Probability, Geometry and Integrable Systems
MSRI Publications
Volume **55**, 2007

Lines on abelian varieties

EMMA PREVIATO

ABSTRACT. We study the function field of a principally polarized abelian variety from the point of view of differential algebra. We implement in a concrete case the following result of I. Barsotti, which he derived from what he called the prostapheresis formula and showed to characterize theta functions: the logarithmic derivatives of the theta function along *one* line generate the function field. We outline three interpretations of the differential algebra of theta functions in the study of commutative rings of partial differential operators.

Henry McKean was one of the earliest contributors to the field of "integrable PDEs", whose origin for simplicity we shall place in the late 1960s. One way in which Henry conveyed the stunning and powerful discovery of a linearizing change of variables was by choosing Isaiah 40:3-4 as an epigram for [McKean 1979]: *The voice of him that crieth in the wilderness, Prepare ye the way of the Lord, make straight in the desert a highway for our God. Every valley shall be exalted and every mountain and hill shall be made low: and the crooked shall be made straight and the rough places plain*. Thus, on this contribution to a volume intended to celebrate Henry's many fundamental achievements on the occasion of his birthday, my title. I use the word line in the extended sense of "linear flow", of course, since no projective line can be contained in an abelian variety — the actual line resides in the universal cover.

This article is concerned primarily with classical theta functions, with an appendix to report on a daring extension of the concept to infinite-dimensional tori, also initiated by Henry. Thirty years (or forty, if you regard the earliest experiments by E. Fermi, J. Pasta and S. Ulam, then M. D. Kruskal and N. J. Zabusky, as more than an inspiration in the discovery of solitons; see [Previato 2008] for references) after the ground was broken in this new field, in my view one of the main remaining questions in the area of theta functions as related to PDEs, is still that of straight lines, both on abelian varieties and on Grassmann manifolds (the two objects of greatest interest to geometers in the nineteenth century!). On a Jacobian $\mathrm{Pic}^0(X)$, where X is a Riemann surface of genus g (which we also call a "curve", for brevity), there is a line which is better than any other. That is, after choosing a point on the curve. Whatever point is chosen, the sequence of hyperosculating vectors to the Abel image of the curve in the Jacobian at that point can be taken as the flows of the KP hierarchy, according the Krichever's inverse spectral theory. As a side remark, also related to KP, on a curve not all points are created equal. For a Weierstrass point, there are more independent functions in the linear systems nP for small n than there are for generic curves, which translates into early vanishing of (combinations of) KP flows, giving rise to n-th KdV-reduction hierarchies of a sort; other special differential-algebraic properties would obtain if $(2g-2)P$ is a canonical divisor K_X [Matsutani and Previato 2008]. However, on a general (principally polarized) abelian variety, "there should be complete democracy".[1] My central question is: What line, or lines, are important to the study of differential equations satisfied by the theta function?

In this paper I put together a number of different proposed constructions and ground them in a common project: use the differential equations for the theta function along a generic line in an abelian variety, to characterize abelian varieties, give in particular generalized KP equations, and interpret these PDEs as geometric constraints that define the image of infinite-dimensional flag manifolds in $\mathbb{P}B$, where B is a bosonic space. These topics are developed section-by-section as follows: Firstly, Barsotti proved (in an essentially algebraic way) that on any abelian variety[2] there exists a direction such that the set of derivatives of sufficiently high order of the logarithm of the theta function along that direction generates the function field of the abelian variety. Moreover, he characterized theta functions by a system of ordinary differential equations, polynomial in that direction. These facts have been found hard to believe by sufficiently many experts to whom I quoted them, that it may be of some value (if only entertainment value), to give a brutally "honest", boring and painstaking proof in this paper, for small dimension. This gives me the excuse for advertising a different line of work on differential equations for theta functions (Section 1). Then, I propose to link this problem of lines and the other outstanding problem of algebro-geometric PDEs, which was the theme of my talk at the workshop reported in this volume: commuting partial differential operators (PDOs). There is a classification of (maximal-)commutative rings of ordinary differential operators, and their isospectral deformations are in fact the KP flows. In more than one variable

[1] I quote this nice catchphrase without attribution, this being the reaction to the assertions of Section 1 evinced by an expert whom I hadn't warned he would be "on record".

[2] Assume for simplicity that it is irreducible; let me also beg forgiveness if in this introduction I do not specify all possible degenerate cases which Barsotti must except in his statements, namely extensions of abelian varieties by a number of multiplicative or additive 1-dimensional groups.

very little is known, though several remarkable examples have appeared. The two theories that I will mention here were proposed by Sato (and implemented by Nakayashiki) and Parshin. Nakayashiki's work produced commuting matrix partial differential operators, but has the advantage of giving differential equations for theta functions. Since Barsotti's equations characterize theta functions, I believe that it would be profitable to identify Nakayashiki's equations, which were never worked out explicitly, among Barsotti's (Section 2). Parshin's construction produces (in principle, though recent work by his students shows that essential constraints must be introduced) deformations of scalar PDOs; in his setting, it is possible to generalize the Krichever map. It is a generalization of the Krichever map which constitutes the last link I would like to propose. Parshin sends a surface and a line bundle on it to a flag manifold; Arbarello and De Concini generalize the Krichever map and embed the general abelian variety and a line bundle on it into a projective space where Sato's Grassmannian is a submanifold, the image of Jacobians. My proposal is to characterize the image of the abelian varieties, in both Parshin's and Arbarello–De Concini's maps, by Barsotti's equations (Section 3). In conclusion, some concrete constructions are touched upon (Section 4). In a much too short Appendix, I reference Henry McKean's contribution on infinite-genus Riemann surfaces.

1. Incomplete democracy

Lines in Jacobians. Jacobians are special among principally polarized abelian varieties (ppav's), in that they contain a curve that generates the torus as a subgroup. For any choice of point on the curve, there is a specific line on the torus, which one expects to have special properties: indeed, the hyperosculating tangents to the embedding of the curve in the Jacobian given by that chosen point, give a sequence of flows satisfying the KP hierarchy. The KP equations provide an analytic proof that the tangent line (more precisely, its projection modulo the period lattice) cannot be contained in the theta divisor (no geometric proof has been given to date), while the order of vanishing of the theta function at the point (first given in connection to the KP equation as a sum of codimensions of a stratification of Sato's Grassmannian) was recently interpreted geometrically [Birkenhake and Vanhaecke 2003].

More geometrically yet, the Riemann approach links linear series on the curve to differential equations on the Jacobian, and again these lines play a very special role. I give two examples only. I choose these because both authors pose specific open problems (concerning indeed the special role of Jacobians among ppav's, known as "Schottky problem"), through the theory of special linear series. The subvarieties of such special linear series are acquiring increasing importance in providing exact solutions to Hamiltonian systems; see [Eilbeck et al. 2007] and references therein.

EXAMPLE. In one among his many contributions to these problems, Gunning [1986] produced in several, essentially different ways, differential equations satisfied by level-two theta functions. These are mainly limits, after J. Fay, of addition formulas, and this depends crucially on the tangent direction to the curve (at any variable point), *the* line. Gunning's focus is the study of the "Wirtinger varieties", roughly speaking, the images under the Kummer map of the W_k ($1 \le k \le g$), which in turn are images in the Jacobian of the k-fold symmetric products of the curve, via differential equations and thetanulls. For example, he proves the following (his notation for level-2 theta functions is ϑ_2):

If S is the subspace of dimension $\dim S = \binom{g+1}{2} + 1$ spanned by the vectors $\vartheta_2(0)$ and $\partial_{jk}\vartheta_2(0)$ for all (j,k), then the projectivization of this subspace contains the Kummer image of the surface $W_1 - W_1$, so it has intersection with the Kummer variety of dimension higher than expected, as soon as $g \ge 4$.

So little is known about these important subvarieties, that Welters [1986] states the following as an open problem: *Does there exist a relationship between* $\{a \in \mathrm{Pic}^0 X \mid a + W_d^r \subset W_d^{r-k}\}$ *and* $W_k^0 - W_k^0$ *(*$0 \le k \le r$*,* $0 \le d \le g-1$*)?* He had previously shown that

$$W_1^0 - W_1^0 = \{a \in \mathrm{Pic}^0 X \mid a + W_{g-1}^1 \subset W_{g-1}^0\},$$

where the notation W_d^r is the classical one for linear series of degree d and (projective) dimension at least r; g_d^r denotes a linear series of degree d and projective dimension r.

EXAMPLE. It is intriguing that Mumford, in his book devoted to applications of theta functions to integrable systems, states as an open problem [1984, Chapter IIIb, § 3]: If V is the vector space spanned by

$$\left\{\vartheta^2(z), \vartheta(z) \cdot \frac{\partial^2 \vartheta}{\partial z_i \partial z_j} - \frac{\partial \vartheta}{\partial z_i} \cdot \frac{\partial \vartheta}{\partial z_j}\right\}$$

and B is the set of "decomposition functions" $\vartheta(z-a)\cdot\vartheta(z+a)$, does the intersection of V and B equal the set $\{\vartheta(z - \int_p^q) \cdot \vartheta(z + \int_p^q)\}$, where p, q are any two points of the curve? As Mumford notes, this is equivalent to asking: If $a \in \mathrm{Jac}\, X$ is such that for all $w \in W_{g-1}^1$, either $w + a$ or $w - a$ is in W_{g-1}^0, does a belong to $W_1 - W_1$? The latter is settled by Welters (*loc. cit.*), showing that indeed, for $g \ge 4$ (for smaller genus the statement should be modified and still holds when it makes sense),

$$X - X = \cap_{\xi \in W_{g-1}^1} \big((W_{g-1}^0)_{-\xi} + (W_{g-1}^0)_{\xi - K_X}\big)$$

(as customary, subscript denotes inverse image under translation in the Picard group and K_X the canonical divisor), unless X is trigonal, for which it was known:

$$\bigcap_{\xi \in W^1_{g-1}} (W^0_{g-1})_{-\xi} + (W^0_{g-1})_{\xi - K_X} = (W^0_3 - g^1_3) \cup (g^1_3 - W^0_3).$$

On the enumerative side, Beauville [1982] shows that the sum of all the divisor classes in W^r_d is a multiple of the canonical divisor, provided r and d satisfy $g = (r+1)(g+r-d)$. The proof uses nontrivial properties of the Chow ring of the Jacobian, and it would be nice to find an interpretation in terms of theta functions.

A line of attack to these problems is suggested in [Jorgenson 1992a; 1992b], where theta functions defined on the W_k's are related to algebraic functions, generalizing the way that the Weierstrass points are defined in terms of ranks of matrices of holomorphic differentials. In a related way, techniques of expansion of the sigma function (associated to theta) along the curve, yield differential equations; see [Eilbeck et al. 2007].

Barsotti lines. However, on a general abelian variety, there should be "complete democracy", the catchphrase, in reaction to my report on Barsotti's result, that I am appropriating. Barsotti showed — in a way which is exquisitely algebraic (and almost, though not quite, valid for any characteristic of the field of coefficients), based on his theory of "hyperfields" for describing abelian varieties (developed in the fifties and only partly translated by his school into standard language), and independent of the periods — that *one* line suffices, to produce the differential field of the abelian variety. Barsotti's approach was aimed at a characterization of functions which he called "theta type", and this means generalized theta, pertaining to a product of tori as well as group extensions by a number of copies of the additive and multiplicative group of the field.

I will phrase this important result, along with a sketch of the proof, reintroducing the period lattice, though aware that Barsotti would disapprove of this naive approach, and I will give an "honest" proof in the (trigonal) case of genus 3, the last case when all (indecomposable) ppav's are Jacobians, yet the first case in which several experts reacted to Barsotti's result with "complete disbelief" (not in the sense of deeming Barsotti wrong, but rather, in intrigued astonishment that the democracy of lines should allow for such a property).

Barsotti is concerned with *abelian group varieties*, our abelian varieties, which he studies locally by rings of formal power series $k\{u_1, u_2, \ldots, u_n] = k\{u]$, which we will take to be the convergent power series in n indeterminates, $\mathbb{C}[\![u]\!]$, as usual abbreviating by u the n-tuple of variables. The context below will accommodate both cases, that u signify an n-tuple or a single variable. We follow

Barsotti's notation for derivatives: $d_i = \partial/\partial u_i$, and in case $r = (r_1, \dots, r_n)$ is a multi-index, $d_r = (r!)^{-1} d^r = (r_1!)^{-1} \cdots (r_n!)^{-1} d_1^{r_1} \cdots d_n^{r_n}$; also, $|r| := \sum_{i=1}^{n} r_i$, n-tuples of indices are ordered componentwise, and if different sets of indeterminates appear, d_{ur} will denote derivatives with respect to the u-variables. The symbol $\mathcal{Q}(-)$ generally associates to an integral domain its field of fractions. The notation is abbreviated: $k\{u\} := \mathcal{Q}(k\{u\})$.

THEOREM 1 [Barsotti 1983, Theorem 3.7]. *A function $\vartheta(u) \in k\{u\}$ is such that*

$$\vartheta(u+v)\vartheta(u-v) \in k\{u] \otimes k\{v] \tag{1}$$

if and only if it has the property

$$F(u, v, w) := \frac{\vartheta(u+v+w)\vartheta(u)\vartheta(v)\vartheta(w)}{\vartheta(u+v)\vartheta(u+w)\vartheta(v+w)} \in \mathcal{Q}(k\{u] \otimes k\{v] \otimes k\{w]). \tag{2}$$

Barsotti regarded this as the main result of [Barsotti 1983]. He had called (1) the prostapheresis formula[3] and (2) the condition for being "theta-type". His ultimate goal was to produce a theory of theta functions that could work over any field, and in doing so, he analyzed the fundamental role of the addition formulas; indeed, H. E. Rauch, in his review of [Barsotti 1970] (MR0302655 – Mathematical Reviews **46** #1799) exclaims, of the fact that (2) characterizes classical theta functions for $k = \mathbb{C}$, "This ... result is, to this reviewer, new and beautiful and crowns a conceptually and technically elegant paper". In order to appreciate the scope of (1) and (2), we have to put them to the use of computing dimensions of vector spaces spanned by their derivatives. To me (I may be missing something more profound, of course) the *segue* from properties of type (1) or (2) into dimensions of spaces of derivatives is this: u (the n-tuple) gives us local coordinates on the abelian variety; we understand analytic functions by computing coefficients of their Taylor expansions (derivatives) and the finite dimensionality corresponds to the fact that, while *a priori* the LHS belongs to $k\{u, v] := k\{u]\overline{\otimes} k\{v]$, which denotes the completion of the tensor product $k\{u] \otimes k\{v]$, only finitely many tensors suffice. The precise statement is this:

LEMMA [Barsotti 1983, 2.1]. *A function $\varphi(u, v)$ in $k\{u, v]$ belongs to*

$$k\{u] \otimes k\{v]$$

if and only if the vector space U spanned over k by the derivatives $d_{vr}\varphi(u, 0)$ has finite dimension. If such is the case, the vector space V spanned over k by the derivatives $d_{ur}\varphi(0, v)$ has the same dimension, and $\varphi(u, v) \in U \otimes V$.

[3] "We are indebted to the Arab mathematician Ibn Jounis for having proposed, in the XIth century a method, called prostapheresis, to replace the multiplication of two sines by a sum of the same functions", according to Papers on History of Science, by Xavier Lefort, Les Instituts de Recherche sur l'Enseignement des Mathématiques, Nantes.

To understand the theta-type functions as analytic functions, we also need to introduce certain numerical invariants.

DEFINITION. We denote by C_ϑ the smallest subfield of $k\{u\}$ containing k and such that $F(u,v,w) \in C_\vartheta\{v,w\}$. Note that C_ϑ is generated over k by the $d_r \log\vartheta$ for $|r| \geq 2$. This fact has already nontrivial content, in the classical case; the function field of an abelian variety is generated by the second and higher logarithmic derivatives of the Riemann theta function. The transcendence degree transcϑ is transc(C_ϑ/k) and the dimension $\dim\vartheta$ is the dimension (in the sense of algebraic varieties) of the smallest local subring of $k\{u]$ whose quotient field contains a theta-type function associated to (namely, as usual, differing from by a quadratic exponential) ϑ. I am giving a slightly inaccurate definition of dimension, for in his algebraic theory Barsotti had introduced more sophisticated objects than subrings; but I will limit myself, for the purposes of the results of this paper, to the case of "nondegenerate" thetas, which Barsotti defines as satisfying $\dim\vartheta = n$. The inequality transc $\vartheta \geq \dim\vartheta$ always holds and Barsotti calls ϑ a "theta function" when equality holds.

The next result is the root of all mystery. Here Barsotti demonstrates that in fact, the function field of the abelian variety could be generated by the derivatives of a theta function along fewer than m directions, m being the dimension of the abelian variety.

THEOREM 2 [Barsotti 1983, 2.4]. *For a nondegenerate theta-type*

$$\vartheta(u) \in k\{u_1, \cdots, u_n\},$$

there exist a nondegenerate theta $\theta(v) \in k\{v_1, \cdots, v_m\}$, $m \geq n$, and $c_{ij} \in k$, $1 \leq i \leq n$; $1 \leq j \leq m$, such that the matrix $[c_{ij}]$ has rank n, and $\vartheta(u) = \theta(x_1, \cdots, x_m)$ where $x_i = \sum_j c_{ij} u_j$. The induced homomorphism of $k\{v]$ onto $k\{u]$ induces an isomorphism between C_θ and C_ϑ. Conversely, given a compact abelian variety A of dimension m, for any $0 < n < m$ there is a holomorphic theta-type $\vartheta(u_1, \cdots u_n)$ such that C_ϑ is the function field of A, and is generated over k by a finite number of d_r $\log\vartheta$ *with* $|r| \geq 2$.

The example. Several experts have suggested (without producing details, as far as I know) that the statement may be believable in the case of a hyperelliptic Jacobian, but is already startling in the $g = 3$, nonhyperelliptic case, and this is the example I report. This is current work which I happen to be involved in for totally unrelated reasons; to summarize the motivation and goals in much too brief a manner, it is work concerned with addition formulae for a function associated to theta over a stratification of the theta divisor related to the abel image of the symmetric powers of the curve. Repeating the preliminaries would be quite lengthy and, more importantly, detract from the focus of this paper, so aside from indispensable notation I take the liberty of referring to [Eilbeck et al. 2007].

The key idea goes back to Klein and was developed by H. F. Baker over a long period (see especially [Baker 1907], where he collected and systematized this work). To generalize the theory of elliptic functions to higher-genus curves, these authors started with curves of special (planar) type, for which they expressed algebraically as many of the abelian objects as possible, differentials of first and second kind, Jacobi inversion formula, and ultimately, equations for the Kummer variety (in terms of theta-nulls) and linear flows on the Jacobian. In the process, they obtained or introduced important PDEs to characterize the abelian functions in question, and anecdotally, even produced, in the late 1800s, exact solutions to the KdV and KP hierarchy, without of course calling them by these names. I just need to quote certain PDEs satisfied by these "generalized abelian functions", but I will mention the methods by which these can be obtained. Firstly, the simplest function to work with, for reasons of local expansion at the origin, is called "sigma", it is associated to Riemann's theta function, and its normalized (almost-)period matrix satisfies generalized Legendre relations, being the matrix of periods of suitable bases of differentials of first and second kind. The definition of sigma is not explicit and considerable computer algebra is involved, genus-by-genus. The $g = 3$ case I need here is explicitly reported in [Eilbeck et al. 2007], but had been obtained earlier (by Ônishi, for instance).

In the suitable normalization, the "last" holomorphic differential ω_g always gives rise to the KP flow, namely the abelian vector (0,. . . ,0,1) in the coordinates $(u_1, \cdots, u_g) = \int_{g\infty}^{\sum_{i=1}^{g}(x_i,y_i)} \omega,\ \omega = (\omega_1, \cdots, \omega_g)$, simply because of the given orders of zero of the basis of differentials at the point ∞ of the curve, in the affine (x, y) plane, which is also chosen as the point of tangency of the KP flow to the abel image of the curve (indeed, in [Eilbeck et al. 2007] the Boussinesq equation is derived, as expected for the cyclic trigonal case). It is for this reason that I choose this direction for the Barsotti variable u.

Now the role of Barsotti's theta is played by $\sigma(u_1, u_2, u_3)$ — associated to a Riemann theta function with half-integer characteristics, and explicitly given in [Eilbeck et al. 2007, (3.8)] — and the role of the Weierstrass $\wp$-function, by the abelian functions $\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u)$; we label the higher derivatives the same way,

$$\wp_{ijk}(u) = \frac{\partial}{\partial u_k} \wp_{ij}(u),\ \wp_{ijk\ell}(u) = \frac{\partial}{\partial u_\ell} \wp_{ijk}(u),$$

(et cetera, but I only need the first four in my proof).

Barsotti's statement now amounts to this: the function $\wp_{33}(0, 0, u_3)$ together with all its derivatives in the u_3 variable, generate the function field of the Jacobian. Here's the boring proof! First, the work in [Eilbeck et al. 2007] (and a series of papers that preceded it): It is straightforward to expand σ in terms of a local parameter on the curve, for example,

$$u_1 = \frac{1}{5}{u_3}^5 + \cdots, \quad u_2 = \frac{1}{2}{u_3}^2 + \cdots$$

and

$$x(u_1, u_2, u_3) = \frac{1}{{u_3}^3} + \cdots, \quad y(u_1, u_2, u_3) = \frac{1}{{u_3}^4} + \cdots.$$

where $P \mapsto \int_\infty^P \omega := u(P)$, so $x(P)$ and $y(P)$ are viewed as functions of $u(P) = (u_1, u_2, u_3)$; the image of the curve implicitly defines any of the three coordinates as functions of one only. Next one expands σ as a function of (u_1, u_2, u_3), and with the aid of computer algebra, obtains PDEs for the abelian functions. For example, the identity

$$\wp_{3333} = 6\wp_{33}^2 - 3\wp_{22}$$

implies the Boussinesq equation for the function $\wp_{33}$, as expected. It is by using these differential equations, worked out in [Eilbeck et al. 2007] up to four indices (Appendix B), that I prove Barsotti's result. As a shorcut, I record a basis of the space 3Θ where Θ (this notation slightly differs from the one chosen in that reference) is the divisor of the σ function. If we can get this basis of abelian functions, we are sure to generate the function field of the Jacobian, since by the classical Lefschetz theorem the 3Θ-divisor map is an embedding. Lemma 8.1 in [Eilbeck et al. 2007] provides the following basis of 27 elements: $\{1, \wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}, Q_{1333}, \wp_{111}, \wp_{112}, \wp_{113}, \wp_{122}, \wp_{123}, \wp_{133}, \wp_{222}, \wp_{223}, \wp_{233}, \wp_{333}, \partial_1 Q_{1333}, \partial_2 Q_{1333}, \partial_3 Q_{1333}, \wp^{[11]}, \wp^{[12]}, \wp^{[13]}, \wp^{[22]}, \wp^{[23]}, \wp^{[33]}\}$, where

$$Q_{ijk\ell}(u) = \wp_{ijk\ell}(u) - 2(\wp_{ij}\wp_{k\ell} + \wp_{ik}\wp_{j\ell} + \wp_{i\ell}\wp_{jk})(u)$$

and $\wp^{[ij]}$ is the determinant of the complementary (i, j)-minor of $[\wp_{ij}]_{3\times 3}$. It is easy, by substituting in the equations given in [Eilbeck et al. 2007], to see that if we can obtain all the 2-index $\wp$ functions, then we can write the necessary 3, 4, and 5-index functions in the given basis. By definition of the Barsotti line, we have $\wp_{33}$, which gives us $\wp_{22}$ by the Boussinesq relation given above (we are allowed to take derivatives with respect to u_3). The one that seemed most difficult to obtain was $\wp_{23}$, and I argued as follows: Denote by F the differential field in the variable u_3 generated over $\mathbb{C}$ by $\wp_{33}$; as we saw it contains $\wp_{333}$ and $\wp_{22}$. Now F_{ij} denotes the field generated over F by adding $\wp_{ij}$. Eliminating $\wp_{13}$ from the two equations

$$\wp_{333}^2 = \wp_{23}^2 + 4\wp_{13} - 4\wp_{33}\wp_{22} + 4\wp_{33}^3$$

and

$$Q_{2233} = 4\wp_{13} + 3\lambda_3\wp_{23} + 2\lambda_2,$$

we see that F_{23} is an extension of degree at most 2 of F; then, either equation says that $\wp_{13}$ belongs to F_{23}. Now, we would like to say that F_{23} is also at most a cubic extension of F, and for that, use the equation

$$\wp_{223}\wp_{233} = 2\wp_{23}^3 + 2\wp_{22}\wp_{23}\wp_{33} \\ + 2\lambda_1 + 4\wp_{23}\wp_{13} + 2\wp_{23}\lambda_2 + 2\lambda_3\wp_{13} + 2\lambda_3\wp_{23}^2 + \lambda_3\wp_{22}\wp_{33}.$$

However, we can't quite control $\wp_{233}$, so we also bring in the equations

$$\wp_{333}\wp_{223} = 2\wp_{33}\wp_{23}^2 + \wp_{33}\lambda_3\wp_{23} - 2\wp_{22}^2 + \tfrac{2}{3}\wp_{1333} + 2\wp_{33}^2\wp_{22}$$

and

$$\wp_{233}^2 = 4\wp_{33}\wp_{23}^2 + 4\wp_{33}\lambda_3\wp_{23} + \wp_{22}^2 - \tfrac{4}{3}\wp_{1333} + 4\wp_{33}\lambda_2 + 8\wp_{33}\wp_{13}.$$

The first says that $\wp_{1333}$ is at most degree two (over F) in $\wp_{23}$; now using the cubic (and substituting for $\wp_{1333}$ in it), we see that $\wp_{23}$ satisfies an equation of degree 3 over $F[\wp_{233}]$, but from the second equation, $\wp_{233}$ is in an extension of degree at most 2 of F_{23}, so if F_{23} and $F[\wp_{233}]$ were disjoint, their join would have degree 4 and $\wp_{23}$ could not have degree 3 over $F[\wp_{233}]$. This shows that $\wp_{233}$ is in F_{23}, and now the cubic together with the quadratic equation yield $\wp_{23} \in F$. The proof that all other $\wp_{ij}$ are also in F is now much easier, again using several of the equations given in [Eilbeck et al. 2007]. If there is an easier proof, it beats me, for now at least.

REMARK. In a letter of reply to my querie (February 6, 1987), which I would translate, were it not for fear of misrepresenting as a conjecture what he only intended to offer as a possibility for my pursuing, Barsotti wrote that it might be that for generic $(c_1, \cdots, c_m)$, suitably high derivatives of $\log \vartheta(c_1 u, \cdots, c_m u)$ generate the function field of the abelian variety. In my view, this would not only restore democracy, but give a beautiful technique for stratifying the moduli space of abelian varieties according to the "special" parameters $c_1, \cdots, c_m$ whose line fails to generate, and which might correspond to tangent vectors to an abelian variety of smaller dimension (I claim all blame for this additional thought, but see Section 4 below). In the case of the "purely trigonal" curve above, we know that "elliptic solitons" can occur [Eilbeck et al. 2001], so does my proof say that even though σ is an elliptic function in the u_3 direction, still u_3 is a Barsotti direction? I don't think so; my proof requires obtaining $\wp_{23}$ from algebraic equations with coefficients in F, for example, but those coefficients depend on the λ_i's and there is no reason why for special values of λ_i's the equations shouldn't become trivial identities (while they patently can be solved for $\wp_{23}$ when the λ_i's are generic).

I can state with certainty at least that at the time of his tragic demise in 1987 Barsotti was very keen on pursuing these ideas [Scorza Dragoni 1988].

2. Barsotti's and Nakayashiki's equations

Barsotti equations include KP. Barsotti [1983; 1985] then proceeded to characterize abelian varieties. Again, I give a sketchy rendition of his results, which glosses over the technical issues of decomposable or degenerate abelian varieties. These are both important and subtle (for instance, the results have to be modified if you take for ϑ a polynomial!) but since this paper does not make substantial use of those exceptional cases, my goal is to give a geometric understanding of the generic situation. Calling "holomorphic" a theta-type function whose divisor divϑ is effective, Barsotti obtains:

THEOREM 3 [Barsotti 1983, 4.1]. *A nonzero function $\vartheta(u) \in \mathbb{C}[[u_1, \dots, u_n]]$ is holomorphic theta-type if and only if all differential polynomials*

$$H_{r,s}(\vartheta(u)) = \sum_{\substack{p+q=s \\ i+j=r}} (-1)^{i+p} d_i(\vartheta) d_p(\vartheta) d_j(\vartheta) d_q(\vartheta)$$

span a finite dimensional $\mathbb{C}$-vector space. In this case, if $\{U_0, \dots, U_h\}$ is a basis, the field $\mathbb{C}(\dots, \vartheta^{-3}U_i, \dots)$ is the same as the field of the abelian variety associated to ϑ, $H_{r,s}$ in turn are holomorphic theta-type and their divisors are linearly equivalent to 3divϑ.

Finally, by Taylor-series expansion, Barsotti writes a set of universal PDEs that characterize abelian varieties, and because of the "incomplete democracy" result, such PDEs can be produced for any positive number of variables less than or equal to the dimension of the abelian variety, in particular, one!

THEOREM 4 [Barsotti 1983, 5.5; Barsotti 1985, 12.2]. *For the universal differential polynomials with rational coefficients $P_{2k}(y_2, y_4, \dots, y_{2k})$ defined by*

$$\vartheta(u+v)\vartheta(u-v) = 2\vartheta^2(u) \sum_{r=0}^{\infty} P_{2r}(\vartheta(u)) v^{2r}$$

the same criterion as the above for $H_{r,s}$ holds. In particular, for the case of one variable ($n = 1$), the $P_0 = \frac{1}{2}, \dots, P_{2k}(y_2, y_4, \dots, y_{2k})$ are given by

$$P_{2r}(\vartheta) = \sum_j 2^{|j|-1} (j!)^{-1} \vartheta_2^{j_1} \vartheta_4^{j_2} \dots \vartheta_{2r}^{j_r},$$

where the sum is over the multi-indices $j \geq 0$ such that $j_1 + 2j_2 + \dots + r j_r = r$.

Barsotti [1985] wrote examples of a PDE version of his result, suggesting that it would be interesting to determine explicitly these PDEs from the ones in one variable and the vector field $\partial/\partial u$, and in [1989] he conjectures that the KP equation in his notation become

$$12P_{400}(\dots,\vartheta_i,\dots)-3P_{020}(\dots,\vartheta_i,\dots)-2P_{101}(\dots,\vartheta_i,\dots)=0.$$

Nakayashiki's generalized KP. A generalization of the KP equation as deformation of commutative rings of PDOs was long sought-after, and Nakayashiki [1991] did in fact produce such rings, in g variables for generic (thus, not Jacobians if $g \geq 4$) abelian varieties A of dimension g (as well as more general cases), as $(g! \times g!)$ matrix operators. He constructed modules over such rings that deform according to a generalized KP hierarchy, though he did not pursue explicit equations for bases of such modules, which have the form

$$N_{ct}(n)=\sum_{s\in\mathbb{Z}^g/n\mathbb{Z}^g}\mathbb{C}[\![t]\!]\frac{\vartheta\left[\begin{smallmatrix}s/n\\0\end{smallmatrix}\right](nz+c-(x'\cdot d-x_1,x'))}{\vartheta^n(z)}$$
$$\times\exp\Biggl(-\sum_{i=1}^{g}\sum_{m\geq\delta_{i1}}t_{m,(i)}\frac{(-1)^m}{m!}\left(u_{m,(i)}+d_i(1-\delta_{i1})u_{m+1,(1)}\right)\Biggr),$$

where we have set $x_1=t_{1,(1)}$, $x_i=t_{0,(i)}$ for $2\leq i\leq g$, and $d=(d_2,\dots,d_g)\in\mathbb{C}^{g-1}$ is such that at the point of the theta divisor we are considering (the elements of the module N_{ct} are in the stalk of a sheaf, defined via a cocycle by the vector $c\in\mathbb{C}^g$, the "initial condition" for the hierarchy) the g-tuple $(\zeta_1^{-1},\zeta_1^{-1}\zeta_i+d_i)_{i=2,\dots,g})$ gives local coordinates; moreover x' denotes the vector $(x_2,\dots,x_g)$ if $x=(x_1,\dots,x_g)$, while (i) denotes the $(g-1)$-tuple $(0,\dots,0,1,0,\dots)$ with a 1 in the $(i-1)$-st position, and $(1)=(0,\dots,0)$; finally, $u_{i_1,\dots,i_g}$ denotes $\partial_{z_1}^{i_1}\dots\partial_{z_g}^{i_g}\log\vartheta(z)$.

The differential equations are obtained as follows. Firstly, we denote by $\mathcal{P}$ the ring of microdifferential operators, defined by Sato [1989] via the choice of a codirection dx_1, which can be taken to correspond to an equation $x_1=0$ for the theta divisor

$$\mathcal{D}=\mathbb{C}[\![t_1,\dots,t_g]\!][\partial_1,\dots,\partial_g]\subset\mathcal{P}=\mathbb{C}[\![t]\!][\![\partial_1^{-1},\partial_1^{-1}\partial_2\dots,\partial_1^{-1}\partial_g]\!][\partial_1]$$

filtered by the order $\alpha_1+\dots+\alpha_g$ of $\partial^\alpha=\partial_1^{\alpha_1}\dots\partial_g^{\alpha_g}$.

Now N_{ct} can be embedded in $\mathcal{P}$ as a $\mathcal{D}$-submodule, $\varphi\in N_{ct}\mapsto\iota(\varphi)=W_\varphi$, in such a way that $W_{\partial\varphi/\partial x_i}=(\partial W_\varphi/\partial x_i)+W_\varphi\partial_i=\partial_i W_\varphi$ for $1\leq i\leq g$ and the $\mathcal{D}$-submodule of $\mathcal{P}$, $\mathcal{J}_{ct}(n)=\iota(N_{ct}(n+1))$, satisfies $\mathcal{P}^{(n)}=\mathcal{J}_{ct}(n)\oplus\mathcal{P}(J_{n,ct})$ where $J_{n,ct}$ is a suitable collection of indices from $\mathbb{Z}\times\mathbb{N}^{g-1}$, and

$$\mathcal{P}(J)=\left\{\sum a_\alpha\partial^\alpha \mid a_\alpha=0 \text{ unless } \alpha\in J\right\}.$$

Lastly, a set of $g!$ suitable $\mathcal{D}$-generators W_α of $\mathcal{J}_{ct}$, $\alpha \notin J_{n,ct}$ for all $n \geq 0$, can be chosen of the form ∂^α+[an operator whose terms have multiindices belonging to $J_{|\alpha|,ct}$] and these satisfy the evolution equations $(\partial W_\alpha/\partial t_\beta) + W_\alpha \partial^\beta \in \mathcal{J}_{ct} = \bigcup_{n=0}^{\infty} \mathcal{J}_{ct}(n)$, for β in the index set $(m+1, (i))$, with $(m, (i))$ defined above. In [Mironov 2002], it is claimed that the functions in

$$N_{ct} \cdot \exp\biggl(-\sum_{i=1}^{g} \sum_{m \geq \delta_{i1}} t_{m,(i)} \frac{(-1)^m}{m!} \bigl(u_{m,(i)} + d_i(1-\delta_{i1}) u_{m+1,(1)}\bigr)\biggr)$$

are independent of the time variables, but I think this is due to a small oversight, since the first g time variables do enter the argument of ϑ, as (x_1, x'), whereas, as correctly asserted in [Mironov 2002], the higher-time variables are stationary. The commutative ring of PDOs does not undergo a deformation beyond the g-dimensional variety $A^\vee$, which indeed is $\operatorname{Pic}^0 A$.

Nakayashiki does not claim that his equations characterize abelian varieties. Nevertheless, it should be possible to produce them from Barsotti's equations, which characterize theta functions, and it would be very interesting to see how Nakayashiki's formulas are given by constraints on Barsotti's universal polynomials (among these, what Barsotti calls "initial conditions" return the moduli of the each specific abelian variety; see §7 of [Barsotti 1983] for the example of elliptic curves).

3. Sato's Grassmannian, Parshin's flag manifold, Arbarello–De Concini's projective space

Grassmannian for a chosen splitting. Nakayashiki's theory was inspired by Sato's programme [1989], a specific splitting $\mathcal{P} = \mathcal{J} \oplus \mathcal{E}_0$ into $\mathcal{D}$-modules. In one variable, there is a natural splitting and the corresponding $\mathcal{J}$ are exactly the cyclic submodules; the deformations are linear flows on the universal Grassmann manifold modeled on the vector space $\mathcal{P}_{\text{const}} : \partial^\alpha \leftrightarrow \partial^\alpha/(\mathcal{P}t_1 + \cdots + \mathcal{P}t_g)$. What is the correct model in several variables? To my knowledge there is no definitive answer known; I provide two different models below, based on Parshin's, respectively, Arbarello–De Concini's constructions, and the project of computing Nakayashiki's flows in both, which should be both doable (in dimension 2) and enlightening.

Parshin's Krichever flag manifold. Parshin [1999] proposed a different construction, based on the theory of higher local fields, in which the commuting partial differential operators are scalar. An n-dimensional local field K (with "last" residue field $\mathbb{C}$) is the field of iterated Laurent series $K = \mathbb{C}((x_1))\dots((x_n))$, with the structure of a complete discrete valuation ring $\mathcal{O} = \mathbb{C}((x_1))\dots((x_{n-1}))[[x_n]]$ having an $(n-1)$-dimensional local field for its residue field. Note that the order of the variables matters, in the sense that $\mathbb{C}((x_1))((x_2))$ does not contain the same elements as $\mathbb{C}((x_2))((x_1))$ — for instance, the former contains elements of unbounded positive degree in x_1 — although they are isomorphic. These are suited to give local coordinates on an n-dimensional manifold, since the inverse of a polynomial in x_1, x_2, say, can be written as the inverse of the highest-order monomial times something entire, so as a Laurent series it is bounded in both variables. Whereas the symbols $\mathbb{C}((x_1, x_2)) = \{\sum_{|i+j|<N} c_{ij} x_1^i x_2^j\}$ cannot be given a ring structure unless we want to define sums of infinitely many complex numbers, because $i + j = N$ involves infinitely many indices unless we bound j (or i) from above. With this definition, Parshin constructs a $2n$-dimensional skew-field $\mathcal{P}$, infinite-dimensional over its center, namely the (formal) pseudodifferential operators

$$\mathcal{P} = \mathbb{C}((x_1)) \dots ((x_n))((\partial_1^{-1})) \dots ((\partial_n^{-1})).$$

The order of the variables is also singled out in the definition of the grading: If $L = \sum_{i \le m} a_i \partial_n^i$ with $a_m \ne 0$, we say that the operator L has order m and write $\operatorname{ord} L = m$. If $P_i = \{L \in \mathcal{P} \mid \operatorname{ord} L \le i\}$, then $\cdots P_{-1} \subset P_0 \subset \cdots$ is a decreasing filtration of $\mathcal{P}$ by subspaces and $\mathcal{P} = P_+ \oplus P_-$, where $P_- = P_{-1}$ and P_+ consists of operators involving only nonnegative powers of ∂_n. The highest term (h.t.) of an operator L is defined by induction on n. If $L = \sum_{i \le m} a_i \partial_n^i$ and $\operatorname{ord} L = m$, then $\text{h.t.}(L) = \text{h.t.}(a_m) \cdot \partial^m$. If $\text{h.t.}(L) = f \partial_1^{m_1} \dots \partial_n^{m_n}$ with $0 \ne f \in \mathbb{C}((x_1)) \dots ((x_n))$, then we let $\nu(L) = (m_1, \dots, m_n)$. We consider also the subring $E = \mathbb{C}[[x_1, \dots, x_n]]((\partial_1^{-1})) \dots ((\partial_n^{-1}))$ of $\mathcal{P}$, and $E_\pm = E \cap \mathcal{P}_\pm$.

In this setting, Parshin's original proposal for a KP hierarchy — which is currently being modified by his former student Dr. A. Zheglov [Zheglov 2005] — makes good on his striking conjugation result, based on [Krichever 1977; Sato 1989] (I omit some technical specifications, for which see [Parshin 1999]):

PROPOSITION. (i) *An operator $L \in E$ is invertible in E if and only if the coefficient f in the highest-order term of L is invertible in the ring $\mathbb{C}[[x_1, \dots, x_n]]$. If f in $L \in \mathcal{P}$ is an m-th power in $\mathbb{C}((x_1)) \dots ((x_n))$ (resp., $\mathbb{C}[[x_1, \dots, x_n]]$ for $L \in E$) then there exists, unique up to multiplication by m-th root of unity, an operator $M \in \mathcal{P}$ (resp. $M \in E$) such that $M^m = L$. Thus, P_0 is a discrete valuation ring in $\mathcal{P}$ with residue field $\mathbb{C}((x_1)) \dots ((x_n))((\partial_1^{-1})) \dots ((\partial_{n-1}^{-1}))$.*

(ii) *Let $L_1 \in \partial_1 + E_-, \dots, L_n \in \partial_n + E_-$. Then $[L_i, L_j] = 0$ for all i, j if and only if there exists an operator $S \in 1 + E_-$ such that $L_i = S^{-1} \partial_i S$, for all i.*

(iii) *For $L = (L_1, \dots, L_n)$ as in* (ii), *the flows*

$$\frac{\partial L}{\partial t_M} = \big([(L_1^{m_1} \cdots L_n^{m_n})_+, L_1] \dots [(L_1^{m_1} \cdots L_n^{m_n})_+, L_n]\big),$$
$$M = (m_1, \dots, m_n) \in \mathbb{Z}_{\ge 0} \times \dots \times \mathbb{Z}_{\ge 0}$$

commute, and if $S \in 1 + \mathcal{P}_-$ *satisfies*

$$\frac{\partial S}{\partial t_M} = -(S\partial_1^{m_1} \cdots \partial_n^{m_n} S^{-1})_- S,$$

then $L = (S\partial_1 S^{-1}, \ldots, S\partial_n S^{-1})$ *evolves according to them.*

REMARK. Barsotti's field ($k\{u\} := \mathcal{Q}(k\{u])$ is larger than Sato's ring of pseudodifferential operators. Parshin's E is much larger than Sato's ring

$$\mathbb{C}[\![x_1, \ldots, x_n]\!][\![\partial_1^{-1}, \partial_1^{-1}\partial_2, \ldots, \partial_1^{-1}\partial_g]\!]$$

when $n > 1$. Finally, Barsotti's and Parshin's rings are different, though both adapted to a local description of a (say, if $g = 2$) surface. Parshin's ring is smaller and it is not symmetric in x, y, for instance, while Barsotti's $k\{x, y]$ is.

Parshin [2001a; 2001b] generalizes the Krichever map, which associates to a local parameter on a curve and other geometric data a point of an infinite-dimensional Grassmannian (via the Baker–Akhiezer function), and to two local parameters, roughly speaking the choice of a curve on a surface and a point on that curve, and geometric data (a sheaf on the surface), associates a point of an infinite-dimensional 2-step flag manifold. This would be an appropriate setting for producing Nakayashiki's (2×2) matrix operators, via a choice of one basis element in a subspace and one in the quotient. This approach has not been taken, but the operators are explicit enough for genus 2 that the plan is concrete. At the same time, the brothers Aloysius and Gerard Helminck [1994a; 1994b; 1995; 2002] put a Fubini–Study metric on the infinite-dimensional projective space of flags, computed the central extension of the restricted linear group that acts on the manifold, and adapted the resulting (Kähler) manifold to flows of completely integrable systems, which include well-known ones. This is a natural setting for linearizing Nakayashiki's and Parshin's generalizations of the KP hierarchy. Sato's result, to the effect that Hirota's bilinear equation is equivalent to the Plücker relations which characterize the image of the Grassmannian in its Plücker embedding, should then be extended to the image of the Parshin flags.

Arbarello–De Concini's Plücker embedding. A different Grassmannian construction for abelian varieties is devised by Arbarello and De Concini [Arbarello and De Concini 1991]. They model a moduli space of abelian varieties on a Grassmannian, making use of one local parameter only, reminiscent of Sato's codirection, though they do not assume that its dual is tangent to the theta divisor, as Sato and Parshin do. They succeed, using classical theta-function theory, in producing enough data to embed the moduli space $\tilde{\mathcal{H}}_g$ (very roughly, a universal family of abelian varieties of dimension g, $\tilde{\mathcal{A}}_g$ extended by a Heisenberg action) in $\mathbb{P}B$, where B is the usual Boson space $\mathbb{C}[\![t_1, \ldots, t_k, \ldots]\!]$; they also give a theta-function formula for the τ function. The significant advantage of this construction is that they can compare this embedding with that of Jacobians via the usual Krichever map and they prove that the diagram

$$\begin{array}{ccccc} \tilde{\mathcal{A}}_g & \leftarrow & \tilde{\mathcal{H}}_g & \longrightarrow & \mathbb{P}B \\ \uparrow & & \uparrow & \nearrow & \uparrow \\ \tilde{\mathcal{M}}_g & \leftarrow & \tilde{\mathcal{F}}_g & \hookrightarrow \mathrm{Gr}H \hookrightarrow & \mathbb{P}F \end{array}$$

is commutative, where $\tilde{\mathcal{M}}_g$ is, again roughly speaking, a moduli space of genus-g curves, and $\tilde{\mathcal{F}}_g$ is fibred over $\tilde{\mathcal{M}}_g$ by the $\mathrm{Pic}^{(g-1)}$'s of the curves, $H = \mathbb{C}((z))$ is the space of formal Laurent series, $F = \Gamma(\mathrm{Gr}H, \det^{-1})^*$.

In their moduli spaces, Arbarello and De Concini use one complex variable z, which suggests that Barsotti's line may provide the embedding equations. This is also the principle of the (formal) work we carried out in [Lee and Previato 2006]. Thanks to Parshin's conjugation result, one "Sato operator" S suffices. One then can write, by the usual boson-fermion correspondence, a Baker function, as done by M. H. Lee and also in [Plaza-Martín 2000]; the function comprises $(z_1, \dots, z_n)$ but essentially records points of a Grassmannian where the variable z_1 plays a distinguished role (as in Parshin's grading), and one can write a formal inversion of the logarithm and a formal τ function in the following way. (I omit as usual technical provisos; see [Lee and Previato 2006] for those.)

In analogy to the Segal–Wilson construction for the one-variable case, we let $H = L^2(T^n)$ be the Hilbert space consisting of all square-integrable functions on the n-torus

$$T^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1\},$$

which can be identified with the product of n copies of the unit circle $S^1 \subset \mathbb{C}^n$. Then the Hilbert space H can be written in the form

$$H = \langle z^\alpha | \alpha \in \mathbb{Z}^n \rangle_{\mathbb{C}}.$$

The multi-index notation is defined as follows: $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ if $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. If $\beta = (\beta_1, \dots, \beta_n)$ is another element of $\mathbb{Z}^n$, we write $\alpha \le \beta$ when $\alpha_i \le \beta_i$ for each $i \in \{1, \dots, n\}$. We define a splitting $H = H_+ \oplus H_-$ adapted to Parshin's filtration [1999; 2001b; 2001a] and the Krichever map. Then, as in the one-variable case, there is a one-to-one correspondence between certain subspaces of H commensurable to $H_+ := \mathbb{C}[[z_1, \dots, z_n]]$ and wave functions, given by $\psi \mapsto W$, where a spanning set for W is given by all derivatives $\partial_1^{j_1} \dots \partial_n^{j_n} \psi$, evaluated at $z = \mathbf{0}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^n$. We take this to be the Grassmannian $\mathrm{Gr}(H)$. We denote by $p_+ : H \to H_+$ and $p_- : H \to H_-$ the natural projection maps. A subspace W of H is said to be transversal to H_- if the restriction $p_+ |_W \colon W \to H_+$ of

p_+ to W is an isomorphism. For a holomorphic function $g : D^n \to \mathbb{C}$ defined on the closed polydisk

$$D^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| \le 1, \dots, |z_n| \le 1\}$$

with $g(\mathbf{0}) = 1$, $g(z) = g(z_1, \dots, z_n)$ can be written in the form

$$g(z) = \exp\biggl(\sum_{\alpha \in \mathbb{Z}^n_+} t_\alpha z^\alpha\biggr)$$

with $t_\alpha \in \mathbb{C}$ for all $\alpha \in \mathbb{Z}^n_+ := \{\alpha \in \mathbb{Z}^n | \alpha \ge \mathbf{0},\ \alpha \ne \mathbf{0}\}$. We define the maps $\mu_g, \mu_{g^{-1}} : H \to H$ by

$$(\mu_g f)(z) = g(z) f(z), \quad (\mu_{g^{-1}} f)(z) = g(z)^{-1} f(z)$$

for all $f \in W$ and $z \in \mathbb{C}^n$. Since $\mu_{g^{-1}}(H_+) \subset H_+$, with respect to the decomposition of H, the map $\mu_{g^{-1}}$ can be represented by a block matrix of the form

$$\mu_{g^{-1}} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

whose entries are the maps $a : H_+ \to H_+$, $b : H_- \to H_+$, $c : H_- \to H_-$. Given $W \in \mathrm{Gr}(H)$, we set

$$\Gamma^W_+ = \{g \in \Gamma_+ \mid \mu_{g^{-1}} W \text{ is transversal to } H_-\}.$$

Thus g belongs to Γ^W_+ if and only if the map $p_+|_{\mu_{g^{-1}} W} : \mu_{g^{-1}} W \to H_+$ is an isomorphism.

Let $\mathcal{S}$ be the complex vector space of formal Laurent series in $z_1^{-1}, \dots, z_n^{-1}$ consisting of series of the form

$$v = \sum_{\alpha \le \nu} f_\alpha(t) z^\alpha$$

for some $\nu \in \mathbb{Z}^n$ with $t = (t_\alpha)_{\alpha \in \mathbb{Z}^n_+}$. We consider the subspace $\mathcal{S}_-$ of $\mathcal{S}$ consisting of the series which can be written as

$$v = \sum_{k=-\infty}^{k_0} f_k(t; z_1, \dots, z_{n-1}) z_n^k$$

for some $k_0 \in \mathbb{Z}$ with $k_0 \le -1$, so that there is a decomposition of the form $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_+$ consists of the series of the form

$$\sum_{k=0}^{\ell_0} f_k(t; z_1, \dots, z_{n-1}) z_n^k$$

for some nonzero integer ℓ_0. Given an element W of the Grassmannian $\mathrm{Gr}(H)$, the associated Baker function $w_W(g,z)$ is the function defined for $g \in \Gamma_+^W$ and $z \in T^n$ satisfying the conditions

$$w_W(g,z) \in W, \quad \mu_{g^{-1}} w_W(g,z) = 1 + u$$

with $u \in \mathcal{S}_-$. Since each element $g \in \Gamma_+^W$ can be written in the exponential form, the Baker function $w_W(g,z)$ may be regarded as a function for $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ and $z \in T^n$. Thus we may write $w_W(g,z) = w_W(t,z)$, $t = (t_\alpha)_{\alpha \in \mathbb{Z}_+^n}$.

Let $W \in \mathrm{Gr}(H)$ be transversal to H_-, so that the map $p_+|_W : W \to H_+$ is an isomorphism, and let g be an element of Γ_+^W. We consider the sequence

$$H_+ \xrightarrow{(p_+|_W)^{-1}} W \xrightarrow{\mu_{g^{-1}}} \mu_{g^{-1}} W \xrightarrow{p_+} H_+ \xrightarrow{\mu_g} H_+$$

of complex linear maps. Given $g \in \Gamma_+^W$ and an element $W \in \mathrm{Gr}(H)$ transversal to H_-, the associated τ-function $\tau_W(g) = \tau_W(t) = \tau_W((t_\alpha)_{\alpha \in \mathbb{Z}_+^n})$ is the function

$$\tau_W(g) = \det\bigl(\mu_g \circ p_+ \circ \mu_{g^{-1}} \circ (p_+|_W)^{-1}\bigr)$$

given by the determinant of the composite of the linear maps above. Let $\Lambda : H_+ \to H_-$ be the linear map given by $\Lambda = p_- \circ (p_+|_W)^{-1}$. Then the τ-function can be written in the form

$$\tau_W(g) = \det(1 + a^{-1} b \Lambda),$$

where a and b are as above and 1 denotes the identity map on H_+. We define the rational numbers ε_α for $\alpha \in \mathbb{Z}_+^n$ by requiring

$$\sum_{\alpha \in \mathbb{Z}_+^n} \varepsilon_\alpha x^\alpha = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \biggl(\sum_{\beta \in \mathbb{Z}_+^n} x^\beta \biggr)^k,$$

where $x = (x_1, \dots, x_n)$ is a multivariable.

THEOREM 5 [Lee and Previato 2006]. *Let $W \in \mathrm{Gr}(H)$ be transversal, and let $g : D^n \to \mathbb{C}$ be an exponential. Then the associated τ-function*

$$\tau_W(g) = \tau_W((t_\alpha)_{\alpha \in \mathbb{Z}_+^n})$$

satisfies

$$\mu_{g^{-1}} w_W(g,z) = \frac{\tau_W\bigl((t_\alpha + \varepsilon_\alpha z^{-\alpha})_{\alpha \in \mathbb{Z}_+^n}\bigr)}{\tau_W((t_\alpha)_{\alpha \in \mathbb{Z}_+^n})},$$

where $w_W(g,z)$ is the Baker function.

The next step would be to write this formula in terms of theta functions after Arbarello–De Concini.

4. Reducible cases

A project which I believe less trivial than it seems, occurs in the case of reducible abelian varieties. For example, the Schrödinger operator

$$\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}-\wp(x)\wp(y)$$

has commutator which must be isomorphic to the ring $\mathcal{C}(\partial/\partial x^2-\wp(x))\otimes\mathcal{C}(\partial/\partial y^2-\wp(y))$, whose associated "spectral" variety is $E_0\times E_0$, where E_0 is the spectral variety of $\mathcal{C}(\partial/\partial x^2-\wp(x))$.

While this case can be regarded as trivial, by analogy, the elliptic (or rational) curve case of the Hitchin system for vector bundles, in which the moduli of vector bundles are simply a product of copies of the curve, is still the only one in which the solutions can be given explicitly ([Nekrasov 1996] is just one earliest reference). In the reducible-potential case, Parshin's flows for L_1, L_2 and Nakayashiki's equations can be written explicitly, but they have not yet been compared; in this case there is a KP hierarchy.

The differential resultant for this case, in which the variety is known, can serve as a toy model for a truly generalized theory. (The model used in [Kasman and Previato 2001], by analogy with the algebraic definition of resultant of polynomial equations in several variables given by Macaulay [1916], falls short because, due to the additional variables at infinity, it is often identically zero.) It can also serve for testing the conjecture made in [Kasman and Previato 2001] that the resultant is independent of the operator variables, up to a factor whose numerical nature (degrees in the variables, e.g., for the case of the Weyl algebra), should be the same in general as in the reducible case. Moreover this reducible case provides a nonexample for Barsotti's theorem.[4] Indeed, if an abelian surface is isogenous to the product $E_1\times E_2$ of two elliptic curves, as is the case for the Jacobian of a genus-2 curve that covers an elliptic curve, then we can take u to be the direction that projects to one of the tori; the derivatives in the u direction will only produce the elliptic functions in one variable; an explicit calculation is known classically and was reproduced by J. C. Eilbeck (unpublished notes) to input the parametrization of all the genus-2 elliptic covers whose Jacobian is isomorphic (without principal polarization) to the product of two elliptic curves. I briefly provide some motivation and the formula (which does not do sufficient justice to Eilbeck's considerable work in implementing two reduction algorithms, on Siegel matrices and Fourier expansions, given in theory by H. H. Martens and J.-I. Igusa). The motivation was a recent result

[4] "Quite often in mathematics, a "nonexample" is as helpful in understanding a concept as an example" — J. A. Gallian, *Contemporary abstract algebra*, Chapter 4.

of C. Earle [2006], who described all the 2×2 matrices in the Siegel upper-half space that correspond to genus-2 curves whose Jacobian is isomorphic to a product of elliptic curves. Note that the expected parameter count should be one and not two (Jacobians that split up to isogeny) since Martens had shown that in the isomorphic case the two elliptic curves must be isomorphic. I asked Eilbeck whether we could do the effectivization of the KdV solutions for all these matrices. The "Earle matrix"[5]

$$Z = \tau \begin{bmatrix} na & nb \\ nb & d \end{bmatrix}$$

with τ in the upper-half plane, and a, b, d, n positive integers such that $ad - nb^2 = 1$, nonsymplectically equivalent to a diagonal:

$$(I, z) = (I, \tau V) \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix},$$

where $V := \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}$, $T := \tau \begin{bmatrix} a & b \\ nb & d \end{bmatrix}$ can be symplectically transformed into

$$\begin{bmatrix} -\dfrac{1}{\tau na} & -\dfrac{b}{a} \\ -\dfrac{b}{a} & 1 + \dfrac{\tau}{a} \end{bmatrix}.$$

Eilbeck implemented (by creating Maple routines) a special case, the 2-dimensional abelian variety being (2:1)-isogenous to $E_1 \times E_2$, the two elliptic curves E_i having invariants τ_i, and decomposed the theta function Θ (with characteristics) of A thus: For the matrix

$$\tilde{\tau} = \begin{bmatrix} \frac{1}{2}\tau_1 & \frac{1}{2} \\ \frac{1}{2} & -\dfrac{1}{2(2+\tau_2)} \end{bmatrix},$$

we have

$$\Theta \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \left(\begin{bmatrix} -\frac{1}{2} v_1 \\ -\frac{v_1 - 2v_2}{2(2+\tau_2)} \end{bmatrix}, \tilde{\tau} \right) = \Theta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \left(-\frac{v_1}{2}, \frac{\tau_1}{2} \right) \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\frac{v_1 - 2v_2}{2+\tau_2}, -\frac{2}{2+\tau_2} \right) + \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(-\frac{v_1}{2}, \frac{\tau_1}{2} \right) \Theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \left(\frac{v_1 - 2v_2}{2+\tau_2}, -\frac{2}{2+\tau_2} \right).$$

[5] Not all matrices of this form are period matrices of genus-2 curves; Earle gives a criterion. Also, not all the matrices of this form that are period matrices correspond to different curves, as they may come from the same curve via a different choice of homology basis; in two further theorems Earle gives criteria to tell curves apart.

This shows that higher derivatives of Θ in the directions τ_i are elliptic functions in τ_i, thus cannot generate the function field of A.

Appendix

I cannot help mentioning the theory of infinite-genus Riemann surfaces that Henry McKean, originally in collaboration with Eugene Trubowitz, developed for KdV spectral varieties attached to a periodic potential that is not "finite-gap". In a rigorous analytic way, this extends the theory of the Jacobian, and the theta function.[6] There is still scope for a theory of reduction, elliptic solitons, and differential operators with elliptic coefficients; there are many more "Variations on a Theme of Jacobi", in other words, awaiting for Henry's face-altering contributions to the field: one more reason to say, Henry, many, many happy returns!

Acknowledgements

As always, it is a privilege to participate in an MSRI workshop. The invitation extended to me by the organizers, Björn Birnir, Darryl Holm, Charles Newman, Mark Pinsky, Kirill Vaninsky, and Lai-Sang Young, was both humbling and profoundly rewarding. I am also most grateful to J. C. Eilbeck for his permission to reproduce the formula of Section 4.

References

[Arbarello and De Concini 1991] E. Arbarello and C. De Concini, "Abelian varieties, infinite-dimensional Lie algebras, and the heat equation", pp. 1–31 in *Complex geometry and Lie theory* (Sundance, UT, 1989), edited by J. A. Carlson et al., Proc. Sympos. Pure Math. **53**, Amer. Math. Soc., Providence, RI, 1991.

[Baker 1907] H. F. Baker, *An introduction to the theory of multiply-periodic functions*, Cambridge Univ. Press, Cambridge, 1907.

[Barsotti 1970] I. Barsotti, "Considerazioni sulle funzioni theta", pp. 247–277 in *Symposia Mathematica* (Rome, 1968/69), vol. 3, Academic Press, London, 1970.

[Barsotti 1983] I. Barsotti, "Differential equations of theta functions", *Rend. Accad. Naz. Sci. XL Mem. Mat.* (5) **7** (1983), 227–276.

[Barsotti 1985] I. Barsotti, "Differential equations of theta functions. II", *Rend. Accad. Naz. Sci. XL Mem. Mat.* (5) **9**:1 (1985), 215–236.

[Barsotti 1989] I. Barsotti, "A new look for thetas", pp. 649–662 in *Theta functions* (Brunswick, ME, 1987), vol. I, edited by L. Ehrenpreis and R. C. Gunning, Proc. Sympos. Pure Math. **49**, Amer. Math. Soc., Providence, RI, 1989.

[6] See [Ercolani and McKean 1990; Feldman et al. 1996; 2003; McKean 1980; 1989; McKean and Trubowitz 1976; 1978; Schmidt 1996.]

[Beauville 1982] A. Beauville, “Diviseurs spéciaux et intersection de cycles dans la jacobienne d’une courbe algébrique”, pp. 133–142 in *Enumerative geometry and classical algebraic geometry* (Nice, 1981), edited by P. L. Barz and Y. Hervier, Progr. Math. **24**, Birkhäuser, Mass., 1982.

[Birkenhake and Vanhaecke 2003] C. Birkenhake and P. Vanhaecke, “The vanishing of the theta function in the KP direction: a geometric approach”, *Compositio Math.* **135**:3 (2003), 323–330.

[Earle 2006] C. J. Earle, “The genus two Jacobians that are isomorphic to a product of elliptic curves”, pp. 27–36 in *The geometry of Riemann surfaces and abelian varieties*, Contemp. Math. **397**, Amer. Math. Soc., Providence, RI, 2006.

[Eilbeck et al. 2001] J. C. Eilbeck, V. Z. Enolskii, and E. Previato, “Varieties of elliptic solitons”, *J. Phys. A* **34**:11 (2001), 2215–2227.

[Eilbeck et al. 2007] J. C. Eilbeck, V. Z. Enolski, S. Matsutani, Y. Ônishi, and E. Previato, “Abelian functions for trigonal curves of genus three”, *Int. Math. Res. Not.* (2007). Available at arxiv.org/abs/math.AG/0610019.

[Ercolani and McKean 1990] N. Ercolani and H. P. McKean, “Geometry of KdV, IV: Abel sums, Jacobi variety, and theta function in the scattering case”, *Invent. Math.* **99**:3 (1990), 483–544.

[Feldman et al. 1996] J. Feldman, H. Knörrer, and E. Trubowitz, “Infinite genus Riemann surfaces”, pp. 91–111 in *Canadian Mathematical Society. 1945–1995*, vol. 3, Canadian Math. Soc., Ottawa, ON, 1996.

[Feldman et al. 2003] J. Feldman, H. Knörrer, and E. Trubowitz, *Riemann surfaces of infinite genus*, CRM Monograph Series **20**, American Mathematical Society, Providence, RI, 2003.

[Gunning 1986] R. C. Gunning, “Some identities for abelian integrals”, *Amer. J. Math.* **108**:1 (1986), 39–74.

[Helminck and Helminck 1994a] A. G. Helminck and G. F. Helminck, “Holomorphic line bundles over Hilbert flag varieties”, pp. 349–375 in *Algebraic groups and their generalizations: quantum and infinite-dimensional methods* (University Park, PA, 1991), vol. 2, edited by W. J. Haboush and B. Parshall, Proc. Sympos. Pure Math. **56**, Amer. Math. Soc., Providence, RI, 1994.

[Helminck and Helminck 1994b] G. F. Helminck and A. G. Helminck, “The structure of Hilbert flag varieties”, *Publ. Res. Inst. Math. Sci.* **30**:3 (1994), 401–441.

[Helminck and Helminck 1995] G. F. Helminck and A. G. Helminck, “Infinite-dimensional flag manifolds in integrable systems”, *Acta Appl. Math.* **41**:1-3 (1995), 99–121.

[Helminck and Helminck 2002] G. F. Helminck and A. G. Helminck, “Hilbert flag varieties and their Kähler structure”, *J. Phys. A* **35**:40 (2002), 8531–8550.

[Jorgenson 1992a] J. Jorgenson, “On directional derivatives of the theta function along its divisor”, *Israel J. Math.* **77**:3 (1992), 273–284.

[Jorgenson 1992b] J. Jorgenson, “Some uses of analytic torsion in the study of Weierstrass points”, pp. 255–281 in *Curves, jacobians, and abelian varieties* (Amherst, MA, 1990), edited by R. Donagi, Contemp. Math. **136**, Amer. Math. Soc., Providence, RI, 1992.

[Kasman and Previato 2001] A. Kasman and E. Previato, “Commutative partial differential operators”, *Phys. D* **152/153** (2001), 66–77.

[Krichever 1977] I. M. Krichever, “The methods of algebraic geometry in the theory of nonlinear equations”, *Uspekhi Mat. Nauk* **32**:6 (1977), 183–208.

[Lee and Previato 2006] M. H. Lee and E. Previato, “Grassmannians of higher local fields and multivariable tau functions”, pp. 311–319 in *The ubiquitous heat kernel* (Boulder, CO, 2003), edited by J. Jorgenson, Contemp. Math. **398**, Amer. Math. Soc., Providence, RI, 2006.

[Macaulay 1916] F. S. Macaulay, *The algebraic theory of modular systems*, Cambridge Univ. Press, Cambridge, 1916. Revised reprint, 1994.

[Matsutani and Previato 2008] S. Matsutani and E. Previato, “A generalized Kiepert formula for C_{ab} curves”, *Israel J. Math.* (2008).

[McKean 1979] H. P. McKean, “Integrable systems and algebraic curves”, pp. 83–200 in *Global analysis* (Calgary, AB, 1978), edited by M. Grmela and J. E. Marsden, Lecture Notes in Math. **755**, Springer, Berlin, 1979.

[McKean 1980] H. P. McKean, “Algebraic curves of infinite genus arising in the theory of nonlinear waves”, pp. 777–783 in *Proceedings of the International Congress of Mathematicians* (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980.

[McKean 1989] H. P. McKean, “Is there an infinite-dimensional algebraic geometry? Hints from KdV”, pp. 27–37 in *Theta functions* (Brunswick, ME, 1987), edited by L. Ehrenpreis and R. C. Gunning, Proc. Sympos. Pure Math. **49**, Amer. Math. Soc., Providence, RI, 1989.

[McKean and Trubowitz 1976] H. P. McKean and E. Trubowitz, “Hill’s operator and hyperelliptic function theory in the presence of infinitely many branch points”, *Comm. Pure Appl. Math.* **29**:2 (1976), 143–226.

[McKean and Trubowitz 1978] H. P. McKean and E. Trubowitz, “Hill’s surfaces and their theta functions”, *Bull. Amer. Math. Soc.* **84**:6 (1978), 1042–1085.

[Mironov 2002] A. E. Mironov, “Real commuting differential operators associated with two-dimensional abelian varieties”, *Sibirsk. Mat. Zh.* **43**:1 (2002), 126–143. In Russian; translated in *Siberian Math. J.* **43**:1, 97–113.

[Mumford 1984] D. Mumford, *Tata lectures on theta, II*, Progress in Mathematics **43**, Birkhäuser, Boston, 1984.

[Nakayashiki 1991] A. Nakayashiki, “Structure of Baker-Akhiezer modules of principally polarized abelian varieties, commuting partial differential operators and associated integrable systems”, *Duke Math. J.* **62**:2 (1991), 315–358.

[Nekrasov 1996] N. Nekrasov, “Holomorphic bundles and many-body systems”, *Commun. Math. Phys.* **180**:3 (1996), 587–603.

[Parshin 1999] A. N. Parshin, "On a ring of formal pseudo-differential operators", *Tr. Mat. Inst. Steklova* **224**:Algebra. Topol. Differ. Uravn. i ikh Prilozh. (1999), 291–305. In Russian; translated in *Proc. Steklov Inst. Math.* **224**:1 (1999), 266–280.

[Parshin 2001a] A. N. Parshin, "Integrable systems and local fields", *Comm. Algebra* **29**:9 (2001), 4157–4181.

[Parshin 2001b] A. N. Parshin, "The Krichever correspondence for algebraic surfaces", *Funktsional. Anal. i Prilozhen.* **35**:1 (2001), 88–90. In Russian; translated in *Funct. Anal. Appl.* **35**:1 (2001), 74–76.

[Plaza-Martín 2000] F. Plaza-Martín, "Generalized KP hierarchy for several variables", preprint, 2000. Available at arxiv.org/abs/math.AG/0008004.

[Previato 2008] E. Previato, "Review of *Soliton equations and their algebro-geometric solutions*, vol. I, by Fritz Gesztesy and Helge Holden, Cambridge University Press, 2003", *Bull. Amer. Math. Soc.* (2008). To appear.

[Sato 1989] M. Sato, "The KP hierarchy and infinite-dimensional Grassmann manifolds", pp. 51–66 in *Theta functions* (Brunswick, ME, 1987), edited by L. Ehrenpreis and R. C. Gunning, Proc. Sympos. Pure Math. **49**, Amer. Math. Soc., Providence, RI, 1989.

[Schmidt 1996] M. U. Schmidt, "Integrable systems and Riemann surfaces of infinite genus", *Mem. Amer. Math. Soc.* **122**:581 (1996), viii+111.

[Scorza Dragoni 1988] G. Scorza Dragoni, "Iacopo Barsotti in memoriam", *Rend. Accad. Naz. Sci. XL Mem. Mat.* (5) **12**:1 (1988), 1–2.

[Welters 1986] G. E. Welters, "The surface $C - C$ on Jacobi varieties and 2nd order theta functions", *Acta Math.* **157**:1-2 (1986), 1–22.

[Zheglov 2005] A. Zheglov, "Two dimensional KP systems and their solvability", preprint, 2005. Available at http://www.arxiv.org/abs/math-ph/0503067.

Emma Previato
Department of Mathematics and Statistics
Boston University
Boston, MA 02215-2411
United States
ep@bu.edu