This is ensured since both sides satisfies the same differential equation

$$
\frac{d}{dt} f_t = (\nu + w_*^2){*} f_t, \quad f_0 = \frac{1}{2\pi i}\int_{\widetilde{C}} s^{-2k}{:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}\, ds.
$$

The radius of $\widetilde{C}$ can be infinitesimally small by Cauchy's integral theorem. Hence if $t \neq 0$, then $t+\tau^{-1}$ is outside the path of integration. Thus it must vanish. □

Apparently, this is caused that $\widetilde{C}$ is chosen infinitesimally small. Therefore, if $\widetilde{C}$ is big enough, the integral

$$
\frac{1}{2\pi i}\int_{\widetilde{C}} s^{-2k}{:}e_*^{(a+\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau\, ds
$$

is defined to give a_{2k-1}. Thus, to avoid possible confusion, it is better to fix the definition of the residue by

$$\mathrm{Res}_{s=0} f(s) = \lim_{r\to 0}\int_{C(r)} f(s)\,ds, \tag{5-8}$$

where $C(r)$ is a circle of radius r with the center at $s=0$.

Though Proposition 5.4 shows that $:(\nu+w_*^2):_\tau *_\tau \mathrm{Res}_{z=\tau^{-1}}(:e_*^{z(\nu+w_*^2)}:_\tau) \neq 0$ in general, the case $\nu=0$ is rather special.

By (5-2), we see that $:w_*^2:_\tau *_\tau \mathrm{Res}_{z=\tau^{-1}}(:e_*^{zw_*^2}:_\tau)=0$. Hence, there must be a constant α such that

$$\mathrm{Res}_{z=\tau^{-1}}\big(:e_*^{zw_*^2}:_\tau\big) = \alpha\Phi_0(w^2,\tau),$$

where α is given by the value at $w^2=0$. Hence, we have an equality

$$\begin{aligned}\mathrm{Res}_{z=\tau^{-1}}(:e_*^{zw_*^2}:_\tau) &= \mathrm{Res}_{z=\tau^{-1}}\left(\frac{1}{\sqrt{1-z\tau}}\right)\Phi_0(w^2,\tau)\\ &= \frac{1}{\sqrt{-\tau}}\Phi_0(w^2,\tau).\end{aligned} \tag{5-9}$$

This is strange, for the right-hand side of (5-9) satisfies

$$:e_*^{tw_*^2}:_\tau *_\tau \frac{1}{\sqrt{-\tau}}\Phi_0(w^2,\tau) = \frac{1}{\sqrt{-\tau}}\Phi_0(w^2,\tau),$$

but Proposition 5.4 shows

$$:e_*^{tw_*^2}:_\tau *_\tau \mathrm{Res}_{z=\tau^{-1}}\big(:e_*^{zw_*^2}:_\tau\big) = \mathrm{Res}_{z=\tau^{-1}}\big(:e_*^{(t+z)w_*^2}:_\tau\big)=0,$$

for $t\neq 0$ by the computations as residues. Recall that $\Phi_0(w^2,\tau)$ is defined by the differential equation, while $\mathrm{Res}_{z=\tau^{-1}}(:e_*^{zw_*^2}:_\tau)$ is defined by the integral on an infinitesimally small circuit. The equality

$$\mathrm{Res}_{z=\tau^{-1}}(:e_*^{zw_*^2}:_\tau) = \frac{1}{\sqrt{-\tau}}\Phi_0(w^2,\tau)$$

holds only on some restricted stage.

It is convenient to use the notion of formal distributions to avoid such a strange impression. We regard $\mathrm{Res}_{z=\tau^{-1}} :e_*^{z(w_*^2+\nu)}:_\tau$ as a *formal distribution* supported only on the surface S_*: $z=\tau^{-1}$. This is the notion based on the calculations of residues by regarding Laurent polynomials as *test functions*. Formal distributions are used extensively in conformal field theory.

Covariant differentials and $*$-product integrals. In general, the Laurent coefficient a_{2k-1} of $:e_*^{(z+s^2)(\nu+w_*^2)}:_\tau$ is obtained in the formula

$$\mathrm{Res}_{s=0}(:s^{-2k}e_*^{(z+s^2)(\nu+w_*^2)}:_\tau) = \begin{cases} a_{2k-1} & z=\tau^{-1},\\ 0 & z\neq\tau^{-1}.\end{cases}$$

This is a formal distribution of $(z,\tau)\in\mathbb{C}^2_*$. We denote this by

$$R_{2k-1}(z,\tau)=\mathrm{Res}_{s=0}\big(s^{-2k}{:}e_*^{(\tau^{-1}+s^2)w_*^2}{:}_\tau\big)\delta(z-\tau^{-1}).$$

If we set $E(z,\tau)(s)={:}e_*^{(z+s^2)(\nu+w_*^2)}{:}_\tau$ for $s\neq 0$, and regard this a formal distribution supported on $z=\tau^{-1}$, the Laurent expansion theorem shows that

$$E(z,\tau)(s)=\sum_{k\in\mathbb{Z}}R_{2k-1}(z,\tau)s^{2k-1},\quad 0<|s|<\infty.$$

Now we are interested only in the function $R_{2k-1}(\tau^{-1},\tau)$ restricted to the surface S_*. The infinitesimal intertwiner is given by

$$\lim_{\delta\to 0}I_{z^{-1}}^{(z+\delta)^{-1}}=-\frac{1}{4z^2}\partial_w^2,$$

for every Hol($\mathbb{C}$)-valued function $f(z,\tau,w)$. We now define

$$\begin{aligned}\nabla_z f(z,z^{-1},w)&=\partial_z f(z,\tau,w)\big|_{\tau=z^{-1}}\\&=\partial_z(f(z,z^{-1},w))+\frac{1}{4z^2}\partial_w^2 f(z,z^{-1},w).\end{aligned}\tag{5-10}$$

This will be called *covariant* or *comoving* differentiation. In other words, we define

$$\nabla_{\tau^{-1}}f(\tau^{-1},\tau,w)=\lim_{\delta\to 0}\frac{1}{\delta}\big(I_\tau^{(\tau^{-1}+\delta)^{-1}}f(\tau^{-1}+\delta,\tau,w)-f(\tau^{-1},\tau,w)\big).\tag{5-11}$$

Noting that $\partial_\tau f(z,\tau,w)=-\frac{1}{4}\tau\partial_w^2$, we extend the notion of covariant derivative to functions $f(z,\tau)$ without w by

$$\nabla_z f(z,z^{-1})=\partial_z f(z,\tau)\big|_{\tau=z^{-1}}.$$

We easily see that $\partial_z((m+k)z^m-m\tau^k z^{m+k})\big|_{\tau=z^{-1}}=0$, for every pair of integers (m,k). Hence setting $f_{k,m}(z,\tau)=(m+k)z^m-m\tau^k z^{m+k}$, one may treat this a *parallel polynomial of degree* k as $\nabla_z f_{k,m}(z,z^{-1})=0$. However, we do not use $\partial_z(\log z-z\tau)\big|_{\tau=z^{-1}}=0$ for $\log z$ is multivalued. Such parallel polynomials forms a commutative algebra. We call these *parallel polynomials on* $z=\tau^{-1}$ and denote this by $\mathcal{P}[S_*]$.

Proposition 5.5. *Any Laurent coefficient* $a_{2k-1}(\nu,w^2)(\tau)$ *of* ${:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau$ *satisfies the differential equation*

$$\nabla_{\tau^{-1}}a_{2k-1}(\nu,w^2)(\tau)={:}(\nu+w_*^2){:}_\tau *_\tau a_{2k-1}(\nu,w^2)(\tau).\tag{5-12}$$

We insist that $\nabla_{\tau^{-1}}$ is the notion of comoving derivative. The equality above may be written as

$$\nabla_{\tau^{-1}} {:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau = {:}(\nu + w_*^2){:}_\tau *_\tau {:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau \quad (s \neq 0). \tag{5-13}$$

The equation $\nabla_{\tau^{-1}} F(\tau^{-1}, \tau) = {:}(\nu + w_*^2){:}_\tau *_\tau F(\tau^{-1}, \tau)$**.** Note that for every parallel polynomial $c(z, \tau)$, $c(\tau^{-1}, \tau)F(\tau^{-1}, \tau)$ must satisfy the original equation. Rewrite the equation

$$\nabla_{\tau^{-1}} F(\tau^{-1}, \tau) = {:}(\nu + w_*^2){:}_\tau *_\tau F(\tau^{-1}, \tau)$$

by using (5-10) on the left-hand side and the product formula on the right-hand side. Then, the highest parts cancel out and the equation becomes a differential equation of first order:

$$\partial_{\tau^{-1}} F(\tau^{-1}, \tau) = \tau w \partial_w F(\tau^{-1}, \tau) + \Big(w^2 + \nu + \frac{\tau}{2}\Big) F(\tau^{-1}, \tau). \tag{5-14}$$

Recalling that $F(\tau^{-1}, \tau)$ involves the variable (generator) w, we can solve (5-14) by a standard manner. First set $F(\tau^{-1}, \tau) = e^{-\tau^{-1}w^2} G(\tau^{-1}, w)$. Then, (5-14) turns out $\partial_{\tau^{-1}} G = \tau w \partial_w G + (\nu + \tau/2)G$. Thus, we have

$$F(\tau^{-1}, \tau) = \sqrt{\tau^{-1}} e^{\tau^{-1}(\nu - w^2)} H(\tau^{-1} w), \tag{5-15}$$

using an arbitrarily holomorphic function $H(z)$. If the initial data is given at $\tau^{-1} = 1$ and $F(1, 1) = 1$, then

$$F(\tau^{-1}, \tau) = \sqrt{\tau^{-1}} e^{\tau^{-1}(\nu - w^2)} e^{-(\nu - \tau^{-2} w^2)}.$$

Proposition 5.6. *If the initial data is not singular*, *then there is no singular point on the solution of*

$$\nabla_{\tau^{-1}} F(\tau^{-1}, \tau) = {:}(\nu + w_*^2){:}_\tau *_\tau F(\tau^{-1}, \tau).$$

On the other hand, there must be a holomorphic function $H(z, s)$ on $\mathbb{C}_* \times \mathbb{C}_*$ such that

$${:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau = \sqrt{\tau^{-1}} e^{\tau^{-1}(\nu - w^2)} H(\tau^{-1} w, s).$$

Putting $\tau^{-1} = -s^2$, we have $1 = i s e^{-s^2(\nu - w^2)} H(-s^2 w, s)$, or, with $z = -s^2 w$,

$$H(z, s) = \frac{1}{is} e^{\nu s^2 - z^2 s^{-2}},$$

and

$${:}e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}{:}_\tau = \frac{1}{\sqrt{\tau}} e^{\frac{1}{\tau}(\nu - w^2)} \frac{1}{is} e^{(\nu s^2 - \frac{1}{\tau^2 s^2} w^2)}.$$

This is nothing but the τ-expression of $e_*^{(\tau^{-1}+s^2)(\nu+w_*^2)}$.

6. Isolated singular points and formal distributions

In this section, we view $E(z,\tau) = \mathrm{Res}_{s=0}\, {:}e_*^{(z+s^2)(w_*^2+\nu)}{:}_\tau$ as a formal distribution. Recall that

$$:e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}:_\tau = e^{\tau^{-1}\nu}\frac{1}{\sqrt{-\tau}}e^{-\frac{1}{\tau}w^2}\ \frac{1}{s}e^{\nu s^2-\frac{1}{s^2}\frac{w^2}{\tau^2}}.$$

For every Laurent polynomial $f(s)\in\mathbb{C}[s,s^{-1}]$, we set

$$\{f(s)\} = f(s){:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau \in \mathbb{C}[s,s^{-1}]{:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau.$$

Note that $\{f(s)+g(s)\}$ and $\{f(s)g(s)\}$ are defined in the usual manner. Moreover by definition, we have

$$f(s)\{g(s)\} = \{f(s)g(s)\}.$$

Define the action of the Lie algebra of vector fields $h(s)\partial_s$, $h\in\mathbb{C}[s,s^{-1}]$, as follows:

$$\begin{aligned} h(s)\partial_s\big(f(s){:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau\big) &= (h(s)\partial_s f(s)){:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau \\ &\quad + h(s)(2s)f(s){:}(w_*^2+\nu){*}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau. \end{aligned}$$

For simplicity, we denote it by

$$h(s)\partial_s\{(f(s)\} = \{h(s)\partial_s f(s)\} + {:}\{h(s)(2s)f(s)\}{*}(w_*^2+\nu){:}_\tau. \tag{6-1}$$

For later use we denote these operations on the generators:

$$\begin{gathered} \{s^m\}+\{s^n\} = \{s^m+s^n\}, \quad s^m\{s^n\} = \{s^m s^n\} \\ s^{n+1}\partial_s\{s^m\} = \{ms^{n+m}\} + {:}\{2s^{n+m+2}\}{*}(w_*^2+\nu){:}_\tau. \end{gathered} \tag{6-2}$$

We use the notation $\{s^0\}$, but we do not use the notation $\{1\}$.

The space $\{\mathbb{C}[s,s^{-1}]\} = \mathbb{C}[s,s^{-1}]{:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau$ is a $\mathbb{C}[s,s^{-1}]$-module, often called a *loop algebra*, on which the Lie algebra $\mathbb{C}[s,s^{-1}]\partial_s$ acts naturally as derivations, where a derivation means that

$$h(s)\partial_s(f(s)\{g(s)\}) = (h(s)\partial_s f(s))\{g(s)\} + f(s)h(s)\partial_s\{g(s)\}.$$

By defining $[V(s), f(s)] = V(s)f(s)$ and $[f(s), g(s)] = 0$, the direct product space $\{\mathbb{C}[s,s^{-1}]\}\oplus\mathbb{C}[s,s^{-1}]\partial_s$ has a Lie algebra structure including $\{\mathbb{C}[s,s^{-1}]\}$ as a commutative Lie ideal.

We denote by V_τ the vector space spanned by

$$\mathrm{Res}_{s=0}\, f(s)\partial_s^k{:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau; \quad k\in\mathbb{N},\ f(s)\in\mathbb{C}[s,s^{-1}].$$

That is, $V_\tau = \mathbb{C}[\tau,\tau^{-1},\nu,w^2]e^{\frac{\nu}{\tau}}e^{-\frac{w^2}{\tau}}$.

We note that the essential part of residue calculus is

$$\mathrm{Res}_{s=0}(\partial_s h(s)) = 0 \quad \text{for all } h \in V_\tau[\![s, s^{-1}]\!]. \tag{6-3}$$

From a basic viewpoint of the conformal field theory, a nontrivial residue gives a violation of the additive structure around $s = 0$. Namely, integration by parts gives

$$\begin{aligned}&\mathrm{Res}_s\{(f'(s)g(s))\} = \\ &\qquad -\mathrm{Res}_s\{f(s)g'(s)\} - \mathrm{Res}_s\big(f(s)(2s)g(s){:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}*(w_*^2+\nu){:}_\tau\big).\end{aligned}$$

Denote the second term by $\mathrm{Res}_s({:}\{f(s)(2s)g(s)\}*(w_*^2+\nu){:}_\tau)$, a symmetric bilinear form. Using this bilinear form, we extend the usual commutative structure on the space $\{\mathbb{C}[s, s^{-1}]\}\oplus V_\tau$ to a noncommutative product by defining

$$\begin{aligned}&(\{f(s)\}, a)\circ(\{g(s)\}, b) = \\ &\quad \big(\{f(s)+g(s)\}, \\ &\qquad\qquad a+b - \mathrm{Res}_{s=0}\big({:}\{f(s)sg(s)\}*(w_*^2+\nu){:}_\tau\big) + \mathrm{Res}_{s=0}\{f'(s)g(s)\}\big).\end{aligned}$$

This gives a noncommutative extension of the usual additive operation. However, we regard this as an extension of the commutative Lie algebra $\{\mathbb{C}[s, s^{-1}]\}\oplus V_\tau$ by introducing a Lie bracket:

$$\big[(\{f(s)\}, a), (\{g(s)\}, b)\big] = \big(0, \mathrm{Res}_s\{f(s)'g(s)\} - \mathrm{Res}_s\{g(s)'f(s)\}\big).$$

We call this a Heisenberg Lie algebra and denote it by $\mathfrak{g}$.

We now make its universal enveloping algebra $\mathfrak{A}_\tau$ of $\mathfrak{g}$, but note here that the multiplicative structure is nothing to do with the original multiplicative structure of $\mathbb{C}[s, s^{-1}]$. For that purpose, we extend first the vector space V_τ to the commutative algebra $\widetilde{V}_\tau$ generated by V_τ.

$$\widetilde{V}_\tau = \mathbb{C}[\tau, \tau^{-1}, \nu, w^2]e^{\mathbb{N}\frac{\nu}{\tau}}e^{-\mathbb{N}\frac{w^2}{\tau}}.$$

We define next

$$\begin{aligned}&\{f(s)\}\bullet\{g(s)\} \\ &\qquad = \{f(s)g(s)\} + \mathrm{Res}_{s=0}\{f'(s)g(s)\} + \mathrm{Res}_{s=0}\big({:}\{f(s)sg(s)\}*(w_*^2+\nu){:}_\tau\big),\end{aligned}$$

$$\begin{aligned}&\{f(s)\}\bullet\big({:}\{g(s)\}*(w_*^2+\nu){:}_\tau\big) \\ &\qquad = {:}\{f(s)g(s)\}*(w_*^2+\nu){:}_\tau + \mathrm{Res}_{s=0}\big({:}\{f(s)g(s)\}*(w_*^2+\nu){:}_\tau\big),\end{aligned}$$

$$\begin{aligned}&\big({:}\{f(s)\}*(w_*^2+\nu){:}_\tau\big)\bullet\{g(s)\} \\ &\qquad = {:}\{f(s)g(s)\}*(w_*^2+\nu){:}_\tau + \mathrm{Res}_{s=0}\big({:}\{f(s)g(s)\}*(w_*^2+\nu){:}_\tau\big).\end{aligned}$$

Furthermore, we define

$$\big(:\{f(s)\}*(w_*^2+\nu)^k:_\tau\big)\bullet\big(:\{g(s)\}*(w_*^2+\nu)^\ell:_\tau\big)\\ \quad = :\{f(s)g(s)\}*(w_*^2+\nu)^{k+\ell}:_\tau + \mathrm{Res}_{s=0}\big(:\{f(s)g(s)\}*(w_*^2+\nu)^{k+\ell}:_\tau\big).$$

All this would yield a commutative product, but for the term $\mathrm{Res}_{s=0}\{f'(s)g(s)\}$ in the definition of $\{f(s)\}\bullet\{g(s)\}$. We call this the Heisenberg vertex algebra and denote it by $\mathfrak{A}_\tau$. Note that

$$A\bullet a = a\bullet A \quad \text{for } A\in\mathfrak{A}_\tau,\ a\in\widetilde{V}_\tau.$$

Recall the action $h(s)\partial_s\{f\} = \{h(s)f'(s)\} + :\{h(s)(2s)f(s)\}*(w_*^2+\nu):_\tau$. This forms an action of the Lie algebra $\mathbb{C}[s,s^{-1}]\partial_s$, called the Witt–Lie algebra. That is, we have

$$h\partial_s(k\partial_s\{f\}) - k\partial_s(h\partial_s\{f\}) = [h\partial_s, k\partial_s]\{f\},$$

where $[h\partial_s, k\partial_s] = (hk' - kh')\partial_s$.

Next, we extend this as a derivation of $\mathfrak{A}_\tau$. Namely, we define

$$\begin{aligned} h\partial_s(\{f\}\bullet\{g\}) &= (h\partial_s\{f\})\bullet\{g\} + \{f\}\bullet(h\partial_s\{g\})\\ &= \big(\{hf'\} + :\{h(2s)f\}*(w_*^2+\nu):_\tau\big)\bullet\{g\}\\ &\qquad + \{f\}\bullet\big(\{hg'\} + :\{h(2s)g\}*(w_*^2+\nu):_\tau\big). \end{aligned}$$

As the residues such as $\mathrm{Res}_{s=0}\{f'(s)g(s)\}$ do not involve the variable s, it looks at a glance that $h\partial_s[\{f\},\{g\}] = 0$, but the term $(w_*^2+\nu):_\tau$ can act on the residue part. Indeed, the action $h\partial_s[\{f\},\{g\}]$ is given as follows:

$$\begin{aligned} h\partial_s(\{f\}\bullet\{g\}) &= \big(\{hf'\} + :\{h(2s)f\}*(w_*^2+\nu):_\tau\big)\bullet\{g\}\\ &\quad + \{f\}\bullet\big(\{hg'\} + :\{h(2s)g\}*(w_*^2+\nu):_\tau\big)\\ &= \{hf'g\} + \mathrm{Res}\{(hf')'g\} + \mathrm{Res}\,:\{shf'g\}*(w_*^2+\nu):_\tau\\ &\quad + :\{2shfg\}*(w_*^2+\nu):_\tau + \mathrm{Res}\,:\{2shfg\}*(w_*^2+\nu):_\tau\\ &\quad + \{fhg'\} + \mathrm{Res}\{f'hg'\} + \mathrm{Res}\,:\{sfhg'\}*(w_*^2+\nu):_\tau\\ &\quad + :\{2sfhg\}*(w_*^2+\nu):_\tau + \mathrm{Res}\,:\{2sfhg\}*(w_*^2+\nu):_\tau. \end{aligned}$$

Hence, by using

$$\mathrm{Res}\{(hf')'g\} + \mathrm{Res}\{(hf'g'\} + \mathrm{Res}\,:\{2shf'g\}*(w_*^2+\nu):_\tau = 0,$$

exchanging f and g gives

$$h\partial_s([\{f\},\{g\}]_\bullet) = \mathrm{Res}\,:\{2sh(f'g - fg')\}*(w_*^2+\nu):_\tau. \tag{6-4}$$

Note that this term arises from terms such as $:\{2shf\}{*}(w_*^2+\nu):_\tau$: hence (6-4) must vanish, if one can eliminate these terms by a change of generators.

Central extensions caused by singularities. To clarify, consider the generators $x_m=\{s^m\}$. The Heisenberg Lie algebra is given by

$$\mathfrak{g}=\Big\{\sum_{n\in\mathbb{Z}}c_nx_n; c_n\in\mathbb{C}; [x_m,x_n]=(m-n)a_{m+n-1}(\tau^{-1},\nu,w)\Big\},$$

where $a_{m+n-1}(\tau^{-1},\nu,w)$ are Laurent coefficients. Next, we make its universal enveloping algebra $\mathfrak{A}_\tau$ by extending the vector space V_τ to an algebra $\widetilde{V}_\tau$ generated by

$$\Big\{e^{\tau^{-1}\nu}\frac{1}{\sqrt{-\tau}}e^{-\frac{1}{\tau}w^2}\Big\}$$

under the ordinary commutative product. $\mathfrak{A}_\tau$ is a noncommutative associative algebra generated by infinitely many generators $\{x_k; k\in\mathbb{Z}\}$ together with commutation relations $[x_m,x_n]=(m-n)a_{m+n-1}(\tau^{-1},\nu,w)$. In the case $\nu=0$ and $w=0$, we see that

$$[x_m,x_n]=2m\delta_{m+n,0}\frac{1}{\sqrt{-\tau}},\quad a_{m+n-1}(\tau^{-1},0,0)=0,\quad a_{-1}(\tau^{-1},0,0)=\frac{1}{\sqrt{-\tau}},$$

but in general $x_m\bullet x_n=x_n\bullet x_m+(m-n)a_{m+n-1}(\tau^{-1},\nu,w)$.

Let $E^{(k)}$ be the linear space spanned by $\widetilde{V}_\tau x_{n_1}\bullet x_{n_2}\bullet\cdots\bullet x_{n_k}$. It is not hard to see that the space $E^{(2)}$ consisting of all quadratic forms such as $\sum c_{mn}x_m\bullet x_n$ forms a Lie algebra acting on $E^{(1)}$ under the commutator bracket product $[a,b]_\bullet=a\bullet b-b\bullet a$; that is, $[E^{(2)},E^{(1)}]=E^{(1)}$. This extends naturally to $\mathfrak{A}_\tau$ as derivations:

$$[E^{(2)},\mathfrak{A}_\tau]\subset\tilde{\mathfrak{A}}_\tau,\quad [A,f\bullet g]=[A,f]\bullet g+f\bullet[A,g].$$

We want to write the extended action $h(s)\partial_s\{f(s)\}$ on the generators. Recalling that

$$s\partial_s\{s^m\}=\{ms^m\}+:\{2s^{m+2}\}{*}(w_*^2+\nu):_\tau,$$

we define

$$[L_0,x_m]=mx_m+2{:}x_{m+2}{*}(w_*^2+\nu):_\tau.$$

Since

$$\begin{aligned}[s^nL_0,x_m]&=s^n(mx_m+2{:}x_{m+2}{*}(w_*^2+\nu):_\tau)\\&=mx_{n+m}+2{:}x_{n+m+2}{*}(w_*^2+\nu):_\tau,\end{aligned}$$

we set $L_n=s^nL_0$ and define

$$[L_n,x_m]=mx_{n+m}+2{:}x_{n+m+2}{*}(w_*^2+\nu):_\tau.$$

Then

$$\big[L_\ell,[L_n,x_m]\big]=m(n+m)x_{n+m+\ell}+2(n+m+2){:}x_{n+m+\ell+2}{*}(w_*^2+\nu){:}_\tau \\ +4(n+m+\ell+4){:}x_{n+m+\ell+4}{*}(w_*^2+\nu)_*^2{:}_\tau.$$

It follows that

$$\big[L_n,[L_\ell,x_m]\big]-\big[L_\ell,[L_n,x_m]\big] \\ =m(n-\ell)x_{n+m+\ell}+2(n-\ell)x_{n+m+\ell+2}{:}{*}(w_*^2+\nu){:}_\tau.$$

Thus, this is an action of the Witt–Lie algebra

$$\big[[L_n,L_\ell],x_m\big]=(n-\ell)[L_{n+\ell},x_m].$$

Direct computation shows the following.

Proposition 6.1. *For every integer m, an element*

$$y_m=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}{:}x_{m+k}{*}(w_*^2+\nu)^k{:}_\tau,$$

written as a formal power series of ${:}(w_*^2+\nu)_*^k{:}_\tau$ satisfies $[L_0,y_m]=my_m$. It follows that

$$[L_n,y_m]=s^n[L_0,y_m]=ms^ny_m=my_{n+m}.$$

Note that

$$y_m=\sum_{k=0}^{\infty}s^{m+k}\frac{(-2)^k}{k!}{:}x_0{*}(w_*^2+\nu)^k{:}_\tau \\ =s^m\sum_{k=0}^{\infty}s^k\frac{(-2)^k}{k!}{:}(w_*^2+\nu)^k{*}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau.$$

Hence this is defined only as a formal power series in general, for this is

$$s^m{:}e_*^{-2s(w_*^2+\nu)}{*}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau=s^m e_*^{(\tau^{-1}+s^2-2s)(w_*^2+\nu)}{:}_\tau,$$

and this diverges at $s=2$. Although the expression seems a slightly confusing, it is convenient to view this as

$$y_m={:}e_*^{-2s(w_*^2+\nu)}s^m{*}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau={:}e_*^{-2s(w_*^2+\nu)}{*}x_m{:}_\tau,$$

as it is inverted easily by

$${:}y_m{*}e_*^{2s(w_*^2+\nu)}{:}_\tau=s^m{:}e_*^{(\tau^{-1}+s^2)(w_*^2+\nu)}{:}_\tau=x_m.$$

The Heisenberg vertex algebra. Since (as is easy to see)

$$\big[{:}x_m{*}(w_*^2+\nu)^k{:}_\tau, {:}x_n{*}(w_*^2+\nu)^\ell{:}_\tau\big] = [x_m, x_n]{*_\tau}{:}(w_*^2+\nu)^{k+\ell}{:}_\tau,$$

the commutator $[y_m, y_n]$ belongs to the space $\widetilde{V}_\tau{*_\tau}[\![{:}(w_*^2+\nu){:}_\tau]\!]$ of all formal power series written in the form

$$\sum_k a_k(\tau^{-1}, \nu, w){*_\tau}{:}(w_*^2+\nu)_*^k{:}_\tau, \quad a_k(\tau^{-1}, \nu, w) \in \widetilde{V}_\tau.$$

Set $[y_m, y_n] = C_{m,n}$. By Proposition 6.1 and by the remark below (6-4), we see $[L_k, C_{m,n}] = 0$. Since L_k acts as a derivation, Jacobi identity of Lie algebra gives restriction to the constants $C_{m,n}$:

$$0=\big[L_k,[y_\ell,y_m]\big]=\big[[L_k,y_\ell],y_m\big]+\big[y_\ell,[L_k,y_m]\big]=\ell[y_{\ell+k},y_m]+m[y_\ell,y_{m+k}].$$

Hence

$$\ell C_{\ell+k,m} + mC_{\ell,m+k} = 0. \tag{6-5}$$

Set $k = 0$ to obtain $C_{\ell,m} = c_m\delta_{\ell+m,0}$. Set $m = 1$ further in (6-5) to obtain $\ell c_1\delta_{\ell+k+1,0} + c_{k+1}\delta_{\ell+k+1,0} = 0$. Hence, we have $c_m = mc_1$, with

$$\begin{aligned} c_1 = C_{-1,1} &= (-2)\sum_{k,\ell}\frac{(-2)^{k+\ell}}{k!\ell!}a_{k+\ell-1}(\nu, \tau^{-1}, w){:}(w_*^2+\nu)^{k+\ell}{:}_\tau \\ &= (-2)\sum_n \frac{4^{2n}}{(2n)!}{:}a_{2n-1}(\nu, \tau^{-1}, w){*}(w_*^2+\nu)_*^{2n}{:}_\tau. \end{aligned}$$

Consequently:

Proposition 6.2. *The system* $\{y_m, m \in \mathbb{Z}\}$ *has the following properties.*

$$\begin{aligned} &[y_m, y_n] = m\delta_{m+n,0}c_1, \quad c_1 \in \widetilde{V}_\tau{*_\tau}[\![{:}(w_*^2+\nu){:}_\tau]\!], \\ &[L_m, y_n] = my_{m+n}, \\ &\big[[L_m, L_n], y_\ell\big] = [(m-n)L_{m+n}, y_\ell]. \end{aligned}$$

It follows in particular $[y_0, y_m] = 0$ *for every* y_m. *In particular, as there is no zero-divisor in* $\widetilde{V}_\tau$, *if* $\sum_k a_k y_k$, $a_k \in \widetilde{V}_\tau$, *satisfies* $[\sum_k a_k y_k, y_m] = 0$ *for every* y_m, *then* $\sum_k a_k y_k = a_0 y_0$.

The set $\{y_m; m \in \mathbb{Z}\}$ forms a standard basis of the Heisenberg vertex algebra over

$$\widetilde{V}_\tau{*_\tau}[\![{:}(w_*^2+\nu){:}_\tau]\!].$$

So far, L_m is not an established element defined only as an adjoint operator $[L_m, \cdot]$ acting on $E^{(1)}$. The following theorem is known as the Sugawara construction:

Theorem 6.1. *Elements of Witt–Lie algebra can be represented by elements of* $E^{(2)}$.

Thus, regarding L_m as an element of $E^{(2)}$, we set

$$[L_m, L_n] = (m-n)L_{m+n} + K_{m,n}.$$

Since

$$[K_{m,n}, y_\ell] = \big[L_m, [L_n, y_\ell]\big] - \big[L_n, [L_m, y_\ell]\big] - (m-n)[L_{m+n}, y_\ell] = 0,$$

the $K_{m,n}$ must be central elements. This central extension of the Witt algebra is called the Virasoro algebra. Such an extended Lie algebra is known to be isomorphic to the one defined by

$$[L_m, L_n] = (m-n)L_{m+n} + c(\nu, \tau^{-1}, w)\,\frac{m(m^2-1)}{12}\,\delta_{m+n,0}, \qquad [L_n, c(\nu, \tau^{-1}, w)] = 0. \tag{6-6}$$

Note that restricting n even integers $\{L_{2n}; n \in \mathbb{Z}\}$ forms a Lie subalgebra.

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