Topology of Stratified Spaces
MSRI Publications
Volume **58**, 2011

Motivic characteristic classes

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ABSTRACT. Motivic characteristic classes of possibly singular algebraic varieties are homology class versions of motivic characteristics, not classes in the so-called motivic (co)homology. This paper is a survey of them, with emphasis on capturing infinitude finitely and on the motivic nature, in other words, the scissor relation or additivity.

1. Introduction

Characteristic classes are usually cohomological objects defined on real or complex vector bundles. For a smooth manifold, for instance, its characteristic classes are defined through the tangent bundle. For real vector bundles, Stiefel–Whitney classes and Pontraygin classes are fundamental; for complex vector bundles, the Chern class is the fundamental one.

When it comes to a non-manifold space, such as a singular real or complex algebraic or analytic variety, one cannot talk about its cohomological characteristic class, unlike the smooth case, because one cannot define its tangent bundle — although one can define some reasonable substitutes, such as the tangent cone and tangent star cone, which are not vector bundles, but stratified vector bundles.

In the 1960s people started to define characteristic classes on algebraic varieties as homological objects — not through vector bundles, but as higher analogues of geometrically important invariants such as the Euler–Poincaré characteristic, the signature, and so on. I suppose that the theory of characteristic classes of singular spaces starts with Thom's L-class for oriented PL-manifolds

This is an expanded version of the author's talk at the workshop "Topology of Stratified Spaces" held at MSRI, Berkeley, from September 8 to 12, 2008.
Partially supported by Grant-in-Aid for Scientific Research (No. 21540088), the Ministry of Education, Culture, Sports, Science and Technology (MEXT), and JSPS Core-to-Core Program 18005, Japan.

[Thom], whereas Sullivan's Stiefel–Whitney classes and the so-called Deligne–Grothendieck conjecture about the existence of Chern homology classes started the whole story of *capturing characteristic classes of singular spaces as natural transformations*, more precisely as a natural transformation from a certain covariant functor to the homology functor.

The Deligne–Grothendieck conjecture seems to be based on Grothendieck's ideas or Deligne's modification of Grothendieck's conjecture on a *Riemann–Roch type formula* concerning the constructible étale sheaves and Chow rings (see [Grot, Part II, note(87_1), p. 361 ff.]) and was made in its well-known current form by P. Deligne later. R. MacPherson [M1] gave a positive answer to the Deligne–Grothendieck conjecture and, motivated by this solution, P. Baum, W. Fulton and R. MacPherson [BFM1] further established the singular Riemann–Roch Theorem, which is a singular version of Grothendieck–Riemann–Roch, which is a functorial extension of the celebrated Hirzebruch–Riemann–Roch (abbreviated HRR) [Hi]. HRR is the very origin of the Atiyah–Singer Index Theorem.

The main results of [BSY1] (announced in [BSY2]) are the following:

- *"Motivic" characteristic classes of algebraic varieties*, which is a class version of the motivic characteristic. (Note that this "motivic class" is *not* a class in the so-called motivic cohomology in algebraic/arithmetic geometry.)
- Motivic characteristic classes in a sense give rise to *a unification of three well-known important characteristic homology classes*:

(1) MacPherson's Chern class transformation [M1] (see also [M2; Schw; BrS]);

(2) Baum, Fulton and MacPherson's Riemann–Roch transformation [BFM1];

(3) Goresky and MacPherson's L-homology class (see [GM]), or Cappell and Shaneson's L-homology class [CS1] (cf. [CS2]).

This unification result can be understood to be good enough to consider our motivic characteristic classes as a positive solution to the following MacPherson's question or comment, written at the end of his survey paper of 1973 [M2]:

"*It remains to be seen whether there is a unified theory of characteristic classes of singular varieties like the classical one outlined above.*"

The current theory unifies "only three" characteristic classes, but so far it seems to be a reasonble one.

The purpose of this paper is mainly to explain the results from [BSY1] mentioned above (also see [SY]) with emphasis on the "motivic nature" of motivic characteristic classes. In particular, we show that our motivic characteristic class is a very natural class version of the so-called motivic characteristic, just like the way A. Grothendieck extended HRR to Grothendieck –Riemann–Roch. For that, we go back all the way to the natural numbers, which would be thought of as the very origin of a *characteristic* or *characteristic class*.

We naïvely start with the counting of finite sets. Then we want to count infinite sets as if we are still doing the same way of counting finite sets, and want to understand motivic characteristic classes as higher-class versions of this unusual "counting infinite sets", where infinite sets are complex algebraic varieties. (The usual counting of infinite sets, forgetting the structure of a variety at all, lead us into the mathematics of infinity.) The key is Deligne's mixed Hodge structures [De1; De2], or more generally Saito's deep theory of mixed Hodge modules [Sa2], etc.

As to mixed Hodge modules (MHM), in [Sch3] Jörg Schürmann gives a very nice introduction and overview about recent developments on the interaction of theories of characteristic classes and mixed Hodge theory for singular spaces in the complex algebraic context with MHM as a crucial and fundamental key. For example, a study of characteristic classes of the intersection homological Hodge modules has been done in a series of papers by Sylvain Cappell, Anatoly Libgober, Laurentiu Maxim, Jörg Schürmann and Julius Shaneson [CLMS1; CLMS2; CMS1; CMS2; CMSS; MS1; MS2] (in connection with this last one, see also [Y8]).

The very recent book by C. Peters and J. Steenbrink [PS] seems to be a most up-to-date survey on mixed Hodge structures and Saito's mixed Hodge modules. The Tata Lecture Notes by C. Peters [P] (which is a condensed version of [PS]) give a nice introduction to Hodge Theory with more emphasis on the motivic nature.[1]

2. Preliminaries: from natural numbers to genera

We first consider counting the number of elements of finite sets, i.e., natural numbers. Let $\mathcal{FSET}$ be the category of finite sets and maps among them. For an object $X \in \mathcal{FSET}$, let

$$c(X) \in \mathbb{Z}$$

be the number of the elements of X, which is usually denoted by $|X|$ ($\in \mathbb{N}$) and called the cardinal number, or cardinality of X. It satisfies the following four properties on the category $\mathcal{FSET}$ of finite sets:

(1) $X \cong X'$ (bijection or equipotent) $\Longrightarrow c(X) = c(X')$.
(2) $c(X) = c(X \setminus Y) + c(Y)$ for $Y \subset X$.
(3) $c(X \times Y) = c(X) \cdot c(Y)$.

[1] J. Schürmann informed me of the book [PS] and the lecture [P] at the workshop.

(4) $c(pt) = 1$. (Here pt denotes one point.)

REMARK 2-1. Clearly these four properties characterize the counting $c(X)$. Also note that if $c(X) \in \mathbb{Z}$ satisfies (1)–(3) without (4), then we have $c(pt) = 0$ or $c(pt) = 1$. If $c(pt) = 0$, then it follows from (2) (or (1) and (3)) that $c(X) = 0$ for any finite set X. If $c(pt) = 1$, it follows from (2) that $c(X)$ is the number of elements of a finite set X.

REMARK 2-2. When it comes to infinite sets, cardinality still satisfies properties (1)–(4), but the usual rules of computation no longer work. For natural numbers, $a^2 = a$ implies $a = 0$ or $a = 1$. But the infinite cardinal $\aleph = c(\mathbb{R})$ also has the property that $\aleph^2 = \aleph$; in fact, for any natural number n,

$$c(\mathbb{R}^n) = c(\mathbb{R}) \text{, i.e., } \aleph^n = \aleph.$$

This leads into the *mathematics of infinity*.

One could still imagine counting on the bigger category $\mathcal{SET}$ of sets, where a set can be infinite, and $c(X)$ lies in some integral domain. However, one can see that if for such a counting (1), (2) and (3) are satisfied, it follows automatically that $c(pt) = 0$, contradicting property (4).

In other words: if we consider counting with properties (1)–(3) on the category $\mathcal{SET}$ of all sets, the only possibility is the trivial one: $c(X) = 0$ for any set X!

However, if we consider sets having superstructures on the infrastructure (*set*) *and property* (1) *is replaced by the invariance of the superstructures, we do obtain more reasonable countings which are finite numbers*; *thus we can avoid the mysterious "mathematics of infinity" and extend the usual counting* $c(X)$ *of finite sets very naturally and naïvely. This is exactly what the Euler characteristic, the genus, and many other important and fundamental objects in modern geometry and topology are all about*.

Let us consider the following "topological counting" c_{top} on the category $\mathcal{TOP}$ of topological spaces, which assigns to each topological space X a certain integer (or more generally, an element in an integral domain)

$$c_{\text{top}}(X) \in \mathbb{Z}$$

such that it satisfies the following four properties, which are exactly the same as above except for (1):

(1) $X \cong X'$ (homeomorphism = $\mathcal{TOP}$- isomorphism) $\Longrightarrow c_{\text{top}}(X) = c_{\text{top}}(X')$,
(2) $c_{\text{top}}(X) = c_{\text{top}}(X \setminus Y) + c_{\text{top}}(Y)$ for $Y \subset X$ (for the moment no condition),
(3) $c_{\text{top}}(X \times Y) = c_{\text{top}}(X) \cdot c_{\text{top}}(Y)$,
(4) $c_{\text{top}}(pt) = 1$.

REMARK 2-3. As in Remark 2-1, conditions (1) and (3) imply that $c_{\text{top}}(pt) = 0$ or 1. If $c(pt) = 0$, it follows from (1) and (3) that $c_{\text{top}}(X) = 0$ for any topological space X. Thus the last condition, $c(pt) = 1$, means that c_{top} is a nontrivial counting. Hence, topological counting c_{top} can be regarded as *a nontrivial, multiplicative, additive, topological invariant*.

PROPOSITION 2-4. *If such a c_{top} exists, then*

$$c_{\text{top}}(\mathbb{R}^1) = -1, \quad \textit{hence} \quad c_{\text{top}}(\mathbb{R}^n) = (-1)^n.$$

Hence if X is a finite CW-complex with $\sigma_n(X)$ open n-cells, then

$$c_{\text{top}}(X) = \textstyle\sum_n (-1)^n \sigma_n(X) = \chi(X),$$

the Euler–Poincaré characteristic of X.

The equality $c_{\text{top}}(\mathbb{R}^1) = -1$ can be seen by considering

$$\mathbb{R}^1 = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty).$$

Condition (2) implies $c_{\text{top}}(\mathbb{R}^1) = c_{\text{top}}((-\infty, 0)) + c_{\text{top}}(\{0\}) + c_{\text{top}}((0, \infty))$, so

$$-c_{\text{top}}(\{0\}) = c_{\text{top}}((-\infty, 0)) + c_{\text{top}}((0, \infty)) - c_{\text{top}}(\mathbb{R}^1).$$

Since $\mathbb{R}^1 \cong (-\infty, 0) \cong (0, \infty)$, it follows from (1) and (4) that

$$c_{\text{top}}(\mathbb{R}^1) = -c_{\text{top}}(\{0\}) = -1.$$

The existence of a counting c_{top} can be shown using ordinary homology/cohomology theory: symbolically,

topological counting c_{top} : ***ordinary*** (***co***)***homology theory.***

To be more precise, we use Borel–Moore homology theory [BM], the homology theory with closed supports. For a locally compact Hausdorff space X, Borel–Moore homology theory $H_*^{BM}(X; R)$ with a ring coefficient R is isomorphic to the relative homology theory of the pair $(X^c, *)$, with X^c the one-point compactification of X and $*$ the one point added to X:

$$H_*^{BM}(X; R) \cong H_*(X^c, *; R).$$

Hence, if X is compact, Borel–Moore homology theory is the usual homology theory: $H_*^{BM}(X; R) = H_*(X; R)$.

Let $\mathfrak{K}$ be a field, such as $\mathbb{R}$ or $\mathbb{C}$. If the Borel–Moore homology $H_*^{BM}(X; \mathfrak{K})$ is finite-dimensional — say, if X is a finite CW-complex — then the Euler–Poincaré characteristic χ_{BM} using the Borel–Moore homology theory with coefficient field $\mathfrak{K}$, namely

$$\chi_{BM}(X) := \sum_n (-1)^n \dim_{\mathfrak{K}} H_n^{BM}(X; \mathfrak{K}),$$

gives rise to a topological counting χ_{top}, because it satisfies $H_n^{BM}(\mathbb{R}^n, \mathfrak{K}) = \mathfrak{K}$ and $H_k^{BM}(\mathbb{R}^n, \mathfrak{K}) = 0$ for $k \neq n$, and thus

$$\chi_{BM}(\mathbb{R}^n) = (-1)^n.$$

It turns out that for coefficients in a field $\mathfrak{K}$, Borel–Moore homology is *dual* [2] *as a vector space* to the cohomology with compact support, namely

$$H_p^{BM}(X; \mathfrak{K}) = \operatorname{Hom}(H_c^p(X; \mathfrak{K}), \mathfrak{K}).$$

Since $\mathfrak{K}$ is a field, we have

$$H_p^{BM}(X; \mathfrak{K}) \cong H_c^p(X; \mathfrak{K})$$

Hence the Euler-Poincaré characteristic using Borel–Moore homology $\chi_{BM}(X)$ is equal to the Euler-Poincaré characteristic using cohomology with compact support, usually denoted by χ_c:

$$\chi_c(X) = \sum_i (-1)^i \dim_K H_c^i(X; \mathfrak{K}).$$

Since it is quite common to use χ_c, we have

COROLLARY 2-5. *For the category of locally compact Hausdorff spaces*,

$$c_{\text{top}} = \chi_c,$$

the Euler–Poincaré characteristic using cohomology with compact support.

REMARK 2-6. This story could be retold as follows: There might be many ways of "topologically counting" on the category $\mathcal{TOP}$ of topological spaces, but they are *all identical to the Euler–Poincaré characteristic with compact support* when restricted to the subcategory of locally compact Hausdorff spaces with finite dimensional Borel–Moore homologies. Symbolically speaking,

$$c_{\text{top}} = \chi_c.$$

Next consider the following "algebraic counting" c_{alg} on the category $\mathcal{VAR}$ of *complex* algebraic varieties (of finite type over $\mathbb{C}$), which assigns to each complex algebraic variety X a certain element

$$c_{\text{alg}}(X) \in R$$

in a commutative ring R with unity, such that:

(1) $X \cong X'$ ($\mathcal{VAR}$-isomorphism) $\Longrightarrow c_{\text{alg}}(X) = c_{\text{alg}}(X')$.
(2) $c_{\text{alg}}(X) = c_{\text{alg}}(X \setminus Y) + c_{\text{alg}}(Y)$ for a closed subvariety $Y \subset X$.

[2]For an n-dimensional manifold M the Poincaré duality map $\mathcal{PD} : H_c^k(M) \cong H_{n-k}(M)$ is an isomorphism and also $\mathcal{PD} : H^k(M) \cong H_{n-k}^{BM}(M)$ is an isomorphism. Thus they are *Poincaré dual*, but *not dual as vector spaces*.

(3) $c_{\text{alg}}(X \times Y) = c_{\text{alg}}(X) \cdot c_{\text{alg}}(Y)$.
(4) $c_{\text{alg}}(pt) = 1$.

Just like $c(X)$ and $c_{\text{top}}(X)$, the last condition simply means that c_{alg} is a non-trivial counting.

The real numbers $\mathbb{R}$ and in general the Euclidean space $\mathbb{R}^n$ are the most fundamental objects in the category $\mathcal{TOP}$ of topological spaces, and the complex numbers $\mathbb{C}$ and in general complex affine spaces $\mathbb{C}^n$ are the most fundamental objects in the category $\mathcal{VAR}$ of complex algebraic varieties. The decomposition of n-dimensional complex projective space as

$$\mathbb{P}^n = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \cdots \sqcup \mathbb{C}^{n-1} \sqcup \mathbb{C}^n$$

implies the following:

PROPOSITION 2-7. *If c_{alg} exists, then*

$$c_{\text{alg}}(\mathbb{P}^n) = 1 - y + y^2 - y^3 + \cdots + (-y)^n,$$

where $y := -c_{\text{alg}}(\mathbb{C}^1) \in R$.

REMARK 2-8. Proposition 2-7 already indicates that there could exist infinitely many ways — as many as the elements y — to do algebraic counting c_{alg} on the category $\mathcal{VAR}$ of complex algebraic varieties. This is strikingly different from the topological counting c_{top} and the original counting c, which are uniquely determined. This difference of course lies in the complex structure:

set + topological structure + ***complex structure***.

Here there is no question of considering $\mathbb{R}^1$, so the previous argument showing that $c_{\text{top}}(\mathbb{R}^1) = -1$ does not work. In this sense, we should have used the symbol $c_{\text{alg}/\mathbb{C}}$ to emphasize the complex structure, instead of c_{alg}. Since we are dealing with only the category of complex algebraic varieties in this paper, we write just c_{alg}. See Remark 2-11 below for the category of real algebraic varieties.

The existence of a c_{alg} — in fact, of many such ways of algebraically counting — can be shown using *Deligne's theory of mixed Hodge structures* [De1; De2], which comes from the algebraic structure:

set + topological structure + ***complex structure*** + ***algebraic structure***.

Then the Hodge–Deligne polynomial

$$\chi_{u,v}(X) := \sum_{i,p,q \geq 0} (-1)^i (-1)^{p+q} \dim_{\mathbb{C}}(\mathrm{Gr}_F^p \, \mathrm{Gr}_{p+q}^W \, H_c^i(X, \mathbb{C})) u^p v^q$$

satisfies the four properties above with $R = \mathbb{Z}[u, v]$ and $-y := c_{\mathrm{alg}}(\mathbb{C}^1) = uv$, namely any Hodge–Deligne polynomial $\chi_{u,v}$ with $uv = -y$ is a c_{alg}. Here we point out that by Deligne's work only graded terms with $p, q \geq 0$ are nontrivial; otherwise one would have $\chi_{u,v}(X) \in \mathbb{Z}[u, u^{-1}, v, v^{-1}]$.

Similarly one can consider the invariant

$$c_{\mathrm{alg}}(X) := \chi_{y,-1} \in \mathbb{Z}[y],$$

with $c_{\mathrm{alg}}(\mathbb{C}^1) = -y$.

Here we should note that for $(u, v) = (-1, -1)$ we have

$$\chi_{-1,-1}(X) = \chi_c(X) = c_{\mathrm{top}}(X).$$

Further, for a smooth compact variety X, $\chi_{0,-1}(X)$ is the arithmetic genus, while $\chi_{1,-1}(X)$ is the signature. These three cases, $(u, v) = (-1, -1)$, $(0, -1)$ and $(1, -1)$, are very important.

algebraic counting c_{alg}: mixed Hodge theory
= ordinary (co)homology theory + mixed Hodge structures.

REMARK 2-9. (See [DK], for example.) The following description is also fine, but we use the one above in our later discussion on motivic characteristic classes:

$$c_{\mathrm{alg}}(\mathbb{P}^n) = 1 + y + y^2 + y^3 + \cdots + y^n,$$

where $y = c_{\mathrm{alg}}(\mathbb{C}^1) \in \mathbb{Z}[y]$. The Hodge–Deligne polynomial is usually denoted by $E(X; u, v)$ and is defined to be

$$E(X; u, v) := \sum_{i,p,q \geq 0} (-1)^i \dim_{\mathbb{C}}(\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^i(X, \mathbb{C}))u^p v^q.$$

Thus

$$\chi_{u,v}(X) = E(X; -u, -v).$$

The reason why we define $\chi_{u,v}(X)$ to be $E(X; -u, -v)$ rather than $E(X; u, v)$ lies in the definition of Hirzebruch's generalized Todd class and Hirzebruch's χ_y characteristic, which will come below.

The algebraic counting c_{alg} specializes to the topological counting c_{top}. Are there other algebraic countings that specialize to the Hodge–Deligne polynomial $\chi_{u,v}$ (which is sensitive to an algebraic structure)?

CONJECTURE 2-10. *The answer is negative; in other words, there are no extra structures other than Deligne's mixed Hodge structure that contribute more to the algebraic counting c_{alg} of complex algebraic varieties.*

REMARK 2-11. In the category $\mathcal{VAR}(\mathbb{R})$ of *real algebraic varieties*, we can of course consider $c_{\mathrm{alg}\,/\mathbb{R}}(\mathbb{R}^1)$ of the real line $\mathbb{R}^1$; therefore we might be tempted to the hasty conclusion that in the category of real algebraic varieties the topological counting c_{top}, i.e., χ_c, is sufficient. Unfortunately, the argument for $c_{\mathrm{top}}(\mathbb{R}^1) = -1$ does not work in the category $\mathcal{VAR}(\mathbb{R})$, because $\mathbb{R}^1$ and $(-\infty, 0)$ or $(0, \infty)$ are not isomorphic as real algebraic varieties. Even among compact varieties there do exist real algebraic varieties that are homeomorphic but not isomorphic as real algebraic varieties. For instance (see [MP1, Example 2.7]):

Consider the usual *normal crossing* figure eight curve:

$$\mathrm{F8} = \{(x, y) \mid y^2 = x^2 - x^4\}.$$

The proper transform of F8 under the blowup of the plane at the origin is homeomorphic to a circle, and the preimage of the singular point of F8 is two points.

Next take the *tangential* figure eight curve:

$$t\mathrm{F8} = \bigl\{(x, y) \mid \bigl((x+1)^2 + y^2 - 1\bigr)\bigl((x-1)^2 + y^2 - 1\bigr) = 0\bigr\},$$

which is the union of two circles tangent at the origin. Here the preimage of the singular point is a single point. Therefore, in contrast to the category of crude topological spaces, in the category of *real algebraic* varieties an "algebraic counting" $c_{\mathrm{alg}/\mathbb{R}}(\mathbb{R}^1)$ is meaningful, i.e., sensitive to the algebraic structure. Indeed, as such a real algebraic counting $c_{\mathrm{alg}/\mathbb{R}}(\mathbb{R}^1)$ there are

$$\textit{the } i\textit{-th virtual Betti number} \quad \beta_i(X) \in \mathbb{Z}$$

and

$$\textit{the virtual Poincaré polynomial} \quad \beta_t(X) = \textstyle\sum_i \beta_i(X) t^i \in \mathbb{Z}[t].$$

They are both identical to the usual Betti number and Poincaré polynomial on compact nonsingular varieties. For the above two figure eight curves F8 and tF8 we indeed have that

$$\beta_t(\mathrm{F8}) \neq \beta_t(t\mathrm{F8}).$$

For more details, see [MP1] and [To3], and see also Remark 4-12.

Finally, in passing, we also mention the following "cobordism" counting c_{cob} on the category of closed oriented differential manifolds or the category of stably almost complex manifolds:

(1) $X \cong X'$ (cobordant, or bordant) $\Longrightarrow c_{\mathrm{cob}}(X) = c_{\mathrm{cob}}(X')$.
(2) $c_{\mathrm{cob}}(X \sqcup Y) = c_{\mathrm{cob}}(X) + c_{\mathrm{cob}}(Y)$. (Note: in this case $c_{\mathrm{cob}}(X \setminus Y)$ does not make sense, because $X \setminus Y$ has to be a closed oriented manifold.)
(3) $c_{\mathrm{cob}}(X \times Y) = c_{\mathrm{cob}}(X) \cdot c_{\mathrm{cob}}(Y)$.
(4) $c_{\mathrm{cob}}(pt) = 1$.

As in the cases of the previous countings, (1) and (3) imply $c_{\mathrm{cob}}(pt) = 0$ or $c_{\mathrm{cob}}(pt) = 1$. It follows from (3) that $c_{\mathrm{cob}}(pt) = 0$ implies that $c_{\mathrm{cob}}(X) = 0$ for any closed oriented differential manifolds X. Thus the last condition $c_{\mathrm{cob}}(pt) = 1$ means that our c_{cob} is nontrivial. Such a cobordism counting c_{cob} is nothing but a *genus* such as the signature, the $\hat{A}$-genus, or the elliptic genus. As in Hirzebruch's book, a genus is usually defined as a nontrivial counting satisfying properties (1), (2) and (3). Thus, it is the same as the one given above.

Here is a very simple problem on genera of closed oriented differentiable manifolds or stably almost complex manifolds:

PROBLEM 2-12. *Determine all genera.*

Let $\mathrm{Iso}(G)_n$ be the set of isomorphism classes of smooth closed (and oriented) pure n-dimensional manifolds M for $G = O$ (or $G = SO$), or of pure n-dimensional weakly ("= stably") almost complex manifolds M for $G = U$, i.e., $TM \oplus \mathbb{R}^N_M$ is a complex vector bundle (for suitable N, with $\mathbb{R}_M$ the trivial real line bundle over M). Then

$$\mathrm{Iso}(G) := \bigoplus_n \mathrm{Iso}(G)_n$$

becomes a commutative graded semiring with addition and multiplication given by disjoint union and exterior product, with 0 and 1 given by the classes of the empty set and one point space.

Let $\Omega^G := \mathrm{Iso}(G)/\sim$ be the corresponding *cobordism ring* of closed ($G = O$) and oriented ($G = SO$) or weakly ("= stably") almost complex manifolds ($G = U$) as discussed for example in [Stong]. Here $M \sim 0$ for a closed pure n-dimensional G-manifold M if and only if there is a compact pure $(n+1)$-dimensional G-manifold B with boundary $\partial B \simeq M$. This is indeed a ring with $-[M] = [M]$ for $G = O$ or $-[M] = [-M]$ for $G = SO, U$, where $-M$ has the opposite orientation of M. Moreover, for B as above with $\partial B \simeq M$ one has

$$TB|\partial B \simeq TM \oplus \mathbb{R}_M.$$

This also explains the use of the stable tangent bundle for the definition of a stably or weakly almost complex manifold.

The following structure theorems are fundamental (see [Stong, Theorems on p. 177 and p. 110]):

THEOREM 2-13. (1) (Thom) $\Omega^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{P}^2, \mathbb{P}^4, \mathbb{P}^6, \dots, \mathbb{P}^{2n}, \dots]$ *is a polynomial algebra in the classes of the complex even dimensional projective spaces.*

(2) (Milnor) $\Omega^U_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \dots, \mathbb{P}^n, \dots]$ *is a polynomial algebra in the classes of the complex projective spaces.*

So, if we consider a commutative ring R without torsion for a genus $\gamma : \Omega^{SO} \to R$, the genus γ is completely determined by the value $\gamma(\mathbb{P}^{2n})$ of the cobordism class of each even-dimensional complex projective space $\mathbb{P}^{2n}$. Using this value one could consider the related generating "function" or formal power series such as $\sum_n \gamma(\mathbb{P}^{2n})x^n$, or $\sum_n \gamma(\mathbb{P}^{2n})x^{2n}$, and etc. In fact, a more interesting problem is to determine all *rigid* genera such as the signature σ and the A-genus: namely, a genus satisfying the following multiplicativity property stronger than the product property (3):

$(3)_{\text{rigid}}$: $\gamma(M) = \gamma(F)\gamma(B)$ for a fiber bundle $M \to B$ with fiber F and compact connected structure group.

THEOREM 2-14. *Let* $\log_\gamma(x)$ *be the "logarithmic" formal power series in* $R[\![x]\!]$ *given by*

$$\log_\gamma(x) := \sum_n \frac{1}{2n+1}\gamma(\mathbb{P}^{2n})x^{2n+1}.$$

The genus γ *is rigid if and only if it is an elliptic genus*; *i.e., its logarithm* $\log_\gamma$ *is an elliptic integral*; *i.e.,*

$$\log_\gamma(x) = \int_0^x \frac{1}{\sqrt{1-2\delta t^2+\varepsilon t^4}}dt$$

for some $\delta, \varepsilon \in R$.

The "only if" part was proved by S. Ochanine [Oc] and the "if part" was first "physically" proved by E. Witten [Wit] and later "mathematically" proved by C. Taubes [Ta] and also by R. Bott and C. Taubes [BT]. See also B. Totaro's papers [To2; To4].

cobordism counting c_{cob} ***: Thom's Theorem***

rigid genus = elliptic genus : elliptic integral

The oriented cobordism group Ω^{SO} above was extended by M. Atiyah [At] to a generalized cohomology theory, i.e., the oriented cobordism theory $MSO^*(X)$ of a topological space X. The theory $MSO^*(X)$ is defined by the so-called Thom spectra: the infinite sequence of Thom complexes given, for a topological pair (X, Y) with $Y \subset X$, by

$$MSO^k(X,Y) := \lim_{n\to\infty}[\Sigma^{n-k}(X/Y), MSO(n)].$$

Here the homotopy group $[\Sigma^{n-k}(X/Y), MSO(n)]$ is stable.

As a covariant or homology-like version of $MSO^*(X)$, M. Atiyah [At] introduced the bordism theory $MSO_*(X)$ geometrically in quite a simple manner: Let $f_1 : M_1 \to X$, $f_2 : M_2 \to X$ be continuous maps from closed oriented n-dimensional manifolds to a topological space X. f and g are said to be bordant if there exists an oriented manifold W with boundary and a continuous map $g : W \to X$ such that

(1) $g|_{M_1} = f_1$ and $g|_{M_2} = f_2$, and

(2) $\partial W = M_1 \cup -M_2$, where $-M_2$ is M_2 with its reverse orientation.

It turns out that $MSO_*(X)$ is a generalized homology theory and

$$MSO^0(pt) = MSO_0(pt) = \Omega^{SO}.$$

M. Atiyah [At] also showed Poincaré duality for an oriented closed manifold M of dimension n:

$$MSO^k(M) \cong MSO_{n-k}(M).$$

If we replace $SO(n)$ by the other groups $O(n)$, $U(n)$, Spin(n), we get the corresponding cobordism and bordism theories.

REMARK 2-15 (ELLIPTIC COHOMOLOGY). Given a ring homomorphism $\varphi : MSO^*(pt) \to R$, R is an $MSO^*(pt)$-module and

$$MSO^*(X) \otimes_{MSO^*(pt)} R$$

becomes "almost" a generalized cohomology theory (one not necessarily satisfying the Exactness Axiom). P. S. Landweber [L] gave an algebraic criterion (called the Exact Functor Theorem) for it to become a generalized cohomology theory. Applying this theorem, P. E. Landweber, D. C. Ravenel and R. E. Stong [LRS] showed the following theorem:

THEOREM 2-16. *For the elliptic genus* $\gamma : MSO^*(pt) = MSO_*(pt) = \Omega \to \mathbb{Z}\left[\frac{1}{2}\right][\delta, \varepsilon]$, *the following functors are generalized cohomology theories*:

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon][\varepsilon^{-1}],$$
$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon][(\delta^2 - \varepsilon)^{-1}],$$
$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon][\Delta^{-1}],$$

where $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$.

More generally J. Franke [Fr] showed this:

THEOREM 2-17. *For the elliptic genus* $\gamma : MSO^*(pt) = MSO_*(pt) = \Omega^{SO} \to \mathbb{Z}\left[\frac{1}{2}\right][\delta, \varepsilon]$, *the functor*

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon][P(\delta, \varepsilon)^{-1}]$$

is a generalized cohomology theory. Here $P(\delta, \varepsilon)$ *is a homogeneous polynomial of positive degree with* $\deg \delta = 4$, $\deg \varepsilon = 8$.

The generalized cohomology theory

$$MSO^*(X) \otimes_{MSO^*(pt)} \mathbb{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon][P(\delta, \varepsilon)^{-1}]$$

is called *elliptic cohomology theory*. It was recently surveyed by J. Lurie [Lu]. It is defined in an algebraic manner, but not in a topologically or geometrically simpler manner than K-theory or the bordism theory $MSO_*(X)$. So, people have been searching for a reasonable geometric or topological construction for elliptic cohomology (cf. [KrSt]).

REMARK 2-18 (MUMBO JUMBO). In the above we see that if you just count points of a variety simply as a set, we get infinity unless it is a finite set or the trivial counting 0, but that if we count it "respecting" the topological and algebraic structures we get a certain reasonable number which is not infinity. Getting carried away, "zeta function-theoretic" formulae such as

$$\begin{aligned} 1+1+1+\cdots+1+\cdots &= -\tfrac{1}{2} = \zeta(0), \\ 1+2+3+\cdots+n+\cdots &= -\tfrac{1}{12} = \zeta(-1), \\ 1^2+2^2+3^2+\cdots+n^2+\cdots &= \;\;0\;\; = \zeta(-2), \\ 1^3+2^3+3^3+\cdots+n^3+\cdots &= \tfrac{1}{120} = \zeta(-3) \end{aligned}$$

could be considered as based on a counting of infinite sets that respects some kind of "zeta structure" on it, whatever that is. In nature, the equality $1^3+2^3+3^3+\cdots+n^3+\cdots = \frac{1}{120}$ is relevant to the *Casimir effect*, named after the Dutch physicist Hendrik B. G. Casimir. (See [Wil, Lecture 7] for the connection.) So, nature perhaps already knows what the zeta structure is. It would be fun, even nonmathematically, to imagine what a zeta structure would be on the natural numbers $\mathbb{N}$, or the integers $\mathbb{Z}$ or the rational numbers $\mathbb{Q}$, or more generally "zeta structured" spaces or varieties. Note that, like the topological counting $c_{\text{top}} = \chi$, zeta-theoretical counting (denoted by c_{zeta} here) was discovered by Euler!

REMARK 2-19. Regarding "counting", one is advised to read Baez [Ba1; Ba2], Baez and Dolan [BD], and Leinster [Lein].

3. Motivic characteristic classes

Any algebraic counting c_{alg} gives rise to the following naïve ring homomorphism to a commutative ring R with unity:

$$c_{\text{alg}} : \text{Iso}(\mathcal{VAR}) \to R \quad \text{defined by } c_{\text{alg}}([X]) := c_{\text{alg}}(X).$$

Here $\text{Iso}(\mathcal{VAR})$ is the free abelian group generated by the isomorphism classes $[X]$ of complex varieties. The additivity relation

$$c_{\text{alg}}([X]) = c_{\text{alg}}([X \setminus Y]) + c_{\text{alg}}([Y]) \text{ for any closed subvariety } Y \subset X$$

— or, in other words,

$$c_{\text{alg}}([X]-[Y]-[X\setminus Y])=0 \text{ for any closed subvariety } Y\subset X,$$

induces the following finer ring homomorphism:

$$c_{\text{alg}}: K_0(\mathcal{VAR})\to R \quad \text{defined by } c_{\text{alg}}([X]):=c_{\text{alg}}(X).$$

Here $K_0(\mathcal{VAR})$ is the Grothendieck ring of complex algebraic varieties, which is $\operatorname{Iso}(\mathcal{VAR})$ modulo the additivity relation

$$[X]=[X\setminus Y]+[Y] \text{ for any closed subvariety } Y\subset X$$

(in other words, $\operatorname{Iso}(\mathcal{VAR})$ modded out by the subgroup generated by elements of the form $[X]-[Y]-[X\setminus Y]$ for any closed subvariety $Y\subset X$).

The equivalence class of $[X]$ in $K_0(\mathcal{VAR})$ should be written as, $[\![X]\!]$, say, but we just use the symbol $[X]$ for simplicity.

More generally, let y be an indeterminate and consider the following homomorphism $c_{\text{alg}}:=\chi_y:=\chi_{y,-1}$, i.e.,

$$c_{\text{alg}}: K_0(\mathcal{VAR})\to \mathbb{Z}[y] \quad \text{with } c_{\text{alg}}(\mathbb{C}^1)=-y.$$

This will be called a *motivic characteristic*, to emphasize the fact that its domain is the Grothendieck ring of varieties.

REMARK 3-1. In fact, for the category $\mathcal{VAR}(k)$ of algebraic varieties over any field, the above Grothendieck ring $K_0(\mathcal{VAR}(k))$ can be defined in the same way.

What we want to do is an analogue to the way that Grothendieck extended the celebrated Hirzebruch–Riemann–Roch Theorem (which was the very beginning of the Atiyah–Singer Index Theorem) to the Grothendieck–Riemann–Roch Theorem. In other words, we want to solve the following problem:

PROBLEM 3-2. *Let R be a commutative ring with unity such that $\mathbb{Z}\subset R$, and let y be an indeterminate. Do there exist some covariant functor $\diamondsuit$ and some natural transformation* (*here pushforwards are considered for proper maps*)

$$\natural:\diamondsuit(\quad)\to H_*^{BM}(\quad)\otimes R[y]$$

satisfying conditions (1)–(3) *below*?

(1) $\diamondsuit(pt)=K_0(\mathcal{VAR})$.
(2) $\natural(pt)=c_{\text{alg}}$, *i.e.*,

$$\natural(pt)=c_{\text{alg}}:\diamondsuit(pt)=K_0(\mathcal{VAR})\to R[y]=H_*^{BM}(pt)\otimes R[y].$$

(3) *For the mapping* $\pi_X : X \to pt$ *to a point, for a certain distinguished element* $\Delta_X \in \diamondsuit(X)$ *we have*

$$\pi_{X*}(\natural(\Delta_X)) = c_{\mathrm{alg}}(X) \in R[y] \quad \textit{and} \quad \pi_{X*}(\Delta_X) = [X] \in K_0(\mathcal{VAR}).$$

$$\begin{array}{ccc} \diamondsuit(X) & \xrightarrow{\natural(X)} & H_*^{BM}(X) \otimes R[y] \\ {\scriptstyle \pi_{X*}}\downarrow & & \downarrow{\scriptstyle \pi_{X*}} \\ \diamondsuit(pt) = K_0(\mathcal{VAR}) & \xrightarrow[\natural(pt)=c_{\mathrm{alg}}]{} & R[y]. \end{array}$$

(If there exist such $\diamondsuit$ and $\natural$, then $\natural(\Delta_X)$ could be called the *motivic characteristic class* corresponding to the motivic characteristic $c_{\mathrm{alg}}(X)$, just like the Poincaré dual of the total Chern cohomology class $c(X)$ of a complex manifold X corresponds to the Euler–Poincaré characteristic: $\pi_{X*}(c(X) \cap [X]) = \chi(X)$.)

A more concrete one for the Hodge–Deligne polynomial (a prototype of this problem was considered in [Y5]; cf. [Y6]):

PROBLEM 3-3. *Let* R *be a commutative ring with unity such that* $\mathbb{Z} \subset R$, *and let* u, v *be two indeterminates. Do there exist a covariant functor* $\diamondsuit$ *and a natural transformation* (*here pushforwards are considered for proper maps*)

$$\natural : \diamondsuit(\quad) \to H_*BM(\quad) \otimes R[u, v]$$

satisfying conditions (1)–(3) *below*?

(1) $\diamondsuit(pt) = K_0(\mathcal{VAR})$.
(2) $\natural(pt) = \chi_{u,v}$, *i.e.*,

$$\natural(pt) = \chi_{u,v} : \diamondsuit(pt) = K_0(\mathcal{VAR}) \to R[u, v] = H_*^{BM}(pt) \otimes R[u, v].$$

(3) *For the mapping* $\pi_X : X \to pt$ *to a point, for a certain distinguished element* $\Delta_X \in \diamondsuit(X)$ *we have*

$$\pi_{X*}(\sharp(\Delta_X)) = \chi_{u,v}(X) \in R[u, v] \quad \textit{and} \quad \pi_{X*}(\Delta_X) = [X] \in K_0(\mathcal{VAR}).$$

One reasonable candidate for the covariant functor $\diamondsuit$ is the following:

DEFINITION 3-4. (See [Lo2], for example.) *The relative Grothendieck group of* X, denoted by

$$K_0(\mathcal{VAR}/X),$$

is defined to be the free abelian group $\mathrm{Iso}(\mathcal{VAR}/X)$ generated by isomorphism classes $[V \xrightarrow{h} X]$ of morphisms $h : V \to X$ of complex algebraic varieties over X, modulo the additivity relation

$$[V \xrightarrow{h} X] = [V \setminus Z \xrightarrow{h_{|V \setminus Z}} X + [Z \xrightarrow{h_{|Z}} X] \text{ for any closed subvariety } Z \subset V;$$

in other words, $\mathrm{Iso}(\mathcal{VAR}/X)$ modulo the subgroup generated by the elements of the form

$$[V \xrightarrow{h} X] - [Z \xrightarrow{h_{|Z}} X] - [V \setminus Z \xrightarrow{h_{|V \setminus Z}} X]$$

for any closed subvariety $Z \subset V$.

REMARK 3-5. For the category $\mathcal{VAR}(k)$ of algebraic varieties over any field, we can consider the same relative Grothendieck ring $K_0(\mathcal{VAR}(k)/X)$.

NOTE 1. $K_0(\mathcal{VAR}/pt) = K_0(\mathcal{VAR})$.

NOTE 2. $K_0(\mathcal{VAR}/X)$[3] is a covariant functor with the obvious pushforward: for a morphism $f : X \to Y$, the pushforward

$$f_* : K_0(\mathcal{VAR}/X) \to K_0(\mathcal{VAR}/Y)$$

is defined by

$$f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y].$$

NOTE 3. Although we do not need the ring structure on $K_0(\mathcal{VAR}/X)$ in later discussion, the fiber product gives a ring structure on it:

$$[V_1 \xrightarrow{h_1} X] \cdot [V_2 \xrightarrow{h_2} X] := [V_1 \times_X V_2 \xrightarrow{h_1 \times_X h_2} X].$$

NOTE 4. If $\diamondsuit(X) = K_0(\mathcal{VAR}/X)$, the distinguished element Δ_X is the isomorphism class of the identity map:

$$\Delta_X = [X \xrightarrow{\mathrm{id}_X} X].$$

If we impose one more requirement in Problems 3-2 and 3-3, we can find the answer. The newcomer is the *normalization condition* (or *"smooth condition"*) that for nonsingular X we have

$$\natural(\Delta_X) = c\ell(TX) \cap [X]$$

for a certain normalized multiplicative characteristic class $c\ell$ of complex vector bundles. Note that $c\ell$ is a polynomial in the Chern classes such that it satisfies the normalization condition. Here "normalized" means that $c\ell(E) = 1$ for any trivial bundle E and "multiplicative" means that $c\ell(E \oplus F) = c\ell(E)c\ell(F)$, which is called the *Whitney sum formula*. In connection with the Whitney sum formula, in the analytic or algebraic context, one asks for this multiplicativity for a short exact sequence of vector bundles (which splits only in the topological context):

$$c\ell(E) = c\ell(E')c\ell(E'') \quad \text{for} \quad 1 \to E' \to E \to E'' \to 1.$$

[3] According to a recent paper by M. Kontsevich ("Notes on motives in finite characteristic", math.AG/0702206), Vladimir Drinfeld calls an element of $K_0(\mathcal{VAR}/X)$ "poor man's motivic function".

The normalization condition requirement is natural, in the sense that the other well-known/studied characteristic homology classes of possibly singular varieties are formulated as natural transformations satisfying such a normalization condition, as recalled later. Also, as discussed later (see Conjecture 6-1), this seemingly strong requirement of the normalization condition could be eventually dropped.

OBSERVATION 3-6. Let $\pi_X : X \to pt$ be the mapping to a point. It follows from the naturality of $\natural$ and the normalization condition that

$$c_{\mathrm{alg}}([X]) = \natural\big(\pi_{X*}([X \xrightarrow{\mathrm{id}_X} X])\big) = \pi_{X*}\big(\natural([X \xrightarrow{\mathrm{id}_X} X])\big) = \pi_{X*}\big(c\ell(TX)\cap[X]\big).$$

for any nonsingular variety X. Therefore the normalization condition on nonsingular varieties implies that for a nonsingular variety X the algebraic counting $c_{\mathrm{alg}}(X)$ has to be the characteristic number or Chern number [Ful; MiSt]. This is another requirement on c_{alg}, but an inevitable one if we want to capture it functorially (à la Grothendieck–Riemann–Roch) together with the normalization condition above for smooth varieties.

The normalization condition turns out to be essential, and in fact it automatically determines the characteristic class $c\ell$ as follows, if we consider the bigger ring $\mathbb{Q}[y]$ instead of $\mathbb{Z}[y]$:

PROPOSITION 3-7. *If the normalization condition is imposed in Problems* 3-2 *and* 3-3, *the multiplicative characteristic class* $c\ell$ *with coefficients in* $\mathbb{Q}[y]$ *has to be the generalized Todd class, or the Hirzebruch class* T_y, *defined as follows*: *for a complex vector bundle* V,

$$T_y(V) := \prod_{i=1}^{\operatorname{rank} V}\left(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} - \alpha_i y\right)$$

where the α_i *are the Chern roots of the vector bundle*: $c(V) = \prod_{i=1}^{\operatorname{rank} V}(1+\alpha_i)$.

PROOF. The multiplicativity of $c\ell$ guarantees that if X and Y are smooth compact varieties, then

$$\pi_{X\times Y*}(c\ell(T(X\times Y)\cap[X\times Y]) = \pi_{X*}(c\ell(TX)\cap[X])\cdot\pi_{Y*}(c\ell(TY)\cap[Y]).$$

In other words, the Chern number is multiplicative, i.e., it is compatible with the multiplicativity of c_{alg}. Now Hirzebruch's theorem [Hi, Theorem 10.3.1] says that if the multiplicative Chern number defined by a multiplicative characteristic class $c\ell$ with coefficients in $\mathbb{Q}[y]$ satisfies that the corresponding characteristic number of the complex projective space $\mathbb{P}^n$ is equal to $1-y+y^2-y^3+\cdots+(-y)^n$, then the multiplicative characteristic class $c\ell$ has to be the generalized Todd class, i.e., the Hirzebruch class T_y above. □

REMARK 3-8. In other words, in a sense $c_{\text{alg}}(\mathbb{C}^1)$ uniquely determines the class version of the motivic characteristic c_{alg}, i.e., the motivic characteristic class. This is very similar to the fact foreseen that $c_{\text{top}}(\mathbb{R}^1) = -1$ uniquely determines the "topological counting" c_{top}.

The Hirzebruch class T_y specializes to the following important characteristic classes:

$$y = -1: \quad T_{-1}(V) = c(V) = \prod_{i=1}^{\text{rank } V} (1 + \alpha_i) \quad \text{(total Chern class)}$$

$$y = 0: \quad T_0(X) = td(V) = \prod_{i=1}^{\text{rank } V} \frac{\alpha_i}{1 - e^{-\alpha_i}} \quad \text{(total Todd class)}$$

$$y = 1: \quad T_1(X) = L(V) = \prod_{i=1}^{\text{rank } V} \frac{\alpha_i}{\tanh \alpha_i} \quad \text{(total Thom–Hirzebruch class)}$$

Now we are ready to state our answer to Problem 3-2, which is one of the main theorems of [BSY1]:

THEOREM 3-9 (MOTIVIC CHARACTERISTIC CLASSES). *Let y be an indeterminate.*

(1) *There exists a unique natural transformation*

$$T_{y*} : K_0(\mathcal{VAR}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

satisfying the normalization condition that for a nonsingular variety X

$$T_{y*}([X \xrightarrow{\text{id}_X} X]) = T_y(TX) \cap [X].$$

(2) *For $X = pt$, the transformation $T_{y*} : K_0(\mathcal{VAR}) \to \mathbb{Q}[y]$ equals the Hodge–Deligne polynomial*

$$\chi_{y,-1} : K_0(\mathcal{VAR}) \to \mathbb{Z}[y] \subset \mathbb{Q}[y],$$

namely,

$$T_{y*}([X \to pt]) = \chi_{y,-1}([X]) = \sum_{i,p \geq 0} (-1)^i \dim_{\mathbb{C}}(\mathrm{Gr}_F^p H_c^i(X, \mathbb{C}))(-y)^p.$$

$\chi_{y,-1}(X)$ is simply denoted by $\chi_y(X)$.

PROOF. (1) The main part is of course the existence of such a T_{y*}, the proof of which is outlined in a later section. Here we point out only the uniqueness of T_{y*}, which follows from resolution of singularities. More precisely it follows from two results:

(i) Nagata's compactification theorem, or, if we do not wish to use such a fancy result, the projective closure of affine subvarieties. We get the surjective homomorphism

$$A : \mathrm{Iso}^{\mathrm{prop}}(\mathcal{VAR}/X) \twoheadrightarrow K_0(\mathcal{VAR}/X),$$

where $\mathrm{Iso}^{\mathrm{prop}}(\mathcal{VAR}/X)$ is the free abelian group generated by the isomorphism class of *proper* morphisms to X.

(ii) Hironaka's resolution of singularities: it implies, by induction on dimension that any isomorphism class $[Y \xrightarrow{h} X]$ can be expressed as

$$\sum_V a_V [V \xrightarrow{h_V} X],$$

with V nonsingular and $h_V : V \to X$ proper. We get the surjective maps

$$\mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X) \twoheadrightarrow \mathrm{Iso}^{\mathrm{prop}}(\mathcal{VAR}/X);$$

therefore

$$B : \mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X) \twoheadrightarrow K_0(\mathcal{VAR}/X),$$

where $\mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X)$ is the free abelian group generated by the isomorphism class of *proper* morphisms from *smooth varieties* to X.

(iii) The normalization condition ("smooth condition") of page 390.

(iv) The naturality of T_{y*}.

The two surjective homomorphisms A and B also play key roles in the proof of the existence of T_{y*}.

(2) As pointed out in (ii), $K_0(\mathcal{VAR})$ is generated by the isomorphism classes of compact smooth varieties. On a nonsingular compact variety X we have

$$\chi_{y,-1}(X) = \sum_{p,q \geq 0} (-1)^q \dim_{\mathbb{C}} H^q(X; \Omega_X^p) y^p,$$

which is denoted by $\chi_y(X)$ and is called the Hirzebruch χ_y-genus. Next we have the *generalized Hirzebruch–Riemann–Roch Theorem* (gHRR), which says [Hi] that

$$\chi_y(X) = \int_X T_y(TX) \cap [X].$$

Since $\int_X T_y(TX) \cap [X] = \pi_{X*}(T_y(TX) \cap [X]) = T_{y*}([X \to pt])$, we have

$$T_{y*}([X \to pt]) = \chi_{y,-1}([X])$$

on generators of $K_0(\mathcal{VAR})$, and hence on all of $K_0(\mathcal{VAR})$; thus $T_{y*} = \chi_{y,-1}$. □

REMARK 3-10. Problem 3-3 is slightly more general than Problem 3-2 in the sense that it involves two indeterminates u, v. However, the important keys are the normalization condition for smooth compact varieties and the fact that $\chi_{u,v}(\mathbb{P}^1) = 1 + uv + (uv)^2 + \cdots + (uv)^n$, which automatically implies that $c\ell = T_{-uv}$, as shown in the proof above. In fact, we can say more about u and v: either $u = -1$ or $v = -1$, as shown below (see also [Jo] — the arXiv version). Hence, we can conclude that for Problem 3-3 there is *no* such transformation $\sharp : K_0(\mathcal{VAR}/-) \to H_*^{BM}(-) \otimes R[u, v]$ with both intermediates u and v varying.

To show the claim about u and v, suppose that for X smooth and for a certain multiplicative characteristic class $c\ell$ we have

$$\chi_{u,v}(X) = \pi_{X*}(c\ell(TX) \cap [X]).$$

In particular, consider a smooth elliptic curve E and any d-fold covering

$$\pi : \widetilde{E} \to E$$

with $\widetilde{E}$ a smooth elliptic curve. Note that $T\widetilde{E} = \pi^* TE$ and

$$\chi_{u,v}(E) = \chi_{u,v}(\tilde{E}) = 1 + u + v + uv = (1+u)(1+v).$$

Hence we have

$$\begin{aligned}(1+u)(1+v) = \chi_{u,v}(\widetilde{E}) &= \pi_{\widetilde{E}*}(c\ell(T\widetilde{E}) \cap [\widetilde{E}]) = \pi_{\widetilde{E}*}(c\ell(\pi^* TE) \cap [\widetilde{E}]) \\ &= \pi_{E*}\pi_*(c\ell(\pi^* TE) \cap [\widetilde{E}]) = \pi_{E*}(c\ell(TE) \cap \pi_*[\widetilde{E}]) \\ &= \pi_{E*}(c\ell(TE) \cap d[E]) = d \cdot \pi_{E*}(c\ell(TE) \cap [E]) \\ &= d \cdot \chi_{u,v}(E) = d(1+u)(1+v).\end{aligned}$$

Thus we get $(1+u)(1+v) = d(1+u)(1+v)$. Since $d \neq 0$, we must have that $(1+u)(1+v) = 0$, showing that $u = -1$ or $v = -1$.

REMARK 3-11. Note that $\chi_{u,v}(X)$ is symmetric in u and v; thus both special cases $u = -1$ and $v = -1$ give rise to the same $c\ell = T_y$. It suffices to check this for a compact nonsingular variety X. In fact this follows from the Serre duality.

REMARK 3-12. The heart of the mixed Hodge structure is certainly the existence of the weight filtration $W^\bullet$ and the Hodge–Deligne polynomial, i.e., the algebraic counting c_{alg}, involves the mixed Hodge structure, i.e., both the weight filtration $W^\bullet$ and the Hodge filtration $F_\bullet$. However, when one tries to capture c_{alg} *functorially*, only the Hodge filtration $F_\bullet$ gets involved; the weight filtration *does not*, as seen in the Hodge genus χ_y.

DEFINITION 3-13. For a possibly singular variety X, we call

$$T_{y*}(X) := T_{y*}([X \xrightarrow{\mathrm{id}_X} X])$$

the *Hirzebruch class of* X.

COROLLARY 3-14. *The degree of the* 0*-dimensional component of the Hirzebruch class of a compact complex algebraic variety* X *is just the Hodge genus*:

$$\chi_y(X) = \int_X T_{y*}(X).$$

This is another singular analogue of the gHRR theorem ($\chi_y = T_y$), which is a generalization of the famous Hirzebruch–Riemann–Roch Theorem (which was further generalized to the Grothendieck–Riemann–Roch Theorem):

$$\text{Hirzebruch–Riemann–Roch:} \quad p_a(X) = \int_X td(TX) \cap [X],$$

with $p_a(X)$ the arithmetic genus and $td(V)$ the original Todd class. Noticing the above specializations of χ_y and $T_y(V)$, this gHRR is a unification of the following three well-known theorems:

$$y = -1: \quad \chi(X) = \int_X c(X) \cap [X] \quad \text{(Gauss–Bonnet, or Poincaré–Hopf)}$$
$$y = 0: \quad p_a(X) = \int_X td(X) \cap [X] \quad \text{(Hirzebruch–Riemann–Roch)}$$
$$y = 1: \quad \sigma(X) = \int_X L(X) \cap [X] \quad \text{(Hirzebruch's Signature Theorem)}$$

4. Proofs of the existence of the motivic characteristic class T_{y*}

Our motivic characteristic class transformation

$$T_{y*} : K_0(\mathcal{VAR}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

is obtained as the composite

$$T_{y*} = \widetilde{td_{*(y)}^{BFM}} \circ \Lambda_y^{\text{mot}}$$

of the natural transformations

$$\Lambda_y^{\text{mot}} : K_0(\mathcal{VAR}/X) \to G_0(X) \otimes \mathbb{Z}[y]$$

and

$$\widetilde{td_{*(y)}^{BFM}} : G_0(X) \otimes \mathbb{Z}[y] \to H_*^{BM}(X) \otimes \mathbb{Q}[y, (1+y)^{-1}].$$

Here, to describe $\widetilde{td_{*(y)}^{BFM}}$, we need to recall the following Baum–Fulton–MacPherson's Riemann–Roch or Todd class for singular varieties [BFM1]:

THEOREM 4-1. *There exists a unique natural transformation*

$$td_*^{BFM} : G_0(-) \to H_*^{BM}(-) \otimes \mathbb{Q}$$

such that for a smooth X

$$td_*^{BFM}(\mathcal{O}_X) = td(TX) \cap [X].$$

Here $G_0(X)$ *is the Grothendieck group of coherent sheaves on* X*, which is a covariant functor with the pushforward* $f_* : G_0(X) \to G_0(Y)$ *for a proper morhphism* $f : X \to Y$ *defined by*

$$f_!(\mathcal{F}) = \sum_j (-1)^j [R^j f_* \mathcal{F}].$$

Now set

$$td_*^{BFM}(X) := td_*^{BFM}(\mathcal{O}_X);$$

this is called the Baum–Fulton–MacPherson Todd class of X. Then

$$p_a(X) = \chi(X, \mathcal{O}_X) = \int_X td_*^{BFM}(X) \quad \text{(HRR-type theorem).}$$

Let

$$td_{*i}^{BFM} : G_0(X) \xrightarrow{td_*^{BFM}} H_*^{BM}(X) \otimes \mathbb{Q} \xrightarrow{\text{projection}} H_{2i}^{BM}(X) \otimes \mathbb{Q}$$

be the i-th (i.e., $2i$-dimensional) component of td_*^{BFM}. Then the above *twisted BFM-Todd class transformation* or *twisted BFM-RR transformation* (cf. [Y4])

$$\widetilde{td_{*(y)}^{BFM}} : G_0(X) \otimes \mathbb{Z}[y] \to H_*^{BM}(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

is defined by

$$\widetilde{td_{*(y)}^{BFM}} := \sum_{i \geq 0} \frac{1}{(1+y)^i} td_{*i}^{BFM}.$$

In this process, $\Lambda_y^{\text{mot}} : K_0(\mathcal{VAR}/X) \to G_0(X) \otimes \mathbb{Z}[y]$ is the key. This object was denoted by mC_* in our paper [BSY1] and called the *motivic Chern class*. In this paper, we use the notation Λ_y^{mot} to emphasize the following property of it:

THEOREM 4-2 ("MOTIVIC" λ_y-CLASS TRANSFORMATION). *There exists a unique natural transformation*

$$\Lambda_y^{\text{mot}} : K_0(\mathcal{VAR}/X) \to G_0(X) \otimes \mathbb{Z}[y]$$

satisfying the normalization condition that for smooth X

$$\Lambda_y^{\text{mot}}([X \xrightarrow{\text{id}} X]) = \sum_{p=0}^{\dim X} [\Omega_X^p] y^p = \lambda_y(T^*X) \otimes [\mathcal{O}_X].$$

Here $\lambda_y(T^*X) = \sum_{p=0}^{\dim X}[\Lambda^p(T^*X)]y^p$ *and* $\otimes[\mathcal{O}_X] : K^0(X) \cong G_0(X)$ *is an isomorphism for smooth* X, *i.e., taking the sheaf of local sections.*

THEOREM 4-3. *The natural transformation*

$$T_{y*} := \widetilde{td}^{BFM}_{*(y)} \circ \Lambda^{\mathrm{mot}}_y : K_0(\mathcal{VAR}/X) \to H^{BM}_*(X) \otimes \mathbb{Q}[y] \subset H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

satisfies the normalization condition that for smooth X

$$T_{y*}([X \xrightarrow{\mathrm{id}} X]) = T_y(TX) \cap [X].$$

Hence such a natural transformation is unique.

REMARK 4-4. Why is the image of T_{y*} in $H^{BM}_*(X) \otimes \mathbb{Q}[y]$? Even though the target of

$$\widetilde{td}^{BFM}_{*(y)} : G_0(X) \otimes \mathbb{Z}[y] \to H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

is $H^{BM}_*(X) \otimes \underline{\mathbb{Q}[y, (1+y)^{-1}]}$, the image of $T_{y*} = \widetilde{td}^{BFM}_{*(y)} \circ \Lambda^{\mathrm{mot}}_y$ is contained in $H_*(X) \otimes \underline{\mathbb{Q}[y]}$. Indeed, as mentioned, by Hironaka's resolution of singularities, induction on dimension, the normalization condition, and the naturality of T_{y*}, the domain $K_0(\mathcal{VAR}/X)$ is generated by $[V \xrightarrow{h} X]$ with h proper and V smooth. Hence

$$T_{y*}([V \xrightarrow{h} X]) = T_{y*}(h_*[V \xrightarrow{\mathrm{id}_V} V]) = h_*(T_{y*}([V \xrightarrow{\mathrm{id}_V} V]) \in H^{BM}_*(X) \otimes \mathbb{Q}[y].$$

PROOF OF THEOREM 4-3. In [BSY1] we gave a slick way of proving this. Here we give a nonslick, direct one. Let X be smooth.

$$\begin{aligned}
&\widetilde{td}^{BFM}_{*(y)} \circ \Lambda^{\mathrm{mot}}_y([X \xrightarrow{\mathrm{id}} X]) \\
&\quad = \widetilde{td}^{BFM}_{*(y)}(\lambda_y(\Omega_X)) = \sum_{i \ge 0} \frac{1}{(1+y)^i} td^{BFM}_{*i}(\lambda_y(\Omega_X)) \\
&\quad = \sum_{i \ge 0} \frac{1}{(1+y)^i} \big(td^{BFM}_*(\lambda_y(\Omega_X))\big)_i \\
&\quad = \sum_{i \ge 0} \frac{1}{(1+y)^i} \big(td^{BFM}_*(\lambda_y(T^*X) \otimes [\mathcal{O}_X])\big)_i \\
&\quad = \sum_{i \ge 0} \frac{1}{(1+y)^i} \big(ch(\lambda_y(T^*X)) \cap td^{BFM}_*(\mathcal{O}_X)\big)_i \\
&\quad = \sum_{i \ge 0} \frac{1}{(1+y)^i} \big(ch(\lambda_y(T^*X)) \cap (td(TX) \cap [X])\big)_i \\
&\quad = \sum_{i \ge 0} \frac{1}{(1+y)^i} \left(\prod_{j=1}^{\dim X} (1 + ye^{-\alpha_j}) \prod_{j=1}^{\dim X} \frac{\alpha_j}{1-e^{-\alpha_j}}\right)_{\dim X - i} \cap [X].
\end{aligned}$$

Furthermore we have

$$\frac{1}{(1+y)^i}\left(\prod_{j=1}^{\dim X}(1+ye^{-\alpha_j})\prod_{j=1}^{\dim X}\frac{\alpha_j}{1-e^{-\alpha_j}}\right)_{\dim X-i}$$

$$\begin{aligned}
&= \frac{(1+y)^{\dim X}}{(1+y)^i}\left(\prod_{j=1}^{\dim X}\frac{1+ye^{-\alpha_j}}{1+y}\prod_{j=1}^{\dim X}\frac{\alpha_j}{1-e^{-\alpha_j}}\right)_{\dim X-i}\\
&= (1+y)^{\dim X-i}\left(\prod_{j=1}^{\dim X}\frac{1+ye^{-\alpha_j}}{1+y}\prod_{j=1}^{\dim X}\frac{\alpha_j}{1-e^{-\alpha_j}}\right)_{\dim X-i}\\
&= \left(\prod_{j=1}^{\dim X}\frac{1+ye^{-\alpha_j}}{1+y}\prod_{j=1}^{\dim X}\frac{\alpha_j(1+y)}{1-e^{-\alpha_j(1+y)}}\right)_{\dim X-i}\\
&= \left(\prod_{j=1}^{\dim X}\frac{1+ye^{-\alpha_j}}{1+y}\cdot\frac{\alpha_j(1+y)}{1-e^{-\alpha_j(1+y)}}\right)_{\dim X-i}\\
&= \left(\prod_{j=1}^{\dim X}\frac{\alpha_j(1+y)}{1-e^{-\alpha_j(1+y)}}-\alpha_j y\right)_{\dim X-i}\\
&= \big(T_y(TX)\big)_{\dim X-i}.
\end{aligned}$$

Therefore $\widetilde{td^{BFM}_{*(y)}}\circ \Lambda^{\mathrm{mot}}_y([X\xrightarrow{\mathrm{id}}X]) = T_y(TX)\cap[X]$. □

It remains to show Theorem 4-2. There are at least three proofs, each with its own advantages.

FIRST PROOF (using Saito's theory of mixed Hodge modules [Sa1; Sa2; Sa3; Sa4; Sa5; Sa6]).

Even though Saito's theory is very complicated, this approach turns out to be useful and for example has been used in recent works of Cappell, Libgober, Maxim, Schürmann and Shaneson [CLMS1; CLMS2; CMS1; CMS2; CMSS; MS1; MS2], related to intersection (co)homology. Here we recall only the ingredients which we need to define Λ^{mot}_y:

MHM1 : To X one can associate an abelian category of *mixed Hodge modules* $MHM(X)$, together with a functorial pullback f^* and pushforward $f_!$ on the level of bounded derived categories $D^b(MHM(X))$ for any (not necessarily proper) map. These natural transformations are functors of triangulated categories.

MHM2 : Let $i : Y \to X$ be the inclusion of a closed subspace, with open complement $j : U := X\setminus Y \to X$. Then one has for $M \in D^bMHM(X)$ a distinguished triangle

$$j_!j^*M \to M \to i_!i^*M \xrightarrow{[1]} .$$

MHM3 : For all $p \in \mathbb{Z}$ one has a "filtered De Rham complex" functor of triangulated categories

$$\mathrm{gr}_p^F DR : D^b(MHM(X)) \to D^b_{\mathrm{coh}}(X)$$

commuting with proper pushforward. Here $D^b_{\mathrm{coh}}(X)$ is the bounded derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology sheaves. Moreover, $\mathrm{gr}_p^F DR(M) = 0$ for almost all p and $M \in D^b MHM(X)$ fixed.

MHM4 : There is a distinguished element $\mathbb{Q}^H_{pt} \in MHM(pt)$ such that

$$\mathrm{gr}^F_{-p} DR(\mathbb{Q}^H_X) \simeq \Omega^p_X[-p] \in D^b_{\mathrm{coh}}(X)$$

for X smooth and pure-dimensional. Here $\mathbb{Q}^H_X := \pi_X^* \mathbb{Q}^H_{pt}$ for $\pi_X : X \to pt$ a constant map, with $\mathbb{Q}^H_{pt}$ viewed as a complex concentrated in degree zero.

The transformations above are functors of triangulated categories; thus they induce functors even on the level of *Grothendieck groups of triangulated categories*, which we denote by the same name. Note that for these *Grothendieck groups* we have isomorphisms

$$K_0(D^b MHM(X)) \simeq K_0(MHM(X)) \quad \text{and} \quad K_0(D^b_{\mathrm{coh}}(X)) \simeq G_0(X)$$

by associating to a complex its alternating sum of cohomology objects.

Now we are ready for the transformations mH and $\mathrm{gr}^F_{-*} DR$. Define

$$mH : K_0(\mathcal{VAR}/X) \to K_0(MHM(X)) \quad \text{by} \quad mH([V \xrightarrow{f} X]) := [f_! \mathbb{Q}^H_V].$$

In a sense $K_0(MHM(X))$ is like the abelian group of "mixed-Hodge-module constructible functions", with the class of $\mathbb{Q}^H_X$ as a "constant function" on X. The well-definedness of mH, i.e., the additivity relation follows from property (MHM2). By (MHM3) we get the following homomorphism commuting with proper pushforward:

$$\mathrm{gr}^F_{-*} DR : K_0(MHM(X)) \to G_0(X) \otimes \mathbb{Z}[y, y^{-1}]$$

defined by

$$\mathrm{gr}^F_{-*} DR([M]) := \sum_p [\mathrm{gr}^F_{-p} DR(M)] \cdot (-y)^p$$

Then we define our Λ^{mot}_y as the composite of these two natural transformations:

$$\Lambda^{\mathrm{mot}}_y := \mathrm{gr}^F_{-*} DR \circ mH : K_0(\mathcal{VAR}/X) \xrightarrow{mH} K_0(MHM(X)) \xrightarrow{\mathrm{gr}^F_{-*} DR} G_0(X) \otimes \mathbb{Z}[y].$$

By (MHM4), for X smooth and pure-dimensional we have

$$\mathrm{gr}^F_{-*} DR \circ mH([\mathrm{id}_X]) = \sum_{p=0}^{\dim X} [\Omega^p_X] \cdot y^p \in G_0(X) \otimes \mathbb{Z}[y]\,.$$

Thus we get the unique existence of the "motivic" λ_y-class transformation Λ^{mot}_y. □

SECOND PROOF (using the filtered Du Bois complexes [DB]). Recall the surjective homomorphism

$$A : \mathrm{Iso}^{\mathrm{prop}}(\mathcal{VAR}/X) \twoheadrightarrow K_0(\mathcal{VAR}/X).$$

We can describe its kernel as follows:

THEOREM 4-5. *$K_0(\mathcal{VAR}/X)$ is isomorphic to the quotient of* $\mathrm{Iso}^{\mathrm{pro}}(\mathcal{VAR}/X)$ *modulo the "acyclicity" relation*

$$[\varnothing \to X] = 0 \quad \textit{and} \quad [\tilde{X}' \to X] - [\tilde{Z}' \to X] = [X' \to X] - [Z' \to X], \quad \text{(ac)}$$

for any cartesian diagram

$$\begin{array}{ccccc} \tilde{Z}' & \longrightarrow & \tilde{X}' & & \\ \downarrow & & \downarrow{\scriptstyle q} & & \\ Z' & \xrightarrow{\ i\ } & X' & \longrightarrow & X\,, \end{array}$$

with q proper, i a closed embedding, and $q : \tilde{X}' \backslash \tilde{Z}' \to X' \backslash Z'$ an isomorphism.

For a proper map $X' \to X$, consider the filtered Du Bois complex

$$(\underline{\Omega}^*_{X'}, F),$$

which has the following properties:

(1) $\underline{\Omega}^*_{X'}$ is a resolution of the constant sheaf $\mathbb{C}$.
(2) $\mathrm{gr}^p_F(\underline{\Omega}^*_{X'}) \in D^b_{\mathrm{coh}}(X')$.
(3) Let $DR(\mathcal{O}_{X'}) = \Omega^*_{X'}$ be the de Rham complex of X' with σ being the stupid filtration. Then there is a filtered morphism

$$\lambda : (\Omega^*_{X'}, \sigma) \to (\underline{\Omega}^*_{X'}, F).$$

If X' is smooth, this is a filtered quasi-isomorphism.

Note that $G_0(X') \cong K_0(D^b_{\mathrm{coh}}(X'))$. Let us define

$$[\mathrm{gr}^p_F(\underline{\Omega}^*_{X'})] := \sum_i (-1)^i H^i(\mathrm{gr}^p_F(\underline{\Omega}^*_{X'})) \in K_0(D^b_{\mathrm{coh}}(X')) = G_0(X').$$

THEOREM 4-6. *The transformation*

$$\Lambda_y^{\mathrm{mot}} : K_0(\mathcal{VAR}/X) \to G_0(X) \otimes \mathbb{Z}[y]$$

defined by

$$\Lambda_y^{\mathrm{mot}}([X' \xrightarrow{h} X]) := \sum_p h_*[\mathrm{gr}_F^p(\underline{\Omega}^*_{X'})](-y)^p$$

is well-defined and is the unique natural transformation satisying the normalization condition that for smooth X

$$\Lambda_y^{\mathrm{mot}}([X \xrightarrow{\mathrm{id}_X} X]) = \sum_{p=0}^{\dim X} [\Omega_X^p] y^p = \lambda_y(T^*X) \otimes \mathcal{O}_X.$$

PROOF. The well-definedness follows from the fact that Λ_y^{mot} preserves the acyclicity relation above [DB]. Then uniqueness follows from resolution of singularities and the normalization condition for smooth varieties. □

REMARK 4-7. When X is smooth, we have

$$[\mathrm{gr}_\sigma^p(\underline{\Omega}^*_X)] = (-1)^p [\Omega_X^p]\,!$$

That is why we need $(-y)^p$, instead of y^p, in the definition of $\Lambda_y^{\mathrm{mot}}([X' \xrightarrow{h} X])$.

REMARK 4-8. When $y = 0$, we have the natural transformation

$$\Lambda_0^{\mathrm{mot}} : K_0(\mathcal{VAR}/X) \to G_0(X) \quad \text{defined by} \quad \Lambda_0^{\mathrm{mot}}([X' \xrightarrow{h} X]) = h_*[\mathrm{gr}_F^0(\underline{\Omega}^*_{X'})]$$

satisying the normalization condition that for a smooth X

$$\Lambda_0^{\mathrm{mot}}([X \xrightarrow{\mathrm{id}_X} X]) = [\mathcal{O}_X].$$ □

THIRD PROOF (using Bittner's theorem on $K_0(\mathcal{VAR}/X)$ [Bi]). Recall the surjective homomorphism

$$B : \mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X) \twoheadrightarrow K_0(\mathcal{VAR}/X).$$

Its kernel is identified by F. Bittner and E. Looijenga as follows [Bi]:

THEOREM 4-9. *The group $K_0(\mathcal{VAR}/X)$ is isomorphic to the quotient of* $\mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X)$ (*the free abelian group generated by the isomorphism classes of proper morphisms from smooth varieties to X*) *by the "blow-up" relation*

$$[\varnothing \to X] = 0 \quad \textit{and} \quad [\mathrm{Bl}_Y X' \to X] - [E \to X] = [X' \to X] - [Y \to X], \quad \text{(bl)}$$

for any cartesian diagram

$$\begin{array}{ccccc} E & \xrightarrow{i'} & \mathrm{Bl}_Y\, X' & & \\ \big\downarrow{\scriptstyle q'} & & \big\downarrow{\scriptstyle q} & & \\ Y & \xrightarrow{i} & X' & \xrightarrow{f} & X\,, \end{array}$$

with i a closed embedding of smooth (pure-dimensional) spaces and $f : X' \to X$ proper. Here $\mathrm{Bl}_Y\, X' \to X'$ is the blow-up of X' along Y with exceptional divisor E. Note that all these spaces over X are also smooth (and pure-dimensional and/or quasiprojective, if this is the case for X' and Y).

The proof of this theorem requires the Weak Factorization Theorem, due to D. Abramovich, K. Karu, K. Matsuki and J. Włodarczyk [AKMW] (see also [Wlo]). □

COROLLARY 4-10. (1) *Let $B_* : \mathcal{VAR}/k \to \mathcal{AB}$ be a functor from the category var/k of (reduced) separated schemes of finite type over $\mathrm{spec}(k)$ to the category of abelian groups, which is covariantly functorial for proper morphisms, with $B_*(\emptyset) := \{0\}$. Assume we can associate to any (quasiprojective) smooth space $X \in ob(\mathcal{VAR}/k)$ of pure dimension a distinguished element*

$$\phi_X \in B_*(X)$$

such that $h_*(\phi_{X'}) = \phi_X$ for any isomorphism $h : X' \to X$. There exists a unique natural transformation

$$\Phi : \mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/-) \to B_*(-)$$

satisfying the "normalization" condition that for any smooth X

$$\Phi([X \xrightarrow{\mathrm{id}_X} X]) = \phi_X.$$

(2) *Let $B_* : \mathcal{VAR}/k \to \mathcal{AB}$ and ϕ_X be as above and furthermore we assume that*

$$q_*(\phi_{\mathrm{Bl}_Y\, X}) - i_* q'_*(\phi_E) = \phi_X - i_*(\phi_Y) \in B_*(X)$$

for any cartesian blow-up diagram as in the above Bittner's theorem with $f = \mathrm{id}_X$. Then there exists a unique natural transformation

$$\Phi : K_0(\mathcal{VAR}/-) \to B_*(-)$$

satisfying the "normalization" condition that for any smooth X

$$\Phi([X \xrightarrow{\mathrm{id}_X} X]) = \phi_X.$$

We will now use Corollary 4-10(2) to conclude our third proof. Consider the coherent sheaf $\Omega_X^p \in G_0(X)$ of a smooth X as the distinguished element ϕ_X of a smooth X. It follows from M. Gros's work [Gr] or the recent work of Guillén and Navarro Aznar [GNA] that it satisfies the blow-up relation

$$q_*(\Omega^p_{\mathrm{Bl}_Y X}) - i_* q'_*(\Omega^p_E) = \Omega^p_X - i_*(\Omega^p_Y) \in G_0(X),$$

which in turn implies a blow-up relation for the λ_y-class:

$$q_*(\lambda_y(\Omega_{\mathrm{Bl}_Y X})) - i_* q'_*(\lambda_y(\Omega_E)) = \lambda_y(\Omega_X) - i_*(\lambda_y(\Omega_Y)) \in G_0(X) \otimes \mathbb{Z}[y].$$

Therefore Corollary 4-10(2) implies this:

THEOREM 4-11. *The transformation*

$$\Lambda_y^{\mathrm{mot}} : K_0(\mathcal{VAR}/X) \to G_0(X) \otimes \mathbb{Z}[y]$$

defined by

$$\Lambda_y^{\mathrm{mot}}([X' \xrightarrow{h} X]) := h_*\Bigl(\sum_{p \geq 0} [\Omega^p_{X'}] y^p\Bigr),$$

where X' is smooth and $h : X' \to X$ is proper, is well-defined and is a unique natural transformation satisying the normalization condition that for smooth X

$$\Lambda_y^{\mathrm{mot}}([X \xrightarrow{\mathrm{id}_X} X]) = \sum_{p=0}^{\dim X} [\Omega^p_X] y^p = \lambda_y(T^*X) \otimes \mathcal{O}_X.$$

REMARK 4-12. The virtual Poincaré polynomial β_t (Remark 2-11) for the category $\mathcal{VAR}(\mathbb{R})$ of real algebraic varieties is the unique homomorphism

$$\beta_t : K_0(\mathcal{VAR}(\mathbb{R})) \to \mathbb{Z}[t] \quad \text{such that } \beta_t(\mathbb{R}^1) = t$$

and $\beta_t(X) = P_t(X)$ is the classical or usual topological Poincaré polynomial for compact nonsingular varieties. The proof of the existence of β_i, thus β_t, also uses Corollary 4-10(2); see [MP1]. Speaking of the Poincaré polynomial $P_t(X)$, we emphasize that this polynoimal cannot be a topological counting at all in the category of topological spaces, simply because the argument in the proof of Proposition 2-4 does not work! The Poincaré polynomial $P_t(X)$ is certainly a *multiplicative* topological invariant, but not an *additive* one.

REMARK 4-13. The virtual Poincaré polynomial $\beta_t : K_0(\mathcal{VAR}(\mathbb{R})) \to \mathbb{Z}[t]$ is the *unique* extension of the Poincaré polynomial $P_t(X)$ to arbitrary varieties. Note that if we consider complex algebraic varieties, the virtual Poincaré polynomial

$$\beta_t : K_0(\mathcal{VAR}) \to \mathbb{Z}[t]$$

is equal to the following motivic characteristic, using only the weight filtration:

$$w\chi(X) = \sum (-1)^i \dim_{\mathbb{C}}\big(\mathrm{Gr}_q^W H_c^i(X,\mathbb{C})\big)t^q,$$

because on any smooth compact complex algebraic variety X they are all the same: $\beta_t(X) = P_t(X) = w\chi(X)$. These last equalities follow from the fact that the Hodge structures on $H^k(X,\mathbb{Q})$ are of pure weight k.

This "weight filtration" motivic characteristic $w\chi(X)$ is equal to the specialization $\chi_{-t,-t}$ of the Hodge–Deligne polynomial for $(u,v) = (-t,-t)$. This observation implies that there is ***no class version*** *of the complex virtual Poincaré polynomial* $\beta_t : K_0(\mathcal{VAR}) \to \mathbb{Z}[t]$. In other words, there is no natural transformation

$$\natural : K_0(\mathcal{VAR}/-) \to H_*^{BM}(-) \otimes \mathbb{Z}[t]$$

satisfying the conditions that

- if X is smooth and compact, then $\natural([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(TX) \cap [X]$ for some multiplicative characteristic class of complex vector bundles; and
- $\natural(pt) = \beta_t : K_0(\mathcal{VAR}) \to \mathbb{Z}[t]$.

This is because $\beta_t(X) = \chi_{-t,-t}(X)$ for a smooth compact complex algebraic variety X (hence for all X), and so, as in Remark 3-10, one can conclude that $(-t,-t) = (-1,-1)$. Thus t has to be equal to 1 and cannot be allowed to vary. In other words, the only chance for such a class version is when $t = 1$, which gives the Euler–Poincaré characteristic $\chi : K_0(\mathcal{VAR}) \to \mathbb{Z}$. In that case, we do have the Chern class transformation

$$c_* : K_0(\mathcal{VAR}/-) \to H_*^{BM}(-;\mathbb{Z}).$$

This follows again from Corollary 4-10(2) and the blow-up formula of Chern class [Ful].

REMARK 4-14. The same discussion as in Remark 4-13 can be applied to the context of real algebraic varieties, i.e., the same example for real elliptic curves leads us to the conclusion that $t = 1$ for β_t satisfying the corresponding normalization condition for a normalized multiplicative characteristic class. This class has to be a polynomial in the Stiefel–Whitney classes, and we end up with the Stiefel–Whitney homology class w_*, which also satisfies the corresponding blow-up formula.

REMARK 4-15 (POOR MAN'S MOTIVIC CHARACTERISTIC CLASS). If we use the much simpler covariant functor $\mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X)$ above (the abelian group of "poor man's motivic functions"), we can get the following "poor man's motivic characteristic class" for any characteristic class $c\ell$ of vector bundles: Let $c\ell$ be *any* characteristic class of vector bundles with coefficient ring K. There exists a unique natural transformation

$$c\ell_* : \mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/-) \to H_*^{BM}(-) \otimes K$$

satisfying the normalization condition that for any smooth variety X,

$$c\ell_*([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(TX) \cap [X].$$

There is a bivariant theoretical version of $\mathrm{Iso}^{\mathrm{prop}}(\mathcal{SM}/X)$ (see [Y7]); a good reference for it is Fulton and MacPherson's AMS memoir [FM].

5. Chern class, Todd class and L-class of singular varieties: towards a unification

Our next task is to describe another main theorem of [BSY1], to the effect that our motivic characteristic class T_{y*} is, in a sense, a unification of MacPherson's Chern class, the Todd class of Baum, Fulton, and MacPherson (discussed in the previous section), and the L-class of singular varieties of Cappell and Shaneson. Let's briefly review these classes:

MacPherson's Chern class [M1]

THEOREM 5-1. *There exists a unique natural transformation*

$$c_*^{\mathrm{Mac}} : F(-) \to H_*^{BM}(-)$$

such that, for smooth X,

$$c_*^{\mathrm{Mac}}(\mathbb{1}_X) = c(TX) \cap [X].$$

Here $F(X)$ is the abelian group of constructible functions, which is a covariant functor with the pushforward $f_* : F(X) \to F(Y)$ for a proper morphism $f : X \to Y$ defined by

$$f_*(\mathbb{1}_W)(y) = \chi_c(f^{-1}(y) \cap W).$$

We call $c_*^{\mathrm{Mac}}(X) := c_*^{\mathrm{Mac}}(\mathbb{1}_X)$ the MacPherson's Chern class of X, or the Chern–Schwartz–MacPherson class. We have

$$\chi(X) = \int_X c_*^{\mathrm{Mac}}(X).$$

The Todd class of Baum, Fulton, and MacPherson [BFM1]

THEOREM 5-2. *There exists a unique natural transformation*

$$td_*^{BFM} : G_0(-) \to H_*^{BM}(-) \otimes \mathbb{Q}$$

such that, *for smooth* X,

$$td_*^{BFM}(\mathcal{O}_X) = td(TX) \cap [X].$$

Here $G_0(X)$ *is the Grothendieck group of coherent sheaves on* X, *which is a covariant functor with the pushforward* $f_* : G_0(X) \to G_0(Y)$ *for a proper morphism* $f : X \to Y$ *defined by*

$$f_!(\mathcal{F}) = \sum_j (-1)^j [R^j f_* \mathcal{F}].$$

We call $td_*^{BFM}(X) := td_*^{BFM}(\mathcal{O}_X)$ the Baum–Fulton–MacPherson Todd class of X, and we have

$$p_a(X) = \chi(X, \mathcal{O}_X) = \int_X td_*^{BFM}(X).$$

The L-class of Cappell and Shaneson [CS1; Sh] (cf. [Y4])

THEOREM 5-3. *There exists a unique natural transformation*

$$L_*^{CS} : \Omega(-) \to H_*^{BM}(-) \otimes \mathbb{Q}$$

such that, *for smooth* X,

$$L_*^{CS}(\mathcal{IC}_X) = L(TX) \cap [X].$$

Here $\Omega(X)$ *is the abelian group of Youssin's cobordism classes of self-dual constructible complexes of sheaves on* X.

We call $L_*^{GM}(X) := L_*^{CS}(\mathcal{IC}_X)$ the Goresky–MacPherson homology L-class of X. The Goresky–MacPherson theorem [GM] says that

$$\sigma^{GM}(X) = \int_X L_*^{GM}(X).$$

We now explain in what sense our motivic characteristic class transformation

$$T_{y*} : K_0(\mathcal{VAR}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

unifies these three characteristic classes of singular varieties, providing a kind of partial positive answer to MacPherson's question[4] of *whether there is a unified theory of characteristic classes of singular varieties*.

[4] Posed in his survey talk [M2] at the Ninth Brazilian Mathematics Colloquium in 1973.

THEOREM 5-4 (UNIFIED FRAMEWORK FOR CHERN, TODD AND HOMOLOGY L-CLASSES OF SINGULAR VARIETIES).

$y = -1$: *There exists a unique natural transformation* $\varepsilon : K_0(\mathcal{VAR}/-) \to F(-)$ *such that, for* X *nonsingular,* $\varepsilon([\mathrm{id} : X \to X]) = \mathbb{1}_X$, *and the following diagram commutes*:

$$\begin{array}{ccc} K_0(\mathcal{VAR}/X) & \xrightarrow{\quad\varepsilon\quad} & F(X) \\ & {\scriptstyle T_{-1*}} \searrow \quad \swarrow {\scriptstyle c_*^{\mathrm{Mac}} \otimes \mathbb{Q}} & \\ & H_*^{BM}(X) \otimes \mathbb{Q} & \end{array}$$

$y = 0$: *There exists a unique natural transformation* $\gamma : K_0(\mathcal{VAR}/-) \to G_0(-)$ *such that, for* X *nonsingular,* $\gamma([\mathrm{id} : X \to X]) = [\mathcal{O}_X]$, *and the following diagram commutes*:

$$\begin{array}{ccc} K_0(\mathcal{VAR}/X) & \xrightarrow{\quad\gamma\quad} & G_0(X) \\ & {\scriptstyle T_{0*}} \searrow \quad \swarrow {\scriptstyle td_*^{BFM}} & \\ & H_*^{BM}(X) \otimes \mathbb{Q} & \end{array}$$

$y = 1$: *There exists a unique natural transformation* $sd : K_0(\mathcal{VAR}/-) \to \Omega(-)$ *such that, for* X *nonsingular,* $sd([\mathrm{id} : X \to X]) = [\mathbb{Q}_X[2\dim X]]$, *and the following diagram commutes*:

$$\begin{array}{ccc} K_0(\mathcal{VAR}/X) & \xrightarrow{\quad sd\quad} & \Omega(X) \\ & {\scriptstyle T_{1*}} \searrow \quad \swarrow {\scriptstyle L_*^{CS}} & \\ & H_*^{BM}(X) \otimes \mathbb{Q}\,. & \end{array}$$

The first two claims are straightforward; the third, the case $y = 1$, is anything but. In particular, the existence of $sd : K_0(\mathcal{VAR}/-) \to \Omega(-)$ is not obvious at all. The only way we know to prove it is by going through some details of Youssin's work [You] and using Corollary 4-10(2) again. This is done in [BSY1]; see also [BSY2; SY].

REMARK 5-5. $y = -1$: $T_{-1*}(X) = c_*^{\mathrm{Mac}}(X) \otimes \mathbb{Q}$.
$y = 0$: In general, for a singular variety X we have

$$\Lambda_0^{\mathrm{mot}}([X \xrightarrow{\mathrm{id}_X} X]) \neq [\mathcal{O}_X].$$

Therefore, in general, $T_{0*}(X) \neq td_*^{BFM}(X)$. So, our $T_{0*}(X)$ shall be called the Hodge–Todd class and denoted by $td_*^H(X)$. However, if X is a Du Bois variety, i.e., every point of X is a Du Bois singularity (note a nonsingular point is also a Du Bois singularity), we DO have

$$\Lambda_0^{\mathrm{mot}}([X \xrightarrow{\mathrm{id}_X} X]) = [\mathcal{O}_X].$$

This is because of the definition of Du Bois variety: X is called a Du Bois variety if we have

$$\mathcal{O}_X = \mathrm{gr}_\sigma^0(DR(\mathcal{O}_X)) \cong \mathrm{gr}_F^0(\underline{\Omega}_X^*).$$

Hence, for a Du Bois variety X we have $T_{0*}(X) = td_*^{BFM}(X)$. For example, S. Kovács [Kov] proved Steenbrink's conjecture that rational singularities are Du Bois, thus for the quotient X of any smooth variety acted on by a finite group we have that $T_{0*}(X) = td_*^{BFM}(X)$.

$y=1$: In general, $sd([X \xrightarrow{\mathrm{id}_X} X])$ is distinct from $\mathcal{IC}_X$, so $T_{1*}(X) \neq L_*^{GM}(X)$. We therefore call $T_{1*}(X)$ the *Hodge L-class* and denote it, alternatively, by $L_*^H(X)$. It is conjectured that $T_{1*}(X) \neq L_*^{GM}(X)$ for a rational homology manifold X.

6. A few more conjectures

CONJECTURE 6-1. *Any natural transformation*

$$T : K_0(\mathcal{VAR}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

without the normalization condition is a linear combination of components of the form $td_{y*i} : K_0(\mathcal{VAR}/X) \to H_{2i}^{BM}(X) \otimes \mathbb{Q}[y]$:

$$T = \sum_{i \geq 0} r_i(y)\, td_{y*i} \quad (r_i(y) \in \mathbb{Q}[y]).$$

This conjecture means that the normalization condition for smooth varieties imposed to get our motivic characteristic class can be basically *dropped*. This conjecture is motivated by the following theorems:

THEOREM 6-2 [Y1]. *Any natural transformation*

$$T : G_0(-) \to H_*^{BM}(-) \otimes \mathbb{Q}$$

without the normalization condition is a linear combination of components

$$td_{*\,i}^{BFM} : G_0(-) \to H_{2i}^{BM}(-) \otimes \mathbb{Q},$$

that is,

$$T = \sum_{i \geq 0} r_i\, td_{*\,i}^{BFM} \quad (r_i \in \mathbb{Q}).$$

THEOREM 6-3 [KMY]. *Any natural transformation*

$$T : F(-) \to H_*^{BM}(-) \otimes \mathbb{Q}$$

without the normalization condition is a linear combination of components

$$c^{\text{Mac}}_{*\ i} \otimes \mathbb{Q} : G_0(-) \to H_{2i}^{BM}(-) \otimes \mathbb{Q}$$

of the ***rationalized*** *MacPherson's Chern class* $c_*^{\text{Mac}} \otimes \mathbb{Q}$ (*i.e.*, *a linear combination of* $c^{\text{Mac}}_{*\ i}$ *mod torsion*):

$$T = \sum_{i \geq 0} r_i \, c^{\text{Mac}}_{*\ \ i} \otimes \mathbb{Q} \quad (r_i \in \mathbb{Q}).$$

REMARK 6-4. This theorem certainly implies the uniqueness of such a transformation $c_*^{\text{Mac}} \otimes \mathbb{Q}$ satisfying the normalization. The proof of Theorem 6-3 *does not* appeal to the resolution of singularities at all, therefore modulo torsion the uniqueness of the MacPherson's Chern class transformation c_*^{Mac} is proved without using resolution of singularities. However, in the case of integer coefficients, as shown in [M1], the uniqueness of c_*^{Mac} uses the resolution of singualrities and as far as the author knows, there is no proof available without using this result. Does there exist any mysterious connection between resolution of singularities and finite torsion? (In this connection we quote a comment by J. Schürmann:

> There is indeed a relation between resolution of singularities and torsion information: in [To1] B. Totaro shows by resolution of singularities that the fundamental class $[X]$ of a complex algebraic variety X lies in the image from the complex cobordism $\Omega^U(X) \to H_*(X, \mathbb{Z})$. And this implies some nontrivial topological restrictions: for example, all odd-dimensional elements of the Steenrod algebra vanish on $[X]$ viewed in $H_*(X, \mathbb{Z}_p)$.)

Furthermore, hinted by these two theorems, it would be natural to speculate the following "linearity" on the Cappell–Shaneson L-class also:

CONJECTURE 6-5. *Any natural transformation without the normalization condition*

$$T : \Omega(-) \to H_*^{BM}(-) \otimes \mathbb{Q}$$

is a linear combination of components $L^{CS}_{*\ i} : \Omega(-) \to H_{2i}^{BM}(-) \otimes \mathbb{Q}$:

$$T = \sum_{i \geq 0} r_i \, L^{CS}_{*\ \ i} \quad (r_i \in \mathbb{Q}).$$

7. Some more remarks

For complex algebraic varieties there is another important homology theory. That is Goresky–MacPherson's *intersection homology theory* IH, introduced in [GM] (see also [KW]). It satisfies all the properties which the ordinary (co)homology theory for nonsingular varieties have, in particular the Poincaré duality holds, in contrast to the fact that in general it fails for the ordinary (co)homology theory of singular varieties. In order that the Poincaré duality theorem holds, one needs to control cycles according to *perversity*, which is sensitive to, or "control", complexity of singularities. M. Saito showed that IH satisfies pure Hodge structure just like the cohomology satisfies the pure Hodge structure for compact smooth manifolds (see also [CaMi1; CaMi2]). In this sense, IH is a convenient gadget for possibly singular varieties, and using the IH, we can also get various invariants which are sensitive to the structure of given possibly singular varieties. For the history of IH, see Kleiman's survey article [Kl], and for L_2-*cohomology* — very closely related to the intersection homology — see [CGM; Go; Lo1; SS; SZ], for example. Thus for the category of compact complex algebraic varieties two competing machines are available:

ordinary (co)homology $+$ ***mixed Hodge structures***

intersection homology $+$ ***pure Hodge structures***

Of course, they are the same for the subcategory of compact smooth varieties.

So, for singular varieties one can introduce the similar invariants using IH; in other words, one can naturally think of the IH-version of the Hirzebruch χ_y genus, because of the pure Hodge structure, denote by χ_y^{IH}: Thus we have invariants χ_y-genus and χ_y^{IH}-genus. As to the class version of these, one should go through the derived category of mixed Hodge modules, because the intersection homology sheaf lives in it. Then obviously the difference between these two genera or between the class versions of these two genera should come from the singularities of the given variety. For this line of investigation, see the articles by Cappell, Libgober, Maxim, and Shaneson [CMS1; CMS2; CLMS1; CLMS2].

The most important result is the *Decomposition Theorem* of Beilinson, Bernstein, Deligne, and Gabber [BBD], which was conjectured by I. M. Gelfand and R. MacPherson. A more geometric proof of this is given in the above mentioned paper [CaMi1] of M. de Cataldo and L. Migliorini.

Speaking of the intersection homology, the general category for IH is the category of pseudomanifolds and the canonical and well-studied invariant for pseudomanifolds is the signature, because of the Poincaré duality of IH. Banagl's monograph [Ba1] is recommended on this topic; see also [Ba2; Ba3; Ba4; BCS; CSW; CW; Wei]. Very roughly, T_{y*} is a kind of deformation or perturbation of Baum–Fulton–MacPherson's Riemann–Roch. It would be interesting to consider a similar kind of deformation of L-class theory defined on the (co)bordism theory of pseudomanifolds. Again we quote J. Schürmann:

> A deformation of the L-class theory seems not reasonable. Only the signature = χ_1-genus factorizes over the oriented cobordism ring Ω^{SO}, so that this invariant is of more topological nature related to stratified spaces. For the other desired ("deformation") invariants one needs a complex algebraic or analytic structure. So what is missing up to now is a suitable theory of almost complex stratified spaces.

Finally, since we started the present paper with counting, we end with posing the following question: how about counting pseudomanifolds respecting the structure of pseudomanifolds:

Does "stratified counting" c_{stra} ***make sense?***

For complex algebraic varieties, which are pseudomanifolds, algebraic counting c_{alg} (using mixed Hodge theory = ordinary (co)homology theory + mixed Hodge structure) in fact ignores the stratification. So, in this possible problem, one should consider intersection homology + pure Hodge structure, although intersection homology *is a topological invariant, and hence independent of the stratification.*

J. Schürmann provides one possible answer to the highlighted question above:

> One possible answer would be to work in the complex algebraic context with a fixed (Whitney) stratification $X_\bullet$, so that the closure of a stratum S is a union of strata again. Then one can work with the Grothendieck group $K_0(X_\bullet)$ of $X_\bullet$-constructible sets, i.e., those which are a union of such strata. The topological additive counting would be related again to the Euler characteristic and the group $F(X_\bullet)$ of $X_\bullet$-constructible functions. A more sophisticated version is the Grothendieck group $K_0(X_\bullet)$ of $X_\bullet$-constructible sheaves (or sheaf complexes). These are generated by classes $j_! L_S$ for $j : S \to X$, the inclusion of a stratum S, and L_S a local system on S, and also by the intermediate extensions $j_{!*} L_S$, which are perverse sheaves. In relation to signature and duality, one can work with the corresponding cobordism group $\Omega(X_\bullet)$ of Verdier self-dual $X_\bullet$-constructible sheaf complexes. These are generated by $j_{!*} L_S$, with L_S a self-dual local system on S. Finally one can also work with the Grothendieck group $K_0(MHM(X_\bullet))$ of mixed Hodge modules, whose underlying rational complex is $X_\bullet$-constructible. This last group is of course not a topological invariant.

We hope to come back to the problem of a possible "stratified counting" c_{stra}.

Acknowledgements

This paper is based on my talk at the workshop "Topology of Stratified Spaces" held at MSRI, Berkeley, from September 8 to 12, 2008. I thank the organizers (Greg Friedman, Eugénie Hunsicker, Anatoly Libgober, and Laurentiu Maxim) for their invitation to the workshop. I also thank the referee and Jörg Schürmann for their careful reading of the paper and valuable comments and suggestions, and G. Friedmann and L. Maxim for their much-appreciated feedback on an earlier version of the paper. Finally I thank also Silvio Levy, the editor of MSRI Book Series, for his careful reading and editing of the final version.

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