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Backsliding Toads and Frogs

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ABSTRACT. Backsliding Toads and Frogs is a variant of Toads and Frogs in which virtually all positions are loopy. The game is an excellent case study of Conway's theory of sides. In this paper, we completely characterize the values of all natural starting positions. We also exhibit positions with the familiar values n and 2^{-n}, as well as positions with *temperatures* n and 2^{-n}, for all n.

1. Introduction

The game of Toads and Frogs was introduced in *Winning Ways* [Berlekamp et al. 2001]. It is played on a $1 \times n$ strip, populated by some number of toads and frogs. Left plays by moving any toad one space to the right; Right by moving any frog one space to the left. If either player's move is blocked by the opponent, he may choose to leap over her, provided the next square is empty. Jumps do not result in capture. As usual, the winner is the player who makes the last move.

The variant Backsliding Toads and Frogs was also introduced in *Winning Ways*. Here both players have the additional option of retreating by one space, though reverse jumps are still prohibited. Unlike standard Toads and Frogs, the backsliding variant is loopy. As we will see, this additional rule has a monumental effect on the play of the game.

Figure 1 shows a typical position shortly after the start of the game. Each player has one advancing move and one backsliding move available, and Left has the additional option of leaping over Right's frog.

Standard Toads and Frogs was studied extensively by Erickson [1996], whose results include an analysis of certain natural starting positions, as well as the observation that other starting positions have great canonical complexity. However,

Figure 1. A typical position in Backsliding Toads and Frogs.

aside from a few very small positions analyzed in *Winning Ways*, the backsliding version has been scarcely investigated. This is likely due to the enormous difficulty in calculating its values; a direct analysis by hand is exceedingly difficult, and until very recently the tools for a machine analysis were not available.

The present research relied upon a large database of positions assembled using *CGSuite*'s implementations of the algorithms introduced in [Siegel 2009b]. (See http://www.cgsuite.org/ for *CGSuite*.) However, all of the results presented in this paper, with the exception of the museum pieces in Section 6, are fully verifiable by hand, and their proofs, as presented here, do not rely in any way on the computer's output. The transition from calculations to mathematical proofs followed a familiar pattern: a careful analysis of the database led first to a series of promising conjectures, and then ruled out many misdirections and false hypotheses, until the solutions could be isolated.

One striking result is that, in contrast to the standard version, *all* natural starting positions have simple values. Nonetheless, Backsliding Toads and Frogs is quite an interesting game if one considers arbitrary starting positions. Many typical values occur, including n, 2^{-n}, $\uparrow$ and **over**, as well as values with temperature n and 2^{-n}.

In Section 2, we introduce some notation and prove a key lemma. In Section 3, we analyze positions with just one frog, and in Section 4, those where the groups of toads and frogs are initially separated. Section 5 contains positions with the familiar values mentioned above. Finally, Section 6 lists some of the more interesting values obtained by computer search.

By the end of this paper we will be able to solve this problem:

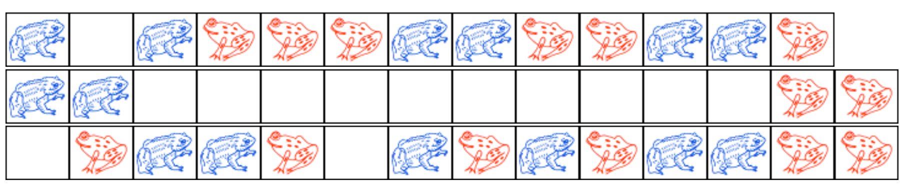

Figure 2. What is the outcome if Left plays first? If Right plays first?

2. Preliminaries

We assume familiarity with the theory of loopy games as presented in Chapter 11 of *Winning Ways*. See [Siegel 2009a] for a gentle introduction.

It is convenient to use Erickson's notation for Toads and Frogs positions. A T represents a toad, an F a frog, and an open box □ an empty space on the board. Superscripts indicate repetition, so for example,

$$\mathrm{T}^3\square^2\mathrm{F}^3 \text{ is the position } \mathrm{TTT}\square\square\mathrm{FFF}.$$

The Ts and Fs in a position will occasionally be subscripted, as in

$$\mathrm{T}\mathrm{T}_1\mathrm{T}_2\square^2\mathrm{F}_1\mathrm{F}_2\mathrm{F}.$$

The subscripts do not affect the actual composition of the position; they are merely labels used to reference specific toads and frogs in the discussion that follows. Additionally, we will use the symbol ⊛ to represent an arbitrary sequence of zero or more empty spaces. For instance, the generality $⊛\mathrm{T}^3⊛\mathrm{F}^3$ would include the previous example.

Define the *configuration* of a position to be that position with all empty spaces removed. Thus the configuration depends only on the relative locations of the toads and frogs, and not on the number of spaces that separate them. For example, the configuration of the position noted above is TTTFFF. Note that sliding moves do not affect a game's configuration, while jumps change it irrevocably.

In many of the proofs that follow, the goal is to show that $X \geq 0$ for a certain position X. Elsewhere, however, we wish to show that $X = \mathbf{on}$, or that the *onside* of X is **on**. In virtually all cases, the necessary relation is established by exhibiting an explicit winning strategy for Left. However, the shapes of the strategies differ in subtle ways depending on the specific goal. The differences are worth highlighting here:

(a) To show that $X \geq 0$ for some position X, we consider X played in isolation, and show that Left, playing second, can get the last move in finite time.
(b) To show that the onside of X is **on**, we allow Right infinitely many pass moves, and show that Left can play so as never to run out of moves.
(c) To show that $X = \mathbf{on}$, we proceed as in (b), with the further restriction that Right must be forced to pass infinitely many times.

Note that (c) does *not* require Left to reach a state where Right is permanently out of moves in X. Indeed, in some cases where $X = \mathbf{on}$, it's possible for Right to make infinitely many moves in X. In such cases, Left can ensure that Right is *temporarily* out of moves infinitely often; but for Left to claim a free move, he must mobilize Right for some finite amount of time.

We close this section with a key result:

LEMMA 1 (THE DECOMPOSITION LEMMA). Let X and Y be arbitrary positions. Then:

(a) $X\mathrm{TT}Y \geq X + Y$ and $X\mathrm{FF}Y \leq X + Y$.
(b) If X contains no empty spaces, then $X\mathrm{T}Y \geq Y$ and $Y\mathrm{F}X \leq Y$.

PROOF. If Left never moves his pair of toads in $X\mathrm{TT}Y$, he can guarantee that X and Y never interact. This establishes (a), and (b) is similar: if Left never moves his extra toad in $X\mathrm{T}Y$, then the entire subposition X is immobilized. □

3. Positions with one Frog

With just one toad and one frog, the position always has value 0 or $*$, and the value depends only on the relative position of the toad and frog (and not on the size of the board):

LEMMA 2.

$$\boxast\mathrm{F}\square^k\mathrm{T}\boxast = 0 \text{ if } k \text{ is even}, * \text{ if } k \text{ is odd};$$

$$\boxast\mathrm{T}\square^k\mathrm{F}\boxast = * \text{ if } k \text{ is even}, 0 \text{ if } k \text{ is odd},$$

except for the trivial case where there are no moves available to either player.

PROOF. We first show that $\boxast\mathrm{F}\square^k\mathrm{T}\boxast = 0$ if k is even. By symmetry, it suffices to show that Left can win playing second. Since no jumps are possible, every move reverses the parity of the distance k. Therefore, the distance will always be odd when Left has the move, so he can always slide toward the left end of the board. Eventually the position will reach $\mathrm{FT}\boxast$, and Right will be without a move. If k is initially odd, then moving to 0 is the only option available to either player, so the value is $*$.

Next we show that $\boxast\mathrm{T}\square^k\mathrm{F}\boxast = 0$ if k is odd. As before, it suffices to show that Left can win playing second. He begins by advancing until the toad and frog are adjacent. Since every sliding move reverses the parity of k, the meeting must occur immediately following an advance by Right, resulting in the position $\boxast\mathrm{TF}\square\boxast$. At this point Left jumps, and since the toad and frog remain adjacent, the resulting position has value 0. If k is initially even, then moving to 0 is the only option available to either player, so the value is $*$ (except in the trivial case when no moves are available to either player). $\square$

With several toads against just one frog, the position always has value **on** except in a few pathological cases:

LEMMA 3. Suppose $m \geq 2$. Then:

$$\boxast\mathrm{FT}^m = 0;$$

$$\boxast\mathrm{F}\square\mathrm{T}^m = \{\mathbf{on} \mid 0\};$$

All other positions involving m toads and one frog have value **on**, except for the trivial case where there are no moves available to either player.

PROOF. *Case 1*: The frog is to the left of all toads, so that no further jumps are possible. If at least one toad has an empty space to its right, the value is **on**, as follows. On his move, Left advances his left-most toad toward the frog. If this is not possible, he moves any *other* toad arbitrarily. Eventually the left-most toad will trap the frog at the end of the board, and Left's remaining toads will still be free to move about indefinitely.

The two special cases in the statement of the lemma follow immediately.

Case 2: The frog is between two toads. We will show that Left can achieve infinitely many free moves against Right. Note that Right can jump only finitely many times; after the last jump, we are in a Case 1 position with an empty space available to Left (the one just vacated by the frog).

Left plays as follows. If there is intervening space between the frog and its adjacent toads, Left moves a surrounding toad toward the frog. Within finite time the frog will be sandwiched between two toads. Right's only move from such a position (if any) is to jump. If Left is to move from such a position, he simply makes *any* available move. This might give Right the opportunity to make an extra sliding move, but Left can reverse this by tightening the gap again. In that event Left makes two moves to Right's one, gaining a free move.

Case 3: The frog is to the right of all toads. Here Left simply advances the rightmost toad toward the frog. If the rightmost toad is adjacent to the frog, Left makes any other move (jumping permitted). Eventually Right's only move will be to jump. Any jump leads to a Case 2 position. □

4. Natural starting positions

In this section we consider positions of the form $\mathrm{T}^m\Box^k\mathrm{F}^n$, where the toads and frogs form two disconnected armies. These were termed $(m,n)_k$-positions in *Winning Ways*. The main result is the chart of Figure 3.

These values are, on the whole, much simpler than those for ordinary Toads and Frogs. The basic reason is that either player, if undisturbed, can assure himself infinitely many free moves by maneuvering just three of his amphibians into a "fortress":

$$\mathrm{T}\Box\mathrm{TT}X$$

Notice that it does not matter whether X contains zero or a hundred frogs. The Decomposition Lemma implies that the value of this position is at least $\mathrm{T}\Box + X$; and since $\mathrm{T}\Box = \mathbf{on}$, the overall position must have onside **on**. This fundamental strategy accounts for the prevalence of **dud**s in the table.

The cases $m = 1$ and $n = 1$ were established in section 3 for all k. We verify the rest of the table with a series of lemmas. We study the easier limiting cases first, and then go back and fill in the gaps.

The first lemma establishes the **dud** values in the $k \geq 3$ section of the chart:

LEMMA 4. If $m, k \geq 3$, then for all n, $\mathrm{T}^m\Box^k\mathrm{F}^n$ has onside **on**.

PROOF. Left, on his first two moves, advances each of the two front toads. Since $k \geq 3$, Right is powerless to interfere even if she moves first, so Left establishes a fortress:

$$\mathrm{T}^{m-2}\Box\mathrm{T}^2\cdots$$

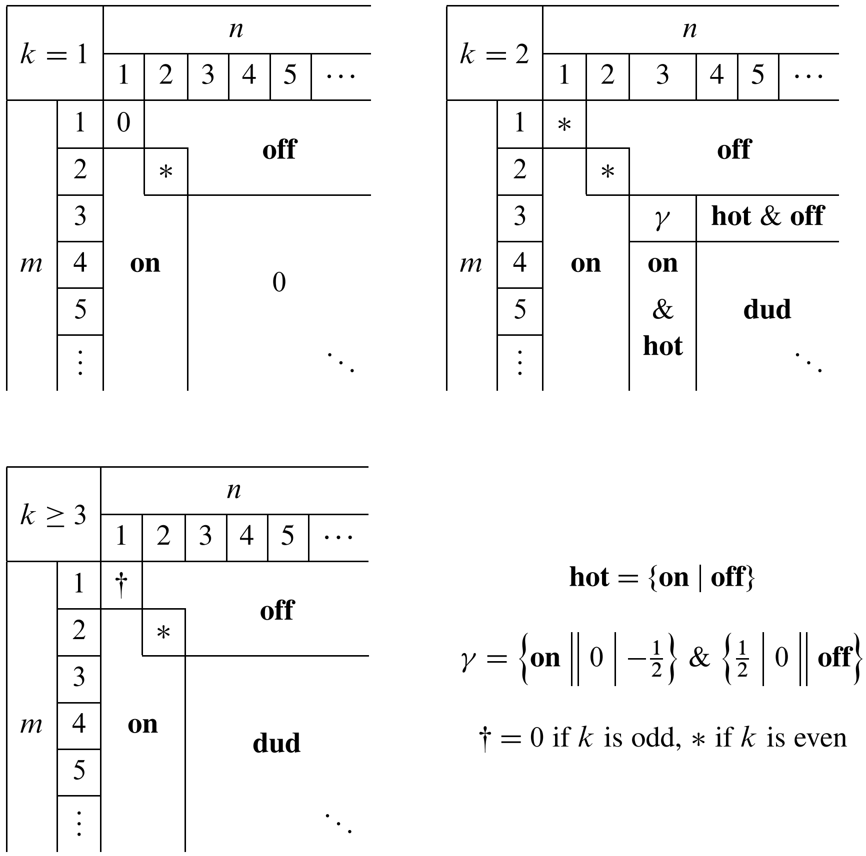

Figure 3. The value of $T^m\square^k F^n$, for all k, m, n.

Since $m \geq 3$, Left has at least one toad remaining in the rear, which he is now free to shuttle indefinitely. □

By symmetry, we know that if $n, k \geq 3$, then the offside of $T^m\square^k F^n$ is **off**, for all m. Therefore, if $m, n, k \geq 3$, we may conclude that $T^m\square^k F^n =$ **dud**. A similar theme establishes the **dud** values for $k = 2$:

LEMMA 5. If $m \geq 4$, then for all n, $T^m\square^2 F^n$ has onside **on**.

PROOF. There are no problems if Left moves first: he can establish a fortress before Right can interfere. The remaining difficulty is Right's immediate move to $T^m\square F\square F^{n-1}$, which Left counters with a move to

$$T^{m-1}\square TF\square F^{n-1}.$$

If Right does anything other than jump, then Left can establish a fortress immediately. If Right jumps, Left responds by advancing his toad, leaving the position

$$\mathrm{T}^{m-1}\mathrm{F}_1\square\mathrm{T}\square\mathrm{F}^{n-1}.$$

There are now two possibilities:

- If Right takes any action other than backsliding F_1, Left jumps to the position

  $$\mathrm{T}^{m-2}\square\mathrm{F}_1\mathrm{TT}X.$$

  By the Decomposition Lemma, this position is $\geq \mathrm{T}^{m-2}\square\mathrm{F}_1 + X$. But since $m-2 \geq 2$, we know from Lemma 3 that $\mathrm{T}^{m-2}\square\mathrm{F}_1 = \mathbf{on}$. So the onside of the sum must be **on**.
- If Right backslides F_1, then Left can advance to

  $$\mathrm{T}^{m-2}\square\mathrm{T}_1\mathrm{F}_1\mathrm{T}\square\mathrm{F}^{n-1}.$$

  From this position Left can shuttle his rear toads indefinitely. If Right ever jumps with F_1, Left responds by advancing T_1, establishing a virtual fortress just as before. □

When $k = 1$, all values with $m, n \geq 3$ are zero:

LEMMA 6. If $m \geq 3$, then for all n, $\mathrm{T}^m\square\mathrm{F}^n \geq 0$.

PROOF. We can assume that $m = 3$, since Left can simply ignore any additional toads. With Left playing second, all Right moves in the following sequence are forced:

$$\begin{array}{ll}
\mathrm{TTT}\square\mathrm{FF}^{n-1} & \\
\quad\searrow & \\
 & \mathrm{TTTF}\square\mathrm{F}^{n-1} \\
\quad\swarrow & \\
\mathrm{TT}\square\mathrm{FTF}^{n-1} & \\
\quad\searrow & \\
 & \mathrm{TTF}\square\mathrm{TF}^{n-1} \\
\quad\swarrow & \\
\mathrm{T}\square\mathrm{FTTF}^{n-1} & \\
\quad\searrow & \\
 & \mathrm{TF}\square\mathrm{TTF}^{n-1} \\
\quad\swarrow & \\
\square\mathrm{FTTTF}^{n-1} & \\
\quad\searrow & \\
 & \mathrm{F}\square\mathrm{TTTF}^{n-1} \\
\quad\swarrow & \\
\mathrm{FT}\square\mathrm{TTF}^{n-1} &
\end{array}$$

whereupon Right cannot move. □

Next we study those positions where Right has exactly two frogs. The following lemma verifies that $\mathrm{T}^2\square^k\mathrm{F}^2 = *$ for $k \geq 2$. The remaining case $k = 1$ is analyzed in *Winning Ways* and can easily be checked by showing that the sum $* + \mathrm{T}^2\square\mathrm{F}^2$ is a second-player win.

LEMMA 7. If $k \geq 1$, then $\mathrm{TT}\square^k\mathrm{F}\square\mathrm{F} = 0$.

PROOF. We show that either player, moving second, can force a win in finite time. Clearly, by symmetry, it suffices to show that Left can win either $\mathrm{TT}\square^k\mathrm{F}\square\mathrm{F}$ or $\mathrm{T}\square\mathrm{T}\square^k\mathrm{FF}$ playing second; we will exhibit a strategy that succeeds in both cases.

Let p be the sum of the distance between the two toads and the distance between the two frogs. The following facts are apparent: p is initially odd; each sliding move changes the parity of p; and each jump maintains the parity of p. So on Left's move, p will be even just if an even number of jumps have occurred.

The remainder of the proof exhaustively describes Left's winning strategy. The strategy is broken down by configuration: at each stage, Left guarantees that the next configuration will be reached within a finite number of moves. Eventually the position will reach the configuration FTFT, whereupon we will see that Left can force a win.

Configuration TTFF: Left either jumps or advances a toad. One of these options must be available unless the position has the form ⊛TTFF⊛. Since $k > 0$, such a position cannot be reached on Right's opening move. So Left has had an opportunity to move, and therefore there is an empty space behind the pair of toads. Left backslides, jumping on the next move if Right does not.

Configuration TFTF: Left either jumps with the front toad or advances either toad. If neither option is available, the position must be ⊛TF⊛TF; but that position has even p, and since there has been exactly one jump, it cannot occur on Left's move.

Configuration TFFT: Left either jumps, advances the rear toad, or backslides the front toad. One of these options must be available unless the position is ⊛TFFT⊛, in which case Left moves either toad and jumps on the next move if Right does not.

Configuration FTTF: If Left is following our strategy, he will never jump to this configuration, so it must have been reached by a Right jump. Therefore, at the outset, there must be at least one empty space between the toads. Left plays to maintain this space: On his move, if possible, he either advances the front toad, backslides the rear toad, or jumps to a new configuration. If none of these options is available, the position must be $⊛\mathrm{F}_1\mathrm{T}_1⊛\square\mathrm{T}_2\mathrm{F}_2$. In this case, Left backslides T_2, reaching $⊛\mathrm{F}_1\mathrm{T}_1⊛\mathrm{T}_2\square\mathrm{F}_2$. Left now counters each F_1 move by backsliding T_1. Eventually Right will be forced to advance F_2 and permit a jump.

Configuration FTFT: If Left jumps into this configuration, then Right's *first* move cannot be a jump, so Left is guaranteed at least one move in this configuration. Left always backslides the rear toad if possible; otherwise he backslides the front toad. This guarantees that Right will never be allowed to jump from this configuration. The only positions from which Left cannot backslide either toad are those of the form $\boxast$FT$\boxast$FT$\boxast$; but those have even p, and so cannot occur on Left's move (as this configuration can only be reached after exactly three jumps). So play continues like this until the position reaches FTFT$\boxast$, whereupon Right is without a move, and Left has won. $\square$

Note: It is *not* true that all positions with two toads, two frogs, and two empty spaces have value 0 or $*$. For example, $\square$F$\square$TFT has the value $* \; \& \; \{0 \mid \textbf{off}\}$.

Three toads are sufficient to overpower Right's two frogs:

LEMMA 8. If $m \geq 3$, then $\mathrm{T}^m\square^k\mathrm{F}^2 = \textbf{on}$.

PROOF. It suffices to prove that $\mathrm{T}^3\square^k\mathrm{F}^2 = \textbf{on}$, since Left can ignore any additional toads. We will exhibit a strategy for Left that forces Right to make infinitely many pass moves. The strategy is broken down into two major phases. In the first phase, Left ignores his rear toad completely. For each configuration K of the four remaining amphibians, our strategy will guarantee that, if Right passes only finitely many times at K, then the next configuration will eventually be reached. When the configuration reaches FTFT, the second phase of the strategy begins, and Left mobilizes his third toad.

We begin by describing the first phase, broken down by configuration.

Configuration TTFF: On his move, Left jumps if possible; otherwise he advances either toad. If neither option is available, the position must be $\boxast\mathrm{T}_1\mathrm{TFF}\boxast$. Since Left is guaranteed at least one move in this configuration, there must be an empty space behind T_1, so he backslides. If Right responds by passing, Left advances T_1, returning to an earlier position with an intervening pass move. Right's only other options are to jump or to allow a jump.

Configuration TFTF: Left jumps with the *front* toad if possible (never the rear toad); otherwise he advances either toad. If neither option is available, the position must be $\boxast\mathrm{T}_1\mathrm{F}_1\boxast\mathrm{T}_2\mathrm{F}_2$. If there is space between the toad/frog pairs, Left backslides T_2, reaching $\boxast\mathrm{T}_1\mathrm{F}_1\boxast\mathrm{T}_2\square\mathrm{F}_2$. From this position Left counters each F_1 backslide by advancing T_1. Eventually Right must either jump with F_1, advance F_2, or pass. If he advances F_2, Left jumps immediately with T_2; while if he passes, Left advances T_2, having just gained a move.

Finally, if Left is ever to move from $\boxast\mathrm{T}_1\mathrm{F}_1\mathrm{T}_2\mathrm{F}_2$, he backslides T_1. If Right passes, Left advances again, gaining a move; if Right advances F_1, Left backslides T_2 to $\boxast\mathrm{T}_1\mathrm{F}_1\mathrm{T}_2\square\mathrm{F}_2$. Again Right must either jump, allow a jump with T_2, or pass, granting Left a free move.

Configuration FTTF: If Left is following our strategy, he will never jump into this configuration. So when the configuration is first reached, Right has just leapt into it, and there is at least one space between the toads. Left's moves are, in order of preference: jump; advance the front toad; backslide the rear toad; backslide the front toad. By following this strategy, Left guarantees that the position ⧆FTTF will never arise, so at least one of these options is always available. The analysis is similar to the above.

Configuration TFFT: Left jumps if possible; otherwise he moves either toad toward the frogs. If neither option is available, the position must be ⧆TFFT⧆. In this case Left slides either toad (it's possible that only one is mobile), and Right must either pass, jump, or permit a jump. If he passes, then Left returns to ⧆TFFT⧆.

Once the configuration reaches FTFT, the second phase of Left's strategy begins. The full configuration, including Left's extra toad, is $\mathrm{T}_1\mathrm{F}_1\mathrm{T}_2\mathrm{F}_2\mathrm{T}_3$. If Right moves first in this configuration, then Left's previous move must have been a jump with T_2 or T_3. So Right's first move cannot be to jump with F_2. Likewise, since T_1 begins on the far left-hand side of the board, Right's first move cannot be to jump with F_1. Therefore Left is guaranteed at least one move in this configuration.

On his move, Left picks one of the following options, listed in order of preference.

1. Backslide T_2 or T_3, preferring T_2 *except* from the position $\mathrm{T}_1\mathrm{F}_1\square\mathrm{T}_2\boxast\mathrm{F}_2\boxast\mathrm{T}_3\boxast$.
2. Shuttle T_1 between the two squares at the far left-hand side of the board.
3. Advance T_2.
4. Advance T_3.

There are three possible ways play might continue:

- Right never jumps again. Here a careful check of Left's strategy reveals that Right is forced to pass infinitely many times.
- Right eventually jumps with F_1. Since T_1 never leaves the two left-hand squares, the resulting position must be FTX for some X containing two toads and one frog. From Lemma 3 we know that $X = \mathbf{on}$; but by the Decomposition Lemma, FT$X \geq X$.
- Right eventually jumps with F_2. As we have observed, this cannot happen on Right's first move. Consider Left's previous move. It was not a T_2-backslide, since T_2 and F_2 must now be adjacent. So either Left was *unable* to backslide, or he *chose* not to. If he was *unable* to, then it is because F_1 and T_2 were adjacent; since they no longer are, Left must have just advanced T_2. This means T_1 is immobile and the position (before Right's jump) is exactly $Z = \mathrm{T}_1\mathrm{F}_1\square\mathrm{T}_2\mathrm{F}_2\boxast\mathrm{T}_3\boxast$. If Left *chose* not to backslide, then again the position is exactly Z, since otherwise backsliding T_2 is top priority. So Right's jump is

to the position $\mathrm{T}_1\mathrm{F}_1\mathrm{F}_2\mathrm{T}_2\boxast\mathrm{T}_3\boxast$. By the Decomposition Lemma, this position has value **on**. □

This covers all cases with two frogs (or, by symmetry, two toads). All that remains now are the peculiar values along the $k = 2$, m or $n = 3$ band. The specific case $k = 2$, $m = n = 3$ can be verified computationally. A final lemma completes the analysis.

LEMMA 9. If $m \geq 4$, then $\mathrm{T}^m\Box^2\mathrm{F}^3 =$ **on** & **hot**.

PROOF. The onside is given by Lemma 5. If Right moves first in the offside, he can establish a fortress, so we know the offside is $\{H \mid \textbf{off}\}$ for some H. Finally, a quick computation establishes that $\mathrm{T}^3\Box\mathrm{T}\Box\mathrm{F}^3 =$ **on**. Increasing the number of toads cannot reduce this value, so this verifies that $H =$ **on** in all cases. □

5. Some familiar values

In ordinary Toads and Frogs, it is easy to construct positions of positive integer value n: simply place a single toad at the far left of an otherwise-empty $(n+1)$-length board. Naive constructions fail in the backsliding version, however: if $n > 0$ then such a position has value **on**.

With somewhat more effort, though, it is possible to construct positions of value n and 2^{-n} in Backsliding Toads and Frogs. Further, from these we can derive positions of temperature n and 2^{-n}.

THEOREM 10.

$$(\mathrm{TFFT})^n\Box = n;$$
$$\Box(\mathrm{TF})^n\mathrm{TTFF} = 2^{-n}.$$

A few Lemmas are needed to prove Theorem 10:

LEMMA 11. If a Backsliding Toads and Frogs position contains just one empty space, then its value is a stopper.

PROOF. We need to show that there are no infinite alternating lines of play from any such position. Since each jump changes the position irrevocably, it suffices to show that there can be no infinite alternating sequence of sliding moves. But after any such move, the only sliding options are to return to the previous position, or to slide in the same direction as the previous move. The first is only available to the same player who just moved. So all moves in any *alternating* sequence of sliding moves must be in the same direction. Therefore any such sequence must terminate. □

Lemma 11 is fundamentally important, because it is relatively easy to compare two stoppers γ and δ. To show that $\gamma \leq \delta$, we just need to check that Left can play so as never to run out of moves in $\delta - \gamma$. (See [Berlekamp et al. 2001] for a proof of this fact.)

Our first application of this technique is the following lemma, which concerns "dead pairs" of toads and frogs. These occur in positions of the form FTX and XFT, where a toad and a frog face away from each other at the far edge of the board. The key result is that dead pairs do not change the value of positions with just one empty space.

LEMMA 12 (DEAD PAIRS LEMMA). Let X be any position with just one empty space. Then

$$\mathrm{FT}X = X = X\mathrm{FT}.$$

PROOF. By symmetry it suffices to prove just the first equality. Decomposition implies that FT$X \geq X$. To show $X \geq$ FTX, it suffices to show that Left, playing second, never runs out of moves in $X -$ FTX (since by Lemma 11 both games are stoppers).

Left's strategy for playing second from $X + (-X)$FT is summarized as follows. Left copies Right's move in the opposite component until Right moves the dead frog. If Right *jumps* with the dead frog, then the second component becomes $\cdots$FT□T, with no empty spaces except the one indicated. This clearly has value **on**, guaranteeing Left an infinite supply of moves. Suppose instead that Right slides the dead frog. This necessarily leaves the position

$$\square Y + (-Y)\mathrm{F}\square\mathrm{T}$$

for some sequence Y with no empty spaces, whereupon Left can backslide his dead toad:

$$\square Y + (-Y)\mathrm{FT}\square.$$

Now write $Y = \mathrm{F}^n Z$ with n maximal. Right's only possible move is to

$$\mathrm{F}\square\mathrm{F}^{n-1}Z + (-Y)\mathrm{FT}\square,$$

which Left can answer by moving to

$$\mathrm{F}\square\mathrm{F}^{n-1}Z + (-Y)\mathrm{F}\square\mathrm{T}.$$

Now there are three possible options for Right:

- If Right backslides her previous move, Left does the same, returning to a prior position.
- If Right moves to

  $$\mathrm{F}\square\mathrm{F}^{n-1}Z + (-Y)\square\mathrm{FT},$$

  then Left simply responds with

  $$\mathrm{F}\square\mathrm{F}^{n-1}Z + (-Z)\mathrm{T}^{n-1}\square\mathrm{TFT},$$

and resumes his initial strategy of mirroring Right's moves until the next time Right activates the dead frog.

- Finally, suppose Right has another frog available:

$$\mathrm{FF}\square\mathrm{F}^{n-2}Z + (-Y)\mathrm{F}\square\mathrm{T}.$$

Then necessarily $n \geq 2$, so the second component is $\cdots\mathrm{TTF}\square\mathrm{T}$. By Decomposition this is $\geq \mathrm{TF}\square\mathrm{T}$. But Lemma 3 showed that $\mathrm{TF}\square\mathrm{T} = \mathbf{on}$, guaranteeing Left an infinite supply of moves. □

LEMMA 13.

$$(\mathrm{TF})^n\square = 0 \text{ if } n \text{ is even}, * \text{ if } n \text{ is odd}.$$

PROOF. $n = 0$ is trivial. For even $n > 0$, it suffices to see that $(\mathrm{TF})^n\square$ is a second-player win. By induction and the Dead Pairs Lemma, Left's only move is to jump to a position of value $*$, which clearly loses. If Right moves first, then Left's moves are all forced until Right chooses to jump, reaching:

$$(\mathrm{TF})^k\mathrm{F}_0\mathrm{T}_0\square(\mathrm{TF})^{k'}$$

with $k + k' = n - 1$. Left's strategy now depends on the parity of k'. If k' is odd, then by induction (and symmetry) $\square(\mathrm{TF})^{k'} = *$. By Decomposition the full position has value $\geq *$; so Left, with the move, has won. If k' is even, then Left advances T_0 immediately, and after Right's forced response the position is

$$(\mathrm{TF})^k\square\mathrm{F}_0\mathrm{T}_0(\mathrm{TF})^{k'}.$$

Now since k' is even and $n - 1$ is odd, k must be odd. So $k \geq 1$ and Left can respond by jumping to

$$(\mathrm{TF})^{k-1}\square\mathrm{F}_1\mathrm{T}_1\mathrm{F}_0\mathrm{T}_0(\mathrm{TF})^{k'}.$$

By induction, $(\mathrm{TF})^{k-1} = 0$. Henceforth Left follows his winning strategy for $(\mathrm{TF})^{k-1}\square$, until Right chooses to move F_1. Then Left backslides T_1, and after Right's forced move backslides T_0, reaching

$$X\mathrm{T}\square(\mathrm{TF})^{k'}$$

for some sequence X. Since k' is even, $\square(\mathrm{TF})^{k'} = 0$. By Decomposition the position has value ≥ 0, and since it is Right's move, Left has won.

When n is odd, Left's only move is to jump to a position of value 0. Right's only move is to

$$Z = (\mathrm{TF})^{n-1}\mathrm{T}\square\mathrm{F}_0,$$

so the proof is completed by showing that Z is a second-player win. Right's only move from Z is to return to $(\mathrm{TF})^n\square$, from which Left can move to 0 (as we already observed). Finally, if Left makes his only move from Z, then Right leaps with F_0, and after a pair of forced moves the position reaches:

$$(\mathrm{TF})^{n-1}\square\mathrm{F}_0\mathrm{T}.$$

By induction and the Dead Pairs Lemma, this is a zero position. $\square$

PROOF OF THEOREM 10.. For the first sequence, observe that in $(\mathrm{TFFT})^n\square$ Right has no legal move, and if Left moves first then the following three-move sequence is forced:

$$(\mathrm{TFFT})^n\square$$
$$\swarrow$$
$$(\mathrm{TFFT})^{n-1}\mathrm{TFF}\square\mathrm{T}$$
$$\searrow$$
$$(\mathrm{TFFT})^{n-1}\mathrm{TF}\square\mathrm{FT}$$
$$\swarrow$$
$$(\mathrm{TFFT})^{n-1}\square\mathrm{FTFT}$$

By induction and the Dead Pairs Lemma, the result has value $n-1$. Note that it is disastrous for Right to ignore Left's opening move, since Left can then backslide for two free moves.

Next we show that $\square(\mathrm{TF})^n\mathrm{TTFF} = 2^{-n}$. The proof is by induction on n. The base case is easily verified: $\square\mathrm{TTFF} = 1$. For the general case, we show that

$$\square(\mathrm{TF})^n\mathrm{TTFF} - 2^{-n}$$

is a second-player win. Suppose first that Left is playing second. If Right jumps with his only mobile frog, Left reduces to

$$\mathrm{FT}\square(\mathrm{TF})^{n-1}\mathrm{TTFF} - 2^{-n+1}.$$

By induction and the Dead Pairs Lemma, this is a zero position. If instead Right reduces -2^{-n} to 0, Left makes backsliding moves until the position

$$(\mathrm{TF})^n\square\mathrm{TTFF}$$

is reached. Now Left's move depends on the parity of n:

- If n is even, Left backslides, and after a forced sequence the position $X = (\mathrm{TF})^{n+1}\square\mathrm{TF}$ is reached with Right to move. But Lemma 13 showed that $(\mathrm{TF})^{n+2}\square = 0$, and since X occurs after Left's only response to Right's opening move, we must have $X \geq 0$.
- If n is odd, then by Decomposition $(\mathrm{TF})^n\square\mathrm{TTFF} \geq (\mathrm{TF})^n\square = *$, so Left must have a winning move.

If Right plays second from $\square(\mathrm{TF})^n\mathrm{TTFF} - 2^{-n}$, his opening strategy is similar: he counters any Left backslides with advancing moves. There are three possibilities.

- *Case 1*: The position $(\mathrm{TF})^n\mathrm{T}\square\mathrm{TF}_0\mathrm{F} - 2^{-n}$ is reached with Right to move. Then Right jumps with F_0; by Lemma 13 the resulting position has value -2^{-n} or $-2^{-n}*$, a win for Right.
- *Case 2*: Left jumps at some point before the above position is reached, to

$$(\mathrm{TF})^k\square\mathrm{FT}(\mathrm{TF})^{k'}\mathrm{TTFF} - 2^{-n}.$$

By Decomposition this is $\le (\mathrm{TF})^k\square{-2^{-n}}$. Since $(\mathrm{TF})^k\square = 0$ or $*$ by Lemma 13, Right has won.
- *Case 3*: Left plays from -2^{-n} to -2^{-n+1} before either of the above occur. Then Right backslides until reaching $\square(\mathrm{TF})^n\mathrm{TTFF} - 2^{-n+1}$, and jumps to

$$\mathrm{FT}\square(\mathrm{TF})^{n-1}\mathrm{TTFF} - 2^{n+1}.$$

By induction and the Dead Pairs Lemma, this position is exactly equal to 0. Note that if Left prematurely moves to -2^{n+2}, Right can return to his opening strategy, having gained appreciably. $\square$

Essentially as a corollary, we have:

THEOREM 14. There exist positions with arbitrarily large finite temperature, and with arbitrarily small positive temperature. In particular:

(a) For $n \ge 1$, $(\mathrm{TFFT})^n\mathrm{TTF}\square\mathrm{F} = \{n* \mid 0\}$.
(b) For $n \ge 0$, $\mathrm{TTF}\square\mathrm{F}(\mathrm{TF})^n\mathrm{TTFF} = \left\{2^{-n-2}* \,\middle|\, 0\right\}$.

PROOF. (a) We first show that $(\mathrm{TFFT})^n\mathrm{T}\square\mathrm{FTF} = n*$. Left's only move is to the position $A = (\mathrm{TFFT})^n\square\mathrm{TFTF}$ and Right's is to $B = (\mathrm{TFFT})^n\mathrm{TF}\square\mathrm{TF}$; we will show that $A = B = n$. First we describe Left's strategy as second player in $A - n$. He plays just as if the position were $(\mathrm{TFFT})^n\square - n$ (following Theorem 10) until Right chooses to move one of the extra frogs, which necessarily gives the position $Y\mathrm{FT}\square\mathrm{T}_0\mathrm{F} - m$, where $Y\square - m \ge 0$. Then Left backslides T_0, initiating a forced sequence that leads to the position $Y\square\mathrm{FTFT} - (m-1)$, with Left to move. Since $Y > m - 1$, the Dead Pairs Lemma implies that Left has a winning move.

If Left plays second in $B - n$, he reduces immediately to $A - n$ unless Right's first move is to jump. In that case the opening sequence forces the position $(\mathrm{TFFT})^n\square\mathrm{FTFT} - n$, with Right to move, so by the Dead Pairs Lemma Left has won.

Next suppose Right plays second in $B - n$. If Left begins by backsliding, then a forcing sequence ensues ending in $(\mathrm{TFFT})^n\square\mathrm{FTFT} - n$, and Right has won. If Left jumps instead, Right makes his only available move, to

$$(\mathrm{TFFT})^n\mathrm{F}\square\mathrm{T}_0\mathrm{TF} - n.$$

Now if Left backslides T_0, then a typical forcing sequence leads to a win for Right. His only other move is to $(\text{TFFT})^{n-1}\text{TFF}\square\text{FTTTF} - n$; but then Right can simply move in $-n$, leaving Left without a move.

Finally, suppose Right plays second in

$$A - n = (\text{TFFT})^{n-1}\text{TFF}_1\text{T}_1\square\text{T}_2\text{F}_2\text{TF} - n.$$

If Left backslides T_2, Right advances F_2, reducing the position to $B - n$. If Left advances T_1, Right backslides F_1 and follows the strategy outlined in the proof of Theorem 10 for winning $(\text{TFFT})^n - n$. This guarantees that F_1 will never move again, thereby immobilizing T_2 and ensuring that Left's extra toads do not break the strategy.

This establishes that $(\text{TFFT})^n\text{T}\square\text{FTF} = n*$. Thus $(\text{TFFT})^n\text{TTFF}\square = 0$, since Left has no move from that position, and Right's only move allows Left a response to $n*$ ($n \geq 1$). This suffices to confirm (a).

(b) As a simple corollary of Theorem 10, Left's only move is to a position of value $2^{-n-2}*$. Right's only move is to the position

$$Z = \text{TTFF}\square(\text{TF})^n\text{TTFF}.$$

The proof is completed by showing that $Z = 0$. To see that $Z \geq 0$: If Right begins with a jump, it must be to $\text{TTFFFT}\square(\text{TF})^{n-1}\text{TTFF}$. By Decomposition and Theorem 10, this position has value $\geq 2^{-(n-1)}$, so Left has won. If instead Right's first move is to backslide, then Left jumps to $\text{T}\square\text{FTF}(\text{TF})^n\text{TTFF}$. Theorem 10 implies that this position has value $2^{-n-2}*$, so again Left has won.

To see that $Z \leq 0$: Right begins by countering each backslide with an advancing move. Notice that, by symmetry, $-Z = \text{TTFF}(\text{TF})^n\square\text{TTFF}$. Since $Z \geq 0$, we know that $-Z \leq 0$, so if Left does not jump before the position reaches $-Z$, then Right has won. But Left can only jump to

$$\text{TTFF}(\text{TF})^k\square\text{FT}(\text{TF})^{k'}\text{TTFF}$$

for some k, k', and by Decomposition and Lemma 13, this position has value $\leq -2^{-k}$. Again Right has won. □

6. A little museum

In this section we show some positions with particularly interesting values. All of the museum pieces were obtained by an exhaustive computer search using *CGSuite*. There are a number of surprises, including a position with offside $\uparrow$ on a board of length 14. This seems to be the smallest board on which $\uparrow$ appears in any context.

$$\text{TFF}\square\text{TTF} = \pm(1*) \qquad \text{TTFFF}\square\text{TTTFF} = \pm 1$$

Some simple switches of temperature 1.

$$\text{F}\square\text{TTTFFF}\square\text{T} = \pm(\{1 \mid *\}, \{1 \mid 0\})$$

A pure infinitesimal other than $0, *$.

$$\text{T}\square\text{F}\square\text{TTFFFT} = 1 \;\&\; \textbf{over} \qquad \text{T}\square\text{TT}\square\text{FFFFT} = \textbf{on} \;\&\; \textbf{over}$$

Some positions with offside **over**.

$$\text{F}\square\text{TTTFFTFT}\square\text{TFF} =$$
$$\big\{\{1 \mid *\}, \{1 \mid 0\} \mid \{* \mid -\tfrac{1}{8}*\}, \{0 \mid -\tfrac{1}{8}*\}\big\} \;\&\; \pm\big(\{\tfrac{1}{8}* \mid *\}, \{\tfrac{1}{8}* \mid 0\}\big)$$

A pure infinitesimal with distinct sides.

$$\square\text{FTTF}\square\text{TFTFTTFF} = \tfrac{1}{8} \;\&\; \uparrow \qquad \square\text{FTTFTFTTFT}\square\text{FTFFFT} = * \;\&\; -1\Uparrow$$
$$\square\text{FTTF}\square\text{TFTFTFTTFF} = \tfrac{1}{16} \;\&\; 1/2$$

On large boards, familiar values mysteriously arise.

$$\text{TTT}\square\text{FFFFT}\square\text{TFFT}\square = \textbf{upon}* \;\&\; * \qquad \text{TF}\square\text{FT}\square\text{TF}\square\text{TTFFFT} = 1 \;\&\; \textbf{upon}$$

Some higher-order loopy infinitesimals.

$$\text{F}\square\text{TTTFTFTF}\square\text{TFF} =$$
$$\Big\{\{1 \mid *\}, \{1 \,|||\, \tfrac{1}{8}* \mid 0 \,\|\, -\tfrac{1}{8}\} \,\Big|\, \{* \mid -\tfrac{1}{2}*\}, \{\tfrac{1}{8}* \mid 0 \,\|\, -\tfrac{1}{8} \,|||\, -\tfrac{1}{2}*\}\Big\}$$
$$\&\; \Big\{\{\tfrac{1}{8}* \mid *\}, \{\tfrac{1}{8}* \,||||\, \tfrac{1}{8} \mid \uparrow \,\|\, 0 \,|||\, -\tfrac{1}{8}\} \,\Big|\, \{* \mid -\tfrac{1}{2}*\}, \{\tfrac{1}{8} \mid \uparrow \,\|\, 0 \,|||\, -\tfrac{1}{8} \,||||\, -\tfrac{1}{2}*\}\Big\}$$

The most complicated value known.

7. A solution

We can now solve the problem presented at the beginning of this paper. By Theorem 14 we know that the first position has value $\{0 \mid -2*\}$. The analysis of Section 4 demonstrates that the middle position has value $*$. Finally, the last position is one of the special values reported in Section 6: $\frac{1}{8} \;\&\; \uparrow$.

Adding these together gives a value of

$$\{\tfrac{1}{8}* \mid -\tfrac{15}{8}\} \;\&\; \{\uparrow* \mid -2\uparrow\}$$

for the overall position. The onside is fuzzy, and the offside is negative: Right can win playing first; while if Left plays first he holds the game to a draw.

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