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Flexible Weinstein manifolds

KAI CIELIEBAK AND YAKOV ELIASHBERG

To Alan Weinstein with admiration.

This survey on flexible Weinstein manifolds, which is essentially an extract from [Cieliebak and Eliashberg 2012], provides to an interested reader a shortcut to theorems on deformations of flexible Weinstein structures and their applications.

1. Introduction

The notion of a *Weinstein manifold* was introduced in [Eliashberg and Gromov 1991], formalizing the symplectic handlebody construction from Alan Weinstein's paper [1991] and the Stein handlebody construction from [Eliashberg 1990]. Since then, the notion of a Weinstein manifold has become one of the central notions in symplectic and contact topology. The existence question for Weinstein structures on manifolds of dimension > 4 was settled in [Eliashberg 1990]. The past five years have brought two major breakthroughs on the uniqueness question: From [McLean 2009] and other work we know that, on any manifold of dimension > 4 which admits a Weinstein structure, there exist infinitely many Weinstein structures that are pairwise nonhomotopic (but formally homotopic). On the other hand, Murphy's h-principle for loose Legendrian knots [Murphy 2012] has led to the notion of *flexible* Weinstein structures, which are unique up to homotopy in their formal class. In this survey, which is essentially an extract from [Cieliebak and Eliashberg 2012], we discuss this uniqueness result and some of its applications.

1A. *Weinstein manifolds and cobordisms.*

Definition. A *Weinstein structure* on an open manifold V is a triple (ω, X, ϕ), where

- ω is a symplectic form on V,
- $\phi : V \to \mathbb{R}$ is an exhausting generalized Morse function,
- X is a complete vector field which is Liouville for ω and gradient-like for ϕ.

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The quadruple (V, ω, X, ϕ) is then called a *Weinstein manifold.*

Let us explain all the terms in this definition. A *symplectic form* is a nondegenerate closed 2-form ω. A *Liouville field* for ω is a vector field X satisfying $L_X\omega = \omega$; by Cartan's formula, this is equivalent to saying that the associated *Liouville form*

$$\lambda := i_X\omega$$

satisfies $d\lambda = \omega$. A function $\phi : V \to \mathbb{R}$ is called *exhausting* if it is proper (i.e., preimages of compact sets are compact) and bounded from below. It is called *Morse* if all its critical points are nondegenerate, and *generalized Morse* if its critical points are either nondegenerate or *embryonic*, where the latter condition means that in some local coordinates $x_1, \dots, x_m$ near the critical point p the function looks like the function ϕ_0 in the *birth–death family*

$$\phi_t(x) = \phi_t(p) \pm tx_1 + x_1^3 - \sum_{i=2}^{k} x_i^2 + \sum_{j=k+1}^{m} x_j^2.$$

A vector field X is called *complete* if its flow exists for all times. It is called *gradient-like* for a function ϕ if

$$d\phi(X) \geq \delta(|X|^2 + |d\phi|^2),$$

where $\delta : V \to \mathbb{R}_+$ is a positive function and the norms are taken with respect to any Riemannian metric on V. Note that away from critical points this just means $d\phi(X) > 0$. Critical points p of ϕ agree with zeroes of X, and p is nondegenerate (resp. embryonic) as a critical point of ϕ if and only if it is nondegenerate (resp. embryonic) as a zero of X. Here a zero p of a vector field X is called embryonic if X agrees near p, up to higher order terms, with the gradient of a function having p as an embryonic critical point.

It is not hard to see that any Weinstein structure (ω, X, ϕ) can be perturbed to make the function ϕ Morse. However, in 1-parameter families of Weinstein structures embryonic zeroes are generically unavoidable. Since we wish to study such families, we allow for embryonic zeroes in the definition of a Weinstein structure.

We will also consider Weinstein structures on a *cobordism*, that is, a compact manifold W with boundary $\partial W = \partial_+ W \amalg \partial_- W$. The definition of a *Weinstein cobordism* (W, ω, X, ϕ) differs from that of a Weinstein manifold only in replacing the condition that ϕ is exhausting by the requirement that $\partial_\pm W$ are regular level sets of ϕ with $\phi|_{\partial_- W} = \min \phi$ and $\phi|_{\partial_+ W} = \max \phi$, and completeness of X by the condition that X points inward along $\partial_- W$ and outward along $\partial_+ W$.

A Weinstein cobordism with $\partial_- W = \varnothing$ is called a *Weinstein domain*. Thus any Weinstein manifold (V, ω, X, ϕ) can be exhausted by Weinstein domains $W_k = \{\phi \le c_k\}$, where $c_k \nearrow \infty$ is a sequence of regular values of the function ϕ.

The Liouville form $\lambda = i_X\omega$ induces contact forms $\alpha_c := \lambda|_{\Sigma_c}$ and contact structures $\xi_c := \ker(\alpha_c)$ on all regular level sets $\Sigma_c := \phi^{-1}(c)$ of ϕ. In particular, the boundary components of a Weinstein cobordism carry contact forms which make $\partial_+ W$ a symplectically convex and $\partial_- W$ a symplectically concave boundary (i.e., the orientation induced by the contact form agrees with the boundary orientation on $\partial_+ W$ and is opposite to it on $\partial_- W$). Contact manifolds which appear as boundaries of Weinstein domains are called *Weinstein fillable*.

A Weinstein manifold (V, ω, X, ϕ) is said to be of *finite type* if ϕ has only finitely many critical points. By attaching a cylindrical end

$$\left(\mathbb{R}_+ \times \partial W, d(e^r \lambda|_{\partial W}), \frac{\partial}{\partial r}, f(r)\right)$$

(i.e., the positive half of the symplectization of the contact structure on the boundary) to the boundary, any Weinstein domain (W, ω, X, ϕ) can be completed to a finite type Weinstein manifold, called its *completion*. Conversely, any finite type Weinstein manifold can be obtained by attaching a cylindrical end to a Weinstein domain.

Here are some basic examples of Weinstein manifolds:

(1) $\mathbb{C}^n$ with complex coordinates $x_j + iy_j$ carries the canonical Weinstein structure

$$\left(\sum_j dx_j \wedge dy_j,\ \frac{1}{2}\sum_j \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right),\ \sum_j (x_j^2 + y_j^2)\right).$$

(2) The cotangent bundle T^*Q of a closed manifold Q carries a canonical Weinstein structure which in canonical local coordinates (q_j, p_j) is given by

$$\left(\sum_j dp_j \wedge dq_j,\ \sum_j p_j \frac{\partial}{\partial p_j},\ \sum_j p_j^2\right).$$

(As it stands, this is not yet a Weinstein structure because $\sum_j p_j^2$ is not a generalized Morse function, but a perturbation can easily be constructed to make the function Morse.)

(3) The product of two Weinstein manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ has a canonical Weinstein structure $(V_1 \times V_2, \omega_1 \oplus \omega_2, X_1 \oplus X_2, \phi_1 \oplus \phi_2)$. The product $V \times \mathbb{C}$ with its canonical Weinstein structure is called the *stabilization* of the Weinstein manifold (V, ω, X, ϕ).

In a Weinstein manifold (V, ω, X, ϕ), there is an intriguing interplay between Morse theoretic properties of ϕ and symplectic geometry: the stable manifold W_p^- (with respect to the vector field X) of a critical point p is *isotropic* in the symplectic sense (i.e., $\omega|_{W_p^-} = 0$), and its intersection with every regular level set $\phi^{-1}(c)$ is *isotropic* in the contact sense (i.e., it is tangent to ξ_c). In particular, the Morse indices of critical points of ϕ are $\leq \frac{1}{2} \dim V$.

1B. ***Stein–Weinstein–Morse.*** Weinstein structures are related to several other interesting structures as shown in the following diagram:

$$\begin{array}{ccccc} \mathfrak{Stein} & \xrightarrow{\mathfrak{W}} & \mathfrak{Weinstein} & \xrightarrow{\mathfrak{M}} & \mathfrak{Morse} \\ & & \downarrow & & \\ & & \mathfrak{Liouville}. & & \end{array}$$

Here $\mathfrak{Weinstein}$ denotes the space of Weinstein structures and $\mathfrak{Morse}$ the space of generalized Morse functions on a fixed manifold V or a cobordism W. As before, we require the function ϕ to be exhausting in the manifold case, and to have $\partial_\pm W$ as regular level sets with $\phi|_{\partial_- W} = \min \phi$ and $\phi|_{\partial_+ W} = \max \phi$ in the cobordism case. The map $\mathfrak{M} : \mathfrak{Weinstein} \to \mathfrak{Morse}$ is the obvious one $(\omega, X, \phi) \mapsto \phi$.

The space $\mathfrak{Liouville}$ of *Liouville structures* consists of pairs (ω, X) of a symplectic form ω and a vector field X (the *Liouville field*) satisfying $L_X \omega = \omega$. Moreover, in the cobordism case we require that the Liouville field X points inward along $\partial_- W$ and outward along $\partial_+ W$, and in the manifold case we require that X is complete and there exists an exhaustion $V_1 \subset V_2 \subset \cdots$ of $V = \cup_k V_k$ by compact sets with smooth boundary ∂V_k along which X points outward. The map $\mathfrak{Weinstein} \to \mathfrak{Liouville}$ sends (ω, X, ϕ) to (ω, X). Note that to each Liouville structure (ω, X) we can associate the *Liouville form* $\lambda := i_X \omega$, and (ω, X) can be recovered from λ by the formulas $\omega = d\lambda$ and $i_X d\lambda = \lambda$.

The space $\mathfrak{Stein}$ of *Stein structures* consists of pairs (J, ϕ) of an integrable complex structure J and a generalized Morse function ϕ (exhausting resp. constant on the boundary components) such that $-dd^{\mathbb{C}}\phi(v, Jv) > 0$ for all nonzero $v \in TV$, where $d^{\mathbb{C}}\phi := d\phi \circ J$. If (J, ϕ) is a Stein structure, then $\omega_\phi := -dd^{\mathbb{C}}\phi$ is a symplectic form compatible with J. Moreover, the Liouville field X_ϕ defined by

$$i_{X_\phi} \omega_\phi = -d^{\mathbb{C}}\phi$$

is the gradient of ϕ with respect to the Riemannian metric $g_\phi := \omega_\phi(\cdot, J\cdot)$. In the manifold case, completeness of X_ϕ can be arranged by replacing ϕ by $f \circ \phi$ for a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ with $f'' \geq 0$ and $\lim_{x \to \infty} f'(x) = \infty$; we will suppress the function f from the notation. So we have a canonical map

$$\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}, \quad (J, \phi) \mapsto (\omega_\phi, X_\phi, \phi).$$

It is interesting to compare the homotopy types of these spaces. For simplicity, let us consider the case of a compact domain W and equip all spaces with the C^∞ topology. The results which we discuss below remain true in the manifold case, but one needs to define the topology more carefully; see Section 4C. Since all the spaces have the homotopy types of CW complexes, any weak homotopy equivalence between them is a homotopy equivalence.

The spaces $\mathfrak{Liouville}$ and $\mathfrak{Weinstein}$ are very different: there exist many examples of Liouville domains that admit no Weinstein structure, and of contact manifolds that bound a Liouville domain but no Weinstein domain. The first such example was constructed in [McDuff 1991]: the manifold $[0,1] \times \Sigma$, where Σ is the unit cotangent bundle of a closed oriented surface of genus > 1, carries a Liouville structure, but its boundary is disconnected and hence cannot bound a Weinstein domain. Many more such examples are discussed in [Geiges 1994].

By contrast, the spaces of Stein and Weinstein structures turn out to be closely related. One of the main results of [Cieliebak and Eliashberg 2012] is this:

Theorem 1.1. *The map* $\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}$ *induces an isomorphism on* π_0 *and a surjection on* π_1.

It lends evidence to the conjecture that $\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}$ is a homotopy equivalence.

The relation between the spaces $\mathfrak{Morse}$ and $\mathfrak{Weinstein}$ is the subject of this article. Note first that, since for a Weinstein domain (W, ω, X, ϕ) of real dimension $2n$ all critical points of ϕ have index $\le n$, one should only consider the subset $\mathfrak{Morse}_n \subset \mathfrak{Morse}$ of functions all of whose critical points have index $\le n$. Moreover, one should restrict to the subset $\mathfrak{Weinstein}_\eta^{\text{flex}} \subset \mathfrak{Weinstein}$ of Weinstein structures (ω, X, ϕ) with ω in a fixed given homotopy class η of nondegenerate 2-forms which are *flexible* in the sense of Section 2 below. The following sections are devoted to the proof of the next theorem.

Theorem 1.2 [Cieliebak and Eliashberg 2012]. *Let* η *be a nonempty homotopy class of nondegenerate* 2*-forms on a domain or manifold of dimension* $2n > 4$. *Then*:

(a) *Any Morse function* $\phi \in \mathfrak{Morse}_n$ *can be lifted to a flexible Weinstein structure* (ω, X, ϕ) *with* $\omega \in \eta$.

(b) *Given two flexible Weinstein structures* (ω_0, X_0, ϕ_0) *and* (ω_1, X_1, ϕ_1) *in* $\mathfrak{Weinstein}_\eta^{\text{flex}}$, *any path* $\phi_t \in \mathfrak{Morse}_n$, $t \in [0,1]$, *connecting* ϕ_0 *and* ϕ_1 *can be lifted to a path of flexible Weinstein structures* (ω_t, X_t, ϕ_t) *connecting* (ω_0, X_0, ϕ_0) *and* (ω_1, X_1, ϕ_1).

In other words, the map $\mathfrak{M} : \mathfrak{Weinstein}_\eta^{\rm flex} \to \mathfrak{Morse}_n$ has the following properties:

- $\mathfrak{M}$ is surjective;
- the fibers of $\mathfrak{M}$ are path connected;
- $\mathfrak{M}$ has the path lifting property.

This motivates the following:

Conjecture. On a domain or manifold of dimension $2n > 4$, the map

$$\mathfrak{M} : \mathfrak{Weinstein}_\eta^{\rm flex} \to \mathfrak{Morse}_n$$

is a Serre fibration with contractible fibers.

2. Flexible Weinstein structures

Roughly speaking, a Weinstein structure is "flexible" if all its attaching spheres obey an h-principle. More precisely, note that each Weinstein manifold or cobordism can be cut along regular level sets of the function into Weinstein cobordisms that are elementary in the sense that there are no trajectories of the vector field connecting different critical points. An elementary $2n$-dimensional Weinstein cobordism (W, ω, X, ϕ), $n > 2$, is called *flexible* if the attaching spheres of all index n handles form in $\partial_- W$ a *loose* Legendrian link in the sense of Section 2C below. A Weinstein cobordism or manifold structure (ω, X, ϕ) is called flexible if it can be decomposed into elementary flexible cobordisms.

A $2n$-dimensional Weinstein structure (ω, X, ϕ), $n \geq 2$, is called *subcritical* if all critical points of the function ϕ have index $< n$. In particular, any subcritical Weinstein structure in dimension $2n > 4$ is flexible.

The notion of flexibility can be extended to dimension 4 as follows. We call a 4-dimensional Weinstein cobordism *flexible* if it is either subcritical, or the contact structure on $\partial_- W$ is overtwisted (or both); see Section 2B below. In particular, a 4-dimensional Weinstein *manifold* is then flexible if and only if it is subcritical.

Remark 2.1. The property of a Weinstein structure being subcritical is not preserved under Weinstein homotopies because one can always create index n critical points (see Proposition 4.7 below). We do not know whether flexibility is preserved under Weinstein homotopies. In fact, it is not even clear to us whether every decomposition of a flexible Weinstein cobordism W into elementary cobordisms consists of flexible elementary cobordisms. Indeed, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of W into elementary cobordisms and $\mathcal{P}_2$ is finer than $\mathcal{P}_1$, then flexibility of $\mathcal{P}_1$ implies flexibility of $\mathcal{P}_2$ (in particular the partition for which each elementary cobordism contains only one critical value is then flexible), but we do not know whether flexibility of $\mathcal{P}_2$ implies flexibility of $\mathcal{P}_1$.

The remainder of this section is devoted to the definition of loose Legendrian links and a discussion of the relevant h-principles.

2A. *Gromov's h-principle for subcritical isotropic embeddings.* Consider a contact manifold $(M, \xi = \ker\alpha)$ of dimension $2n-1$ and a manifold Λ of dimension $k-1 \le n-1$. A *monomorphism* $F : T\Lambda \to TM$ is a fiberwise injective bundle homomorphism covering a smooth map $f : \Lambda \to M$. It is called *isotropic* if it sends each $T_x\Lambda$ to a symplectically isotropic subspace of $\xi_{f(x)}$ (with respect to the symplectic form $d\alpha|_\xi$). A *formal isotropic embedding* of Λ into (M, ξ) is a pair (f, F^s), where $f : \Lambda \hookrightarrow M$ is a smooth embedding and $F^s : T\Lambda \to TM$, $s \in [0, 1]$, is a homotopy of monomorphisms covering f that starts at $F^0 = df$ and ends at an isotropic monomorphism $F^1 : T\Lambda \to \xi$. In the case $k = n$ we also call this a *formal Legendrian embedding*.

Any genuine isotropic embedding can be viewed as a formal isotropic embedding $(f, F^s \equiv df)$. We will not distinguish between an isotropic embedding and its canonical lift to the space of formal isotropic embeddings. A homotopy of formal isotropic embeddings (f_t, F_t^s), $t \in [0, 1]$, will be called a *formal isotropic isotopy*. Note that the maps f_t underlying a formal isotropic isotopy form a smooth isotopy.

In the *subcritical* case $k < n$, Gromov proved the following h-principle.

Theorem 2.2 (h-principle for subcritical isotropic embeddings [Gromov 1986; Eliashberg and Mishachev 2002]). *Let (M, ξ) be a contact manifold of dimension $2n-1$ and Λ a manifold of dimension $k-1 < n-1$. Then the inclusion of the space of isotropic embeddings $\Lambda \hookrightarrow (M, \xi)$ into the space of formal isotropic embeddings is a weak homotopy equivalence. In particular*:

(a) *Given any formal isotropic embedding (f, F^s) of Λ into (M, ξ), there exists an isotropic embedding $\tilde f : \Lambda \hookrightarrow M$ which is C^0-close to f and formally isotropically isotopic to (f, F^s).*

(b) *Let (f_t, F_t^s), $t \in [0, 1]$, be a formal isotropic isotopy connecting two isotropic embeddings $f_0, f_1 : \Lambda \hookrightarrow M$. Then there exists an isotropic isotopy $\tilde f_t$ connecting $\tilde f_0 = f_0$ and $\tilde f_1 = f_1$ which is C^0-close to f_t and is homotopic to the formal isotopy (f_t, F_t^s) through formal isotropic isotopies with fixed endpoints.*

Let us discuss what happens with this theorem in the critical case $k = n$. Part (a) remains true in all higher dimensions $k = n > 2$:

Theorem 2.3 (existence theorem for Legendrian embeddings for $n > 2$ [Eliashberg 1990; Cieliebak and Eliashberg 2012][1]). *Let (M, ξ) be a contact manifold of dimension $2n - 1 \geq 5$ and Λ a manifold of dimension $n - 1$. Then given any formal Legendrian embedding (f, F^s) of Λ into (M, ξ), there exists a Legendrian embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is C^0-close to f and formally Legendrian isotopic to (f, F^s).*

Part (b) of Theorem 2.2 does not carry over to the critical case $k = n$: For any $n \geq 2$, there are many examples of pairs of Legendrian knots in $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ which are formally Legendrian isotopic but not Legendrian isotopic; see, for example, [Chekanov 2002; Ekholm et al. 2005].

2B. ***Legendrian knots in overtwisted contact manifolds.*** Finally, we consider Theorem 2.2 in the case $k = n = 2$, that is, for Legendrian knots (or links) in contact 3-manifolds. Recall that in dimension 3 there is a dichotomy between tight and overtwisted contact structures, which was introduced in [Eliashberg 1989]. A contact structure ξ on a 3-dimensional manifold M is called *overtwisted* if there exists an embedded disc $D \subset M$ which is tangent to ξ along its boundary ∂D. Equivalently, one can require the existence of an embedded disc with Legendrian boundary ∂D which is transverse to ξ along ∂D. A disc with such properties is called an *overtwisted disc*.

Part (a) of Theorem 2.2 becomes false for $k = n = 2$ due to Bennequin's inequality. Let us explain this for $\mathbb{R}^3$ with its standard (tight) contact structure $\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}$, $\alpha_{\mathrm{st}} = dz - p\,dq$. To any formal Legendrian embedding (f, F^s) of S^1 into $(\mathbb{R}^3, \xi_{\mathrm{st}})$ we can associate two integers as follows. Identifying ξ_{st} to $\mathbb{R}^2$ via the projection $\mathbb{R}^3 \to \mathbb{R}^2$ onto the (q, p)-plane, the fiberwise injective bundle homomorphism $F^1 : TS^1 \cong S^1 \times \mathbb{R} \to \xi_{\mathrm{st}} \cong \mathbb{R}^2$ gives rise to a map $S^1 \to \mathbb{R}^2 \setminus 0$, $t \mapsto F^1(t, 1)$. The winding number of this map around $0 \in \mathbb{R}^2$ is called the *rotation number* $r(f, F^1)$. On the other hand, (F^1, iF^1, ∂_z) defines a trivialization of the bundle $f^*T\mathbb{R}^3$, where i is the standard complex structure on $\xi_{\mathrm{st}} \cong \mathbb{R}^2 \cong \mathbb{C}$. Using the homotopy F^s, we homotope this to a trivialization (e_1, e_2, e_3) of $f^*T\mathbb{R}^3$ with $e_1 = \dot{f}$ (unique up to homotopy). The *Thurston–Bennequin invariant* $\mathrm{tb}(f, F^s)$ is the linking number of f with a push-off in direction e_2. It is not hard to see that the pair of invariants (r, tb) yields a bijection between homotopy classes of formal Legendrian embeddings covering a fixed smooth embedding f and $\mathbb{Z}^2$. In particular, the pair (r, tb) can take arbitrary values on formal Legendrian embeddings, while for genuine Legendrian embeddings $f : S^1 \hookrightarrow (\mathbb{R}^3, \xi_{\mathrm{st}})$ the

[1] The hypothesis in [Cieliebak and Eliashberg 2012] that Λ is simply connected can be easily removed.

values of (r, tb) are constrained by *Bennequin's inequality* [1983]

$$\mathrm{tb}(f) + |r(f)| \leq -\chi(\Sigma),$$

where Σ is a Seifert surface for f.

Bennequin's inequality, and thus the failure of part (a), carry over to all tight contact 3-manifolds. On the other hand, Bennequin's inequality fails, and except for the C^0-closeness Theorem 2.2 remains true, on overtwisted contact 3-manifolds:

Theorem 2.4 [Dymara 2001; Eliashberg and Fraser 2009]. *Let (M, ξ) be a closed connected overtwisted contact* 3*-manifold*, *and $D \subset M$ an overtwisted disc.*

(a) *Any formal Legendrian knot (f, F^s) in M is formally Legendrian isotopic to a Legendrian knot $\tilde{f} : S^1 \hookrightarrow M \setminus D$.*

(b) *Let (f_t, F_t^s), $s, t \in [0, 1]$, be a formal Legendrian isotopy in M connecting two Legendrian knots $f_0, f_1 : S^1 \hookrightarrow M \setminus D$. Then there exists a Legendrian isotopy $\tilde{f_t} : S^1 \hookrightarrow M \setminus D$ connecting $\tilde{f_0} = f_0$ and $\tilde{f_1} = f_1$ which is homotopic to (f_t, F_t^s) through formal Legendrian isotopies with fixed endpoints.*

Although Theorem 2.2 (b) generally fails for knots in tight contact 3-manifolds, there are some remnants for special classes of Legendrian knots:

- any two formally Legendrian isotopic *unknots* in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ are Legendrian isotopic [Eliashberg and Fraser 2009];
- any two formally Legendrian isotopic knots become Legendrian isotopic after sufficiently many stabilizations (whose number depends on the knots) [Fuchs and Tabachnikov 1997].

E. Murphy [2012] discovered that the situation becomes much cleaner for $n > 2$: on any contact manifold of dimension ≥ 5 there exists a class of Legendrian knots, called *loose*, which satisfy both parts of Theorem 2.2. Let us now describe this class.

2C. ***Murphy's h-principle for loose Legendrian knots.*** In order to define loose Legendrian knots we need to describe a local model. Throughout this section we assume $n > 2$.

Consider a Legendrian arc λ_0 in the standard contact space $(\mathbb{R}^3, dz - p_1 dq_1)$ with front projection as shown in Figure 1, for some $a > 0$. Suppose that the slopes at the self-intersection point, as well as at end points of the interval are ± 1, and the slope is everywhere in the interval $[-1, 1]$, so the Legendrian arc λ_0

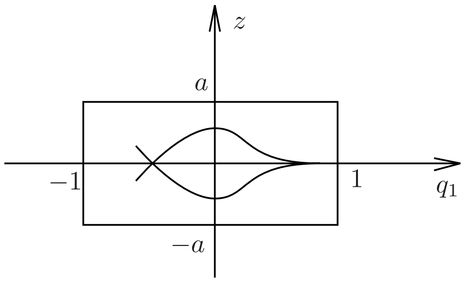

Figure 1. Front of the Legendrian arc λ_0.

is contained in the box

$$Q_a := \{|q_1|, |p_1| \le 1, |z| \le a\}$$

and $\partial\lambda_0 \subset \partial Q_a$. Take the standard contact space $\left(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i\right)$, which we view as the product of the contact space $(\mathbb{R}^3, dz - p_1 dq_1)$ and the Liouville space $\left(\mathbb{R}^{2n-4}, -\sum_{i=2}^{n-1} p_i dq_i\right)$. With $q' := (q_2, \dots, q_{n-1})$ and similarly for p', we set

$$|p'| := \max_{2\le i\le n-2} |p_i| \quad \text{and} \quad |q'| := \max_{2\le i\le n-2} |q_i|.$$

For $b, c > 0$ we define

$$P_{bc} := \{|q'| \le b,\ |p'| \le c\} \subset \mathbb{R}^{2n-4},$$
$$R_{abc} := Q_a \times P_{bc} = \{|q_1|, |p_1| \le 1,\ |z| \le a,\ |q'| \le b,\ |p'| \le c\}.$$

Let the Legendrian solid cylinder $\Lambda_0 \subset \left(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i\right)$ be the product of $\lambda_0 \subset \mathbb{R}^3$ with the Lagrangian disc $\{p' = 0, |q'| \le b\} \subset \mathbb{R}^{2n-4}$. Note that $\Lambda_0 \subset R_{abc}$ and $\partial\Lambda_0 \subset \partial R_{abc}$. The front of Λ_0 is obtained by translating the front of λ_0 in the q'-directions; see Figure 2. The pair (R_{abc}, Λ_0) is called a *standard loose Legendrian chart* if

$$a < bc.$$

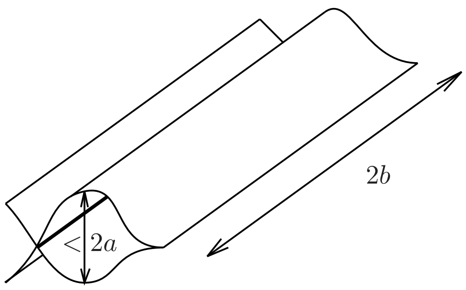

Figure 2. Front of the Legendrian solid cylinder Λ_0.

Given any contact manifold (M^{2n-1}, ξ), a Legendrian submanifold $\Lambda \subset M$ with connected components $\Lambda_1, \dots, \Lambda_k$ is called *loose* if there exist Darboux charts $U_1, \dots, U_k \subset M$ such that $\Lambda_i \cap U_j = \varnothing$ for $i \neq j$ and each pair $(U_i, \Lambda_i \cap U_i)$, $i = 1, \dots, k$, is isomorphic to a standard loose Legendrian chart (R_{abc}, Λ_0). A Legendrian embedding $f : \Lambda \hookrightarrow M$ is called loose if its image is a loose Legendrian submanifold.

Remarks 2.5. (1) By the contact isotopy extension theorem, looseness is preserved under Legendrian isotopies within a fixed contact manifold. Since any two Legendrian discs are Legendrian isotopic, any Legendrian disc is isotopic to its own stabilization (see Section 2D below), and, therefore, *loose*.

(2) By rescaling q' and p' with inverse factors one can always achieve $c = 1$ in the definition of a standard loose Legendrian chart. However, the inequality $a < bc$ is absolutely crucial in the definition. Indeed, it follows from Gromov's isocontact embedding theorem that around *any* point in *any* Legendrian submanifold Λ one can find a Darboux neighborhood U such that the pair $(U, \Lambda \cap U)$ is isomorphic to (R_{1b1}, Λ_0) for some sufficiently small $b > 0$.

(3) Figure 3, taken from [Murphy 2012], shows that the definition of looseness does not depend on the precise choice of the standard loose Legendrian chart (R_{abc}, Λ_0): Given a standard loose Legendrian chart with $c = 1$, the condition $a < b$ allows us to shrink its front in the q'-directions, keeping it fixed near the boundary and with all partial derivatives in $[-1, 1]$ (so the deformation remains in the Darboux chart R_{ab1}), to another standard loose Legendrian chart $(R_{a'b'1}, \Lambda'_0)$ with $b' \geq (b-a)/2$ and arbitrarily small $a' > 0$. Moreover, we can arbitrarily prescribe the shape of the cross section λ'_0 of Λ'_0 in this process. So if a Legendrian submanifold is loose for some model (R_{abc}, Λ_0), then it is also loose for any other model. In particular, fixing b, c we can make a arbitrarily

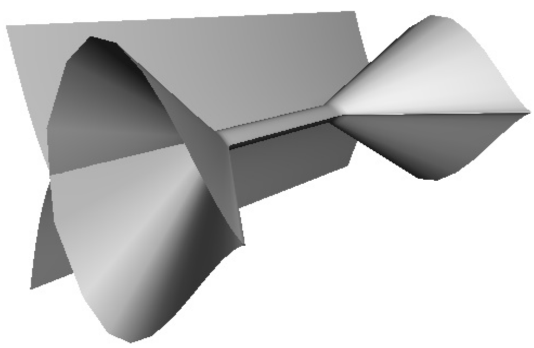

Figure 3. Shrinking a standard loose Legendrian chart (courtesy of E. Murphy).

small, and we can create arbitrarily many disjoint standard loose Legendrian charts.

Now we can state the main result from [Murphy 2012].

Theorem 2.6 (Murphy's h-principle for loose embeddings). *Let (M,ξ) be a contact manifold of dimension $2n-1 \geq 5$ and Λ a manifold of dimension $n-1$.*

(a) *Given any formal Legendrian embedding (f, F^s) of Λ into (M,ξ), there exists a loose Legendrian embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is C^0-close to f and formally Legendrian isotopic to (f, F^s).*

(b) *Let $f_t : \Lambda \hookrightarrow M$, $t \in [0,1]$ be a smooth isotopy which begins with a loose Legendrian embedding f_0. Then there exists a loose Legendrian isotopy $\tilde{f}_t$ starting at f_0 and a C^0-small smooth isotopy f'_t connecting $\tilde{f}_1$ to f_1 such that the concatenation of $\tilde{f}_t$ and f'_t is homotopic to f_t through smooth isotopies with fixed endpoints.*

(c) *Let (f_t, F^s_t), $s,t \in [0,1]$, be a formal Legendrian isotopy connecting two loose Legendrian knots f_0 and f_1. Then there exists a Legendrian isotopy $\tilde{f}_t$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is homotopic to the formal isotopy (f_t, F^s_t) through formal isotopies with fixed endpoints.*

Note that (b) is a direct consequence of (c) and a 1-parametric version of (a). Part (a) is a consequence of Theorem 2.3 and the stabilization construction which we describe next.

2D. ***Stabilization of Legendrian submanifolds.*** Consider a Legendrian submanifold Λ_0 in a contact manifold (M,ξ) of dimension $2n-1$. Near a point of Λ_0, pick Darboux coordinates $(q_1, p_1, \dots, q_{n-1}, p_{n-1}, z)$ in which

$$\xi = \ker\Big(dz - \sum_j p_j \, dq_j\Big)$$

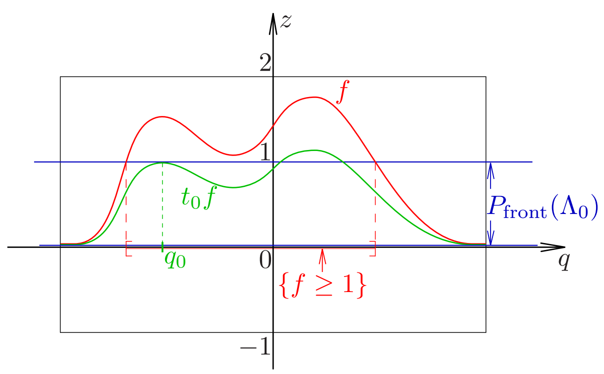

Figure 4. Stabilization of a Legendrian submanifold.

and the front projection of Λ_0 is a standard cusp $z^2 = q_1^3$. Deform the two branches of the front to make them parallel over some open ball $B^{n-1} \subset \mathbb{R}^{n-1}$. After rescaling, we may thus assume that the front of Λ_0 has two parallel branches $\{z = 0\}$ and $\{z = 1\}$ over B^{n-1}; see Figure 4.

Pick a nonnegative function $\phi : B^{n-1} \to \mathbb{R}$ with compact support and 1 as a regular value, so $N := \{\phi \geq 1\} \subset B^{n-1}$ is a compact manifold with boundary. Replacing for each $t \in [0, 1]$ the lower branch $\{z = 0\}$ by the graph $\{z = t\phi(q)\}$ of the function $t\phi$ yields the fronts of a path of Legendrian immersions $\Lambda_t \subset M$ connecting Λ_0 to a new Legendrian submanifold Λ_1. Note that Λ_t has a self-intersection for each critical point of $t\phi$ on level 1.

We count the self-intersections with signs as follows. Consider the immersion $\Gamma := \bigcup_{t\in[0,1]} \Lambda_t \times \{t\} \subset M \times [0, 1]$. After a generic perturbation, we may assume that Γ has finitely many transverse self-intersections and define its *self-intersection index*

$$I_\Gamma := \sum_p I_\Gamma(p) \in \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ \mathbb{Z}_2 & \text{if } n \text{ is odd,} \end{cases}$$

as the sum over the indices of all self-intersection points p. Here the index $I_\Gamma(p) = \pm 1$ is defined by comparing the orientations of the two intersecting branches of Γ to the orientation of $M \times [0, 1]$. For n even this does not depend on the order of the branches and thus gives a well-defined integer, while for n odd it is only well-defined mod 2. By a theorem of Whitney [1944], for $n > 2$, the regular homotopy Λ_t can be deformed through regular homotopies fixed at $t = 0, 1$ to an isotopy if and only if $I_\Gamma = 0$.

Proposition 2.7 [Murphy 2012]. *For $n > 2$, the Legendrian regular homotopy Λ_t obtained from the stabilization construction over a nonempty domain $N \subset B^{n-1}$ has the following properties*:

(a) Λ_1 *is* loose.

(b) *If $\chi(N) = 0$, then Λ_1 is formally Legendrian isotopic to Λ_0.*

(c) *The regular homotopy $(\Lambda_t)_{t\in[0,1]}$ has self-intersection index*

$$(-1)^{(n-1)(n-2)/2}\chi(N).$$

Proof. (a) Recall that in the stabilization construction we choose a Darboux chart in which the front of Λ_0 consists of the two branches $\{z = \pm q_1^{3/2}\}$ of a standard cusp, and then deform the lower branch to the graph of a function ϕ which is bigger than $q_1^{3/2}$ over a domain $N \subset \mathbb{R}^{n-1}$; see Figure 5. Performing this construction sufficiently close to the cusp edge, we can keep the values and the differential of the function ϕ arbitrarily small. Then the deformation is

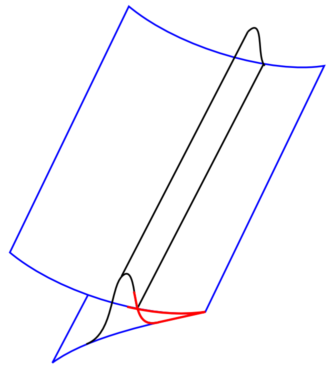

Figure 5. A standard loose Legendrian chart appears in the stabilization procedure.

localized within the chosen Darboux neighborhood, and comparing Figures 5 and 2 we see that Λ_1 is loose.

(b) Consider again the stabilization construction on the two parallel branches $\{z = 0\}$ and $\{z = 1\}$ of Λ_0 over the domain $N = \{\phi \geq 1\}$. Since $\chi(N) = 0$, there exists a nowhere vanishing vector field v on N which agrees with $\nabla\phi$ near ∂N. Linearly interpolating the p-coordinate of Λ_1 from $\nabla\phi(q)$ to $v(q)$ (keeping (q, z) fixed), then pushing the z-coordinate down to 0 (keeping (q, p) fixed), and finally linearly interpolating $v(q)$ to 0 (keeping (q, z) fixed) defines a smooth isotopy $f_t : \Lambda_0 \hookrightarrow M$ from $f_0 = \mathbb{1} : \Lambda_0 \to \Lambda_0$ to a parametrization $f_1 : \Lambda_0 \to \Lambda_1$. On the other hand, the graphs of the functions $t\phi$ define a Legendrian regular homotopy from f_0 to f_1, so their differentials give a path of Legendrian monomorphisms F_t from $F_0 = df_0$ to $F_1 = df_1$. Now note that over the region N all the df_t and F_t project as the identity onto the q-plane, so linearly connecting df_t and F_t yields a path of monomorphisms F_t^s, $s \in [0, 1]$, and hence the desired formal Legendrian isotopy (f_t, F_t^s) from f_0 to f_1.

To prove (c), make the function ϕ Morse on N and apply the Poincaré–Hopf index theorem. □

Since for $n > 2$ there exist domains $N \subset \mathbb{R}^{n-1}$ of arbitrary Euler characteristic $\chi(N) \in \mathbb{Z}$, we can apply Proposition 2.7 in two ways: Choosing $\chi(N) = 0$, we can C^0-approximate every Legendrian submanifold Λ_0 by a loose one which is formally Legendrian homotopic to Λ_0. Combined with Theorem 2.3, this proves Theorem 2.6(a). Choosing $\chi(N) \neq 0$, we can connect each Legendrian submanifold Λ_0 to a (loose) Legendrian submanifold Λ_1 by a Legendrian regular homotopy Λ_t with any prescribed self-intersection index. This will be a crucial ingredient in the proof of existence of Weinstein structures.

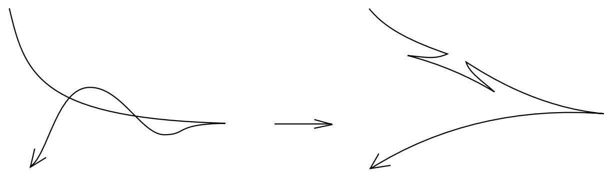

Figure 6. Stabilization in dimension 3.

Remark 2.8. For $n = 2$ we can still perform the stabilization construction. However, since every domain $N \subset \mathbb{R}$ is a union of intervals, the self-intersection index $\chi(N)$ in Proposition 2.7 is now always *positive* and hence cannot be arbitrarily prescribed. Figure 6 shows two front projections of the stabilization over an interval (related by Legendrian Reidemeister moves; see [Etnyre 2005], for example). Thus our stabilization introduces a downward and an upward zigzag, which corresponds to a *positive and a negative stabilization* in the usual terminology of 3-dimensional contact topology. It leaves the rotation number unchanged and *decreases* the Thurston–Bennequin invariant by 2, in accordance with Bennequin's inequality. In particular, stabilization in dimension 3 never preserves the formal Legendrian isotopy class.

2E. *Totally real discs attached to a contact boundary.* The following Theorems 2.9 and 2.10 are combinations of the h-principles discussed in Sections 2A and 2C with Gromov's h-principle [1986] for totally real embeddings.

Let (V, J) be an almost complex manifold and $W \subset V$ a domain with smooth boundary ∂W. Let L be a compact manifold with boundary. Let

$$f : L \hookrightarrow V \setminus \operatorname{Int} W$$

be an embedding with $f(\partial L) = f(L) \cap \partial W$ which is transverse to ∂W along ∂L. We say in this case that f *transversely* attaches L to W along ∂L. If, in addition, $J df(TL|_{\partial L}) \subset T(\partial W)$, then we say that f attaches L to W *J-orthogonally*. Note that this implies that $df(\partial L)$ is tangent to the *maximal J-invariant distribution* $\xi = T(\partial W) \cap JT(\partial W)$. In particular, if the distribution ξ is a contact structure, then $f(\partial L)$ is an isotropic submanifold for the contact structure ξ.

Theorem 2.9. *Let (V, J) be an almost complex manifold of real dimension $2n$, and $W \subset V$ a domain such that the distribution $\xi = T(\partial W) \cap JT(\partial W)$ is contact. Suppose that an embedding $f : D^k \hookrightarrow V$, $k \le n$, transversely attaches D^k to W along ∂D^k. If $k = n = 2$ we assume, in addition, that the induced contact structure on ∂W is overtwisted. Then there exists an isotopy $f_t : D^k \hookrightarrow V$, $t \in [0, 1]$, through embeddings transversely attaching D^k to W, such that $f_0 = f$, and f_1 is totally real and J-orthogonally attached to W. Moreover, in the case*

$k = n > 2$ we can arrange that the Legendrian embedding

$$f_1|_{\partial D^k} : \partial D^k \hookrightarrow \partial W$$

is loose, while for $k = n = 2$ we can arrange that the complement $\partial W \setminus f_t(\partial D^2)$ is overtwisted for all $t \in [0, 1]$.

We will also need the following 1-parametric version of Theorem 2.9 for totally real discs attached along loose knots.

Theorem 2.10. *Let J_t, $t \in [0, 1]$, be a family of almost complex structures on a $2n$-dimensional manifold V. Let $W \subset V$ be a domain with smooth boundary such that the distribution $\xi_t = T(\partial W) \cap J_t T(\partial W)$ is contact for each $t \in [0, 1]$. Let*

$$f_t : D^k \hookrightarrow V \setminus \operatorname{Int} W, \quad t \in [0, 1],\ k \leq n,$$

be an isotopy of embeddings transversely attaching D^k to W along ∂D^k. Suppose that for $i = 0, 1$ the embedding f_i is J_i-totally real and J_i-orthogonally attached to W. Suppose that either $k < n$ or $k = n > 2$ and the Legendrian embeddings $f_i|_{\partial D}$, $i = 0, 1$ are loose. *Then there exists a 2-parameter family of embeddings $f_t^s : D^k \hookrightarrow V \setminus \operatorname{Int} W$ with the following properties*:

- *f_t^s is transversely attached to W along ∂D^k and C^0-close to f_t for all $t, s \in [0, 1]$.*
- *$f_t^0 = f_t$ for all $t \in [0, 1]$ and $f_0^s = f_0$, $f_1^s = f_1$ for all $s \in [0, 1]$.*
- *f_t^1 is J_t-totally real and J_t-orthogonally attached to W along ∂D^k for all $t \in [0, 1]$.*

3. Morse preliminaries

In this section we gather some notions and results from Morse theory that are needed for our main results. We omit most of the proofs and refer the reader to the corresponding chapter of [Cieliebak and Eliashberg 2012]. Throughout this section, V denotes a smooth manifold and W a cobordism, both of dimension m.

3A. ***Gradient-like vector fields.*** A smooth function $\phi : V \to \mathbb{R}$ is called *Lyapunov* for a vector field for X, and X is called *gradient-like* for ϕ, if

$$X \cdot \phi \geq \delta(|X|^2 + |d\phi|^2) \tag{1}$$

for a positive function $\delta : V \to \mathbb{R}_+$, where $|X|$ is the norm with respect to some Riemannian metric on V and $|d\phi|$ is the dual norm. By the Cauchy–Schwarz inequality, condition (1) implies

$$\delta|X| \leq |d\phi| \leq \frac{1}{\delta}|X|. \tag{2}$$

In particular, zeroes of X coincide with critical points of ϕ.

Lemma 3.1. (a) *If X_0, X_1 are gradient-like vector fields for ϕ, then so is $f_0X_0 + f_1X_1$ for any nonnegative functions f_0, f_1 with $f_0 + f_1 > 0$.*

(b) *If ϕ_0, ϕ_1 are Lyapunov functions for X, then so is $\lambda_0\phi_0 + \lambda_1\phi_1$ for any nonnegative constants λ_0, λ_1 with $\lambda_0 + \lambda_1 > 0$.*

In particular, the following spaces are convex cones and hence contractible:

- *the space of Lyapunov functions for a given vector field X;*
- *the space of gradient-like vector fields for a given function ϕ.*

Proof. Consider two vector fields X_0, X_1 satisfying $X_i \cdot \phi \geq \delta_i(|X_i|^2 + |d\phi|^2)$ and nonnegative functions f_0, f_1 with $f_0 + f_1 > 0$. Then the vector field $X = f_0X_0 + f_1X_1$ satisfies (1) with $\delta := \min\{\delta_0/2f_0, \delta_1/2f_1, f_0\delta_0 + f_1\delta_1\}$:

$$\begin{aligned} X \cdot \phi &\geq f_0\delta_0|X_0|^2 + f_1\delta_1|X_1|^2 + (f_0\delta_0 + f_1\delta_1)|d\phi|^2 \\ &\geq 2\delta(|f_0X_0|^2 + |f_1X_1|^2) + \delta|d\phi|^2 \\ &\geq \delta(|X|^2 + |d\phi|^2). \end{aligned}$$

Positive combinations of functions are treated analogously. □

3B. ***Morse and Smale cobordisms.*** A *(generalized) Morse cobordism* is a pair (W, ϕ), where W is a cobordism and $\phi : W \to \mathbb{R}$ is a (generalized) Morse function which has $\partial_\pm W$ as its regular level sets such that $\phi|_{\partial_- W} < \phi|_{\partial_+ W}$. A *(generalized) Smale cobordism* is a triple (W, ϕ, X), where (W, ϕ) is a (generalized) Morse cobordism and X is a gradient-like vector field for ϕ. Note that X points inward along $\partial_- W$ and outward along $\partial_+ W$. A generalized Smale cobordism (W, ϕ, X) is called *elementary* if there are no X-trajectories between different critical points of ϕ.

If (W, ϕ, X) is an elementary generalized Smale cobordism, then the stable manifold of each nondegenerate critical point p is a disc D_p^- which intersects $\partial_- W$ along a sphere $S_p^- = \partial D_p^-$. We call D_p^- and S_p^- the *stable disc (resp. sphere)* of p. Similarly, the unstable manifolds and their intersections with $\partial_+ W$ are called *unstable discs and spheres*. For an embryonic critical point p, the closure of the (un)stable manifold is the *(un)stable half-disc* $\widehat{D}_p^\pm$ intersecting $\partial_\pm W$ along the hemisphere $\widehat{S}_p^\pm$.

An *admissible partition* of a generalized Smale cobordism (W, ϕ, X) is a finite sequence $m = c_0 < c_1 < \cdots < c_N = M$ of regular values of ϕ, where we denote $\phi|_{\partial_- W} = m$ and $\phi|_{\partial_+ W} = M$, such that each subcobordism

$$W_k = \{c_{k-1} \leq \phi \leq c_k\}, \quad k = 1, \ldots, N,$$

is elementary. The following lemma is straightforward.

Lemma 3.2. *Any generalized Smale cobordism admits an admissible partition into elementary cobordisms. Similarly, for any exhausting generalized Morse function ϕ and gradient-like vector field X on a noncompact manifold V, one can find regular values $c_0 < \min\phi < c_1 < \cdots \to \infty$ such that the cobordisms $W_k = \{c_{k-1} \le \phi \le c_k\}$, $k = 1, \dots,$ are elementary. If ϕ has finitely many critical points, then all but finitely many of these cobordisms have no critical points.*

3C. ***Morse and Smale homotopies.*** A smooth family (W, ϕ_t), $t \in [0,1]$, of generalized Morse cobordism structures is called a *Morse homotopy* if there is a finite set $A \subset (0,1)$ with the following properties:

- for each $t \in A$ the function ϕ_t has a unique birth-death type critical point e_t such that $\phi_t(e_t) \ne \phi_t(q)$ for all other critical points q of ϕ_t;
- for each $t \notin A$ the function ϕ_t is Morse.

A *Smale homotopy* is a smooth family (W, X_t, ϕ_t), $t \in [0,1]$, of generalized Smale cobordism structures such that (W, ϕ_t) a Morse homotopy. A Smale homotopy $\mathfrak{S}_t = (W, X_t, \phi_t)$, $t \in [0,1]$ is called an *elementary Smale homotopy* of Type I, IIb, IId, respectively, if the following holds:

- Type I. $\mathfrak{S}_t$ is an elementary Smale cobordism for all $t \in [0,1]$.
- Type IIb (birth). There is $t_0 \in (0,1)$ such that for $t < t_0$ the function ϕ_t has no critical points, ϕ_{t_0} has a birth type critical point, and for $t > t_0$ the function ϕ_t has two critical points p_t and q_t of index i and $i-1$, respectively, connected by a unique X_t-trajectory.
- Type IId (death). There is $t_0 \in (0,1)$ such that for $t > t_0$ the function ϕ_t has no critical points, ϕ_{t_0} has a death type critical point, and for $t < t_0$ the function ϕ_t has two critical points p_t and q_t of index i and $i-1$, respectively, connected by a unique X_t-trajectory.

We will also refer to an elementary Smale homotopy of Type IIb (resp. IId) as a *creation* (*resp. cancellation*) *family*.

An *admissible partition* of a Smale homotopy $\mathfrak{S}_t = (W, X_t, \phi_t)$, $t \in [0,1]$, is a sequence $0 = t_0 < t_1 < \cdots < t_p = 1$ of parameter values, and for each $k = 1, \dots, p$ a finite sequence of functions

$$m(t) = c_0^k(t) < c_1^k(t) < \cdots < c_{N_k}^k(t) = M(t), \quad t \in [t_{k-1}, t_k],$$

where $m(t) := \phi_t(\partial_- W)$ and $M(t) := \phi_t(\partial_+ W)$, such that $c_j^k(t)$, $j = 0, \dots, N_k$ are regular values of ϕ_t and each Smale homotopy

$$\mathfrak{S}_j^k := \big(W_j^k(t) := \{c_{j-1}^k(t) \le \phi_t \le c_j^k(t)\}, X_t|_{W_j^k(t)}, \phi_t|_{W_j^k(t)}\big)_{t \in [t_{k-1}, t_k]}$$

is elementary.

Lemma 3.3. *Any Smale homotopy admits an admissible partition.*

Proof. Let $A \subset (0,1)$ be the finite subset in the definition of a Smale homotopy. Using Lemma 3.2, we now first construct an admissible partition on $\mathcal{O}p\, A$ and then extend it over $[0,1] \setminus \mathcal{O}p\, A$. □

3D. *Equivalence of elementary Smale homotopies.* We define the *profile* (or Cerf diagram) of a Smale homotopy $\mathfrak{S}_t = (W, X_t, \phi_t)$, $t \in [0,1]$, as the subset $C(\{\phi_t\}) \subset \mathbb{R} \times \mathbb{R}$ such that $C(\{\phi_t\}) \cap (t \times \mathbb{R})$ is the set of critical values of the function ϕ_t. We will use the notion of profile only for elementary homotopies.

The following two easy lemmas are proved in [Cieliebak and Eliashberg 2012]. The first one shows that if two elementary Smale homotopies have the same profile, then their functions are related by diffeomorphisms.

Lemma 3.4. *Let $\mathfrak{S}_t = (W, X_t, \phi_t)$ and $\tilde{\mathfrak{S}}_t = (W, \tilde{X}_t, \tilde{\phi}_t)$, $t \in [0,1]$, be two elementary Smale homotopies with the same profile such that $\mathfrak{S}_0 = \tilde{\mathfrak{S}}_0$. Then there exists a diffeotopy $h_t : W \to W$ with $h_0 = \mathbb{1}$ such that $\phi_t = \tilde{\phi}_t \circ h_t$ for all $t \in [0,1]$. Moreover, if $\phi_t = \tilde{\phi}_t$ near $\partial_+ W$ and/or $\partial_- W$ we can arrange $h_t = \mathbb{1}$ near $\partial_+ W$ and/or $\partial_- W$.*

The second lemma provides elementary Smale homotopies with prescribed profile.

Lemma 3.5. *Let (W, X, ϕ) be an elementary Smale cobordism with $\phi|_{\partial_\pm W} = a_\pm$ and critical points $p_1, \dots, p_n$ of values $\phi(p_i) = c_i$. For $i = 1, \dots, n$ let $c_i : [0,1] \to (a_-, a_+)$ be smooth functions with $c_i(0) = c_i$. Then there exists a smooth family ϕ_t, $t \in [0,1]$, of Lyapunov functions for X with $\phi_0 = \phi$ and $\phi_t = \phi$ on $\mathcal{O}p\, \partial W$ such that $\phi_t(p_i) = c_i(t)$.*

Here we use Gromov's notation $\mathcal{O}p\, A$ for an unspecified neighborhood of a subset $A \subset W$.

3E. *Holonomy of Smale cobordisms.* Let (W, X, ϕ) be a Smale cobordism such that the function ϕ has no critical points. The *holonomy* of X is the diffeomorphism

$$\Gamma_X : \partial_+ W \to \partial_- W,$$

which maps $x \in \partial_+ W$ to the intersection of its trajectory under the flow of $-X$ with $\partial_- W$.

Consider now a Morse cobordism (W, ϕ) without critical points. Denote by $\mathcal{X}(W, \phi)$ the space of all gradient-like vector fields for ϕ. Note that the holonomies of all $X \in \mathcal{X}(W, \phi)$ are isotopic. We denote by $\mathcal{D}(\partial_+ W, \partial_- W)$ the corresponding path component in the space of diffeomorphisms from $\partial_+ W$ to $\partial_- W$. All spaces are equipped with the C^∞-topology.

Recall that a continuous map $p : E \to B$ is a *Serre fibration* if it has the homotopy lifting property for all closed discs D^k, that is, given a homotopy $h_t : D^k \to B$, $t \in [0,1]$, and a lift $\tilde{h}_0 : D^k \to E$ with $p \circ \tilde{h}_0 = h_0$, there exists a homotopy $\tilde{h}_t : D^k \to E$ with $p \circ \tilde{h}_t = h_t$. We omit the proof of the following easy lemma.

Lemma 3.6. *Let (W, ϕ) be a Morse cobordism without critical points. Then the map $\mathscr{X}(W, \phi) \to \mathscr{D}(\partial_+ W, \partial_- W)$ that assigns to X its holonomy Γ_X is a Serre fibration. In particular*:

- (i) *Given $X \in \mathscr{X}(W, \phi)$ and an isotopy $h_t \in \mathscr{D}(\partial_+ W, \partial_- W)$, $t \in [0,1]$, with $h_0 = \Gamma_X$, there exists a path $X_t \in \mathscr{X}(W, \phi)$ with $X_0 = X$ such that $\Gamma_{X_t} = h_t$ for all $t \in [0,1]$.*
- (ii) *Given a path $X_t \in \mathscr{X}(W, \phi)$, $t \in [0,1]$, and a path $h_t \in \mathscr{D}(\partial_+ W, \partial_- W)$ which is homotopic to Γ_{X_t} with fixed endpoints, there exists a path $\tilde{X}_t \in \mathscr{X}(W, \phi)$ with $\tilde{X}_0 = X_0$ and $\tilde{X}_1 = X_1$ such that $\Gamma_{\tilde{X}_t} = h_t$ for all $t \in [0,1]$.*

As a consequence, we obtain:

Lemma 3.7. *Let X_t, Y_t be two paths in $\mathscr{X}(W, \phi)$ starting at the same point $X_0 = Y_0$. Suppose that for a subset $A \subset \partial_+ W$, one has $\Gamma_{X_t}(A) = \Gamma_{Y_t}(A)$ for all $t \in [0,1]$. Then there exists a path $\widehat{X}_t \in \mathscr{X}(W, \phi)$ such that*

- (i) $\widehat{X}_t = X_{2t}$ *for* $t \in \left[0, \frac{1}{2}\right]$;
- (ii) $\widehat{X}_1 = Y_1$;
- (iii) $\Gamma_{\widehat{X}_t}(A) = \Gamma_{Y_1}(A)$ *for* $t \in \left[\frac{1}{2}, 1\right]$.

Proof. Consider the path $\gamma : [0,1] \to \mathscr{D}(\partial_+ W, \partial_- W)$ given by the formula

$$\gamma(t) := \Gamma_{X_1} \circ \Gamma_{X_t}^{-1} \circ \Gamma_{Y_t}.$$

We have $\gamma(0) = \Gamma_{X_1}$ and $\gamma(1) = \Gamma_{Y_1}$. The path γ is homotopic with fixed endpoints to the concatenation of the paths $\Gamma_{X_{1-t}}$ and Γ_{Y_t}. Hence by Lemma 3.6 we conclude that there exists a path $X'_t \in \mathscr{X}(W, \phi)$ such that $X'_0 = X_1$, $X'_1 = Y_1$, and $\Gamma_{X'_t} = \gamma(t)$ for all $t \in [0,1]$. Since

$$\Gamma_{X'_t}(A) = \Gamma_{X_1}\big(\Gamma_{X_t}^{-1}(\Gamma_{Y_t}(A))\big) = \Gamma_{X_1}(A) = \Gamma_{Y_1}(A),$$

the concatenation $\widehat{X}_t$ of the paths X_t and X'_t has the required properties. □

4. Weinstein preliminaries

In this section we collect some facts about Weinstein structures needed for the proofs of our main results. Most of the proofs are omitted and we refer the reader to [Cieliebak and Eliashberg 2012] for a more systematic treatment of the subject.

4A. ***Holonomy of Weinstein cobordisms.*** In this subsection we consider Weinstein cobordisms $\mathfrak{W} = (W, \omega, X, \phi)$ *without critical points* (of the function ϕ). Its holonomy along trajectories of X defines a contactomorphism

$$\Gamma_{\mathfrak{W}} : (\partial_+ W, \xi_+) \to (\partial_- W, \xi_-)$$

for the contact structures $\xi_\pm$ on $\partial_\pm W$ induced by the Liouville form $\lambda = i_X \omega$.

We say that two Weinstein structures $\mathfrak{W} = (\omega, X, \phi)$ and $\tilde{\mathfrak{W}}$ agree *up to scaling* on a subset $A \subset W$ if $\tilde{\mathfrak{W}}|_A = (C\omega, X, \phi)$ for a constant $C > 0$. Note that in this case $\tilde{\mathfrak{W}}|_A$ has Liouville form $C\lambda$.

Let us fix a Weinstein cobordism $\overline{\mathfrak{W}} = (W, \overline{\omega}, \overline{X}, \phi)$ without critical points. We denote by $\mathcal{W}(\overline{\mathfrak{W}})$ the space of all Weinstein structures $\mathfrak{W} = (W, \omega, X, \phi)$ with the same function ϕ such that

- $\mathfrak{W}$ coincides with $\overline{\mathfrak{W}}$ on $\mathcal{O}p\, \partial_- W$ and up to scaling on $\mathcal{O}p\, \partial_+ W$;
- $\mathfrak{W}$ and $\overline{\mathfrak{W}}$ induce the same contact structures on level sets of ϕ.

Equivalently, $\mathcal{W}(\overline{\mathfrak{W}})$ can be viewed as the space of Liouville forms $\lambda = f\bar{\lambda} + g\, d\phi$ with $f \equiv 1$ near $\partial_- W$, $f \equiv C$ near $\partial_+ W$, and $g \equiv 0$ near ∂W, where $\overline{\lambda}$ denotes the Liouville form of $\overline{\mathfrak{W}}$.

Denote by $\mathfrak{D}(\overline{\mathfrak{W}})$ the space of contactomorphisms $(\partial_+ W, \xi_+) \to (\partial_- W, \xi_-)$, where $\xi_\pm$ is the contact structure induced on $\partial_\pm W$ by $\overline{\mathfrak{W}}$. Note that $\Gamma_{\mathfrak{W}} \in \mathfrak{D}(\overline{\mathfrak{W}})$ for any $\mathfrak{W} \in \mathcal{W}(\overline{\mathfrak{W}})$. The following lemma is the analogue of Lemma 3.6 in the context of Weinstein cobordisms.

Lemma 4.1. *Let $\overline{\mathfrak{W}}$ be a Weinstein cobordism without critical points. Then the map $\mathcal{W}(\overline{\mathfrak{W}}) \to \mathfrak{D}(\overline{\mathfrak{W}})$ that assigns to $\mathfrak{W}$ its holonomy $\Gamma_{\mathfrak{W}}$ is a Serre fibration. In particular*:

(i) *Given $\mathfrak{W} \in \mathcal{W}(\overline{\mathfrak{W}})$ and an isotopy $h_t \in \mathfrak{D}(\overline{\mathfrak{W}})$, $t \in [0, 1]$, with $h_0 = \Gamma_{\mathfrak{W}}$, there exists a path $\mathfrak{W}_t \in \mathcal{W}(\overline{\mathfrak{W}})$ with $\mathfrak{W}_0 = \mathfrak{W}$ such that $\Gamma_{\mathfrak{W}_t} = h_t$ for all $t \in [0, 1]$.*

(ii) *Given a path $\mathfrak{W}_t \in \mathcal{W}(\overline{\mathfrak{W}})$, $t \in [0, 1]$, and a path $h_t \in \mathfrak{D}(\overline{\mathfrak{W}})$ which is homotopic to $\Gamma_{\mathfrak{W}_t}$ with fixed endpoints, there exists a path $\tilde{\mathfrak{W}}_t \in \mathcal{W}(\overline{\mathfrak{W}})$ with $\tilde{\mathfrak{W}}_0 = \mathfrak{W}_0$ and $\tilde{\mathfrak{W}}_1 = \mathfrak{W}_1$ such that $\Gamma_{\tilde{\mathfrak{W}}_t} = h_t$ for all $t \in [0, 1]$.*

4B. ***Weinstein structures near stable discs.*** The following two lemmas concern the construction of Weinstein structures near stable discs of Smale cobordisms.

Lemma 4.2. *Let $\mathfrak{S} = (W, X, \phi)$ be an elementary Smale cobordism and ω a nondegenerate 2-form on W. Let $D_1, \dots, D_k$ be the stable discs of critical points of ϕ, and set $\Delta := \bigcup_{j=1}^k D_j$. Suppose that the discs $D_1, \dots, D_k$ are ω-isotropic and the pair (ω, X) is Liouville on $\mathcal{O}p\,(\partial_- W)$. Then for any neighborhood U of $\partial_- W \cup \Delta$ there exists a homotopy (ω_t, X_t), $t \in [0, 1]$, with these properties*:

(i) *X_t is a gradient-like vector field for ϕ and ω_t is a nondegenerate 2-form on W for all $t \in [0,1]$;*

(ii) *$(\omega_0, X_0) = (\omega, X)$, and $(\omega_t, X_t) = (\omega, X)$ outside U and on $\Delta \cup \mathcal{O}p\,(\partial_- W)$ for all $t \in [0,1]$;*

(iii) *(ω_1, X_1) is a Liouville structure on $\mathcal{O}p\,(\partial_- W \cup \Delta)$.*

Lemma 4.2 has the following version for homotopies.

Lemma 4.3. *Let $\mathfrak{S}_t = (W, X_t, \phi_t)$, $t \in [0,1]$, be an elementary Smale homotopy and ω_t, $t \in [0,1]$, a family of nondegenerate 2-forms on W. Let Δ_t be the union of the stable (half-)discs of zeroes of X_t and set*

$$\Delta := \bigcup_{t \in [0,1]} \{t\} \times \Delta_t \subset [0,1] \times W.$$

Suppose that Δ_t is ω_t-isotropic for all $t \in [0,1]$, the pair (ω_t, X_t) is Liouville on $\mathcal{O}p\,(\partial_- W)$ for all $t \in [0,1]$, and (ω_0, X_0) and (ω_1, X_1) are Liouville on all of W. Then, for any open neighborhood $V = \bigcup_{t \in [0,1]} \{t\} \times V_t$ of Δ, there exists an open neighborhood $U = \bigcup_{t \in [0,1]} \{t\} \times U_t \subset V$ of Δ and a 2-parameter family (ω_t^s, X_t^s), $s, t \in [0,1]$, with the following properties:

(i) *X_t^s is a gradient-like vector field for ϕ_t and ω_t^s is a nondegenerate 2-form on W for all $s, t \in [0,1]$.*

(ii) *$(\omega_t^0, X_t^0) = (\omega_t, X_t)$ for all $t \in [0,1]$, $(\omega_0^s, X_0^s) = (\omega_0, X_0)$ and $(\omega_1^s, X_1^s) = (\omega_1, X_1)$ for all $s \in [0,1]$, and $(\omega_t^s, X_t^s) = (\omega_t, X_t)$ outside V_t and on $\Delta_t \cup \mathcal{O}p\,(\partial_- W)$ for all $s, t \in [0,1]$.*

(iii) *(ω_t^1, X_t^1) is a Liouville structure on U_t for all $t \in [0,1]$.*

4C. ***Weinstein homotopies.*** A smooth family $(W, \omega_t, X_t, \phi_t)$, $t \in [0,1]$, of Weinstein cobordism structures is called a *Weinstein homotopy* if the family (W, X_t, ϕ_t) is a Smale homotopy in the sense of Section 3C. Recall that this means in particular that the functions ϕ_t have $\partial_\pm W$ as regular level sets, and they are Morse except for finitely many $t \in (0,1)$ at which a birth-death type critical point occurs.

The definition of a Weinstein homotopy on a *manifold V* requires more care. Consider first a smooth family $\phi_t : V \to \mathbb{R}$, $t \in [0,1]$, of exhausting generalized Morse functions such that there exists a finite set $A \subset (0,1)$ satisfying the conditions stated at the beginning of Section 3C. We call ϕ_t a *simple Morse homotopy* if there exists a sequence of smooth functions $c_1 < c_2 < \cdots$ on the interval $[0,1]$ such that for each $t \in [0,1]$, $c_i(t)$ is a regular value of the function ϕ_t and $\bigcup_k \{\phi_t \le c_k(t)\} = V$. A *Morse homotopy* is a composition of finitely many simple Morse homotopies. Then a *Weinstein homotopy* on the manifold

V is a family of Weinstein manifold structures $(V, \omega_t, X_t, \phi_t)$ such that the associated functions ϕ_t form a Morse homotopy.

The main motivation for this definition of a Weinstein homotopy is the following result from [Eliashberg and Gromov 1991] (see also [Cieliebak and Eliashberg 2012]).

Proposition 4.4. *Any two Weinstein manifolds* $\mathfrak{W}_0 = (V, \omega_0, X_0, \phi_0)$ *and* $\mathfrak{W}_1 = (V, \omega_1, X_1, \phi_1)$ *that are Weinstein homotopic are symplectomorphic. More precisely, there exists a diffeotopy* $h_t : V \to V$ *with* $h_0 = \mathbb{1}$ *such that* $h_1^*\lambda_1 - \lambda_0$ *is exact, where* $\lambda_i = i_{X_i}\omega_i$ *are the Liouville forms. If* $\mathfrak{W}_0$ *and* $\mathfrak{W}_1$ *are the completions of homotopic Weinstein domains, then we can achieve* $h_1^*\lambda_1 - \lambda_0 = 0$ *outside a compact set.*

Remark 4.5. Without the hypothesis on the functions $c_k(t)$ in the definition of a Weinstein homotopy, Proposition 4.4 would fail. Indeed, it is not hard to see that without this hypothesis all Weinstein structures on $\mathbb{R}^{2n}$ would be homotopic. But according to McLean [2009], there are infinitely many Weinstein structures on $\mathbb{R}^{2n}$ which are pairwise nonsymplectomorphic.

Remark 4.6. It is not entirely obvious but true (see [Cieliebak and Eliashberg 2012]) that any two exhausting Morse functions on the same manifold can be connected by a Morse homotopy.

The notion of Weinstein (or Stein) homotopy can be formulated in more topological terms. Let us denote by $\mathfrak{Weinstein}$ the space of Weinstein structures on a fixed manifold V. For any $\mathfrak{W}_0 \in \mathfrak{Weinstein}$, $\varepsilon > 0$, $A \subset V$ compact, $k \in \mathbb{N}$, and any unbounded sequence $c_1 < c_2 < \cdots$, we define the set

$$\mathcal{U}(\mathfrak{W}_0, \varepsilon, A, k, c) := \{\mathfrak{W} = (\omega, X, \phi) \in \mathfrak{Weinstein} \mid \|\mathfrak{W} - \mathfrak{W}_0\|_{C^k(A)} < \varepsilon,\ c_i \text{ regular values of } \phi\}.$$

It is easy to see that these sets are the basis of a topology on $\mathfrak{Weinstein}$, and a smooth family of Weinstein structures satisfying the conditions at the beginning of Section 3C defines a continuous path $[0, 1] \to \mathfrak{Weinstein}$ with respect to this topology if and only if (possibly after target reparametrization of the functions) it is a Weinstein homotopy according to the definition above. A topology on the space $\mathfrak{Morse}$ of exhausting generalized Morse functions can be defined similarly.

4D. ***Creation and cancellation of critical points of Weinstein structures.*** A key ingredient in Smale's proof of the h-cobordism theorem is the creation and cancellation of pairs of critical points of a Morse function. The following two propositions describe analogues of these operations for Weinstein structures.

Proposition 4.7 (creation of critical points). *Let (W, ω, X, ϕ) be a $2n$-dimensional Weinstein cobordism without critical points. Given any point $p \in \mathrm{Int}\, W$ and any integer $k \in \{1, \dots, n\}$, there exists a Weinstein homotopy (ω, X_t, ϕ_t) with the following properties*:

(i) $(X_0, \phi_0) = (X, \phi)$ *and* $(X_t, \phi_t) = (X, \phi)$ *outside a neighborhood of* p.

(ii) ϕ_t *is a creation family such that* ϕ_1 *has a pair of critical points of index* k *and* $k - 1$.

Proposition 4.8 (cancellation of critical points). *Let (W, ω, X, ϕ) be a Weinstein cobordism with exactly two critical points p, q of index k and $k-1$, respectively, which are connected by a unique X-trajectory along which the stable and unstable manifolds intersect transversely. Let Δ be the skeleton of (W, X), that is, the closure of the stable manifold of the critical point p. Then there exists a Weinstein homotopy (ω, X_t, ϕ_t) with the following properties*:

(i) $(X_0, \phi_0) = (X, \phi)$, *and* $(X_t, \phi_t) = (X, \phi)$ *near* ∂W *and outside a neighborhood of* Δ.

(ii) ϕ_t *has no critical points outside* Δ.

(iii) ϕ_t *is a cancellation family such that* ϕ_1 *has no critical points.*

5. Existence and deformations of flexible Weinstein structures

In this section we prove Theorem 1.2 from Section 1B and some other results about flexible Weinstein manifolds and cobordisms. For simplicity, we assume that individual functions are Morse and 1-parameter families are Morse homotopies in the sense of Section 3C. The more general case of arbitrary (1-parameter families of) generalized Morse functions is treated similarly.

5A. ***Existence of Weinstein structures.*** The next two theorems imply Theorem 1.2(a) from Section 1B.

Theorem 5.1 (Weinstein existence theorem). *Let (W, ϕ) be a $2n$-dimensional Morse cobordism such that ϕ has no critical points of index $> n$. Let η be a nondegenerate (not necessarily closed) 2-form on W and Y a vector field near $\partial_- W$ such that (η, Y, ϕ) defines a Weinstein structure on $\mathcal{O}p\, \partial_- W$. Suppose that either $n > 2$, or $n = 2$ and the contact structure induced by the Liouville form $\lambda = i_Y \eta$ on $\partial_- W$ is overtwisted. Then there exists a Weinstein structure (ω, X, ϕ) on W with the following properties*:

(i) $(\omega, X) = (\eta, Y)$ *on* $\mathcal{O}p\, \partial_- W$.

(ii) *The nondegenerate 2-forms ω and η on W are homotopic* rel $\mathcal{O}p\, \partial_- W$.

Moreover, we can arrange that (ω, X, ϕ) is flexible.

Theorem 5.1 immediately implies the following version for manifolds.

Theorem 5.2. *Let (V,ϕ) be a $2n$-dimensional manifold with an exhausting Morse function ϕ that has no critical points of index $> n$. Let η be a nondegenerate (not necessarily closed) 2-form on V. Suppose that $n > 2$. Then there exists a Weinstein structure (ω, X, ϕ) on V such that the nondegenerate 2-forms ω and η on V are homotopic. Moreover, we can arrange that (ω, X, ϕ) is flexible.*
□

Proof of Theorem 5.1. By decomposing the Morse cobordism $\mathfrak{M} = (W,\phi)$ into elementary ones, $W = W_1 \cup \cdots \cup W_N$, and inductively extending the Weinstein structure over $W_1,\ldots,W_N$, it suffices to consider the case of an elementary cobordism. To simplify the notation, we will assume that ϕ has a unique critical point p; the general case is similar. Let us extend Y to a gradient-like vector field for ϕ on W and denote by Δ the stable disc of p.

Step 1. We first show that, after a homotopy of (η, Y) fixed on $\mathcal{O}p\,\partial_- W$, we may assume that Δ is η-isotropic.

The Liouville form $\lambda = i_Y\eta$ on $\mathcal{O}p\,\partial_- W$ defines a contact structure $\xi := \ker(\lambda|_{\partial_- W})$ on $\partial_- W$. We choose an auxiliary η-compatible almost complex structure J on W which preserves ξ and maps Y along $\partial_- W$ to the Reeb vector field R of $\lambda|_{\partial_- W}$. We apply Theorem 2.9 to find a diffeotopy $f_t : W \to W$ such that the disc $\Delta' = f_1(\Delta)$ is J-totally real and J-orthogonally attached to $\partial_- W$. This is the only point in the proof where the overtwistedness assumption for $n = 2$ is needed. Moreover, according to Theorem 2.9, in the case $\dim \Delta = n$ we can arrange that the Legendrian sphere $\partial\Delta'$ in $(\partial_- W, \xi)$ is loose (meaning that $\partial_- W \setminus \partial\Delta'$ is overtwisted in the case $n = 2$).

Next we will modify the homotopy $f_t^* J$ to keep it fixed near $\partial_- W$. Because of J-orthogonality, $\partial\Delta'$ is tangent to the maximal J-invariant distribution $\xi \subset T(\partial_- W)$ and thus $\lambda|_{\partial\Delta'} = 0$. Since the spaces $T\Delta'$ and $\mathrm{span}\{T\partial\Delta', Y\}$ are both totally real and J-orthogonal to $T(\partial_- W)$, we can further adjust the disc Δ' (keeping $\partial\Delta'$ fixed) to make it tangent to Y in a neighborhood of $\partial\Delta'$. It follows that we can modify f_t such that it preserves the function ϕ and the vector field Y on a neighborhood U of $\partial_- W$ (extend f_t from $\partial_- W$ to U using the flow of Y).

Hence, there exists a diffeotopy $g_t : W \to W$, $t \in [0,1]$, which equals f_t on $W \setminus U$, the identity on $\mathcal{O}p\,\partial_- W$, and preserves ϕ (but not Y!) on U; see Figure 7. Then the diffeotopy $k_t := f_t^{-1} \circ g_t$ equals the identity on $W \setminus U$, f_t^{-1} on $\mathcal{O}p\,\partial_- W$, and preserves ϕ on all of W. Thus the vector fields $Y_t := k_t^* Y$ are gradient-like for $\phi = k_t^*\phi$ and coincide with Y on $(W \setminus U) \cup \mathcal{O}p\,\partial_- W$. The nondegenerate 2-forms $\eta_t := g_t^*\eta$ are compatible with $J_t := g_t^* J$ and coincide with η on $\mathcal{O}p\,\partial_- W$. Moreover, since Δ' is J-totally real, the stable

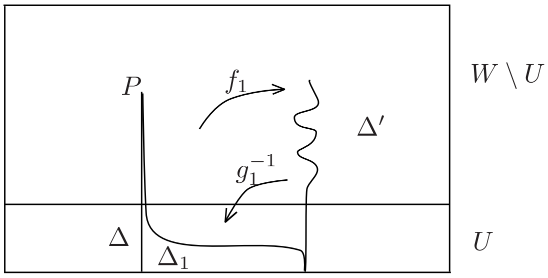

Figure 7. Deforming the disc Δ to one which is totally real and J-orthogonally attached.

disc $\Delta_1 := k_1^{-1}(\Delta) = g_1^{-1}(\Delta')$ of p with respect to Y_1 is J_1-totally real and J_1-orthogonally attached to $\partial_- W$.

After renaming (η_1, Y_1, Δ_1) back to (η, Y, Δ), we may hence assume that Δ is J-totally real and J-orthogonally attached to $\partial_- W$ for some η-compatible almost complex structure J on W which preserves ξ and maps Y to the Reeb vector field R along $\partial_- W$. In particular, $\partial\Delta$ is λ-isotropic and $\Delta \cap \mathcal{O}p\, \partial_- W$ is η-isotropic. Since the space of nondegenerate 2-forms compatible with J is contractible, after a further homotopy of η fixed on $\mathcal{O}p\, \partial_- W$ and outside a neighborhood of Δ, we may assume that Δ is η-isotropic.

Step 2. By Lemma 4.2 there exists a homotopy (η_t, Y_t), $t \in [0, 1]$, of gradient-like vector fields for ϕ and nondegenerate 2-forms on W, fixed on $\Delta \cup \mathcal{O}p\, \partial_- W$ and outside a neighborhood of Δ, such that $(\eta_0, Y_0) = (\eta, Y)$ and (η_1, Y_1) is Liouville on $\mathcal{O}p\, (\partial_- W \cup \Delta)$. After renaming (η_1, Y_1) back to (η, Y), we may hence assume that (η, Y) is Liouville on a neighborhood U of $\partial_- W \cup \Delta$.

Step 3. Pushing down along trajectories of Y, we construct an isotopy of embeddings $h_t : W \hookrightarrow W$, $t \in [0, 1]$, with $h_0 = \mathbb{1}$ and $h_t = \mathbb{1}$ on $\mathcal{O}p\, (\partial_- W \cup \Delta)$, which preserves trajectories of Y and such that $h_1(W) \subset U$. Then $(\eta_t, Y_t) := (h_t^* \eta, h_t^* Y)$ defines a homotopy of nondegenerate 2-forms and vector fields on W, fixed on $\mathcal{O}p\, (\partial_- W \cup \Delta)$, from $(\eta_0, Y_0) = (\eta, Y)$ to the Liouville structure $(\eta_1, Y_1) =: (\omega, X)$. Since the Y_t are proportional to Y, they are gradient-like for ϕ for all $t \in [0, 1]$.

The Weinstein structure (ω, X, ϕ) will be flexible if we choose the stable sphere $\partial\Delta$ in Step 1 to be loose, so Theorem 5.1 is proved. ☐

5B. ***Homotopies of flexible Weinstein structures.*** Theorems 5.3 and 5.4 for cobordisms, and Theorems 5.5 and 5.6 for manifolds, are our main results concerning deformations of flexible Weinstein structures. They imply Theorem 1.2(b).

Theorem 5.3 (first Weinstein deformation theorem). *Let $\mathfrak{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism of dimension $2n$. Let ϕ_t, $t \in [0, 1]$, be a Morse* *homotopy without critical points of index* $> n$ *with* $\phi_0 = \phi$ *and* $\phi_t = \phi$ *near* ∂W. *In the case* $2n = 4$ *assume that either* $\partial_- W$ *is overtwisted, or* ϕ_t *has no critical points of index* > 1. *Then there exists a homotopy* $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0, 1]$, *of flexible Weinstein structures, starting at* $\mathfrak{W}_0 = \mathfrak{W}$, *which is fixed near* $\partial_- W$ *and fixed up to scaling near* $\partial_+ W$.

Theorem 5.4 (second Weinstein deformation theorem). *Let* $\mathfrak{W}_0 = (\omega_0, X_0, \phi_0)$ *and* $\mathfrak{W}_1 = (\omega_1, X_1, \phi_1)$ *be two flexible Weinstein structures on a cobordism* W *of dimension* $2n$. *Let* ϕ_t, $t \in [0, 1]$, *be a Morse homotopy without critical points of index* $> n$ *connecting* ϕ_0 *and* ϕ_1. *In the case* $2n = 4$ *assume that either* $\partial_- W$ *is overtwisted, or* ϕ_t *has no critical points of index* > 1. *Let* η_t, $t \in [0, 1]$, *be a homotopy of nondegenerate (not necessarily closed)* 2*-forms connecting* ω_0 *and* ω_1 *such that* (η_t, Y_t, ϕ_t) *is Weinstein near* $\partial_- W$ *for a homotopy of vector fields* Y_t *on* $\mathcal{O}p\, \partial_- W$ *connecting* X_0 *and* X_1.

Then $\mathfrak{W}_0$ *and* $\mathfrak{W}_1$ *can be connected by a homotopy* $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in [0, 1]$, *of flexible Weinstein structures, agreeing with* (η_t, Y_t, ϕ_t) *on* $\mathcal{O}p\, \partial_- W$, *such that the paths of nondegenerate* 2*-forms* $t \mapsto \eta_t$ *and* $t \mapsto \omega_t$, $t \in [0, 1]$, *are homotopic rel* $\mathcal{O}p\, \partial_- W$ *with fixed endpoints.*

Theorems 5.3 and 5.4 will be proved in Sections 5C and 5D. They have the following analogues for deformations of flexible Weinstein *manifolds*, which are derived from the cobordism versions by induction over sublevel sets.

Theorem 5.5. *Let* $\mathfrak{W} = (V, \omega, X, \phi)$ *be a flexible Weinstein manifold of dimension* $2n$. *Let* ϕ_t, $t \in [0, 1]$, *be a Morse homotopy without critical points of index* $> n$ *with* $\phi_0 = \phi$. *In the case* $2n = 4$ *assume that* ϕ_t *has no critical points of index* > 1. *Then there exists a homotopy* $\mathfrak{W}_t = (V, \omega_t, X_t, \phi_t)$, $t \in [0, 1]$, *of flexible Weinstein structures such that* $\mathfrak{W}_0 = \mathfrak{W}$.

If the Morse homotopy ϕ_t *are fixed outside a compact set, then the Weinstein homotopy* $\mathfrak{W}_t$ *can be chosen fixed outside a compact set.* □

Theorem 5.6. *Let* $\mathfrak{W}_0 = (\omega_0, X_0, \phi_0)$ *and* $\mathfrak{W}_1 = (\omega_1, X_1, \phi_1)$ *be two flexible Weinstein structures on the same manifold* V *of dimension* $2n$. *Let* ϕ_t, $t \in [0, 1]$, *be a Morse homotopy without critical points of index* $> n$ *connecting* ϕ_0 *and* ϕ_1. *In the case* $2n = 4$, *assume that* ϕ_t *has no critical points of index* > 1. *Let* η_t *be a homotopy of nondegenerate* 2*-forms on* V *connecting* ω_0 *and* ω_1. *Then there exists a homotopy* $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$ *of flexible Weinstein structures connecting* $\mathfrak{W}_0$ *and* $\mathfrak{W}_1$ *such that the paths* ω_t *and* η_t *of nondegenerate* 2*-forms are homotopic with fixed endpoints.* □

5C. *Proof of the first Weinstein deformation theorem.* The proof of Theorem 5.3 is based on the following three lemmas.

Lemma 5.7. *Let $\mathfrak{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism and Y a gradient-like vector field for ϕ such that the Smale cobordism (W, Y, ϕ) is* elementary. *Then there exists a family X_t, $t \in [0, 1]$, of gradient-like vector fields for ϕ and a family ω_t, $t \in \left[0, \frac{1}{2}\right]$, of symplectic forms on W such that*

- $\mathfrak{W}_t = (W, \omega_t, X_t, \phi)$, $t \in \left[0, \frac{1}{2}\right]$, *is a flexible Weinstein homotopy with $\mathfrak{W}_0 = \mathfrak{W}$, fixed on $\mathcal{O}p\, \partial_- W$ and fixed up to scaling on $\mathcal{O}p\, \partial_+ W$;*
- $X_1 = Y$ *and the Smale cobordisms (W, X_t, ϕ), $t \in \left[\frac{1}{2}, 1\right]$, are elementary.*

Proof. *Step 1.* Let $c_1 < \cdots < c_N$ be the critical values of the function ϕ. Set $c_0 := \phi|_{\partial_- W}$ and $c_{N+1} := \phi|_{\partial_+ W}$. Choose

$$\varepsilon \in \left(0, \min_{j=0,\dots,N} \frac{c_{j+1} - c_j}{2}\right)$$

and define (see Figure 8)

$$\begin{aligned}
W_1 &:= \{\phi \le c_1 + \varepsilon\}, \\
W_j &:= \{c_j - \varepsilon \le \phi \le c_j + \varepsilon\}, \qquad j = 2, \dots, N-1, \\
W_N &:= \{\phi \ge c_N - \varepsilon\}, \\
V_j &:= \{c_j + \varepsilon \le \phi \le c_{j+1} - \varepsilon\}, \quad j = 1, \dots, N-1, \\
\Sigma_j^{\pm} &:= \{\phi = c_j \pm \varepsilon\}, \qquad j = 1, \dots, N;
\end{aligned}$$

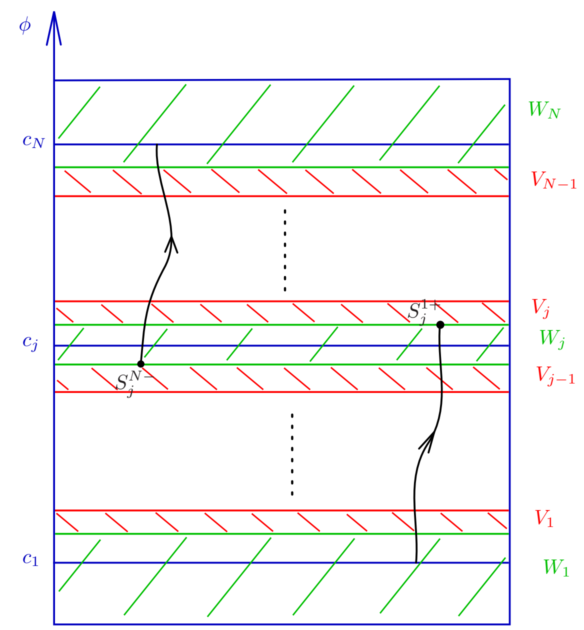

Figure 8. The partition of W into subcobordisms.

Thus we have

$$\Sigma_j^+ = \partial_- V_j = \partial_+ W_j \qquad \text{for } j = 1, \dots, N-1,$$
$$\Sigma_j^- = \partial_+ V_{j-1} = \partial_- W_j \qquad \text{for } j = 2, \dots, N.$$

We denote by $\xi_j^\pm$ the contact structure induced by the Liouville form $i_X\omega$ on $\Sigma_j^\pm$, $j = 1, \dots, N$.

For $k \geq j$ we denote by S_j^{k-} the intersection of the union of the Y-stable manifolds of the critical points on level c_k with the hypersurface Σ_j^-. Similarly, for $i \leq j$ we denote by S_j^{i+} the intersection of the union of the Y-unstable manifolds of the critical points on level c_i with the hypersurface Σ_j^+; see Figure 8. Set

$$S_j^- := \bigcup_{k \geq j} S_j^{k-}, \quad S_j^+ := \bigcup_{i \leq j} S_j^{i+}.$$

The assumption that the Smale cobordism (Y, ϕ) is elementary implies that $S_j^\pm$ is a union of spheres in $\Sigma_j^\pm$.

Consider on $\bigcup_{j=1}^N W_j$ the gradient-like vector fields $Y_t := (1-t)Y + tX$, $t \in [0,1]$, for ϕ. Let us pick ε so small that for all $t \in [0,1]$ the Y_t-unstable spheres in Σ_j^+ of the critical points on level c_j do not intersect the Y-stable spheres in Σ_j^+ of any critical points on higher levels. By Lemma 3.1 we can extend the Y_t to gradient-like vector fields for ϕ on W such that $Y_0 = Y$ and $Y_t = Y$ outside $\mathcal{O}p \bigcup_{j=1}^N W_j$ for all $t \in [0,1]$. By Lemma 3.6, this can be done in such a way that the intersection of the Y_t-stable manifold of the critical point locus on level c_i with the hypersurface Σ_j^+ remains unchanged. This implies that the cobordisms (W, Y_t, ϕ) are elementary for all $t \in [0,1]$. After renaming Y_1 back to Y and shrinking the W_j, we may hence assume that $Y = X$ on $\mathcal{O}p \bigcup_{j=1}^N W_j$.

We will construct the required homotopies X_t, $t \in [0,1]$, and ω_t, $t \in \left[0, \frac{1}{2}\right]$, separately on each V_j, $j = 1, \dots, N-1$, in such a way that X_t is fixed near ∂V_j for all $t \in [0,1]$ and ω_t is fixed up to scaling near ∂V_j for $t \in \left[0, \frac{1}{2}\right]$. This will allow us then to extend the homotopies X_t and ω_t to $\bigcup_{j=1}^N W_j$ as constant (resp. constant up to scaling).

Step 2. Consider V_j for $1 \leq j \leq N-1$. To simplify the notation, we denote the restriction of objects to V_j by the same symbol as the original objects, omitting the index j. Let us denote by $\mathcal{X}(V_j, \phi)$ the space of all gradient-like vector fields for ϕ on V_j that agree with X near ∂V_j. We connect X and Y by the path $Y_t := (1-t)X + tY$ in $\mathcal{X}(V_j, \phi)$.

Denote by $\Gamma_{Y_t} : \Sigma_{j+1}^- \to \Sigma_j^+$ the holonomy of the vector field Y_t on V_j and consider the isotopy $g_t := \Gamma_{Y_t}|_{S_{j+1}^-} : S_{j+1}^- \hookrightarrow \Sigma_j^+$. Suppose for the moment that $S^-_{j+1} \subset \Sigma^-_{j+1}$ *is isotropic and loose* (this hypothesis will be satisfied below when we perform induction on descending values of j).

Since $\Gamma_{Y_0} = \Gamma_X$ is a contactomorphism, this implies that the embedding g_0 is loose isotropic. Hence, by Theorem 2.2 for the subcritical case, Theorem 2.4 for the Legendrian overtwisted case in dimension 4, and Theorem 2.6 in the Legendrian loose case in dimension $2n > 4$, there exists a (loose) isotropic isotopy $\tilde{g}_t$ starting at g_0 and a C^0-small smooth isotopy g'_t connecting $\tilde{g}_1$ to g_1 such that the concatenation of $\tilde{g}_t$ and g'_t is homotopic to g_t through smooth isotopies with fixed endpoints. More precisely, by the isotopy extension theorem, we find diffeotopies $\delta_t, \widehat{\delta}_t : \Sigma^+_j \to \Sigma^+_j$ with the following properties:

- $\delta_0 = \widehat{\delta}_0 = \mathrm{Id}$ and $\delta_1 = \widehat{\delta}_1$;
- the diffeotopies δ_t and $\widehat{\delta}_t$ are homotopic with fixed endpoints;
- $\widehat{\delta}_t$ is C^0-small and $\widehat{\delta}_t \circ g_1(S^-_{j+1}) \cap S^+_j = \varnothing$ for all $t \in [0,1]$;
- $\delta_t \circ g_t$ is loose isotropic in Σ^+_j with respect to the contact structure ξ^+_j;
- $\delta_1 \circ g_1$ is loose isotropic in $\Sigma^+_j \setminus S^+_j$.

The path Γ_{Y_t}, $t \in [0,1]$, in $\mathrm{Diff}(\Sigma^-_{j+1}, \Sigma^+_j)$ is homotopic with fixed endpoints to the concatenation of the paths $\delta_t \circ \Gamma_{Y_t}$ (from Γ_{Y_0} to $\delta_1 \circ \Gamma_{Y_1}$) and $\widehat{\delta}_t^{-1} \circ \delta_1 \circ \Gamma_{Y_1}$ (from $\delta_1 \circ \Gamma_{Y_1}$ to Γ_{Y_1}). Hence by Lemma 9.41 we find paths Y'_t and Y''_t, $t \in [0,1]$, in $\mathscr{X}(V_j, \phi)$ such that

- $Y'_0 = X$, $Y'_1 = Y''_0$ and $Y''_1 = Y$;
- $\Gamma_{Y'_t} = \delta_t \circ \Gamma_{Y_t}$ and $\Gamma_{Y''_t} = \widehat{\delta}_t^{-1} \circ \delta_1 \circ \Gamma_{Y_1}$, $t \in [0,1]$.

Note that $\Gamma_{Y'_t}|_{S^-_{j+1}}$ is loose isotropic. Moreover, $\Gamma_{Y''_t}(S^-_{j+1}) \cap S^+_j = \varnothing$ in Σ^+_j and $\Gamma_{Y'_1}(S^-_{j+1})$ is loose in $\Sigma^+_j \setminus S^+_j$. So the image of $\Gamma_Y(S^-_{j+1})$ under the holonomy of the elementary Weinstein cobordism $(W_j, \omega, X = Y, \phi)$ is loose isotropic in Σ^-_j. Since the union S^-_j of the stable spheres of (W_j, Y) are loose by the flexibility hypothesis on $\mathfrak{W}$, this implies that $S^-_j \subset \Sigma^-_j$ is loose isotropic.

Now we perform this construction inductively in *descending* order over V_j for $j = N-1, N-2, \dots, 1$, always renaming the new vector fields back to Y. The resulting vector field Y is then connected to X by a homotopy Y_t such that the manifolds $S^-_{j+1} \subset \Sigma^-_{j+1}$ and the isotopies $\Gamma_{Y_t}|_{S^-_{j+1}} : S^-_{j+1} \hookrightarrow \Sigma^+_j$, $t \in [0,1]$, are loose isotropic for all $j = 1, \dots, N-1$.

Step 3. Let Y and Y_t be as constructed in Step 2. Now we construct the desired homotopies X_t and ω_t separately on each V_j, $j = 1, \dots, N-1$, keeping them fixed near ∂V_j. We keep the notation from Step 2. By the contact isotopy extension theorem, we can extend the isotropic isotopy $\Gamma_{Y_t}|_{S^-_{j+1}} : S^-_{j+1} \hookrightarrow \Sigma^+_j$

to a contact isotopy

$$G_t : (\Sigma^-_{j+1}, \xi^-_{j+1}) \to (\Sigma^+_j, \xi^+_j)$$

starting at $G_0 = \Gamma_{Y_0} = \Gamma_X$. By Lemma 4.1, we find a Weinstein homotopy $\tilde{\mathfrak{W}}_t = (V_j, \tilde{\omega}_t, \tilde{X}_t, \phi)$ beginning at $\tilde{\mathfrak{W}}_0 = \mathfrak{W}$ with holonomy $\Gamma_{\tilde{\mathfrak{W}}_t} = G_t$ for all $t \in [0, 1]$. Now Lemma 3.7 provides a path $X_t \in \mathscr{X}(V_j, \phi)$ such that

(i) $X_t = \tilde{X}_{2t}$ for $t \in \left[0, \frac{1}{2}\right]$;

(ii) $X_1 = Y_1 = Y$;

(iii) $\Gamma_{X_t}(S^-_{j+1}) = \Gamma_Y(S^-_{j+1})$ for $t \in \left[\frac{1}{2}, 1\right]$.

Over the interval $\left[0, \frac{1}{2}\right]$ the Smale homotopy $\mathfrak{S}_t = (V_j, X_t, \phi)$ can be lifted to the Weinstein homotopy $\mathfrak{W}_t = (V_j, \omega_t, X_t, \phi)$, where $\omega_t := \tilde{\omega}_{2t}$.

Condition (iii) implies that $\Gamma_{X_t}(S^-_{j+1}) \cap S^+_j = \varnothing$ for all $t \in \left[\frac{1}{2}, 1\right]$, so the resulting Smale homotopy on W is elementary over the interval $\left[\frac{1}{2}, 1\right]$. □

The following lemma is the analogue of Lemma 5.7 in the case that the Smale cobordism (W, Y, ϕ) is not elementary, but has exactly two critical points connected by a unique trajectory.

Lemma 5.8. *Let $\mathfrak{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism and Y a gradient-like vector field for ϕ. Suppose that the function ϕ has exactly two critical points connected by a unique Y-trajectory along which the stable and unstable manifolds intersect transversely. Then there exists a family X_t, $t \in [0, 1]$, of gradient-like vector fields for ϕ and a family ω_t, $t \in \left[0, \frac{1}{2}\right]$, of symplectic forms on W such that*

- *$\mathfrak{W}_t = (W, \omega_t, X_t, \phi)$, $t \in \left[0, \frac{1}{2}\right]$, is a flexible Weinstein homotopy with $\mathfrak{W}_0 = \mathfrak{W}$, fixed on $\mathcal{O}p\, \partial_- W$ and fixed up to scaling on $\mathcal{O}p\, \partial_+ W$;*
- *$X_1 = Y$ and for $t \in \left[\frac{1}{2}, 1\right]$ the critical points of the function ϕ are connected by a unique X_t-trajectory.*

Proof. Let us denote the critical points of the function ϕ by p_1 and p_2 and the corresponding critical values by $c_1 < c_2$. As in the proof of Lemma 5.7, for sufficiently small $\varepsilon > 0$, we split the cobordism W into three parts:

$$W_1 := \{\phi \le c_1 + \varepsilon\}, \quad V := \{c_1 + \varepsilon \le \phi \le c_2 - \varepsilon\}, \quad W_2 := \{\phi \ge c_2 - \varepsilon\}.$$

Arguing as in Step 1 of the proof of Lemma 5.7, we reduce to the case that $Y = X$ on $\mathcal{O}p\,(W_1 \cup W_2)$.

On V consider the gradient-like vector fields $Y_t := (1-t)X + tY$ for ϕ. Let $\Sigma := \{\phi = c_1 + \varepsilon\} = \partial_- V$. Denote by $S_t \subset \Sigma$ the Y_t-stable sphere of p_2 and by $S^+ \subset \Sigma$ the Y-unstable sphere of p_1. Note that S^+ is coisotropic, S_0 is isotropic, and S_1 intersects S^+ transversely in a unique point q. We deform S_1 to S_1' by a

C^0-small deformation, keeping the unique transverse intersection point q with S^+, such that S_1' is isotropic near q. Connect S_0 to S_1' by an isotopy S_t' which is C^0-close to S_t. Due to the flexibility hypothesis on $\mathfrak{W}$, the isotropic sphere $S_0' = S_0$ is loose. Hence by Theorems 2.2, 2.4, and 2.6, we find an isotropic isotopy $\tilde{S}_t$ with the following properties:

- $\tilde{S}_0 = S_0' = S_0$;
- $\tilde{S}_1$ is connected to S_1' by a C^0-small smooth isotopy that coincides with S_1' near q and has q as its unique transverse intersection point with S^+.

Arguing as in Steps 2 and 3 of the proof of Lemma 5.7, we now construct a Weinstein homotopy $\mathfrak{W}_t = (V, \omega_t, X_t, \phi)$, $t \in \left[0, \frac{1}{2}\right]$, fixed near $\partial_- V$ and fixed up to scaling near $\partial_+ V$, and Smale cobordisms (V, X_t, ϕ), $t \in \left[\frac{1}{2}, 1\right]$, fixed near ∂V, such that

- $\mathfrak{W}_0 = \mathfrak{W}|_V$ and $X_1 = Y|_V$;
- the X_t-stable sphere of p_2 in Σ equals $\tilde{S}_{2t}$ for $t \in \left[0, \frac{1}{2}\right]$, and $\tilde{S}_1$ for $t \in \left[\frac{1}{2}, 1\right]$.

In particular, for $t \in \left[\frac{1}{2}, 1\right]$ the X_t-stable sphere of p_2 in Σ intersects S^+ transversely in the unique point q, so the two critical points p_1, p_2 are connected by a unique X_t-trajectory for $t \in \left[\frac{1}{2}, 1\right]$. $\square$

The following lemma will serve as induction step in proving Theorem 5.3.

Lemma 5.9. *Let $\mathfrak{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism of dimension $2n$. Let $\mathfrak{S}_t = (W, Y_t, \phi_t)$, $t \in [0, 1]$, be an* elementary *Smale homotopy without critical points of index $> n$ such that $\phi_0 = \phi$ on W and $\phi_t = \phi$ near ∂W (but not necessarily $Y_0 = X$!). If $2n = 4$ and $\mathfrak{S}_t$ is of Type IIb assume that either $\partial_- W$ is overtwisted, or ϕ_t has no critical points of index > 1. Then there exists a homotopy $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0, 1]$, of flexible Weinstein structures, starting at $\mathfrak{W}_0 = \mathfrak{W}$, which is fixed near $\partial_- W$ and fixed up to scaling near $\partial_+ W$.*

Proof. Type I. Consider first the case when the homotopy $\mathfrak{S}_t$ is elementary of Type I. We point out that (W, X, ϕ) need not be elementary. To remedy this, we apply Lemma 5.7 to construct families X_t and ω_t such that

- $\mathfrak{W}_t = (W, \omega_t, X_t, \phi)$, $t \in \left[0, \frac{1}{2}\right]$, is a Weinstein homotopy with $\mathfrak{W}_0 = \mathfrak{W}$, fixed on $\mathcal{O}p\, \partial_- W$ and fixed up to scaling on $\mathcal{O}p\, \partial_+ W$;
- $X_1 = Y_0$ and the Smale cobordisms (W, X_t, ϕ), $t \in \left[\frac{1}{2}, 1\right]$, are elementary.

Thus it suffices to prove the lemma for the Weinstein cobordism $(\omega_{1/2}, X_{1/2}, \phi)$ instead of $\mathfrak{W}$, and the concatenation of the Smale homotopies $(X_t, \phi)_{t \in [\frac{1}{2}, 1]}$ and $(Y_t, \phi_t)_{t \in [0,1]}$ instead of (Y_t, ϕ_t). To simplify the notation, we rename the new

Weinstein cobordism and Smale homotopy back to $\mathfrak{W} = (\omega, X, \phi)$ and (Y_t, ϕ_t). So in the new notation we now have $X = Y_0$.

According to Lemma 3.5 there exists a family $\tilde{\phi}_t$, $t \in [0, 1]$, of Lyapunov functions for X with the same profile as the family ϕ_t, and such that $\tilde{\phi}_0 = \phi$ and $\tilde{\phi}_t = \phi_t$ on $\mathcal{O}p\, \partial W$. Then Lemma 3.4 provides a diffeotopy $h_t : W \to W$, $t \in [0, 1]$, such that $h_0 = \mathrm{Id}$, $h_t|_{\mathcal{O}p\, \partial W} = \mathrm{Id}$, and $\phi_t = \tilde{\phi}_t \circ h_t$ for all $t \in [0, 1]$. Thus the Weinstein homotopy $(W, \omega_t = h_t^*\omega, X_t = h_t^* X, \phi_t = h_t^*\tilde{\phi}_t)$, $t \in [0, 1]$, has the desired properties. It is flexible because the X_t-stable spheres in $\partial_- W$ are loose for $t = 0$ and moved by an isotropic isotopy, so they remain loose for all $t \in [0, 1]$.

Type IId. Suppose now that the homotopy $\mathfrak{S}_t$ is of Type IId. Let $t_0 \in [0, 1]$ be the parameter value for which the function ϕ_t has a death-type critical point. In this case the function ϕ has exactly two critical points p and q connected by a unique Y_0-trajectory.

Arguing as for Type I, but using Lemma 5.8 instead of Lemma 5.7, we can again reduce to the case that $X = Y_0$. Then Proposition 4.8 provides an elementary Weinstein homotopy $(W, \omega, \tilde{X}_t, \tilde{\phi}_t)$ of Type IId starting from $\mathfrak{W}$ and killing the critical points p and q at time t_0. One can also arrange that $(\tilde{X}_t, \tilde{\phi}_t)$ coincides with (X, ϕ) on $\mathcal{O}p\, \partial W$, and (by composing $\tilde{\phi}_t$ with suitable functions $\mathbb{R} \to \mathbb{R}$) that the homotopies $\tilde{\phi}_t$ and ϕ_t have equal profiles. Then Lemma 3.4 provides a diffeotopy $h_t : W \to W$, $t \in [0, 1]$, such that $h_0 = \mathrm{Id}$, $h_t|_{\mathcal{O}p\, \partial W} = \mathrm{Id}$, and $\phi_t = \tilde{\phi}_t \circ h_t$ for all $t \in [0, 1]$. Thus the Weinstein homotopy $(W, \omega_t = h_t^*\omega, X_t = h_t^*\tilde{X}, \phi_t = h_t^*\tilde{\phi}_t)$, $t \in [0, 1]$, has the desired properties. It is flexible because the intersections of the X_t-stable manifolds with regular level sets remain loose for $t \in [0, t_0]$ and there are no critical points for $t > t_0$.

Type IIb. The argument in this case is similar, except that we use Proposition 4.7 instead of Proposition 4.8 and we do not need a preliminary homotopy. However, the flexibility of $\mathfrak{W}_t$ for $t \geq t_0$ requires an additional argument.

Consider first the case $2n > 4$. Suppose ϕ_1 has critical points p and q of index n and $n - 1$, respectively (if they have smaller indices flexibility is automatic). Then the closure Δ of the X_1-stable manifold of the point p intersects $\partial_- W$ along a Legendrian disc $\partial_- \Delta$ The boundary S_q^- of this disc is the intersection with $\partial_- W$ of the X_1-stable manifold D_q^- of q. According to Remark 2.5(1) all Legendrian discs are loose, or more precisely, $\partial_- \Delta \setminus S_q^-$ is loose in $\partial_- W \setminus S_q^-$. Let c be a regular value of ϕ_1 which separates $\phi_1(q)$ and $\phi_1(p)$ and consider the level set $\Sigma := \{\phi_1 = c\}$. Flowing along X_1-trajectories defines a contactomorphism $\partial_- W \setminus S_q^- \to \Sigma \setminus D_q^+$ mapping $\partial_- \Delta \setminus S_q^-$ onto $\Delta \cap \Sigma \setminus \{r\}$, where r is the unique intersection point of Δ and the X_1-unstable manifold D_q^+ in the level set

Σ. It follows that $\Delta \cap \Sigma \setminus \{r\}$ is loose in $\Sigma \setminus \{r\}$, and hence $\Delta \cap \Sigma$ is loose in Σ. This proves flexibility of $\mathfrak{W}_1$, and thus of $\mathfrak{W}_t$ for $t \geq t_0$.

Finally, consider the case $2n = 4$. If the critical points have indices 1 and 0, flexibility is automatic. If they have indices 2 and 1 and $\partial_- W$ is overtwisted, we can arrange that $\partial_- \Delta \subset \partial_- W$ (in the notation above) has an overtwisted disc in its complement, hence so does the intersection of Δ with the regular level set $\{\phi = c\}$. □

Proof of Theorem 5.3. Let us pick gradient-like vector fields Y_t for ϕ_t with $Y_0 = X$ and $Y_t = X$ near ∂W to get a Smale homotopy $\mathfrak{S}_t = (W, Y_t, \phi_t)$, $t \in [0,1]$. By Lemma 3.3 we find an admissible partition for the Smale homotopy $\mathfrak{S}_t$. Thus we get a sequence $0 = t_0 < t_1 < \cdots < t_p = 1$ of parameter values and smooth families of partitions

$$W = \bigcup_{j=1}^{N_k} W_j^k(t), \quad W_j^k(t) := \{c_{j-1}^k(t) \leq \phi_t \leq c_j^k(t)\}, \quad t \in [t_{k-1}, t_k],$$

such that each Smale homotopy

$$\mathfrak{S}_j^k := \big(W_j^k(t), Y_t|_{W_j^k(t)}, \phi_t|_{W_j^k(t)}\big)_{t \in [t_{k-1}, t_k]}$$

is elementary. We will construct the Weinstein homotopy (ω_t, X_t, ϕ_t) on the cobordisms $\bigcup_{t \in [t_{k-1}, t_k]} W_j^k(t)$ inductively over $k = 1, \dots, p$, and for fixed k over $j = 1, \dots, N_k$.

Suppose the required Weinstein homotopy is already constructed on W for $t \leq t_{k-1}$. To simplify the notation we rename $\phi_{t_{k-1}}$ to ϕ, the vector fields X_{t_k} and Y_{t_k} to X and Y, and the symplectic form $\omega_{t_{k-1}}$ to ω. We also write N instead of N_k, W_j and $W_j(t)$ instead of $W_j^k(t_{k-1})$ and $W_j^k(t)$, and replace the interval $[t_{k-1}, t_k]$ by $[0,1]$.

There exists a diffeotopy $f_t : W \to W$, fixed on $\mathcal{O}p\, \partial W$, with $f_0 = \mathrm{Id}$ and such that $f_t(W_j) = W_j(t)$ for all $t \in [0,1]$. Moreover, we can choose f_t and a diffeotopy $g_t : \mathbb{R} \to \mathbb{R}$ with $g_0 = \mathbb{1}$ such that the function $\widehat{\phi}_t := g_t \circ \phi_t \circ f_t$ coincides with ϕ on $\mathcal{O}p\, \partial W_j$ for all $t \in [0,1]$, $j = 1, \dots, N$. Set $\widehat{Y}_t := f_t^* Y_t$. So we have a flexible Weinstein cobordism

$$\mathfrak{W} = \left(W = \bigcup_{j=1}^{N} W_j, \omega, X, \phi = \widehat{\phi}_0\right)$$

and a Smale homotopy $(\widehat{Y}_t, \widehat{\phi}_t)$, $t \in [0,1]$, whose restriction to each W_j is elementary. (But the restriction of $\mathfrak{W}$ to W_j need not be elementary.)

Now we apply Lemma 5.9 inductively for $j = 1, \dots, N$ to construct Weinstein homotopies $\widehat{\mathfrak{W}}_t^j = (W_j, \widehat{\omega}_t, \widehat{X}_t, \widehat{\phi}_t)$, fixed near $\partial_- W_j$ and fixed up to scaling near $\partial_+ W_j$, with $\widehat{\mathfrak{W}}_0^j = \mathfrak{W}|_{W_j}$. Thus the $\mathfrak{W}_t^j$ fit together to form a Weinstein homotopy $\widehat{\mathfrak{W}}_t = (\widehat{\omega}_t, \widehat{X}_t, \widehat{\phi}_t)$ on W. The desired Weinstein homotopy on W is now given by

$$\mathfrak{W}_t := \big(f_{t*}\widehat{\omega}_t, f_{t*}\widehat{X}_t, g_t^{-1} \circ \widehat{\phi}_t \circ f_t^{-1}\big). \qquad \square$$

5D. ***Proof of the second Weinstein deformation theorem.*** Let us extend the vector fields Y_t from $\mathcal{O}p\, \partial_- W$ to a path of gradient-like vector fields for ϕ_t on W connecting X_0 and X_1. We will deduce Theorem 5.4 from Theorem 5.3 and the following special case, which is just a 1-parametric version of the Weinstein existence theorem (Theorem 5.1).

Lemma 5.10. *Theorem 5.4 holds under the additional hypothesis that $\phi_t = \phi$ is independent of $t \in [0, 1]$ and the Smale homotopy (W, Y_t, ϕ) is elementary.*

Proof. The proof is just a 1-parametric version of the proof of Theorem 5.1, using Theorem 2.10 and Lemma 4.3 instead of Theorem 2.9 and Lemma 4.2. $\square$

Lemma 5.11. *Theorem 5.4 holds under the additional hypothesis that $\phi_t = \phi$ is independent of $t \in [0, 1]$.*

Proof. Let us pick regular values

$$\phi|_{\partial_- W} = c_0 < c_1 < \cdots < c_N = \phi|_{\partial_+ W}$$

such that each (c_{k-1}, c_k) contains at most one critical value. Then the restriction of the homotopy (Y_t, ϕ), $t \in [0, 1]$, to each cobordism $W^k := \{c_{k-1} \le \phi \le c_k\}$ is elementary.

We apply Lemma 5.10 to the restriction of the homotopy (η_t, Y_t, ϕ) to W^1. Hence $\mathfrak{W}_0|_{W^1}$ and $\mathfrak{W}_1|_{W^1}$ are connected by a homotopy $\mathfrak{W}_t^1 = (\omega_t^1, X_t^1, \phi)$, $t \in [0, 1]$, of flexible Weinstein structures on W^1, agreeing with (η_t, Y_t, ϕ_t) on $\mathcal{O}p\, \partial_- W$, such that the paths $t \mapsto \omega_t^1$ and $t \mapsto \eta_t$, $t \in [0, 1]$, of nondegenerate 2-forms on W^1 are connected by a homotopy η_t^s, $s, t \in [0, 1]$ rel $\mathcal{O}p\, \partial_- W$ with fixed endpoints. We use the homotopy ω_t^s to extend ω_t^1 to nondegenerate 2-forms η_t^1 on W such that $\eta_0^1 = \omega_0$, $\eta_1^1 = \omega_1$, $\eta_t^1 = \eta_t$ outside a neighborhood of W^1, and the paths $t \mapsto \eta_t^1$ and $t \mapsto \eta_t$, $t \in [0, 1]$, of nondegenerate 2-forms on W are homotopic rel $\mathcal{O}p\, \partial_- W$ with fixed endpoints. By Lemma 3.1, we can extend X_t^1 to gradient-like vector fields Y_t^1 for ϕ on W such that $Y_0^1 = X_0$ and $Y_1^1 = X_1$. Now we can apply Lemma 5.10 to the restriction of the homotopy (η_t^1, Y_t^1, ϕ) to the elementary cobordism W^2 and continue inductively to construct homotopies (η_t^k, Y_t^k, ϕ) on W which are Weinstein on W^k, so (η_t^N, Y_t^N, ϕ) is the desired Weinstein homotopy. Note that (η_t^N, Y_t^N, ϕ) is flexible because its restriction to each W^k is flexible. $\square$

Proof of Theorem 5.4. Let us reparametrize the given homotopy (η_t, Y_t, ϕ_t), $t \in [0, 1]$, to make it constant for $t \in \left[\frac{1}{2}, 1\right]$. After pulling back (η_t, Y_t, ϕ_t) by a diffeotopy and target reparametrizing ϕ_t, we may further assume that ϕ_t is independent of t on $\mathcal{O}p\,\partial W$.

By Theorem 5.3, $\mathfrak{W}_0$ can be extended to a homotopy $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in \left[0, \frac{1}{2}\right]$, of flexible Weinstein structures on W, fixed on $\mathcal{O}p\,\partial_- W$. We can modify $\mathfrak{W}_t$ to make it agree with (η_t, Y_t, ϕ_t) on $\mathcal{O}p\,\partial_- W$. Note that $\mathfrak{W}_{1/2}$ and $\mathfrak{W}_1$ share the same function $\phi_{1/2} = \phi_1$. We connect $\omega_{1/2}$ and ω_1 by a path η'_t, $t \in [\frac{1}{2}, 1]$ of nondegenerate 2-forms by following the path ω_t backward and then η_t forward. Since $\omega_t = \eta_t$ on $\mathcal{O}p\,\partial_- W$ for $t \in \left[0, \frac{1}{2}\right]$, we can modify the path η'_t to make it constant equal to $\omega_{1/2} = \omega_1$ on $\mathcal{O}p\,\partial_- W$. By Lemma 3.1, we can connect $X_{1/2}$ and X_1 by a homotopy Y'_t, $t \in \left[\frac{1}{2}, 1\right]$, of gradient-like vector fields for ϕ_1 which agree with $X_{1/2} = X_1$ on $\mathcal{O}p\,\partial_- W$.

So we can apply Lemma 5.11 to the homotopy (η'_t, Y'_t, ϕ_1), $t \in \left[\frac{1}{2}, 1\right]$. Hence $\mathfrak{W}_{1/2}$ and $\mathfrak{W}_1$ are connected by a homotopy $\mathfrak{W}_t = (\omega_t, X_t, \phi_1)$, $t \in \left[\frac{1}{2}, 1\right]$, of flexible Weinstein structures, agreeing with (ω_1, X_1, ϕ_1) on $\mathcal{O}p\,\partial_- W$, such that the paths of nondegenerate 2-forms $t \mapsto \omega_t$ and $t \mapsto \eta'_t$, $t \in \left[\frac{1}{2}, 1\right]$, are homotopic rel $\mathcal{O}p\,\partial_- W$ with fixed endpoints. It follows from the definition of η'_t that the concatenated path ω_t, $t \in [0, 1]$, is homotopic to η_t, $t \in [0, 1]$. Thus the concatenated Weinstein homotopy $\mathfrak{W}_t$, $t \in [0, 1]$, has the desired properties. □

6. Applications

6A. *The Weinstein h-cobordism theorem.* Most of our applications are based on the following result, which is a direct consequence of the two-index theorem of Hatcher and Wagoner; see [Cieliebak and Eliashberg 2012] for its formal derivation from the results in [Hatcher and Wagoner 1973; Igusa 1988].

Theorem 6.1. *Any two Morse functions without critical points of index $> n$ on a cobordism or a manifold of dimension $2n > 4$ can be connected by a Morse homotopy without critical points of index $> n$ (where, as usual, functions on a cobordism W are required to have $\partial_\pm W$ as regular level sets and functions on a manifold are required to be exhausting).*

Corollary 6.2. *In the case $2n > 4$, we can remove the hypothesis on the existence of a Morse homotopy ϕ_t from Theorems 5.3, 5.4, 5.5 and 5.6 and still conclude the existence of the stated Weinstein homotopies.* □

In particular, we have the following Weinstein version of the h-cobordism theorem.

Corollary 6.3 (Weinstein h-cobordism theorem). *Any flexible Weinstein structure on a product cobordism $W = Y \times [0, 1]$ of dimension $2n > 4$ is homotopic to a Weinstein structure (W, ω, X, ϕ), where $\phi : W \to [0, 1]$ is a function without critical points.* □

6B. ***Symplectomorphisms of flexible Weinstein manifolds.*** Theorem 5.6 has the following consequence for symplectomorphisms of flexible Weinstein manifolds.

Theorem 6.4. *Let $\mathfrak{W} = (V, \omega, X, \phi)$ be a flexible Weinstein manifold of dimension $2n > 4$, and $f : V \to V$ a diffeomorphism such that $f^*\omega$ is homotopic to ω through nondegenerate 2-forms. Then there exists a diffeotopy $f_t : V \to V$, $t \in [0, 1]$, such that $f_0 = f$, and f_1 is an exact symplectomorphism of (V, ω).*

Proof. By Theorem 5.6 and Corollary 6.2, there exists a Weinstein homotopy $\mathfrak{W}_t$ connecting $\mathfrak{W}_0 = \mathfrak{W}$ and $\mathfrak{W}_1 = f^*\mathfrak{W}$. Thus Proposition 4.4 provides a diffeotopy $h_t : V \to V$ such that $h_0 = \mathbb{1}$ and $h_1^* f^*\lambda - \lambda$ is exact, where λ is the Liouville form of $\mathfrak{W}$. Now $f_t = f \circ h_t$ is the desired diffeotopy. □

Remark 6.5. Even if $\mathfrak{W}$ is of finite type and $f = \mathbb{1}$ outside a compact set, the diffeotopy f_t provided by Theorem 6.4 will in general *not* equal the identity outside a compact set.

6C. ***Symplectic pseudo-isotopies.*** Let us recall the basic notions of pseudo-isotopy theory from [Cerf 1970; Hatcher and Wagoner 1973]. For a manifold W (possibly with boundary) and a closed subset $A \subset W$, we denote by $\mathrm{Diff}(W, A)$ the space of diffeomorphisms of W fixed on $\mathcal{O}p\,(A)$, equipped with the C^∞-topology. For a cobordism W, the restriction map to $\partial_+ W$ defines a fibration

$$\mathrm{Diff}(W, \partial W) \to \mathrm{Diff}(W, \partial_- W) \to \mathrm{Diff}_{\mathcal{P}}(\partial_+ W),$$

where $\mathrm{Diff}_{\mathcal{P}}(\partial_+ W)$ denotes the image of the restriction map $\mathrm{Diff}(W, \partial_- W) \to \mathrm{Diff}(\partial_+ W)$. For the product cobordism $I \times M$, $I = [0, 1]$, $\partial M = \varnothing$,

$$\mathcal{P}(M) := \mathrm{Diff}(I \times M, 0 \times M)$$

is the group of *pseudo-isotopies* of M. Denote by $\mathrm{Diff}_{\mathcal{P}}(M)$ the group of diffeomorphisms of M that are *pseudo-isotopic to the identity*, that is, that appear as the restriction to $1 \times M$ of an element in $\mathcal{P}(M)$. Restriction to $1 \times M$ defines the fibration

$$\mathrm{Diff}(I \times M, \partial I \times M) \to \mathcal{P}(M) \to \mathrm{Diff}_{\mathcal{P}}(M),$$

and thus a homotopy exact sequence

$$\cdots \to \pi_0 \mathrm{Diff}(I \times M, \partial I \times M) \to \pi_0 \mathcal{P}(M) \to \pi_0 \mathrm{Diff}_{\mathcal{P}}(M) \to 0.$$

We will use the following alternative description of $\mathcal{P}(M)$; see [Cerf 1970]. Denote by $\mathcal{E}(M)$ the space of all smooth functions $f : I \times M \to I$ without critical points and satisfying $f(r, x) = r$ on $\mathcal{O}p\,(\partial I \times M)$. We have a homotopy equivalence

$$\mathcal{P}(M) \to \mathcal{E}(M), \qquad F \mapsto p \circ F,$$

where $p : I \times M \to I$ is the projection. A homotopy inverse is given by fixing a metric and sending $f \in \mathscr{E}(M)$ to the unique diffeomorphism F mapping levels of f to levels of p and gradient trajectories of f to straight lines $I \times \{x\}$. Note that the last map in the homotopy exact sequence

$$\cdots \to \pi_0 \mathrm{Diff}(I \times M, \partial I \times M) \to \pi_0 \mathscr{E}(M) \to \pi_0 \mathrm{Diff}_{\mathscr{P}}(M)$$

associates to $f \in \mathscr{E}(M)$ the flow from $0 \times M$ to $1 \times M$ along trajectories of a gradient-like vector field (whose isotopy class does not depend on the gradient-like vector field).

For the symplectic version of the pseudo-isotopy spaces, it will be convenient to replace $I \times M$ by $\mathbb{R} \times M$ as follows: We replace $\mathscr{E}(M)$ by the space of functions $f : \mathbb{R} \times M \to \mathbb{R}$ without critical points and satisfying $f(r, x) = r$ outside a compact set; $\mathrm{Diff}(I \times M, \partial I \times M)$ by the space $\mathrm{Diff}_c(\mathbb{R} \times M)$ of diffeomorphisms that equal the identity outside a compact set; and $\mathscr{P}(M)$ by the space of diffeomorphisms of $\mathbb{R} \times M$ that equal the identity near $\{-\infty\} \times M$ and have the form $(r, x) \mapsto (r + f(x), g(x))$ near $\{+\infty\} \times M$. The last map in the exact sequence

$$\cdots \to \pi_0 \mathrm{Diff}_c(\mathbb{R} \times M) \to \pi_0 \mathscr{E}(M) \to \pi_0 \mathrm{Diff}_{\mathscr{P}}(M)$$

then associates to $f \in \mathscr{E}(M)$ the flow from $\{-\infty\} \times M$ to $\{+\infty\} \times M$ along trajectories of a gradient-like vector field which equals ∂_r outside a compact set.

We endow the spaces $\mathscr{P}(M)$, $\mathscr{E}(M)$ and $\mathrm{Diff}_c(\mathbb{R} \times M)$ with the topology of uniform C^∞-convergence on $\mathbb{R} \times M$ (and *not* the topology of uniform C^∞-convergence on compact sets), with respect to the product of the Euclidean metric on $\mathbb{R}$ and any Riemannian metric on M. In other words, a sequence $F_n \in \mathscr{P}(M)$ converges to $F \in \mathscr{P}(M)$ if and only if $\|F_n - F\|_{C^k(\mathbb{R} \times M)} \to 0$ for every $k = 0, 1, \ldots$. For example, consider any nonidentity element $F \in \mathscr{P}(M)$ and the translations $\tau_c(r, x) = (r + c, x)$, $c \in \mathbb{R}$, on $\mathbb{R} \times M$. Then the sequence $F_n := \tau_n \circ F \circ \tau_{-n}$ *does not converge* as $n \to \infty$ to the identity in $\mathscr{P}(M)$, although it does converge uniformly on compact sets. With this topology, the obvious inclusion maps from the spaces on $I \times M$ to the corresponding spaces on $\mathbb{R} \times M$ are weak homotopy equivalences.

Remark 6.6. Cerf [1970] proved that $\pi_0 \mathscr{P}(M)$ is trivial if $\dim M \geq 5$ and M is simply connected. In other words, under these assumptions pseudo-isotopy implies isotopy. In the nonsimply connected case and for $\dim M \geq 6$, Hatcher and Wagoner [1973] (see also [Igusa 1988]) have expressed $\pi_0 \mathscr{P}(M)$ in terms of algebraic K-theory of the group ring of $\pi_1(M)$. In particular, there are many fundamental groups for which $\pi_0 \mathscr{P}(M)$ is not trivial.

Let us now fix a contact manifold (M^{2n-1}, ξ) and denote by $(SM, \lambda_{\mathrm{st}})$ its symplectization with its canonical Liouville structure $(\omega_{\mathrm{st}} = d\lambda_{\mathrm{st}}, X_{\mathrm{st}})$. Any choice of a contact form α for ξ yields an identification of SM with $\mathbb{R} \times M$ and the Liouville structure $\lambda_{\mathrm{st}} = e^r\alpha$, $\omega_{\mathrm{st}} = d\lambda_{\mathrm{st}}$, $X_{\mathrm{st}} = \partial_r$. However, the following constructions do not require the choice of a contact form. We will refer to the two ends of SM as $\{\pm\infty\} \times M$.

We define the group of *symplectic pseudo-isotopies* of (M, ξ) as

$$\mathcal{P}(M,\xi) := \big\{F \in \mathrm{Diff}(SM) \mid F^*\omega_{\mathrm{st}} = \omega_{\mathrm{st}},\ F = \mathbb{1} \text{ near } \{-\infty\} \times M,\\ F^*\lambda_{\mathrm{st}} = \lambda_{\mathrm{st}} \text{ near } \{+\infty\} \times M\big\}.$$

Moreover, we introduce the space

$$\mathcal{E}(M,\xi) := \{(\lambda, \phi) \text{ Weinstein structure on SM without critical points } \mid\\ d\lambda = \omega_{\mathrm{st}},\ (\lambda, \phi) = (\lambda_{\mathrm{st}}, \phi_{\mathrm{st}}) \text{ outside a compact set}\}$$

and its image $\bar{\mathcal{E}}(M,\xi)$ under the projection $(\lambda, \phi) \mapsto \lambda$. We endow the spaces $\mathcal{P}(M,\xi)$, $\mathcal{E}(M,\xi)$, and $\bar{\mathcal{E}}(M,\xi)$ with the topology of uniform C^∞-convergence on $SM = \mathbb{R} \times M$ as explained above.

Lemma 6.7. *The map*

$$\mathcal{E}(M,\xi) \to \bar{\mathcal{E}}(M,\xi), \quad (\lambda,\phi) \mapsto \lambda,$$

is a homotopy equivalence and the map

$$\mathcal{P}(M,\xi) \to \bar{\mathcal{E}}(M,\xi), \quad F \mapsto F^*\lambda_{\mathrm{st}},$$

is a homeomorphism.

Proof. The first map defines a fibration whose fiber over λ is the contractible space of Lyapunov functions for X which are standard at infinity. The inverse of the second map associates to λ the unique $F \in \mathrm{Diff}(SM)$ satisfying $F_*X = X_{\mathrm{st}}$ on SM and $F = \mathbb{1}$ near $\{-\infty\} \times M$ (which implies $F^*\lambda_{\mathrm{st}} = \lambda$ on SM). ☐

Since $F \in \mathcal{P}(M,\xi)$ satisfies $F^*\lambda_{\mathrm{st}} = \lambda_{\mathrm{st}}$ near $\{+\infty\} \times M$, it descends there to a contactomorphism $F_+ : M \to M$. By construction, F_+ belongs to the group $\mathrm{Diff}_{\mathcal{P}}(M)$ of diffeomorphisms that are pseudo-isotopic to the identity, so it defines an element in

$$\mathrm{Diff}_{\mathcal{P}}(M,\xi) := \{F_+ \in \mathrm{Diff}_{\mathcal{P}}(M) \mid F_+^*\xi = \xi\}.$$

Moreover, $F_+ = \mathbb{1}$ if and only if F belongs to the space

$$\mathrm{Diff}_c(SM, \omega_{\mathrm{st}}) := \{F \in \mathrm{Diff}_c(SM) \mid F^*\omega_{\mathrm{st}} = \omega_{\mathrm{st}}\}$$

of compactly supported symplectomorphisms of $(SM, \omega_{\mathrm{st}})$. Thus we have a fibration

$$\mathrm{Diff}_c(SM, \omega_{\mathrm{st}}) \to \mathcal{P}(M, \xi) \to \mathrm{Diff}_{\mathcal{P}}(M, \xi).$$

The corresponding homotopy exact sequence fits into a commuting diagram

$$\begin{array}{ccccccc} \pi_0\mathrm{Diff}_c(SM, \omega_{\mathrm{st}}) & \longrightarrow & \pi_0\mathcal{P}(M, \xi) & \longrightarrow & \pi_0\mathrm{Diff}_{\mathcal{P}}(M, \xi) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_0\mathrm{Diff}_c(\mathbb{R} \times M) & \longrightarrow & \pi_0\mathcal{P}(M) & \longrightarrow & \pi_0\mathrm{Diff}_{\mathcal{P}}(M) & \longrightarrow & 0 \end{array} \tag{3}$$

where the vertical maps are induced by the obvious inclusions.

We now state the main result of this section.

Theorem 6.8. *For any closed contact manifold (M, ξ) of dimension $2n-1 \geq 5$, the map $\pi_0\mathcal{P}(M, \xi) \to \pi_0\mathcal{P}(M)$ is surjective.*

Proof. By the discussion above, it suffices to show that the map $\pi_0\mathcal{E}(M, \xi) \to \pi_0\mathcal{E}(M)$ induced by the projection $(\lambda, \phi) \mapsto \phi$ is surjective. So let $\psi \in \mathcal{E}(M)$, that is, $\psi : \mathbb{R} \times M \to \mathbb{R}$ is a function without critical points which agrees with $\phi_{\mathrm{st}}(r, x) = r$ outside a compact set $W = [a, b] \times M$. By Theorem 6.1, there exists a Morse homotopy $\phi_t : \mathbb{R} \times M \to \mathbb{R}$ without critical points of index $> n$ connecting $\phi_0 = \phi_{\mathrm{st}}$ with $\phi_1 = \psi$ such that $\phi_t = \phi_{\mathrm{st}}$ outside W for all $t \in [0, 1]$. We apply Theorem 5.3 to the Weinstein cobordism $\mathfrak{W} = (W, \omega_{\mathrm{st}}, X_{\mathrm{st}}, \phi_{\mathrm{st}})$ and the homotopy $\phi_t : W \to \mathbb{R}$. Hence there exists a Weinstein homotopy $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$, fixed on $\mathcal{O}p\, \partial_- W$ and fixed up to scaling on $\mathcal{O}p\, \partial_+ W$, such that $\mathfrak{W}_0 = \mathfrak{W}$. Note that $\lambda_t = c_t \lambda_{\mathrm{st}}$ on $\mathcal{O}p\, \partial_+ W$ for constants c_t with $c_0 = 1$. So we can extend $\mathfrak{W}_t$ over the rest of $\mathbb{R} \times M$ by the function ϕ_{st} and Liouville forms $f_t(r)\lambda_{\mathrm{st}}$ such that $\mathfrak{W}_t = \mathfrak{W}$ on $\{r \leq a\}$ and on $\{r \geq c\}$ for some sufficiently large $c > b$. By Moser's stability theorem, we find a diffeotopy $h_t : SM \to SM$ with $h_0 = \mathbb{1}$, $h_t = \mathbb{1}$ outside $[a, c] \times M$, and $h_t^* \mathfrak{W}_t = \mathfrak{W}$. Thus $h_1^* \mathfrak{W}_1 = (\lambda, \phi)$ with the function $\phi := \psi \circ h_1$ and a Liouville form λ which agrees with λ_{st} outside $[a, c] \times M$ and satisfies $d\lambda = \omega_{\mathrm{st}}$. Hence $(\lambda, \phi) \in \mathcal{E}(M, \xi)$ and ϕ is homotopic (via $\psi \circ h_t$) to ψ in $\mathcal{E}(M)$, that is, $[\phi] = [\psi] \in \pi_0\mathcal{E}(M)$. □

By Theorem 6.8, the second vertical map in the diagram (3) is surjective and we obtain:

Corollary 6.9. *Let (M, ξ) be a closed contact manifold of dimension $2n-1 \geq 5$. Then every diffeomorphism of M that is pseudo-isotopic to the identity is smoothly isotopic to a contactomorphism of (M, ξ).*

Remark 6.10. Considering in the diagram (3) elements in $\pi_0\mathcal{P}(M)$ that map to $\mathbb{1} \in \pi_0\mathrm{Diff}_{\mathcal{P}}(M)$, we obtain the following (nonexclusive) dichotomy for a contact manifold (M, ξ) of dimension at least 5 for which the map $\pi_0\, \mathrm{Diff}_c(\mathbb{R} \times M) \to \pi_0\mathcal{P}(M)$ is nontrivial: either there exists a contactomorphism of (M, ξ) that is smoothly but not contactly isotopic to the identity, or there exists a compactly supported symplectomorphism of $(SM, \omega_{\mathrm{st}})$ that represents a nontrivial smooth pseudo-isotopy class in $\mathcal{P}(M)$. Unfortunately, we cannot decide which of the two cases occurs.

6D. ***Equidimensional symplectic embeddings of flexible Weinstein manifolds.*** Finally, let us mention a recent result concerning equidimensional symplectic embeddings of flexible Weinstein manifolds. Its proof goes beyond the methods discussed in this paper.

Theorem 6.11 [Eliashberg and Murphy 2013]. *Let (W, ω, X, ϕ) be a flexible Weinstein domain with Liouville form λ. Let Λ be any other Liouville form on W such that the symplectic forms ω and $\Omega := d\Lambda$ are homotopic as nondegenerate (not necessarily closed) 2-forms. Then there exists an isotopy $h_t : W \hookrightarrow W$ such that $h_0 = \mathrm{Id}$ and $h_1^*\Lambda = \varepsilon\lambda + dH$ for some small $\varepsilon > 0$ and some smooth function $H : W \to \mathbb{R}$. In particular, h_1 defines a symplectic embedding $(W, \varepsilon\omega) \hookrightarrow (W, \Omega)$.*

Corollary 6.12 [Eliashberg and Murphy 2013]. *Let (W, ω, X, ϕ) be a flexible Weinstein domain and (X, Ω) any symplectic manifold of the same dimension. Then any smooth embedding $f_0 : W \hookrightarrow X$ such that the form $f_0^*\Omega$ is exact and the differential $df : TW \to TX$ is homotopic to a symplectic homomorphism is isotopic to a symplectic embedding $f_1 : (W, \varepsilon\omega) \hookrightarrow (X, \Omega)$ for some small $\varepsilon > 0$. Moreover, if $\Omega = d\Lambda$, then the embedding f_1 can be chosen in such a way that the 1-form $f_1^*\Lambda - i_X\omega$ is exact. If, moreover, the Liouville vector field dual to Λ is complete, then the embedding f_1 exists for arbitrarily large constant ε.* □

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Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany
kai.cieliebak@math.uni-augsburg.de

Department of Mathematics, Stanford University, Building 380, Stanford, CA 94305, United States
eliash@math.stanford.edu