Topology of Stratified Spaces
MSRI Publications
Volume **58**, 2011

Intersection homology Wang sequence

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ABSTRACT. We prove the existence of a Wang-like sequence for intersection homology. A result is given on vanishing of the middle dimensional intersection homology group of "generalized Thom spaces", which naturally occur in the decomposition formula of S. Cappell and J. Shaneson. Based upon this result, consequences for the signature are drawn.

For non-Witt spaces X, signature and L-classes are defined via the hypercohomology groups $\mathcal{H}^i(X;\mathbf{IC}^\bullet_{\mathcal{L}})$, introduced in [Ban02]. A hypercohomology Wang sequence is deduced, connecting $\mathcal{H}^i(-;\mathbf{IC}^\bullet_{\mathcal{L}})$ of the total space with that of the fibre. Also here, a consequence for the signature under collapsing sphere-singularities is drawn.

1. Introduction

The goal of this article is to add to the intersection homology toolkit another useful long exact sequence. In [Wan49], H. C. Wang, calculating the homology of the total space of a fibre bundle over a sphere, actually proved an exact sequence, which is named after him today. It is a useful tool for dealing with fibre bundles over spheres and it is natural to ask: Is there a Wang sequence for intersection homology?

Given an appropriate notion of a stratified fibration, the natural framework for dealing with a question of the kind above would be an intersection homology analogue of a Leray–Serre spectral sequence. Greg Friedman has investigated this and established an appropriate framework in [Fri07]. For a simplified setting of a stratified bundle, however, i.e., a locally trivial bundle over a manifold with a stratified fibre, it seems more natural to explore the hypercohomology spectral sequence directly. In the following we are going to demonstrate this approach.

Mathematics Subject Classification: 55N33.
Keywords: intersection homology, Wang sequence, signature.

Section 3 is a kind of foretaste of what is to come. We prove the monodromy case by hand using only elementary intersection homology and apply it to calculate the intersection homology groups of neighbourhoods of circle singularities with toric links in a 4-dimensional pseudomanifold.

We recall the construction of induced maps in Section 4.1. Because of their central role in the application, the Cappell–Shaneson decomposition formula is explained in Section 4.2. Section 5 contains a proof of the Wang sequence for fibre bundles over simply connected spheres. It is shown that under a certain assumption the middle-dimensional middle perversity intersection homology of generalized Thom spaces of bundles over spheres vanish. The formula of Cappell and Shaneson then implies, that in this situation the signature does not change under the collapsing of the spherical singularities.

In Section 6, we demonstrate a second, concise proof — this is merely the sheaf-theoretic combination of the relative long exact sequence and the suspension isomorphism. However, this proof is mimicked in Section 7 to derive a Wang-like sequence for hypercohomology groups $\mathcal{H}^*(X;\mathbf{S}^\bullet)$ with values in a self-dual perverse sheaf complex $\mathbf{S}^\bullet \in SD(X)$. In Section 8, finally, together with Novikov additivity, this enables us to identify situations when collapsing spherical singularities in non-Witt spaces does not change the signature.

2. Basic notions

We will work in the framework of [GM83]. In the following $X = X_n \supset X_{n-2} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing$ will denote an oriented *n-dimensional stratified topological pseudomanifold*. The intersection homology groups of X with respect to perversity $\bar{p}$ are denoted by $IH_i^{\bar{p}}(X)$, and the analogous compact-support homology groups by $IH_i^{c\bar{p}}(X)$. The indexing convention is also that of [GM83]. Most of the fibre bundles to be considered in the following are going to be *stratified* bundles in the following sense (see also [Fri07, Definition 5.6]):

DEFINITION 2.1. A projection $E \to B$ to a manifold is called a stratified bundle if for each point $b \in B$ there exist a neighbourhood $U \subset B$ and a stratum-preserving trivialization $p^{-1}(U) \cong U \times F$, where F is a topological stratified pseudomanifold.

We will also restrict the automorphism group of F to stratum preserving automorphisms and work with the corresponding fibre bundles in the usual sense. Since we will basically need the local triviality, Definition 2.1 is mostly sufficient. When we pass to applications for Whitney stratified pseudomanifolds, however, the considered bundles will actually be fibre bundles — this follows from the theory of Whitney stratifications. A stratification of the fibre induces an obvious stratification of the total space with the same l-codimensional links, namely by E_{k+n-l} — the total spaces of bundles with fibre F_{k-l} and n the dimension of B.

3. Mapping torus

PROPOSITION 3.1 (INTERSECTION HOMOLOGY WANG SEQUENCE FOR S^1). *Let $F = F_n \supset F_{n-2} \supset \cdots \supset F_0$ be a topological stratified pseudomanifold, $\phi : F \to F$ a stratum and codimension preserving automorphism, i.e., a stratum preserving homeomorphism with stratum preserving inverse such that both maps respect the codimension. Let M_ϕ be the mapping torus of ϕ, i.e., the quotient space $F \times I/(y,1) \sim (\phi(y),0)$. Denote by $i : F = F \times 0 \hookrightarrow M_\phi$ the inclusion. Then the sequence*

$$\cdots \longrightarrow IH_k^{c\bar{p}}(F) \xrightarrow{\mathrm{id}-\phi_*} IH_k^{c\bar{p}}(F) \xrightarrow{i_*} IH_k^{c\bar{p}}(M_\phi) \xrightarrow{\partial} IH_{k-1}^{c\bar{p}}(F) \longrightarrow \cdots$$

is exact.

PROOF. The proof is analogous to the one for ordinary homology. Start with the quotient map $q : (F \times I, F) \to (M_\phi, F)$ and look at the corresponding diagram of long exact sequences of pairs. The boundary of $F \times I$ is a codimension 1 stratum and hence not a pseudomanifold. We have either to introduce the notion of a pseudomanifold with boundary here or work with intersection homology for cs-sets [Kin85; HS91]. However, we can also manage with a work-around: Define

$$I_\varepsilon := (-\varepsilon, 1+\varepsilon), \quad \partial I_\varepsilon := (-\varepsilon,\varepsilon) \cup (1-\varepsilon, 1+\varepsilon), \quad F_\varepsilon := F \times (-\varepsilon,\varepsilon).$$

We extend the identification $(y,1) \sim (\phi(y),0)$ to $F \times I_\varepsilon$ by introducing the quotient map $q : F \times I_\varepsilon \to M_\phi$,

$$q(y,t) = \begin{cases} (\phi^{-1}(y), 1+t) & \text{if } t \in (-\varepsilon, 0] \\ (y,t) & \text{if } t \in (0,1) \\ (\phi(y), t-1) & \text{if } t \in [1, 1+\varepsilon). \end{cases}$$

Evidently, $M_\phi = q(F \times I_\varepsilon)$. Now $F \times \partial I_\varepsilon = (F \times \partial I_\varepsilon)_{n+1} \supset (F \times \partial I_\varepsilon) \supset \cdots \supset (F \times \partial I_\varepsilon)_0$ is an open *sub*pseudomanifold of $F \times I_\varepsilon$ and $F = F_n \supset F_{n-2} \supset \cdots \supset F_0$ sits normally nonsingular in M_ϕ. Hence the inclusions induce morphisms on intersection homology and we get a morphism of the corresponding exact sequences of pairs:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{0} & IH_k^{c\bar{p}}(F\times I_\varepsilon, F\times\partial I_\varepsilon) & \xrightarrow{\partial} & IH_{k-1}^{c\bar{p}}(F\times\partial I_\varepsilon) & \xrightarrow{j_*} & IH_{k-1}^{c\bar{p}}(F\times I_\varepsilon) & \xrightarrow{0} & \cdots \\
 & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \\
\cdots & \longrightarrow & IH_k^{c\bar{p}}(M_\phi, F_\varepsilon) & \xrightarrow{\partial} & IH_{k-1}^{c\bar{p}}(F_\varepsilon) & \xrightarrow{i_*} & IH_{k-1}^{c\bar{p}}(M_\phi) & \longrightarrow & \cdots
\end{array}$$

The "boundary" of $F \times I_\varepsilon$ is the disjoint union of two components of the form $F \times \mathbb{R}$, so j_* is surjective and the outer arrows are zero maps. The connecting morphism ∂ is injective and therefore an isomorphism onto its image, i.e., onto

$$\begin{aligned}
&\ker j_* \\
&= \big\{(\alpha, \beta) \mid \alpha \in IH_k^{c\bar{p}}(F \times (-\varepsilon, +\varepsilon)),\ \beta \in IH_k^{c\bar{p}}(F \times (1-\varepsilon, 1+\varepsilon)),\ [\alpha+\beta] = 0\big\} \\
&= \{(\alpha, -\alpha)\} \cong IH_k^{c\bar{p}}(F \times \mathbb{R}) \cong IH_k^{c\bar{p}}(F).
\end{aligned}$$

The middle q_* maps $(\alpha, -\alpha)$ to $(\alpha - \phi_*(\alpha)) \in IH_k^{c\bar{p}}(F_\varepsilon) \cong IH_k^{c\bar{p}}(F)$. Since q commutes with ∂, one has

$$\partial \circ q_* \circ \partial|^{-1} = q_*|_{\ker j_* \cong IH_k^{c\bar{p}}(F)} = \mathrm{id} - \phi_*.$$

Hence, we have the diagram

$$\begin{array}{ccccccc}
\cdots \to IH_k^{c\bar{p}}(F) & \xrightarrow{\quad q_*|_{\ker j_*} \quad} & IH_k^{c\bar{p}}(F) & \xrightarrow{i_*} & IH_k^{c\bar{p}}(M_\phi) & \to & IH_k^{c\bar{p}}(M_\phi, F_\varepsilon) \\
\| \wr & & \uparrow & & & & \\
\ker j_* & & \Big| \partial & & & & \\
\wr \big\downarrow \partial|^{-1} & & \Big| & & & & \\
IH_{k+1}^{c\bar{p}}(F \times I_\varepsilon, F \times \partial I_\varepsilon) & \xrightarrow{q_*} & IH_{k+1}^{c\bar{p}}(M_\phi, F_\varepsilon) & & & &
\end{array}$$

where the top sequence is exact and the bottom square is commutative. As in ordinary homology one can show that $q_* : IH_k^{c\bar{p}}(F \times I_\varepsilon, F \times \partial I_\varepsilon) \to IH_k^{c\bar{p}}(M_\phi, F_\varepsilon)$ is an isomorphism. Finally, observe that on the right hand side $IH_k^{c\bar{p}}(M_\phi, F_\varepsilon) \cong IH_{k-1}^{c\bar{p}}(F)$ via $\partial \circ q_*^{-1}$. □

Let $X = X_4 \supset X_1 \supset X_0$ be a compact stratified pseudomanifold with $X_0 = \varnothing$. Then, the stratum of codimension 3 is just a disjoint union of circles $X_1 = S^1 \sqcup \cdots \sqcup S^1$. If we assume X to be PL, the link L at a point $p \in X_1$ is independent of p within a connective component of X_1. Furthermore, in X_4, there is a neighbourhood U of the circle containing p, which is a fibre bundle over S^1 and hence homeomorphic to the mapping torus M_ϕ with

$$\phi : \mathring{c}(L) \xrightarrow{\cong} \mathring{c}(L).$$

Putting this data together and using the Wang sequence we can compute $IH_k^{c\bar{p}}(U)$, with U a neighbourhood of $X_1 \subset X_4$. The group $IH_k^{c\bar{p}}(X)$ can then be computed via the Mayer–Vietoris sequence.

In this section we restrict ourselves to the case of L being a torus T^2. While the orientation preserving mapping class group of the torus is known to be $\mathrm{SL}(2; \mathbb{Z})$, we have to make the following restriction on its cone: In the following, we look only at those automorphisms $\phi : \mathring{c}(T^2) \to \mathring{c}(T^2)$ which are induced by an automorphism of the underlying torus $\psi : T^2 \xrightarrow{\cong} T^2$.[1] It is given by a matrix $\alpha \in \mathrm{SL}(2;\mathbb{Z})$ and by abuse of notation we will again write α for this torus automorphism.

Defining ϕ_k to be the map $IH_k^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{\mathrm{id}-(\mathring{c}(\alpha))_*} IH_k^{c\bar{p}}(\mathring{c}(T^2))$, we obtain the sequence

$$\cdots \longrightarrow IH_k^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{\phi_k} IH_k^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{i_*} IH_k^{c\bar{p}}(M_\alpha) \xrightarrow{\partial} IH_{k-1}^{c\bar{p}}(\mathring{c}(T^2)) \longrightarrow \cdots$$

For the open cone we have

$$IH_k^{c\bar{p}}(\mathring{c}(T^2)) = \begin{cases} IH_0^{c\bar{p}}(T^2) & \text{for } \ k=0, \\ IH_1^{c\bar{p}}(T^2) & \text{for } \ \bar{p}=\bar{0} \text{ and } k=1, \\ 0 & \text{else.} \end{cases}$$

Clearly, $IH_k^{c\bar{p}}(M_\alpha) = 0$ for $k \geq 3$. Now examine the nontrivial part of the sequence

$$0 \longrightarrow IH_2^{c\bar{p}}(M_\alpha) \xrightarrow{\partial} IH_1^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{\phi_1} IH_1^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{i_*} IH_1^{c\bar{p}}(M_\alpha) \xrightarrow{\partial} IH_0^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{\phi_0} IH_0^{c\bar{p}}(\mathring{c}(T^2)) \xrightarrow{i_*} IH_0^{c\bar{p}}(M_\alpha) \longrightarrow 0.$$

Since $\mathring{c}(\alpha)_0$ maps a point to a point, clearly $\mathring{c}(\alpha)_0 = \mathrm{id}$, hence $\phi_0 = \mathrm{id} - \mathrm{id} = 0$. It follows that $IH_0^{c\bar{p}}(M_\alpha) = \mathbb{Z}$.

Note that the only possible perversities in this example are $\bar{0}$ and $\bar{t}$. So far we have not distinguished between them. Due to $IH_1^{c\bar{t}}(\mathring{c}(T^2)) = 0$ there is $IH_2^{c\bar{t}}(M_\alpha) = 0$ and $IH_1^{c\bar{t}}(M_\alpha) = \mathbb{Z}$. For the zero perversity, we have

$$\begin{array}{l} 0 \longrightarrow IH_2^{c\bar{0}}(M_\alpha) \xrightarrow{\partial_*} \underset{\displaystyle \mathbb{Z}\oplus\mathbb{Z}}{\underset{\wr\|}{IH_1^{c\bar{0}}(\mathring{c}(T^2))}} \xrightarrow{\phi_1} IH_1^{c\bar{0}}(\mathring{c}(T^2)) \longrightarrow IH_1^{c\bar{0}}(M_\alpha) \longrightarrow \\ \hfill IH_0^{c\bar{0}}(\mathring{c}(T^2)) \longrightarrow 0 \end{array}$$

The group $IH_1^{c\bar{0}}(\mathring{c}(T^2))$ is isomorphic to $H_1(T^2)$ and is therefore generated by the corresponding homology classes of the torus. Hence, $(\mathring{c}(\alpha))_*$ is just the matrix α. If $\alpha = \mathrm{id}$, $\phi_1 = 0$ and $IH_2^{c\bar{0}}(M_\alpha) \cong \mathbb{Z}\oplus\mathbb{Z}$. In the general case, $IH_2^{c\bar{0}}(M_\alpha)$ is isomorphic to $\mathrm{im}\,\partial_* = \ker\phi_1$. We examine the determinant

[1] I believe that in the PL context this does not constitute a real restriction. With [Hud69, Theorem 3.6C] we can find an admissible triangulation of $\mathring{c}(T^2)$, such that ϕ becomes simplicial. Furthermore, it is not difficult to see, that the simplicial link of the cone point $L(c)$ is preserved under ϕ. Since the geometric realization of $L(c)$ is a torus, we get a candidate for ψ. By linearity, every slice between $L(c)$ and the cone point c is mapped by ψ. If we could extend the argument to the rest of $\mathring{c}(T^2)$ the goal would be achieved.

of the matrix: $\det\phi_1 = \det(\mathrm{id}-\alpha) = p_\alpha(1)$, where $p_\alpha(t)$ is the characteristic polynomial of α, which is

$$\begin{aligned} p_\alpha(t) &= \det\Bigl(t\,\mathrm{id}-\bigl(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\bigr)\Bigr) = (t-a_{11})(t-a_{22})-a_{21}a_{12} \\ &= t^2-(a_{11}+a_{22})t+(a_{11}a_{22}-a_{21}a_{12}) \\ &= t^2-\mathrm{tr}\,\alpha\ t+\det\alpha \\ &= t^2-\mathrm{tr}\,\alpha\ t+1. \end{aligned}$$

Here, $\mathrm{tr}\,\alpha$ is the trace of $\alpha \in \mathrm{SL}(2;\mathbb{Z})$. Thus we have $\det\phi_1 = 2-\mathrm{tr}\,\alpha$ and get

$$IH_2^{c\bar{0}}(M_\alpha) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } \alpha = \mathrm{id}, \\ \mathbb{Z} & \text{if } \mathrm{tr}\,\alpha = 2 \text{ and } \alpha \neq \mathrm{id}, \\ 0 & \text{if } \mathrm{tr}\,\alpha \neq 2. \end{cases}$$

Since $IH_0^{c\bar{0}}(\mathring{c}(T^2))$ is free, the sequence above reduces to a split short exact sequence

$$0 \longrightarrow \mathrm{coker}\,\phi_1 \longrightarrow IH_1^{c\bar{0}}(M_\phi) \longrightarrow IH_0^{c\bar{0}}(\mathring{c}(T^2)) \longrightarrow 0.$$

Hence $IH_1^{c\bar{0}}(M_\phi) \cong \mathbb{Z}\oplus\mathrm{coker}\,\phi_1$. In this final case our interest reduces to a cokernel calculation of the 2×2-matrix $\phi_1 = \mathrm{id}-\alpha$. The image $\mathrm{im}\,\phi_1 \subset \mathbb{Z}\oplus\mathbb{Z}$ is of the form $n\mathbb{Z}\oplus m\mathbb{Z}$, $n, m \in \mathbb{Z}$ and so every group $\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}\oplus\mathbb{Z}/m\mathbb{Z}$ can be realized as $IH_1^{c\bar{0}}(M_\phi)$. In particular a torsion intersection homology group may appear. Using $\det\phi_1 = \det(\mathrm{id}-\alpha) = 2-\mathrm{tr}\,\alpha$ as above, we immediately see that

$$\mathrm{coker}\,\phi_1 \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } \alpha = \mathrm{id}, \\ 0 & \text{if } \mathrm{tr}\,\alpha = 1, 3. \end{cases}$$

Summarizing all these results we get:

PROPOSITION 3.2. *Let M_α be the mapping torus over the open cone $\mathring{c}(T^2)$ of a torus glued via $\alpha : T^2 \xrightarrow{\cong} T^2$. Then its intersection homology groups are*

$$IH_k^{c\bar{t}}(M_\alpha) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 1, \\ 0 & \text{if } k \geq 2, \end{cases}$$

$$IH_k^{c\bar{0}}(M_\alpha) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } k = 1 \text{ and } \alpha = \mathrm{id} \\ \mathbb{Z} & \text{if } k = 1 \text{ and } \mathrm{tr}\,\alpha = 1, 3, \\ \mathbb{Z}\oplus\mathrm{coker}(\mathrm{id}-\alpha) & \text{if } k = 1 \text{ (in general)}, \\ \mathbb{Z} & \text{if } k = 2,\ \mathrm{tr}\,\alpha = 2 \text{ and } \alpha \neq \mathrm{id}, \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } k = 2 \text{ and } \alpha = \mathrm{id}, \\ 0 & \text{if } k = 2 \text{ and } \mathrm{tr}\,\alpha \neq 2, \\ 0 & \text{if } k \geq 3. \end{cases}$$

EXAMPLE 3.3. Let X be the fibre bundle over S^1 with fibre ΣT^2 and monodromy $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. We assume $\Sigma\alpha$ to be orientation preserving, i.e., the suspension points are fixed under it. The space X has a filtration $X_4 \supset X_1 = S^1 \sqcup S^1$ and our situation applies. Let α be $\left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. First, using the ordinary Wang sequence, we compute the homology of the total space E of the fibre bundle $T^2 \to E \to S^1$ with the same monodromy:

$$0 \to H_3(E) \xrightarrow{\partial} H_2(T^2) \xrightarrow{\mathrm{id}-\alpha_*} H_2(T^2) \to H_2(E) \xrightarrow{\partial} H_1(T^2)$$
$$\xrightarrow{\mathrm{id}-\alpha_*} H_1(T^2) \to H_1(E) \xrightarrow{\partial} H_0(T^2) \xrightarrow{\mathrm{id}-\alpha_*} H_0(T^2) \to H_0(E) \to 0.$$

In degrees 2 and 0, the map α_* is the identity, so we substitute zeros for $\mathrm{id}-\alpha_*$ to see that $H_3(E) \cong \mathbb{Z} \cong H_0(E)$. In degree 1, the map α_* is just the matrix α. Using $\mathrm{im}\,\partial_2 = \ker(\mathrm{id}-\alpha_*) \cong \mathbb{Z}$, we get the sequence

$$0 \to H_2(T^2) \to H_2(E) \xrightarrow{\partial} \mathbb{Z} \to 0,$$

which yields $H_2(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ and

$$0 \to \mathrm{coker}(\mathrm{id}-\alpha_*) \to H_1(E) \to \mathbb{Z} \to 0.$$

It follows by $\mathrm{coker}(\mathrm{id}-\alpha_*) \cong \mathbb{Z}$ that $H_1(E) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let us now compute the intersection homology groups of X via the Mayer–Vietoris sequence. The neighbourhoods of the two S^1 are of the desired form, i.e., mapping tori over $\mathring{c}(T^2)$ and their intersection is a fibre bundle over S^1 with fibre $T^2 \times \mathbb{R}$, so that the intersection homology groups are just $H_*(E)$ from above. Looking at the exact sequence

$$0 \longrightarrow IH_4^{c\bar{p}}(X) \to IH_3^{c\bar{p}}(E) \to \underset{\underset{0}{||}}{IH_3^{c\bar{p}}(M_\alpha)} \oplus \underset{\underset{0}{||}}{IH_3^{c\bar{p}}(M_\alpha)} \to \cdots,$$

we see that $IH_4^{c\bar{p}}(X) \cong H_3(E) \cong \mathbb{Z}$. In degree 0 the inclusion of E induces an injection on homology, i.e.,

$$0 \longrightarrow H_0(E) \to IH_0^{c\bar{p}}(M_\alpha) \oplus IH_0^{c\bar{p}}(M_\alpha) \to IH_0^{c\bar{p}}(X) \to 0,$$

and $IH_0^{c\bar{p}}(X) \cong \mathbb{Z}$ as it should be. Turning to the interesting degrees, we look at $\bar{p} = (0, 1, \ldots)$ first. Due to $IH_2^{c\bar{t}}(M_\alpha) = 0$, there is $IH_2^{c\bar{t}}(X) \cong \ker(i_* \oplus i_*)_1$. Finally $IH_1^{c\bar{t}}(X)$ is isomorphic to the cokernel of the inclusion $(i_* \oplus i_*)_1 : H_1(E) \to IH_1^{c\bar{t}}(M_\alpha) \oplus IH_1^{c\bar{t}}(M_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is the diagonal map; hence $IH_1^{c\bar{t}}(X) \cong \mathbb{Z}$ and $IH_2^{c\bar{t}}(X) \cong \ker(i_* \oplus i_*)_1 \cong \mathbb{Z}$. Similarly, for $\bar{p} = (0, 0, \ldots)$ we have $IH_3^{c\bar{0}}(X) \cong \ker(i_* \oplus i_*)_2 \cong \mathbb{Z}$; with $IH_1^{c\bar{0}}(M_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z}$ it follows $IH_1^{c\bar{0}}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. And $IH_2^{c\bar{0}}(X) \cong \mathrm{coker}(i_* \oplus i_*)_2 \cong \mathbb{Z}$. Because all the groups are free, the duality is already seen working with integral coefficients, especially

$$IH_3^{c\bar{0}}(X) \cong IH_1^{c\bar{t}}(X) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$IH_1^{c\bar{0}}(X) \cong IH_3^{c\bar{t}}(X) \cong \mathbb{Z}.$$

4. Some more advanced tools

4A. Normally nonsingular maps. Intersection homology is not a functor on the full subcategory of **Top** consisting of pseudomanifolds, since induced maps do not exist in general. However, on the category of topological pseudomanifolds and normally nonsingular maps, intersection homology is a bivariant theory in the sense of [FM81]. This fact is often suppressed. Since most of the maps which we will encounter are normally nonsingular, we recall in this section how induced maps are constructed. See particularly [GM83, 5.4].

DEFINITION 4.1. A map $f : Y \to X$ between two pseudomanifolds is called normally nonsingular (nns) of relative dimension $c = c_1 - c_2$ if it is a composition of a nns inclusion of dimension c_1 — meanining that Y is sitting in a c_1-dimensional tubular neighbourhood in the target — and a nns projection, i.e., a bundle projection with c_2-dimensional manifold fibre.

EXAMPLE 4.2. An open inclusion $U \hookrightarrow X$ is normally nonsingular. The inclusion of the fibre $F = b \times F \hookrightarrow E$, where E is fibred over a manifold is normally nonsingular. The projection $\mathbb{R}^n \times X \to X$ is normally nonsingular.

PROPOSITION 4.3 [GM83, 5.4.1, 5.4.2]. *Let $f : Y \to X$ be normally nonsingular of codimension c. Then there are isomorphisms*

$$f^* \,\mathbf{IC}^\bullet_{\bar{p}}(X) \cong \mathbf{IC}^\bullet_{\bar{p}}(Y)[c] \qquad \textit{and} \qquad f^! \,\mathbf{IC}^\bullet_{\bar{p}}(X) \cong \mathbf{IC}^\bullet_{\bar{p}}(Y).$$

DEFINITION 4.4. If $f : Y \to X$ is a proper normally nonsingular map of codimension c, we have induced homomorphisms

$$f_* : IH_k^{\bar{p}}(Y) \to IH_k^{\bar{p}}(X) \qquad \text{and} \qquad f^* : IH_k^{\bar{p}}(X) \to IH_{k-c}^{\bar{p}}(Y).$$

They are constructed by considering the adjunction morphisms of the adjoint pairs $(Rf_!, f^!)$ and (f^*, Rf_*),

$$Rf_! f^! \,\mathbf{IC}^\bullet_{\bar{p}}(X) \to \mathbf{IC}^\bullet_{\bar{p}}(X) \qquad \text{and} \qquad \mathbf{IC}^\bullet_{\bar{p}}(X) \to Rf_* f^* \,\mathbf{IC}^\bullet_{\bar{p}}(X),$$

by combining them with the proposition above and by finally applying hypercohomology.

We will also need induced maps on intersection homology with compact supports, which are not discussed in [GM83]. The above construction of Goresky and MacPherson works equally well for $IH_*^{c\bar{p}}$. If f is not proper, the map

f^* still exists. For the case of compact supports, $f_! : IH_k^{c\bar{p}}(Y) \to IH_k^{c\bar{p}}(X)$ can be constructed in the same manner. These different maps are listed in the following table — note that only f_* and f^* for proper f are explicitly mentioned in [GM83, 5.4].

f proper	f not proper
$f_* : IH_k^{\bar{p}}(Y) \to IH_k^{\bar{p}}(X)$	
$f_! : IH_k^{c\bar{p}}(Y) \to IH_k^{c\bar{p}}(X)$	$f_! : IH_k^{c\bar{p}}(Y) \to IH_k^{c\bar{p}}(X)$
$f^* : IH_k^{\bar{p}}(X) \to IH_{k-c}^{\bar{p}}(Y)$	$f^* : IH_k^{\bar{p}}(X) \to IH_{k-c}^{\bar{p}}(Y)$
$f^! : IH_k^{c\bar{p}}(X) \to IH_{k-c}^{c\bar{p}}(Y)$	

4B. Behaviour under stratified maps. Computing intersection homology invariants of one space out of the invariants of the other often relies on the decomposition formula of S. Cappell and J. Shaneson [CS91]. Since we will need it in the application below, we briefly recall it in this section.

Let $f : X^n \to Y^m$ be a stratified map between closed, oriented Whitney stratified sets of even relative dimension $2t = n - m$, Y having only even-codimensional strata. Let $\mathbf{S}^\bullet \in D_c^b(X)$ be a self-dual complex. Denote by $\mathcal{V}$ the set of components of pure strata of Y. For each $y \in V_y \in \mathcal{V}$, define[2]

$$E_y := f^{-1}(c\,L(y)) \cup_{f^{-1}L(y)} c\,f^{-1}(L(y)),$$

where $L(y)$ is the link of the stratum component V_y containing y. If y lies in the top stratum, we set $E_y = f^{-1}(y)$. We have the inclusions

$$E_y \overset{i_y}{\hookleftarrow} f^{-1}(\mathring{N}(y)) \overset{\rho_y}{\hookrightarrow} X,$$

where $N(y)$ is the normal slice of y. Note that $N(y) \cong c\,L$ and $\partial N(y) \cong L(y)$ (see [GM88] or [Ban07, 6.2]). Define now the complex

$$\mathbf{S}^\bullet(y) = \tau_{\le -c-t-1}^{\text{cone}} R i_{y*} \rho_y^! \mathbf{S}^\bullet,$$

where $\tau_\le^{\text{cone}}$ stands for truncation over the cone point[3] of $c\,f^{-1}(L(y))$ and $2c = 2c(V) = n - \dim V$ is the codimension of V.

---

[2] Here, $c\,L$ stands for the closed cone $L \times [0, 1]/L \times \{0\}$.

[3] There is a general notion of truncation over a closed subset in [GM83, 1.14]. Let c be the cone-point. For $\mathbf{A}^\bullet \in D_c^b(E_y)$, the derived stalks are

$$H^i(\tau_{\le p}^{\text{cone}} \mathbf{A}^\bullet)_x = \begin{cases} 0 & \text{if } x = c \text{ and } i > p, \\ H^i(\mathbf{A}^\bullet)_x & \text{otherwise.} \end{cases}$$

For $V \in \mathcal{V}$, let $\mathcal{S}_f^V$ be the local coefficient system over V with stalk $(\mathcal{S}_f^V)_z = \mathcal{H}^{-c-t}(E_z; \mathbf{S}^\bullet(z))$. There is an induced nondegenerate bilinear pairing

$$\phi_z : (\mathcal{S}_f^V)_z \times (\mathcal{S}_f^V)_z \to \mathbb{R}.$$

If $\mathbf{S}^\bullet$ is the intersection chain complex $\mathbf{IC}^\bullet_{\bar{m}}(X)$, the pull-back $\rho_y^! \, \mathbf{IC}^\bullet_{\bar{m}}(X)$ is clearly $\mathbf{IC}^\bullet_{\bar{m}}(E_y \setminus \{c\})$. Because of the stalk vanishing of $\mathbf{IC}^\bullet_{\bar{m}}(E_y \setminus \{c\})$, the truncation $\tau^{\text{cone}}_{\leq -c-t-1}$ is the usual truncation $\tau_{\leq -c-t-1}$ and hence

$$\mathbf{S}^\bullet(y) = \tau_{\leq -c-t-1} R i_{y*} \, \mathbf{IC}^\bullet_{\bar{m}}(E_y \setminus \{c\}),$$

which is simply the Deligne extension $\mathbf{IC}^\bullet_{\bar{m}}(E_y)$ of $\mathbf{IC}^\bullet_{\bar{m}}(E_y \setminus \{c\})$ to the point.

Denoting by $\mathbf{IC}^\bullet_{\bar{m}}(\bar{V}; \mathcal{S}_f^V)$ the lower-middle perversity intersection chain complex on the closure of V with coefficients in the local system $\mathcal{S}_f^V$, we can now formulate the important decomposition formula of Cappell and Shaneson.

THEOREM 4.5 [CS91, Theorem 4.2]. *There is an orthogonal decomposition up to algebraic bordism of self-dual complexes of sheaves*

$$Rf_* \mathbf{S}^\bullet[-t] \sim \bigoplus_{V \in \mathcal{V}} j_* \, \mathbf{IC}^\bullet_{\bar{m}}(\bar{V}, \mathcal{S}_f^V)[c(V)],$$

where $j : \bar{V} \hookrightarrow Y$ is the inclusion.

We abstain from giving the definition of algebraic bordism here and refer to the original paper or to Chapter 8 of [Ban07]. All we need for the application is the following, where for a self-dual sheaf $\mathbf{S}^\bullet$ over X, $\sigma(X, \mathbf{S}^\bullet)$ denotes the signature of the pairing on the middle-dimensional hypercohomology induced by self-duality.

PROPOSITION 4.6. *If two self-dual complexes over X, $\mathbf{S}_1^\bullet$ and $\mathbf{S}_2^\bullet$ are (algebraically) bordant, then $\sigma(X, \mathbf{S}_1^\bullet) = \sigma(X, \mathbf{S}_2^\bullet)$.*

PROOF. See [Ban07, Cor. 8.2.5], for example. □

PROPOSITION 4.7 [CS91, 5.5]. *If, in the setting above, $L_i(X, \mathbf{A}^\bullet)$ denotes the i-th L-class of the self-dual sheaf $\mathbf{A}^\bullet$ over a pseudomanifold X, we have*

$$L_i(Y, Rf_* \mathbf{S}^\bullet[-t]) = f_* L_i(X, \mathbf{S}^\bullet).$$

THEOREM 4.8. *With the notation $L_i(\bar{V}, \mathcal{S}_f^V)$ for $L_i(\bar{V}, \mathbf{IC}^\bullet_{\bar{m}}(\bar{V}; \mathcal{S}_f^V))$ we get*

$$f_* L_i(X, \mathbf{S}^\bullet) = \sum_{V \in \mathcal{V}} j_* L_i(\bar{V}, \mathcal{S}_f^V).$$

and bearing in mind that $\sigma(X) = \varepsilon_* L_0(X)$, where ε_* is the augmentation, we conclude:

COROLLARY 4.9. $$\sigma(X, \mathbf{S}^\bullet) = \sigma(Y, Rf_* \mathbf{S}^\bullet[-t]) = \sum_{V \in \mathcal{V}} \sigma(\bar{V}, \mathcal{S}_f^V).$$

In the case of simply connected components of Y, all the coefficient systems become constant and using multiplicativity formulae [Ban07, 8.2.19, 8.2.20], we get:

THEOREM 4.10. *Assume each $V \in \mathcal{V}$ to be simply connected and choose a basepoint y_V for every $V \in \mathcal{V}$. Then*

$$f_* L_i(X) = \sum_{V \in \mathcal{V}} \sigma(E_{y_V}) j_* L_i(\bar{V}).$$

And finally, for the signature:

COROLLARY 4.11. $$\sigma(X) = \sum_{V \in \mathcal{V}} \sigma(E_{y_V}) \sigma(\bar{V}).$$

5. The general simply connected case

PROPOSITION 5.1 (WANG SEQUENCE FOR $n \geq 2$). *Let $F \longrightarrow E \xrightarrow{\pi} S^n$ be a stratified bundle* (2.1) *with F a topological pseudomanifold with finitely generated cohomology, $n \geq 2$. Let $j : F \hookrightarrow E$ be the inclusion.*

(i) *For intersection homology the sequence*

$$\cdots \longrightarrow IH_k^{\bar{p}}(E) \xrightarrow{j^*} IH_{k-n}^{\bar{p}}(F) \longrightarrow IH_{k-1}^{\bar{p}}(F) \xrightarrow{j_*} IH_{k-1}^{\bar{p}}(E) \longrightarrow \cdots$$

is exact.

(ii) *For intersection homology with compact supports the sequence*

$$\cdots \longrightarrow IH_k^{c\bar{p}}(E) \xrightarrow{j^*} IH_{k-n}^{c\bar{p}}(F) \longrightarrow IH_{k-1}^{c\bar{p}}(F) \xrightarrow{j_*} IH_{k-1}^{c\bar{p}}(E) \longrightarrow \cdots$$

is exact.

(iii) *These sequences are natural with respect to fibre-preserving proper normally nonsingular maps between stratified bundles over S^n, i.e., let $F' \to E' \to S^n$ be another fibre bundle such that there is a commutative triangle*

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S^n & \end{array}$$

with f proper normally nonsingular, then there is a commutative diagram of the corresponding Wang-sequences induced by f — both in a covariant and a contravariant way.

PROOF. (i) We begin with the hypercohomology spectral sequence ([Bry93]) for $\mathbf{A}^\bullet := R\pi_* \mathbf{IC}^\bullet_{\bar{p}}(E)$, which converges to $\mathcal{H}^{p+q}(S^n, \mathbf{A}^\bullet) \cong IH^{\bar{p}}_{-p-q}(E)$. Let $U \subset S^n$ be an open set such that $\pi^{-1}(U) \cong U \times F$, then by 4.3

$$\mathbf{IC}^\bullet_{\bar{p}}(E)|_{\pi^{-1}(U)} \cong \mathbf{IC}^\bullet_{\bar{p}}(U \times F) \cong pr^* \mathbf{IC}^\bullet_{\bar{p}}(F)[n].$$

By IV.7.3 of [Bre97], the sheaf $\mathbf{H}^q(\mathbf{A}^\bullet)$, being the *Leray sheaf* of the fibration, is locally constant. Hence, by the assumption $n \geq 2$, it is constant with stalk

$$\mathcal{H}^q(F, \mathbf{IC}^\bullet_{\bar{p}}(F)[n]) = IH^{\bar{p}}_{-q-n}(F).$$

Finally

$$E_2^{p,q} \cong \begin{cases} IH^{\bar{p}}_{-q-n}(F) & \text{if } p = 0 \text{ or } p = n, \\ 0 & \text{else.} \end{cases}$$

Hence $E_2 \cong \ldots \cong E_n$ and the sequence collapses at $n+1$. Now, the proof can be finished as in the ordinary case (see [Spa66, 8.5], for instance). In order to show that $IH^{\bar{p}}_k(F) \to IH^{\bar{p}}_k(E)$ is induced by the inclusion $j : F = b_0 \times F \hookrightarrow E$, look at the fibration

$$F \to b_0 \times F \xrightarrow{\pi'} b_0$$

for $b_0 \in S^n$ the north pole. We have a commutative diagram

$$\begin{array}{ccc} b_0 \times F & \xrightarrow{\pi'} & b_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_0} \\ E & \xrightarrow{\pi} & S^n \end{array} .$$

For $R(j_0\pi')_* \mathbf{IC}^\bullet_{\bar{p}}(b_0 \times F)$ there is a corresponding spectral sequence converging to

$$\mathcal{H}^{p+q}(b_0 \times F, \mathbf{IC}^\bullet_{\bar{p}}(b_0 \times F)) \cong IH^{\bar{p}}_{-p-q}(F).$$

If we start with

$$R\pi_* Rj_* j^! \mathbf{IC}^\bullet_{\bar{p}}(E) \to R\pi_* \mathbf{IC}^\bullet_{\bar{p}}(E)$$

and use the commutative square above, we get a morphism

$$R(j_0\pi')_* \mathbf{IC}^\bullet_{\bar{p}}(b_0 \times F) \to R\pi_* \mathbf{IC}^\bullet_{\bar{p}}(E).$$

This induces

$$IH^{\bar{p}}_i(F) \to IH^{\bar{p}}_i(E),$$

which is j_* by construction (cf. 4.3 and 4.4). The E_2-term of the spectral sequence associated to $R(j_0\pi')_* \mathbf{IC}^\bullet_{\bar{p}}(b_0 \times F)$ is

$$E_2^{p,q} = H^p(S^n, \mathbf{H}^q(R(j_0\pi')_* \mathbf{IC}^\bullet_{\bar{p}}(b_0 \times F))).$$

Since here j_{0*} is just extension by zero the group on the right is isomorphic to $H^p(b_0, IH^{\bar{p}}_{-q}(F))$. For both sequences, the differentials $E_r^{n,q} \to E_r^{n+r,q-r+1}$ are zero for all $r \geq 2$, and so we have epimorphisms $E_r^{n,q} \twoheadrightarrow E_\infty^{n,q}$. Finally by the commutative diagram (denoting by $'$ the terms of the spectral sequence associated to $R(j_0\pi')_* \,\mathbf{IC}^\bullet_{\bar{p}}(b_0 \times F)$)

$$\begin{array}{ccccc} E_n^{n,-i-n} & \longrightarrow & E_\infty^{n,-i-n} & \longrightarrow & \mathcal{H}^q(E, \mathbf{IC}^\bullet_{\bar{p}}(E)) \\ \uparrow \wr\Vert & & \uparrow & & \uparrow j_* \\ E_n'^{n,-i-n} & \xrightarrow{\cong} & E_\infty'^{n,-i-n} & \xrightarrow{\cong} & \mathcal{H}^q(F, \mathbf{IC}^\bullet_{\bar{p}}(F)) \end{array}$$

we deduce that the upper composition is j_*, as stated. A very similar argument works for $IH^{\bar{p}}_k(E) \to IH^{\bar{p}}_{k-n}(F)$.

(ii) Consider the hypercohomology spectral sequence for the complex $\mathbf{B}^\bullet := R\pi_! \,\mathbf{IC}^\bullet_{\bar{p}}(E)$. The main argument is as before. The spectral sequence converges to $\mathcal{H}^{p+q}(S^n, \mathbf{B}^\bullet) = IH^{c\bar{p}}_{-p-q}(E)$. Being the q^{th} derived functor of $\pi_!$, the stalk of the Leray sheaf $\mathbf{H}^q(\mathbf{B}^\bullet)$ is $\mathcal{H}^q_c(F, \mathbf{IC}^\bullet_{\bar{p}}(E)) \cong IH^{c\bar{p}}_{-q-n}(F)$ (see [Bor84, VI, 2.7], for instance). Hence the E_2-terms are:

$$E_2^{0,q} \cong IH^{c\bar{p}}_{-q-n}(F),$$
$$E_2^{n,q-n+1} \cong IH^{c\bar{p}}_{-q-1}(F).$$

These yield the second sequence. The proof that the maps involved in this sequence are j_* and j^* follows as in (i).

(iii) Again, we use the fact that the hypercohomology spectral sequence is natural with respect to morphisms of sheaves over the base space.
For the covariant case, we have to construct

$$f_* : R\pi'_* \,\mathbf{IC}^\bullet_{\bar{p}}(E') \to R\pi_* \,\mathbf{IC}^\bullet_{\bar{p}}(E) \quad \text{or} \quad f_* : R\pi'_! \,\mathbf{IC}^\bullet_{\bar{p}}(E') \to R\pi_! \,\mathbf{IC}^\bullet_{\bar{p}}(E),$$

as the case may be producing morphisms between the terms of the Wang sequences. Then the corresponding maps will commute. Take the adjunction morphism

$$Rf_! f^! \,\mathbf{IC}^\bullet_{\bar{p}}(E) \to \mathbf{IC}^\bullet_{\bar{p}}(E),$$

apply $R\pi_*$ and use functoriality. The case of $IH^{c\bar{p}}_*$ is analogous. Since f is proper, we have $Rf_! = Rf_*$. Observe that when working with intersection homology with compact supports f need not be proper[4]! In the contravariant case, we proceed as above, using the other adjunction morphism

$$\mathbf{IC}^\bullet_{\bar{p}}(E) \to Rf_* f^* \,\mathbf{IC}^\bullet_{\bar{p}}(E). \qquad \square$$

[4] See also the comment at the end of Section 4A.

Now we are going to use this sequence in a concrete computation.

PROPOSITION 5.2. *Let $F^k \to E \xrightarrow{\pi} S^n$ be a locally trivial fibre bundle with F a topological pseudomanifold, $n \geq 2, n+k+1$ even. Define $M := c\,E \cup_E \bar{E}$, where $\bar{E}$ is the total space of the induced — meaning that the structure group acts levelwise — fibre bundle*

$$c\,F \to \bar{E} \xrightarrow{c\,(\pi)} S^n.$$

Suppose further that the following condition (S) *is fulfilled for the Wang sequence of E:*

(S) *the map $j^* : IH^{c\bar{m}}_{(n+k+1)/2}(E) \twoheadrightarrow IH^{c\bar{m}}_{(-n+k+1)/2}(F)$ is surjective.*

Then

$$IH^{c\bar{m}}_{(n+k+1)/2}(M) = 0.$$

PROOF. Throughout the proof, the perversity shall be the lower middle perversity $\bar{m}$, unless stated otherwise. Assume for now that $n = 2b$, $k = 2a - 1$, with $a, b \geq 1$; and with the cone formula there holds

$$IH^c_i(\mathring{c}F) \cong \begin{cases} IH^c_i(F) & \text{if } i < a, \\ 0 & \text{if } i \geq a. \end{cases}$$

Using the Wang sequence of Proposition 5.1 for $\mathring{c}F \to \mathring{\bar{E}} \xrightarrow{\mathring{c}(\pi)} S^n$,

$$\cdots \to IH^c_{a+b}(\mathring{c}F) \to IH^c_{a+b}(\mathring{\bar{E}}) \to IH^c_{a-b}(\mathring{c}F) \to IH^c_{a+b-1}(\mathring{c}F) \to \cdots$$

we get

$$IH^c_{a+b}(\mathring{\bar{E}}) \cong IH^c_{a-b}(F).$$

For the cone on E, there is:

$$IH^c_i(\mathring{c}E) \cong \begin{cases} IH^c_i(E) & \text{if } i < a+b, \\ 0 & \text{if } i \geq a+b. \end{cases}$$

Now consider the Mayer–Vietoris sequence[5]

$$\cdots \to IH^c_{a+b}(E) \xrightarrow{i_{a+b}} IH^c_{a+b}(\mathring{c}E) \oplus IH^c_{a+b}(\mathring{\bar{E}}) \to IH^c_{a+b}(M) \to \cdots$$

which reduces to

$$\cdots \xrightarrow{i_{a+b}} IH^c_{a-b}(F) \longrightarrow IH^c_{a+b}(M) \longrightarrow$$
$$IH^c_{a+b-1}(E) \xrightarrow{i_{a+b-1}} IH^c_{a+b-1}(E) \oplus IH^c_{a+b-1}(\mathring{\bar{E}}) \longrightarrow \cdots$$

[5] To avoid pseudomanifolds with boundary, we take the open part $\mathring{\bar{E}}$ of the induced bundle $\bar{E}$ in the Mayer–Vietoris decomposition.

The map i_{a+b-1} is easily seen to be injective and due to (S), i_{a+b} is surjective. Finally, let $n = 2b+1, k = 2a, a, b \geq 1$. By the cone formula we have:

$$IH_i^c(\mathring{c}E) \cong \begin{cases} IH_i^c(E) & i < a+b+1 \\ 0 & i \geq a+b+1 \end{cases}$$

and

$$IH_i^c(\mathring{c}F) \cong \begin{cases} IH_i^c(F) & i < a+1 \\ 0 & i \geq a+1. \end{cases}$$

Similarly, the Wang sequence yields

$$IH_{a+b+1}^c(\mathring{\bar{E}}) \cong IH_{a-b}^c(F).$$

Now, as above the Mayer–Vietoris sequence gives

$$\xrightarrow{i_{a+b+1}} IH_{a-b}^c(F) \to IH_{a+b+1}^c(M) \to IH_{a+b}^c(E) \xrightarrow{i_{a+b-1}} IH_{a+b}^c(E) \oplus IH_{a+b}^c(\mathring{\bar{E}})$$

where $\ker i_{a+b} = 0$ and $\operatorname{im} i_{a+b+1} = IH_{a-b}^c(F)$ due to (S). Hence,

$$IH_{a+b+1}^c(M) = 0. \qquad \square$$

COROLLARY 5.3. *If M is a Witt space and $n+k+1$ is divisible by* 4, *the signature $\sigma(M)$ vanishes* (*of course it always vanishes if $n+k+1$ is not divisible by* 4.)

Let us now formulate an important consequence of the observations above:

PROPOSITION 5.4. *Let X be a Whitney stratified Witt space of dimension $4k$, with a disjoint union of spheres as the singular locus $\Sigma = S^{n_1} \sqcup \cdots \sqcup S^{n_l}$. Assume $n_j \geq 2$ for $1 \leq j \leq l$. Let Y be the space obtained from X by collapsing the spheres S^{n_j} to points y_j and let $f : X \to Y$ be the collapsing map. Given a fibre bundle neighbourhood of S^{n_j}, we denote by E_j the corresponding fibre bundle with fibre the link of S^{n_j}. If for all $1 \leq j \leq l$, E_j satisfies* (S), *the signature of X does not change under f, i.e.,*

$$\sigma(X) = \sigma(Y).$$

PROOF. Because of 4.11, we have

$$\sigma(X) = \sum_{V \in \mathcal{V}} \sigma(E_{y_V})\sigma(\bar{V})$$

where the sum is taken over all strata. When we isolate the contribution of the top stratum, this looks like

$$= \sum_{y_j} \sigma(E_{y_{jV}})\sigma(\bar{V}) + \sigma(Y).$$

The $E_{y_{jV}}$, in turn, are of the form M of Proposition 5.2 and since the underlying fibrations satisfy (S), the resulting signatures vanish by 5.3. □

Similarly for the L-classes we have:

PROPOSITION 5.5. *In the situation above,*

$$f_* L(X) = L(Y).$$

The following two examples show, that the introduced condition (S) is indeed fulfilled for certain fibre bundles.

EXAMPLE 5.6. In the setting above let the base sphere be of odd dimension $n = 2b + 1$. If we are interested in computing the signature of E, its dimension has to be divisible by 4 — otherwise it is trivial anyway. In this case the fibre F has even dimension $k = 2a$, so that $(k - n + 1)/2$ is odd. Thus, the vanishing of odd dimensional intersection homology of F would imply (S). See [Roy87] for examples of spaces, for which the intersection homology vanishes in odd degrees.

EXAMPLE 5.7. Let the dimension of the sphere be greater than the dimension of the fibre plus 1, i.e., $k + 1 < n$. Then $(k - n + 1)/2$ is negative and the corresponding homology group is zero, thereby (S) is fulfilled.

Since we have not studied the intersection pairing on M, the condition (S) is clearly only sufficient and not necessary. However, the following ”counterexample” to the proposition is a case where (S) does not hold.

EXAMPLE 5.8. Let X be $\mathbb{C}P^2$ stratified as $\mathbb{C}P^2 \supset \mathbb{C}P^1 = S^2$ and f be the map, collapsing the 2-sphere to a point. So the target is $Y = S^4 \supset [S^2]$. Obviously, $\sigma(X) \neq \sigma(Y)$. The link of $\mathbb{C}P^1$ is a circle and the bundle we have to check (S) for is the Hopf bundle $S^3 \to S^2$. However

$$H_2(S^3) \to H_0(S^2)$$

is not onto and (S) fails.

6. A new proof

The application to the signature in the last section suggests a similar approach in the setting of spaces which no longer satisfy the Witt condition, however still posses a signature and L-classes. The suitable homology groups for defining these invariants are the hypercohomology groups $\mathcal{H}^i(-; \mathbf{IC}^\bullet_{\mathcal{L}})$ of Banagl [Ban02]. In the next section we will establish a Wang-like exact sequence for these groups i.e., for hypercohomology with values in a self-dual sheaf complex arising from a Lagrangian structure along the odd-codimension strata. Compare also [Ban07] for a concise exposition. The proof will be modeled on another elegant proof of the Wang sequence without the usage of the spectral sequence. This will be demonstrated in the following.

Suspension isomorphisms in intersection homology are very familiar. For the functoriality, however, we would like to have explicit maps realizing these isomorphisms:

LEMMA 6.1 (SUSPENSION ISOMORPHISM). *Let F be a pseudomanifold. The inclusion $l : F = 0 \times F \hookrightarrow \mathbb{R}^n \times F$ induces isomorphisms*

(a) $l^* : IH_k^{\bar{p}}(\mathbb{R}^n \times F) \to IH_{k-n}^{\bar{p}}(F)$,
(b) $l_! : IH_k^{c\,\bar{p}}(F) \to IH_k^{c\,\bar{p}}(\mathbb{R}^n \times F)$.

PROOF. (a) Let $p : \mathbb{R}^n \times F \to F$ be the normally nonsingular projection. By [Bor84, V,3.13] $Rp_* \circ p^* \simeq \mathrm{id}$, so the adjunction morphism is an isomorphism

$$\mathbf{IC}^\bullet_{\bar{p}}(F) \xrightarrow{\cong} Rp_* p^* \,\mathbf{IC}^\bullet_{\bar{p}}(F) \cong Rp_* \,\mathbf{IC}^\bullet_{\bar{p}}(\mathbb{R}^n \times F)[-n].$$

Applying hypercohomology we get

$$p^* : IH_k^{\bar{p}}(F) \xrightarrow{\cong} IH_{k+n}^{\bar{p}}(\mathbb{R}^n \times F).$$

Now $p \circ l = \mathrm{id}$ and hence $l^* \circ p^* \simeq \mathrm{id}$. Thereby, l^* is the inverse of p^* and the statement follows.

(b) is similar to (a), but uses the fact that $\mathcal{D}_X \mathcal{D}_X \mathbf{A}^\bullet \cong \mathbf{A}^\bullet$ for $\mathbf{A}^\bullet \in D_c^b(X)$, and the duality between p^* and $p^!$. □

PROPOSITION 6.2. *Let $F \longrightarrow E \xrightarrow{\pi} S^n$ be a stratified bundle with F a topological pseudomanifold. Denote by*

$$j : F = b_0 \times F \hookrightarrow E, \quad i : E \setminus b_0 \times F = U \times F \hookrightarrow E, \quad k : b_1 \times F \hookrightarrow E$$

the inclusions, where b_0 is the north pole and b_1 the south pole. Then there are the following long exact sequences:

$$\cdots \longrightarrow IH_k^{\bar{p}}(F) \xrightarrow{j_*} IH_k^{\bar{p}}(E) \xrightarrow{k^*} IH_{k-n}^{\bar{p}}(F) \longrightarrow IH_{k-1}^{\bar{p}}(F) \longrightarrow \cdots$$
$$\cdots \longrightarrow IH_k^{c\,\bar{p}}(F) \xrightarrow{k_!} IH_k^{c\,\bar{p}}(E) \xrightarrow{j^*} IH_{k-n}^{\bar{p}}(F) \longrightarrow IH_{k-1}^{c\,\bar{p}}(F) \longrightarrow \cdots.$$

PROOF. In the following, trivializations of the fibre bundle E are always involved. However, for every pseudomanifold X, $h : X \xrightarrow{\cong} X$ implies $\mathbf{IC}^\bullet_{\bar{p}}(X) \cong h_* \,\mathbf{IC}^\bullet_{\bar{p}}(X) \cong h^* \,\mathbf{IC}^\bullet_{\bar{p}}(X)$. Therefore, for the proof we can suppress them.

We begin with the distinguished triangle

$$\begin{array}{ccc} Rj_* j^! \, \mathbf{IC}^\bullet_{\bar p}(E) & \longrightarrow & \mathbf{IC}^\bullet_{\bar p}(E) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Ri_* i^* \, \mathbf{IC}^\bullet_{\bar p}(E) & \end{array}$$

and keeping in mind that $j^! \, \mathbf{IC}^\bullet_{\bar p}(E) \cong \mathbf{IC}^\bullet_{\bar p}(F)$, $i^* \, \mathbf{IC}^\bullet_{\bar p}(E) \cong \mathbf{IC}^\bullet_{\bar p}(U \times F)$ we apply hypercohomology to get

$$\cdots \to IH^{\bar p}_k(F) \xrightarrow{j_*} IH^{\bar p}_k(E) \xrightarrow{i^*} IH^{\bar p}_k(U \times F) \to \cdots.$$

The third term is isomorphic to $IH^{\bar p}_{k-n}(F)$ under l^* by the preceding lemma. However by the commutative triangle

$$\begin{array}{ccc} b_1 \times F & \xrightarrow{\quad k \quad} & E \\ {\scriptstyle l}\searrow & & \nearrow{\scriptstyle i} \\ & U \times F & \end{array}$$

we have $k^* \simeq l^* \circ i^*$ and the sequence for closed supports is proven.

Now turn to the case of compact supports.[6] Consider the triangle

$$\begin{array}{ccc} Ri_! i^* \, \mathbf{IC}^\bullet_{\bar p}(E) & \longrightarrow & \mathbf{IC}^\bullet_{\bar p}(E) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Rj_* j^* \, \mathbf{IC}^\bullet_{\bar p}(E) & \end{array}$$

and apply hypercohomology with compact supports to get

$$\cdots \to \mathcal{H}^{-k}_c(E; Ri_! \, \mathbf{IC}^\bullet_{\bar p}(U \times F)) \xrightarrow{i_!} IH^{c\,\bar p}_k(E) \xrightarrow{j^*} \mathcal{H}^{-k}_c(E; Rj_* \, \mathbf{IC}^\bullet_{\bar p}(F)[n]) \to \cdots.$$

Now $Rj_* = Rj_!$ as j is a closed inclusion. Hence

$$\mathcal{H}^{-k}_c(E; Rj_* \, \mathbf{IC}^\bullet_{\bar p}(F)[n]) \cong \mathcal{H}^{-k}_c(F; \mathbf{IC}^\bullet_{\bar p}(F)[n]) \cong IH^{c\,\bar p}_{k-n}(F).$$

For the first term, we have

$$\mathcal{H}^{-k}_c(E; Ri_! \, \mathbf{IC}^\bullet_{\bar p}(U \times F)) \cong \mathcal{H}^{-k}_c(U \times F; \mathbf{IC}^\bullet_{\bar p}(U \times F)) \underset{\cong}{\xleftarrow{l_!}} IH^{c\,\bar p}_k(F)$$

and with $i_! \circ l_! = k_!$ the assertion follows. □

[6] Recall that for $f : X \to Y$, $\mathbf{A}^\bullet \in D^b_c(X)$ and $Z \subset X$, we have $\Gamma_c(Z, f_* \mathbf{A}^\bullet) \ncong \Gamma_c(f^{-1}(Z), \mathbf{A}^\bullet)$. However $\Gamma_c(Z, f_! \mathbf{A}^\bullet) \cong \Gamma_c(f^{-1}(Z), \mathbf{A}^\bullet)$.

7. The non-Witt case

7A. The category SD(X). Originally, Goresky and MacPherson defined the signature for spaces with only even-codimensional strata. In [Sie83], Siegel generalizes the definition to Witt spaces. If $IH_*^{\bar{m}}(X) \ncong IH_*^{\bar{n}}(X)$, there still is a method to define a signature and L-classes for a pseudomanifold X compatible with the old definition. In his work [Ban02], Banagl establishes a corresponding framework and decomposition results similar to those of Cappell and Shaneson are presented in further papers. In this section we merely give the definition.

DEFINITION 7.1. Let $X = X_n \supset \cdots \supset X_0$ be an oriented pseudomanifold with orientation

$$\mathfrak{o} : \mathbb{D}^\bullet_{U_2} \xrightarrow{\cong} \mathbb{R}_{U_2}[n].$$

For $k \geq 2$, we write $U_k := X \setminus X_{n-k}$. Define $SD(X)$ as the full subcategory of $D^b_c(X)$ of those $\mathbf{S}^\bullet \in D^b_c(X)$ satisfying the following:

(SD1) Normalization: There is an isomorphism $\nu : \mathbb{R}_{U_2}[n] \xrightarrow{\cong} \mathbf{S}^\bullet|_{U_2}$.

(SD2) Lower bound: $\mathbf{H}^i(\mathbf{S}^\bullet) = 0$, for $i < -n$.

(SD3) Stalk condition for $\bar{n}$: $\mathbf{H}^i(\mathbf{S}^\bullet|_{U_{k+1}}) = 0$, for $i > \bar{n}(k) - n, k \geq 2$.

(SD4) Self-duality: There is an isomorphism $d : \mathcal{D}_X \mathbf{S}^\bullet[n] \to \mathbf{S}^\bullet$ compatible with the orientation, i.e., such that the square

$$\begin{array}{ccc} \mathbb{R}_{U_2}[n] & \xrightarrow[\cong]{\nu} & \mathbf{S}^\bullet|_{U_2} \\ {\scriptstyle \mathfrak{o}} \uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle d|_{U_2}} \\ \mathbb{D}^\bullet_{U_2} & \xrightarrow[\cong]{\mathcal{D}_X \nu^{-1}[n]} & \mathcal{D}_X \mathbf{S}^\bullet|_{U_2}[n] \end{array} \qquad \text{commutes.}$$

We refer to [Ban02] for results on this category, especially for the structure theorem, establishing the relation between $\mathbf{S}^\bullet \in SD(X)$ and a choice of Lagrangian structures along odd-codimensional strata of X.

REMARK 7.2. If X is a Witt space, $SD(X)$ consists up to isomorphism only of $\mathbf{IC}^\bullet_{\bar{m}}(X)$. On the other hand, $SD(X)$ might be empty — e.g., $SD(\Sigma \mathbb{C}P^2) = \varnothing$.

THEOREM 7.3 [Ban02, Theorem.2.2]. *For* $\mathbf{S}^\bullet \in SD(X)$, *there is a factorization*

$$\mathbf{IC}^\bullet_{\bar{m}}(X) \xrightarrow{\alpha} \mathbf{S}^\bullet \xrightarrow{\beta} \mathbf{IC}^\bullet_{\bar{n}}(X),$$

that is compatible with the normalization (*and is unique with respect to this property*) *and such that*

$$\begin{array}{ccc} \mathbf{IC}^\bullet_{\bar{m}}(X) & \xrightarrow{\alpha} & \mathbf{S}^\bullet \\ \cong \uparrow & & \cong \uparrow d \\ \mathcal{D}_X\, \mathbf{IC}^\bullet_{\bar{n}}(X) & \xrightarrow{\mathcal{D}_X \beta[n]} & \mathcal{D}_X \mathbf{S}^\bullet \end{array} \qquad \textit{commutes.}$$

Thus, an object in $SD(X)$ is in fact a self-dual interpolation between $\mathbf{IC}^\bullet_{\bar{m}}(X)$ and $\mathbf{IC}^\bullet_{\bar{n}}(X)$. It is obvious that in the case of X being a Witt space, $SD(X)$ consists (up to quasi-isomorphism) only of $\mathbf{IC}^\bullet_{\bar{m}}(X)$.

DEFINITION 7.4. Let X^n be a closed stratified topological pseudomanifold, not necessarily Witt and $\mathbf{S}^\bullet \in SD(X)$. In case n is divisible by 4, define $\sigma(X^n, \mathbf{S}^\bullet, d)$ to be the signature on $\mathcal{H}^{-n/2}(X^n, \mathbf{S}^\bullet)$ induced by the self-duality of $\mathbf{S}^\bullet$.

REMARK 7.5. If X happens to be a Witt space, $\sigma(X^n, \mathbf{S}^\bullet, d)$ is the usual intersection homology signature due to Theorem 7.3.

Finally, in order to speak of *the* signature of a pseudomanifold (as long as $SD(X) \neq \varnothing$), we need the following important result:

THEOREM 7.6 [Ban06, 4.1]. *Let X^n be an even-dimensional closed oriented pseudomanifold with $SD(X) \neq \varnothing$. For $(\mathbf{S}^\bullet_1, d_1), (\mathbf{S}^\bullet_2, d_2) \in SD(X)$ one has*

$$\sigma(X^n, \mathbf{S}^\bullet_1, d_1) = \sigma(X^n, \mathbf{S}^\bullet_2, d_2).$$

7B. Hypercohomology Wang sequence. Before we deduce the exact sequence for hypercohomology with values in SD-sheaves, we have to determine what the involved complexes of sheaves are going to be. Starting with a SD complex over the total space E we define a SD complex over the fibre F in a canonical way. We will need the following little lemma.

LEMMA 7.7. *For the inclusion $j : X^n \times 0 \hookrightarrow X^n \times \mathbb{R}^m$ with associated projection $p : X^n \times \mathbb{R}^m \to X^n$ we have*

$$j^! \simeq j^*[-m]$$

and thereby

$$p^! \circ j^! \simeq \mathrm{id}\,.$$

PROOF. By [Ban02, Lemma 5.2], $p^* \circ j^* \simeq \mathrm{id}$. Consequently, using $p^*[m] \simeq p^!$([Ban02, Lemma 4.2, Proof]), we get

$$j^! \simeq j^! \circ p^* \circ j^* \simeq j^! \circ p^! \circ j^*[-m] \simeq (p \circ j)^! \circ j^*[-m] \simeq j^*[-m].$$

The second identity is clear by using $p^*[m] \simeq p^!$ again. $\square$

LEMMA 7.8. *Let $X^n \xrightarrow{j} X^n \times \mathbb{R}^m$ be the standard inclusion. Given $\mathbf{T}^\bullet \in SD(X^n \times \mathbb{R}^m)$, the complex $j^!\mathbf{T}^\bullet$ is in $SD(X)$.*

PROOF. Looking at the commutative square

$$\begin{array}{ccc} \mathbb{R}_{U_2}[n+m] & \xrightarrow{\quad \nu \quad} & \mathbf{T}^\bullet|_{U_2} \\ \big\downarrow j^! & & \big\downarrow j^! \\ j^!\mathbb{R}_{U_2}[n+m] \cong \mathbb{R}_{U_2\cap X}[n] & \xrightarrow{\ j^!(\nu)\ } & j^!\mathbf{T}^\bullet|_{U_2\cap X}, \end{array}$$

one checks that (SD1) is fulfilled because of the functoriality of $j^!$. Now using that the inverse image functor j^* is exact, look at

$$\mathbf{H}^i(j^!\mathbf{T}^\bullet) \cong \mathbf{H}^i(j^*\mathbf{T}^\bullet[-m]) \cong \mathbf{H}^{i-m}(j^*\mathbf{T}^\bullet) \cong j^*\,\mathbf{H}^{i-m}(\mathbf{T}^\bullet).$$

Observe that the last term is zero for $i-m < -(n+m)$ or $i < -n$, and so (SD2) holds. Now let $i > \bar{n}(k) - (n), k \geq 2$. We have

$$\mathbf{H}^i((j^!\mathbf{T}^\bullet)|_{U_{k+1}\cap X}) \cong \mathbf{H}^i((j^*\mathbf{T}^\bullet[-m])|_{U_{k+1}\cap X}) \cong j^*\,\mathbf{H}^{i-m}(\mathbf{T}^\bullet|_{U_{k+1}}),$$

where the last term is zero due to $i-m > \bar{n}(k)-(n+m)$ and hence (SD3) holds as well.
Finally by [Ban07, Proposition 3.4.5] we have an isomorphism

$$\mathcal{D}_X\, j^!\mathbf{T}^\bullet[n] \cong j^*\mathcal{D}_{X\times\mathbb{R}^m}\mathbf{T}^\bullet[n] \cong j^!(\mathcal{D}_{X\times\mathbb{R}^m}\mathbf{T}^\bullet[n+m]) \cong j^!\mathbf{T}^\bullet$$

which is compatible with the orientation, proving (SD4). $\square$

Let us now return to the original context. We start with the total space E of a fibre bundle over S^n — a topological pseudomanifold of dimension $k+n$ — and a complex $\mathbf{T}^\bullet \in SD(E)$. Given a trivializing neighbourhood $U \subset S^n$ of the north pole b_0 resp. the south pole b_1, the restriction of $\mathbf{T}^\bullet$ to $\pi^{-1}(U) \cong U \times F$ is clearly in $SD(U \times F)$ since $\pi^{-1}(U)$ is open. With the preceding lemma we can now define:

DEFINITION 7.9. Let $F \to E \to S^n$ be a fibre bundle as above and $\mathbf{T}^\bullet \in SD(E)$. For $i = 0, 1$, we have inclusions

$$\begin{array}{ccc} b_i \times F & \xrightarrow{\quad j_i \quad} & E \\ \big\downarrow i_i^1 & & \big\uparrow i_i^2 \\ U \times F & \xrightarrow[\phi_i]{\cong} & \pi^{-1}(U) \end{array}$$

where j_i is defined to be the composition $i_i^2 \circ \phi_i \circ i_i^1$.

Set $\mathbf{S}^\bullet := \mathbf{S}_N^\bullet := j_0^!\mathbf{T}^\bullet \cong i_0^{1!}((\phi_0^!\mathbf{T}^\bullet)|_{U\times F}) \in SD(F)$, with $j_0 : b_0 \times F \hookrightarrow$ the inclusion of the north pole fibre for some trivialization $\phi_0 : U \times F \xrightarrow{\cong} \pi^{-1}(U)$. Define $\mathbf{S}_S^\bullet$ in the same way using the inclusion of the south pole fibre.

In order for $\mathbf{S}^\bullet$ to be well defined over $F = b_i \times F$ with $b_i \in U_1 \cap U_2$, we have to make an extra assumption, a kind of homogeneity:

DEFINITION 7.10. We call the structure group G of a fibre bundle of the form above *adapted to* $\mathbf{A}^\bullet \in SD(F)$, if for all $h \in G$, $h^!\mathbf{A}^\bullet \cong h^*\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$.

EXAMPLE 7.11. If F is a Witt space, $\mathbf{IC}^\bullet_{\bar{m}}(F) \in SD(F)$. For every stratum-preserving automorphism $h : F \xrightarrow{\cong} F$, we have $h^*\,\mathbf{IC}^\bullet_{\bar{m}}(F) \cong \mathbf{IC}^\bullet_{\bar{m}}(F)$.

REMARK 7.12. Assume G to be adapted to $\mathbf{S}^\bullet$. What if we are given two trivializing neighbourhoods $U_1, U_2 \subset S^n$ with $\phi_i : U_i \times F \to \pi^{-1}(U_i)$? We have a commutative diagram

$$\begin{array}{ccccccc}
F & \xrightarrow{i^2} & b_0 \times F & \longrightarrow & (U_1 \cap U_2) \times F & & \\
h_{21}(b_0)\Big\uparrow\cong & & \Big\uparrow\cong & & \phi_2^{-1}\phi_1\Big\uparrow & \searrow^{\phi_2} & \\
 & & & & & & \pi^{-1}(U_1\cap U_2) \longrightarrow E \\
 & & & & & \nearrow_{\phi_1} & \\
F & \xrightarrow{i^1} & b_0 \times F & \longrightarrow & (U_1 \cap U_2) \times F & &
\end{array}$$

where the upper composition is j^1 and the lower composition is j^2. We have to show that $(j^1)^!\mathbf{T}^\bullet = (j^2)^!\mathbf{T}^\bullet$. Since G is adapted to $\mathbf{S}^\bullet$, the transition function $h_{21}(b_0)$ preserves $\mathbf{S}^\bullet$ and thereby $(j^1)^!\mathbf{T}^\bullet = h_{21}(b_0)^!(j^2)^!\mathbf{T}^\bullet \cong (j^2)^!\mathbf{T}^\bullet$.

We will need some form of suspension isomorphism for hypercohomology with values in a SD sheaf.

LEMMA 7.13. *Let F be a pseudomanifold and* $\mathbf{S}^\bullet \in SD(F)$. *Let* $p : \mathbb{R}^n \times F \to F$ *be the projection. The inclusion* $l : 0 \times F \hookrightarrow \mathbb{R}^n \times F$ *induces the following isomorphisms on hypercohomology*:

(a) $l^* : \mathcal{H}^k(\mathbb{R}^n \times F; \mathbf{S}^\bullet) \to \mathcal{H}^{k+n}(F; l^!\mathbf{S}^\bullet)$

(b) $l_! : \mathcal{H}_c^k(F; l^!\mathbf{S}^\bullet) \to \mathcal{H}_c^k(\mathbb{R}^n \times F; \mathbf{S}^\bullet)$

PROOF. Same as for Lemma 6.1, using $p^* \simeq p^![-n]$ for (a). Note that by [Bor84, V, 3.13] $Rp_*p^*\mathbf{A}^\bullet \cong \mathbf{A}^\bullet$ for all $\mathbf{A}^\bullet \in D^b(X)$. □

Now we are able to formulate the next proposition and imitate the proof of the Wang sequence given in Section 6.

PROPOSITION 7.14. *Let* $F \to E \to S^n$ *be a fibre bundle with a suitable structure group G of automorphisms of the pseudomanifold* F, *which is adapted to*

$\mathbf{S}^\bullet_N$ *and* $\mathbf{S}^\bullet_S$. *Assume* E *to be canonically stratified*. *Given* $\mathbf{T}^\bullet \in SD(E)$ *there are long exact sequences*

(a) $\quad \cdots \to \mathcal{H}^k(F;\mathbf{S}^\bullet) \to \mathcal{H}^k(E;\mathbf{T}^\bullet) \to \mathcal{H}^{k+n}(F;\mathbf{S}^\bullet) \to \cdots,$

(b) $\quad \cdots \to \mathcal{H}^k_c(F;\mathbf{S}^\bullet) \to \mathcal{H}^k_c(E;\mathbf{T}^\bullet) \to \mathcal{H}^{k+n}_c(F;\mathbf{S}^\bullet) \to \cdots,$

where $\mathbf{S}^\bullet$ *is the self-dual sheaf of Definition* 7.9.

PROOF. Due to Remark 7.12 we need not pay attention to different trivializations. Therefore, in the following proof we will not mention them explicitly.

(a) Denote by j the inclusion of the north pole fibre and by i the inclusion of the complement $V \times F$. We begin with the distinguished triangle

$$\begin{array}{ccc} Rj_*j^!\mathbf{T}^\bullet & \longrightarrow & \mathbf{T}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Ri_*i^*\mathbf{T}^\bullet & \end{array}$$

and apply hypercohomology to get

$$\cdots \to \mathcal{H}^k(F;j^!\mathbf{T}^\bullet) \to \mathcal{H}^k(E;\mathbf{T}^\bullet) \to \mathcal{H}^k(V \times F;i^*\mathbf{T}^\bullet) \to \cdots.$$

By construction, we have $\mathcal{H}^k(F;j^!\mathbf{T}^\bullet) \cong \mathcal{H}^k(F;\mathbf{S}^\bullet)$. If $i : V \times F \to E$ denotes the inclusion, then $i^*\mathbf{T}^\bullet$ is again isomorphic to a self-dual complex over the product bundle $V \times F$ containing the south pole fibre $b_1 \times F$. With the inclusion $j_1 : b_1 \times F \hookrightarrow E$ and using the preceding lemma, we finally get

$$\mathcal{H}^k(V \times F;i^*\mathbf{T}^\bullet) = \mathcal{H}^{k+n}(F;j_1^!\mathbf{T}^\bullet) \cong \mathcal{H}^{k+n}(F;\mathbf{S}^\bullet_S).$$

Now choose a trivializing neighbourhood O containing the north pole b_0 and the south pole b_1. Let l_0 and l_1 be the corresponding inclusions into $O \times F$. We have

$$\mathbf{S}^\bullet = j_0^!\mathbf{T}^\bullet = l_0^!(\mathbf{T}^\bullet|_{O\times F}) \cong l_1^!(\mathbf{T}^\bullet|_{O\times F}) \cong j_1^!\mathbf{T}^\bullet = \mathbf{S}^\bullet_S,$$

where the middle isomorphism holds because G is adapted to $\mathbf{S}^\bullet_N$ and $\mathbf{S}^\bullet_S$. This completes the proof.

(b) Begin with the triangle

$$\begin{array}{ccc} Ri_!i^*\mathbf{T}^\bullet & \longrightarrow & \mathbf{T}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Rj_*j^*\mathbf{T}^\bullet & \end{array}$$

and apply hypercohomology with compact supports to get

$$\cdots \to \mathcal{H}^k_c(V \times F;i^*\mathbf{T}^\bullet) \to \mathcal{H}^k_c(E;\mathbf{T}^\bullet) \to \mathcal{H}^k_c(F;j^*\mathbf{T}^\bullet) \to \cdots.$$

Now, using part (b) of the preceding lemma, we identify

$$\mathcal{H}_c^k(V \times F; i^*\mathbf{T}^\bullet) \cong \mathcal{H}_c^k(F; j_1^! i^*\mathbf{T}^\bullet) \cong \mathcal{H}_c^k(F; \mathbf{S}^\bullet).$$

And again by using

$$\begin{array}{ccc} b_0 \times F & \xrightarrow{\quad j \quad} & E \\ & {\scriptstyle i_1} \searrow \quad \nearrow {\scriptstyle i_2} & \\ & U \times F & \end{array}$$

we obtain

$$j^*\mathbf{T}^\bullet \cong i_1^* i_2^* \mathbf{T}^\bullet \cong i_1^!(\mathbf{T}^\bullet|_{V \times F})[n] \cong \mathbf{S}^\bullet[n].$$

Hence the third term in the sequence is equal to $\mathcal{H}_c^k(F; j^*\mathbf{T}^\bullet) \cong \mathcal{H}_c^{k+n}(F; \mathbf{S}^\bullet)$. The observation $\mathbf{S}^\bullet \cong \mathbf{S}_S^\bullet$ holds as before. □

REMARK 7.15. We can still formulate a similar exact sequence even without G being adapted to $\mathbf{S}_N^\bullet$ and $\mathbf{S}_S^\bullet$ or using the local triviality of the stratifold bundle only. Then, however, the involved SD-complexes over the fibre may be different and depend on the choice of trivializations:

$$\cdots \to \mathcal{H}^k(F; \mathbf{S}_N^\bullet) \to \mathcal{H}^k(E; \mathbf{T}^\bullet) \to \mathcal{H}^{k+n}(F; \mathbf{S}_S^\bullet) \to \cdots,$$

$$\cdots \to \mathcal{H}_c^k(F; \mathbf{S}_S^\bullet) \to \mathcal{H}_c^k(E; \mathbf{T}^\bullet) \to \mathcal{H}_c^{k+n}(F; \mathbf{S}_N^\bullet) \to \cdots.$$

8. Novikov additivity and collapsing of spheres

In [Sie83], Siegel generalizes the classical Novikov additivity of the signature for manifolds to pseudomanifolds satisfying the Witt condition. We want to make a further step forward by dropping the Witt condition in certain cases.[7]

PROPOSITION 8.1. *Let $X = X_{2n} \supset X_{2n-k} \supset \varnothing$, $k \geq 2$, be a Whitney stratified compact pseudomanifold with $SD(X) \neq \varnothing$. Given subspaces $M, E, T \subset X$ with T a closed neighbourhood of X_{2n-k}, such that*

(1) $X = M \cup T$,
(2) $M \cap T = E$,
(3) *E has a collar in T, and*
(4) *(M, E) is a compact manifold with boundary.*

Define $X^1 := M \cup_E c(E)$ and $X^2 := T \cup_E c(E)$. Then we have the identity

$$\sigma(X) = \sigma(X^1) + \sigma(X^2).$$

[7] A similar result is given in Theorem 3 of [Hun07]. Thanks to the referee for pointing this out to me.

PROOF. Once again, consider the formula of Cappell and Shaneson 4.5. Let $f : X \to Y$ be the map collapsing X_{2n-k} to a point. The push forward $Rf_* \mathbf{T}^\bullet$ of $\mathbf{T}^\bullet \in SD(E)$ is algebraically cobordant to

$$\mathbf{IC}^\bullet_{\bar{m}}(X^1, \mathcal{S}_f^{\mathring{M}}) \oplus j_{c*}\, \mathbf{IC}^\bullet_{\bar{m}}(\{c\}, \mathcal{S}_f^{\{c\}})[n].$$

Here, c is the conepoint $f(X_{2n-k})$ and j_c its inclusion into Y. Since $\mathcal{S}_f^{\mathring{M}}$ is constant with rank one, the first term is just equal to $\mathbf{IC}^\bullet_{\bar{m}}(X^1)$. Let us look at $(\mathcal{S}_f^{\{c\}})_c$. The link $L(c)$ of c is E, so we deduce from

$$E_c = f^{-1}(c\, L(c)) \cup_E c\, f^{-1}(L(c)) = X^2$$

and

$$E_c \overset{i_c}{\hookleftarrow} E_c \backslash \{c\} \cong f^{-1}(c\, L(c)) \overset{\rho_c}{\hookrightarrow} X$$

(see Section 4B) that $\mathbf{S}^\bullet(c)$ is a Deligne extension of $\rho_c^! \mathbf{S}^\bullet$ — which is just the restriction of the original self-dual complex. Hence[8] $\mathbf{S}^\bullet(c) \in SD(X_2)$. We have $(\mathcal{S}_f^{\{c\}})_c = \mathcal{H}^{-n}(X^2; \mathbf{S}^\bullet(c))$ and consequently

$$\begin{aligned}\mathcal{H}^{-n}(Y; j_{c*}\, \mathbf{IC}^\bullet_{\bar{m}}(\{c\}, \mathcal{S}_f^{\{c\}})[n]) &\cong \mathcal{H}^{-n}(\{c\}; \mathbf{IC}^\bullet_{\bar{m}}(\{c\}, \mathcal{S}_f^{\{c\}})[n]) \\ &\cong \mathcal{H}^{-n}(X^2; \mathbf{S}^\bullet(c)).\end{aligned}$$

Finally, combining these observations and passing to the signature we get

$$\begin{aligned}\sigma(X) &= \sigma(X, Rf_*(\mathbf{T}^\bullet)) \\ &= \sigma(X^1, \mathbf{IC}^\bullet_{\bar{m}}(X^1)) + \sigma(X^2, \mathbf{S}^\bullet(c)) = \sigma(X^1) + \sigma(X^2). \qquad \square\end{aligned}$$

Let E be the total space of a fibre bundle over S^m as before. We investigate, when the middle hypercohomology group $\mathcal{H}^{-n}(M, \mathbf{S}^\bullet)$ vanishes for a given complex $\mathbf{S}^\bullet$ over a non-Witt $M := c\, E \cup_E \bar{E}$ of dimension $2n$. Since we are only interested in computing the signature, only odd-dimensional spheres are considered here. The strategy is very similar to that of Section 5.

PROPOSITION 8.2. *Let $F^{2a} \to E \xrightarrow{\pi} S^{2b+1}$ be a fibre bundle with F a compact topological pseudomanifold, $a \geq 1$, $b \geq 2$. Define $M := c\, E \cup_E \bar{E}$, where $\bar{E}$ is the total space of the induced fibre bundle*

$$c\, F \to \bar{E} \xrightarrow{c\,(\pi)} S^{2b+1}.$$

Given $\mathbf{S}^\bullet \in SD(M)$, denote by $\mathbf{T}^\bullet$ the induced element in $SD(E)$ (*compare Section* 7). *Assume that the following condition* (S) *is fulfilled for the hypercohomology Wang sequence of E*:

$$\mathcal{H}^{-(a+b+1)}(E; \mathbf{T}^\bullet) \twoheadrightarrow \mathcal{H}^{-(a-b)}(F; \mathbf{S_S}^\bullet) \text{ is surjective,}$$

[8] All the axioms are clearly satisfied. The only "new" stalk to look at is the one at c, but $\{c\}$ has even codimension and the modified Deligne extension of $\rho_c^! \mathbf{S}^\bullet$ explained in 4B ensures that SD1-SD4 remain valid.

Then

$$\mathcal{H}^{-(a+b+1)}(M;\mathbf{S}^\bullet) = 0.$$

We will need the following vanishing lemma for the hypercohomology of the cone.

LEMMA 8.3. *Let X be a $2n$-dimensional or a $(2n+1)$-dimensional, compact Witt space such that the signature of the pairing over the middle-dimensional intersection homology vanishes. For $\mathbf{S}^\bullet \in SD(\mathring{c}X)$, we have*

$$\mathcal{H}_c^{-i}(\mathring{c}X;\mathbf{S}^\bullet) = 0 \quad \textit{for}\ \ i \geq n+1.$$

PROOF. Since $\mathbf{S}^\bullet$ is constructible and $\mathring{c}X$ is a distinguished neighbourhood of the conepoint c, there is (by [Ban07, p. 97], for instance)

$$\mathcal{H}_c^{-i}(\mathring{c}X;\mathbf{S}^\bullet) = H^{-i}(j_c^!\mathbf{S}^\bullet)$$

where the latter is the costalk of $\mathbf{S}^\bullet$ at c. Because of self-duality, however, $\mathbf{S}^\bullet$ satisfies the costalk vanishing condition

$$H^{-i}(j_c^!\mathbf{S}^\bullet) = 0 \text{ for } -i \leq \begin{cases} \bar{m}(2n+1) - \dim \mathring{c}X + 1 = -(n+1), \\ \bar{m}(2n+2) - \dim \mathring{c}X + 1 = -(n+1), \end{cases}$$

which is equivalent to the statement. □

PROOF OF PROPOSITION 8.2.. Look at the hypercohomology Wang sequence for $\mathring{c}F \to \mathring{\bar{E}} \to S^{2b+1}$

$$\cdots \to \mathcal{H}_c^{-(a+b+1)}(\mathring{c}F;\mathbf{U}^\bullet) \to \mathcal{H}_c^{-(a+b+1)}(\mathring{\bar{E}};\mathbf{T}^\bullet) \to \mathcal{H}_c^{-(a-b)}(\mathring{c}F;\mathbf{U}^\bullet) \to \cdots$$

where $\mathbf{T}^\bullet = \mathbf{S}^\bullet|_{\mathring{\bar{E}}}$ is in $SD(\mathring{\bar{E}})$ and $\mathbf{U}^\bullet \in SD(\mathring{c}F)$ is constructed as in 7.9. Since $b \geq 2$, we see from the preceding lemma that

$$\mathcal{H}_c^{-(a+b+1)}(\mathring{c}F;\mathbf{U}^\bullet) = \mathcal{H}_c^{-(a+b)}(\mathring{c}F;\mathbf{U}^\bullet) = 0$$

and hence

$$\mathcal{H}_c^{-(a+b+1)}(\mathring{\bar{E}};\mathbf{T}^\bullet) \cong \mathcal{H}_c^{-(a-b)}(\mathring{c}F;\mathbf{U}^\bullet).$$

Decompose M into the open subsets $\mathring{\bar{E}}$ and $\mathring{c}E$ with $\mathring{\bar{E}} \cap \mathring{c}E = E \times (0,1)$. Consider the Mayer–Vietoris hypercohomology sequence (see [Ive86, III.7.5] or [Bre97, II, § 13], for instance)

$$\begin{aligned} \cdots \longrightarrow \mathcal{H}_c^{-(a+b+1)}(E\times(0,1);\mathbf{S}^\bullet|) &\xrightarrow{i_{a+b+1}} \\ \mathcal{H}_c^{-(a+b+1)}(\mathring{c}E;\mathbf{S}^\bullet|) \oplus \mathcal{H}_c^{-(a+b+1)}(\mathring{\bar{E}};\mathbf{S}^\bullet|) &\longrightarrow \\ \mathcal{H}_c^{-(a+b+1)}(M;\mathbf{S}^\bullet) \longrightarrow \mathcal{H}_c^{-(a+b)}(E\times(0,1);\mathbf{S}^\bullet|) &\xrightarrow{i_{a+b}} \cdots \end{aligned}$$

Let us first show that i_{a+b} is injective. We use the following exact sequence for hypercohomology with compact supports (see [Ive86, III.7.7])

$$\cdots \to \mathcal{H}_c^{-(a+b+1)}(\{c\};\mathbf{S}^\bullet) \to \mathcal{H}_c^{-(a+b)}(\mathring{c}E\setminus\{c\};\mathbf{S}^\bullet) \to \mathcal{H}_c^{-(a+b)}(\mathring{c}E;\mathbf{S}^\bullet) \to \cdots.$$

It suffices to show that

$$\mathcal{H}_c^{-(a+b+1)}(\{c\};\mathbf{S}^\bullet) \cong \mathcal{H}^{-(a+b+1)}(\{c\};\mathbf{S}^\bullet)$$

vanishes. The latter is isomorphic to $\mathbf{H}^{-(a+b+1)}(\mathbf{S}_c^\bullet)$ which is 0 because of (SD3). Due to the preceding lemma $\mathcal{H}_c^{-(a+b+1)}(\mathring{c}E;\mathbf{S}^\bullet|)$ is 0. The surjectivity of i_{a+b+1} follows now from the condition (S) using the naturality of the hypercohomology Wang sequence in completely the same manner as in the proof of Proposition 5.2. □

REMARK 8.4. As you can see, we have only used the vanishing lemma 8.3, which is valid for every $\mathbf{S}^\bullet \in SD(F)$. Hence, in the proposition, the structure group G need not be adapted.

REMARK 8.5. In the case of a Witt fibre F the condition (S) is the one of Section 5. See the examples there, especially 5.6.

COROLLARY 8.6. *Let M be as in* 8.2. *The signature $\sigma(M)$ of M vanishes.*

COROLLARY 8.7. *Let X be as in* 8.1 *such that X_{n-k} is an odd-dimensional sphere. Assume E to satisfy* (S). *Then $\sigma(X) = \sigma(X^1) = \sigma(M, \partial M)$, where the latter is the Novikov signature of M.*

Acknowledgements

The article is based on the author's Diploma thesis at the University of Heidelberg [Lev07] and is motivated by the MSRI workshop on the Topology of Stratified Spaces. I want to express my gratitude to Greg Friedman, Eugénie Hunsicker, Anatoly Libgober and Laurentiu-George Maxim for the organisation of this event, the accompanying support and the opportunity for giving a talk. I am also indebted to my thesis advisor Markus Banagl for his help and advice during the development of the thesis. Last but not least I am grateful to the referee for the careful reading of the preliminary version.

References

[Ban02] M. Banagl, *Extending intersection homology type invariants to non-Witt spaces*, Mem. Amer. Math. Soc. **160** (2002), no. 760.

[Ban06] ______, *The L-class of non-Witt spaces*, Ann. of Math. **163** (2006), no. 3.

[Ban07] ______, *Topological invariants of stratified spaces*, Springer Monographs in Mathematics, Springer, 2007.

[Bor84] A. Borel et al., *Intersection cohomology*, Progr. Math., vol. 50, Birkhäuser, Boston, 1984.

[Bre97] G. E. Bredon, *Sheaf theory*, second ed., Grad. Texts in Math., vol. 170, Springer, New York, 1997.

[Bry93] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progr. Math., vol. 107, Birkhäuser, Boston, 1993.

[CS91] S. E. Cappell and J. Shaneson, *Stratifiable maps and topological invariants*, J. Amer. Math. Soc. **4** (1991), 521–551.

[FM81] W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc. **31** (1981), no. 243.

[Fri07] G. Friedman, *Intersection homology of stratified fibrations and neighborhoods*, Adv. Math. **215** (2007), 24–65.

[GM83] M. Goresky and R. Macpherson, *Intersection homology II*, Invent. Math. **72** (1983), no. 1, 77–129.

[GM88] M. Goresky and R. MacPherson, *Stratified morse theory*, Ergebnisse der Mathematischen Wissenschaften, vol. 14, Springer, Berlin Heidelberg, 1988.

[HS91] N. Habegger and L. Saper, *Intersection cohomology of cs-spaces and Zeeman's filtration*, Invent. Math. **105** (1991), no. 2, 247–272.

[Hud69] J. F. P. Hudson, *Piecewise linear topology*, Mathematics Lecture Notes Series, W. A. Benjamin, New York, 1969.

[Hun07] E. Hunsicker, *Hodge and signature theorems for a family of manifolds with fibration boundary*, Geom. Top. **11** (2007), 1581–1622.

[Ive86] B. Iverson, *Cohomology of sheaves*, Universitext, Springer, Berlin, 1986.

[Kin85] H. King, *Topological invariance of intersection homology without sheaves*, Topology Appl. **20** (1985), 149–160.

[Lev07] F. Levikov, *Wang sequences in intersection homology*, Diploma Thesis, Universität Heidelberg, 2007.

[Roy87] J. Roy, *Vanishing of odd-dimensional intersection cohomology*, Math. Z. **195** (1987), 239–253.

[Sie83] P.H. Siegel, *Witt spaces: A geometric cycle theory for KO-homology at odd primes*, American J. Math. **105** (1983), 1067–1105.

[Spa66] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[Wan49] H. C. Wang, *The homology groups of fibre bundles over a sphere*, Duke Math. J. **16** (1949), 33–38.

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