



## Chapter 8

# Mathematical Proficiency for Citizenship

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### Number and U.S. Democracy

“Numbers have immeasurably altered the character of American society,” wrote Patricia Cohen in 1982 in the concluding pages of her book *A Calculating People*. Cohen continues, “Our modern reliance on numbers and quantification was born and nurtured in the scientific and commercial worlds of the seventeenth century and grew in scope in the early nineteenth century, under the twin impacts of republican ideology and economic development.” In the two decades since Cohen’s book, reliance on numbers and quantification has increased well beyond what could have been imagined, and no end is in sight.

Quantification emerged in seventeenth-century Western Europe as an alternative to the classical Aristotelian systems of classification as a way to make sense of the world [Crosby 1997]. England was becoming a commercial capital and numbers were needed to describe the economy. Arabic numerals finally had been adopted and the first books on arithmetic had appeared. Education in arithmetic was beginning but was an arcane subject, cut into small bits, and not easy to learn. Arithmetic as a commercial subject was disconnected from formal mathematics [Cohen 1982]. To a larger extent than I believe healthy, both these circumstances still exist today.

Inherent demands of a democracy, a large diverse country, and persistently aggressive economic development have fed the historical inclination of U.S. culture toward quantification to its enormity of today. Characteristics of the U.S. democracy that increase need for quantitative reasoning include:

- Constitutional mandate for a census.
- Protection of diverse and minority interests.
- Informing political debate.
- Confirming national identity.

- Free market system with minimally regulated labor markets.
- Recent deregulations of markets and services.
- Environmental protection and occupational safety.
- Emphasis on individual wealth accumulation.

Winston Churchill observed that first we shape our buildings and then they shape us. Driven by the characteristics above, our higher education system has permeated U.S. society with vast storehouses of knowledge and technologies based largely on computers. Consequently, floods of data and numbers are daily fare in the U.S. In turn, as with Churchill's buildings, number, quantification, data, and analysis shape U.S. culture and are more powerful than most realize.

What mathematics is critically important for informed participation in this highly quantitative U.S. society? This question has been considered since colonial times but has never been as difficult to answer as it is now, at the beginning of the twenty-first century. A more difficult question is: How we can design a developmental approach to achieve mastery of this mathematics and assessments to measure progress in this development?

I do not pretend to have complete answers to these questions, but I am quite sure that our traditional introductory college mathematics courses and traditional assessments are inadequate responses.

### **The Emergence of U.S. Higher Education**

A century ago, only about 10% of the 14- to 17-year-olds were enrolled in high school. Half a century ago, only a small fraction of high school graduates went on to higher education. Both have changed. Indeed, college enrollments have grown from 10% of the 18- to 24-year-old cohort in 1945 to just over 50% of that cohort in 1990. Thus, today as many students enroll in postsecondary education as in secondary school. For many reasons, the U.S. public has made college the locus of expectations for leading a decent life — the need for sophisticated quantitative literacy is surely among those reasons [Steen 2004].

As enrollments in higher education have grown nearly four-fold during the last forty years (from approximately 4 to 15 million students), the curriculum has remained relatively stagnant. Our system of higher education was conceived at a time when 2% of the U.S. population went to college and its purpose then was to educate students to become a class apart. But life in the twenty-first century is more complex, so college has replaced high school as the educational standard for a democratic society in the twenty-first century. However, the task of reconceiving the structures of higher education for a democratic purpose — especially in fields that rely on quantitative and mathematical methods — has hardly begun. Indeed, much the reverse has taken place. The curriculum in

mathematics has gradually narrowed, forcing almost everyone through the bottleneck of calculus. It may well be that with regard to a democratic conception of higher education it is undergraduate mathematics that is most out of date [Steen 2004].

In the development of undergraduate education in the U.S., among all the subjects or disciplines, mathematics has some unique characteristics. The present-day model of undergraduate education—that of majors and electives with a “general education” core—can be traced to Charles Eliot’s reform of Harvard’s undergraduate curriculum at the end of the nineteenth century. This new “general education”—actually a reaction to Eliot—became the conveyance for the democratic ideal of higher education—to ensure that graduates of all majors left college equipped with the breadth of general knowledge and skills necessary for full participation in democratic society. Majors were developed in standard college disciplines and all but mathematics, the one survivor of the classical curriculum, developed general education courses. Mathematics served up its standard courses—algebra, trigonometry, solid and analytic geometry—for general education. Unlike professors in other disciplines, a college mathematics professor from 1850 would not find today’s college algebra course at all unfamiliar. In short, the bulk of U.S. collegiate mathematics has not changed in a century in spite of the college population and the U.S. society and culture being drastically different [Steen 2004].

### **Transfer to Contexts**

Several suggestions for college mathematics courses for general education are documented in twentieth-century literature. Courses with titles such as finite mathematics, liberal arts mathematics, and college mathematics have entered the college curriculum, but the bulk of general education mathematics enrollments remain in the traditional courses of algebra, trigonometry, analytic geometry and calculus. None of these courses, as traditionally taught, is effective in educating for the quantitative literacy required in twenty-first-century United States. However, except for probability and statistics, almost all the mathematics needed for quantitative literacy is in these traditional courses or their prerequisites. The difficulty is with transfer of the mathematical knowledge and skills to the multitude of contexts where citizens confront quantitative situations. It is the ability to make that transfer that I believe to be essential for being proficient in mathematics, and this ability is apparently incredibly difficult to assess [Wiggins 2003]. The following examples illustrate what I mean.

**Example 1.** The following article appeared in the *Lincoln (NE) Journal Star* on August 21, 2001.

Forcing fuel efficiency on consumers doesn't work

*By Jerry Taylor*

Although the late, great energy crisis seems to have come and gone, the political fight over yesterday's panic rages on. The big dust-up this fall will be over SUVs, light trucks and minivans. Should the government order Detroit to make them get more miles per gallon? Conservationists say "yes." Economics 101 says "no."

Let's start with a simple question: Why should the government mandate conservation? When fuel becomes scarce, fuel prices go up. When fuel prices go up, people buy less fuel. Economists have discovered over the long run, a 20 percent increase in gasoline costs, for instance, will result in a 20 percent decline in gasoline consumption. No federal tax, mandate or regulatory order is necessary.

Notice the phrase "over the long run." Energy markets are volatile because consumers do not change their buying habits much in the short run.

This has led some critics to conclude that people don't conserve enough when left to their own devices. They do, but consumers need to be convinced that the price increases are real and likely to linger before they'll invest in energy-efficient products or adopt lifestyle changes. But even in the short run, people respond. Last summer was a perfect example: For the first time in a non-recession year, gasoline sales declined in absolute terms in response to the \$2 per gallon that sold throughout much of the nation.

Mandated increases in the fuel efficiency of light trucks, moreover, won't save consumers money. A recent report from the National Academy of Sciences, for instance, notes that the fuel efficiency of a large pickup could be increased from 18.1 miles per gallon to 26.7 miles per gallon at a cost to automakers of \$1,466. But do the math: It would take the typical driver 14 years before he would save enough in gasoline costs to pay for the mandated up-front expenditure. A similar calculation for getting a large SUV up to 25.1 miles per gallon leads to a \$1,348 expenditure and, similarly, more than a decade before buyers would break even.

"Fine with me," you say? But it's one thing to waste your own money on a poor investment; it's entirely another to force your neighbor to do so. You could take that \$1,466, for instance, put it in a checking account yielding 5 percent interest, and make a heck of a lot more money than you could by investing it in automobile fuel efficiency.

Even if government promotion of conservation were a worthwhile idea, a fuel efficiency mandate would be wrong. That's because increasing the mileage a vehicle gets from a gallon of gasoline reduces the cost of driving. The result? People drive more.

Energy economists who've studied the relationship between automobile fuel efficiency standards and driving habits conclude that such mandates are offset by increases in vehicle miles traveled.

If we're determined to reduce gasoline consumption dramatically, the right way to go would be to increase the marginal costs of driving by increasing the tax on gasoline. Now, truth be told, I don't support this idea much either. A recent study by Harvard economist Kip Viscusi demonstrates that the massive fuel taxes already levied on drivers (about 40 cents per gallon) fully "internalize" the environmental damages caused by driving. But conservationists reject this approach for a different reason: Consumers hate gasoline taxes and no Congress or state legislature could possibly increase them.

Look, it's a free country. If you want to buy a fuel-efficient car, knock yourself out. But using the brute force of the government to punish consumers who don't share your taste in automobiles serves no economic or environmental purpose.<sup>1</sup>

*What's involved in analyzing the argument?*

- Glean the relevant information.
- Have confidence to take up the challenge.
- Estimate to see if assertions are reasonable.
- Do the mathematics — linear and exponential functions.
- Generalize the situation.
- Reflect on the results.

*Some relevant information*

- The cost of increasing the fuel efficiency of large pickup trucks from 18.1 miles per gallon to 26.7 miles per gallon is \$1466 per vehicle.
- Claim 1: It would take the typical driver 14 years to make up the \$1466.
- Claim 2: You can make a "heck of a lot" more money by putting the \$1466 in an account yielding 5% interest.

*Ambiguities*

- Typical driver's annual mileage — 10,000 to 20,000 is reasonable.
- Price of gallon of fuel — about \$1.40 at the time the article was written.

<sup>1</sup>Lincoln (NE) Journal Star, 21 Aug 2001, p. 5B. © 2001, Jerry Taylor and Cato Institute. Used with permission.

- Frequency of compounding of interest — annually to continuously.
- How long is the \$1466 left to accumulate interest?
- Are the savings on fuel invested similarly?

*Estimating time required to save \$1466*

Assumptions: Annual mileage is 10,000, cost of gasoline is \$1.50 per gallon, fuel efficiency is 20 or 30 miles per gallon. We use \$1.50 instead of \$1.40 to simplify calculation of the estimate.

Estimated cost of fuel for a large pickup truck for one year:

$$\begin{aligned} \frac{10,000 \text{ miles per year}}{20 \text{ miles per gallon}} &= 500 \text{ gallons per year at } \$1.50 \text{ per gallon} \\ &= \$750 \text{ per year} \end{aligned}$$

Estimated time to save \$1466 by using a more fuel-efficient truck:

If the truck's fuel efficiency increased from 20 miles per gallon to 30 miles per gallon, then the price of gasoline would be multiplied by  $\frac{2}{3}$ , so savings is  $\frac{1}{3}$  of \$750 or \$250 per year.

If \$250 per year is saved on fuel, then the amount of time needed to save enough in gasoline costs to make up the \$1466 in Claim 1 is less than 6 years:

$$\text{Number of years to save } \$1466 = \frac{\$1500}{\$250} = 6.$$

Estimated time to save \$1466 by investing \$1466 in a checking account:

5% per year on \$1500 yields \$75 per year at simple interest. So it takes approximately  $1500/75$  or 20 years to earn \$1466.

*Calculating time required to save \$1466*

Assumptions: Annual mileage is 10,000, price of gasoline is \$1.40 per gallon, fuel efficiency is 18.1 or 26.7 miles per gallon. The number of years required to save \$1466 is

$$\frac{1466}{\left(\frac{10000}{18.1} - \frac{10000}{26.7}\right) \times 1.40} = 5.88$$

This answer of 5.88 years is quite different from the 14 years claimed in the article.

*Calculating comparison of costs over time*

Let  $M$  be miles driven per year,  $G$  the price of gasoline per gallon,  $t$  the time in years.

Let  $C_1$  be the price of gasoline for a large pickup truck with fuel efficiency of 18.1 miles per gallon that is driven  $M$  miles per year for  $t$  years.

Let  $C_2$  be the price of gasoline for a large pickup truck with increased fuel efficiency (26.7 miles per gallon) that is driven  $M$  miles per year for  $t$  years, plus \$1466.

Then, setting  $M$  equal to 10,000 and setting  $G$  equal to 1.4:

$$C_1(t) = \frac{MG}{18.1}t = 773.48t$$

$$C_2(t) = \frac{MG}{26.7}t + 1466 = 524.34t + 1466$$

At year  $t$  the earnings from \$1466 invested at a rate of 5% are

$$1466e^{.05t} - 1466 \text{ if compounded continuously,}$$

$$1466(1.05^t) - 1466 \text{ if compounded annually.}$$

The annual difference in the amount spent on gasoline for the truck with fuel efficiency 18.1 miles per gallon and the truck with fuel efficiency 26.7 miles per gallons is:

$$773.48 - 524.34 = 249.14.$$

If the savings of \$249.14 per year due to increased fuel efficiency are invested in an account at 5% compounded continuously, then after  $t$  years the value of the account will be:

$$\sum_{n=1}^t 249.14e^{.05(t-n)}.$$

Some computations using calculators yield the following information:

Year	Investment earnings on \$1466	Gas savings	Investment earnings on gas savings
10	951	2491	3152
20	2519	4982	8350
30	5104	7474	16918
40	9366	9965	31046
50	16339	12457	54339

These data, based on the stated assumptions, show that it would take approximately 40 years for the \$1466 invested at 5% to return as much as savings on gasoline. If these savings are invested annually, the income far exceeds the returns on the \$1466.

*Generalize the situation*

One direction for generalizing this is analyzing the linear functions  $C_1$  and  $C_2$  with either the price of gasoline or the miles driven per year as parameters. Another is to introduce more complex investment issues such as present value.

*Reflection on the results*

One such reflection is that the article does not reveal some information that is needed to check to see if the author's conclusions are valid or reasonable. Based on the calculations above with assumptions that seem reasonable, the author's conclusions are not valid. Further analysis might reveal possible assumptions on the cost of gasoline and miles driven per year by the "typical driver" that would support the author's conclusions. For example, using \$1.40 per gallon of gasoline and 4200 miles per year, the author's claim of 14 years to save the \$1466 would be valid.

Another reflection is that the author's claim that the 5% return on the checking account would exceed the savings on gasoline will be correct if one is willing to wait long enough. The exponential growth will eventually exceed the linear growth.

**Example 2.** This letter to the editor appeared in the *Arkansas Democrat-Gazette* on April 9, 2002 (p. 7B).

My children asked me how many ancestors and how many acts of these ancestors they are responsible for after reading and listening to the Razor-backs' coaching dilemma.

They have been taught that they are responsible for their own actions and sometimes the actions of their friends or even their parents. They just want to know how far this goes back.

My daughter had visited the slave ship exhibit at one of our downtown museums and recognized a family name as being a builder of slave ships back in the 1500s in Britain. She also knew that another relative brought six slaves over to Jamestown in the 1600s. How much was she going to have to pay in retribution? Was she the only one responsible or were there others?

Before this got even more out of hand, we decided to do the math. Assuming four generations per century and only one child per family, that would be 19 generations. Two to the power of 19 would be 524,288 people who shared the responsibility.

Then we started laughing at the total absurdity of the idea of one person today paying for the sins of another when there had been 524,288 people in between.

And that wasn't even counting brothers and sisters.

Conclusion: Get a life. Forgive and forget all 524,288 of them.<sup>2</sup>

*Analysis of the argument*

Accepting the author's assumption that there are 19 generations between the sixteenth-century ancestor and the daughter, one explores how the author arrives at  $2^{19} = 524,288$  "people in between" the ancestor and the daughter.<sup>3</sup>

One way to arrive at  $2^{19}$  is to assume that the number of descendants of the ancestor doubles every generation. However, this model of exponential growth of population would require two children of the ancestor and each of these children (along with a spouse) would have two children. This doubling-each-generation yields  $2^{19}$  descendants of the ancestor at the daughter's generation. However, this violates the author's assumption of "only one child per family," and most of the descendants of the ancestor would not be considered in between the ancestor and the daughter.

A second way to arrive at  $2^{19}$  is to begin with the daughter and consider her parents, then her grandparents, then great-grandparents, etc. Continuing, one would arrive at  $2^{19}$  ancestors of the daughter in the sixteenth century. Although this can be consistent with the assumption of one child per family, the  $2^{19}$  ancestors are not in between the daughter and the sixteenth-century ancestor. There is a possibility that one would sum up this chain of ancestors of the daughter, two parents plus four grandparents, etc. This would yield a sum as follows:

$$2 + 2^2 + 2^3 + \dots + 2^{19} = 2(2^{19} - 1)$$

or

$$2 + 2^2 + 2^3 + \dots + 2^{18} = 2(2^{18} - 1).$$

However, neither of these is  $2^{19}$ .

Two reasonable ways to count the number of people in between the ancestor and the daughter would be by counting the child of the ancestor, then the grandchild, then the great grandchild, etc. This would yield 18 or 19 (depending on where one begins counting) people in between. Since at each generation, the child takes a spouse, one could count the children and the spouses, giving 36 or 38 people in between.

Extensions of this discussion can move to models of population growth, showing why the simple exponential growth (doubling each generation) has limitations and moving to bounded exponential growth and logistic growth.

<sup>2</sup>© 2002 *Arkansas Democrat-Gazette*. Used with permission.

<sup>3</sup>I have used this letter with more than 130 students in four different classes. When presented with the situation as described in the letter, some students have believed that the counting of generations is flawed, and only a few students independently have detected flaws in the counting and language. Because of the serious nature of the issue discussed in the letter, I have insisted that we accept only that the author is making an argument for a position and focus on the validity of the argument that is given credibility by "doing the math." Because of the nature of the issue and the diverse opinions about it, I only use this item in class or homework rather than on a quiz or examination.

As in the previous example, we see that the author's assumptions about what constitutes a relevant mathematical model shape the conclusions that can be drawn from it. One draws very different conclusions from the same basic data when different models are used. The best defense against being misled by inappropriate arguments is to be able to figure out the right arguments oneself.

**Example 3.** The False Positive Paradox (conditional probability).

Use the following information to answer the question.

- The incidence of breast cancer in women over 50 is approximately 380 per 100,000.
- If cancer is present, a mammogram will detect it approximately 85% of the time (15% false negatives).
- If cancer is not present, the mammogram will indicate cancer approximately 7% of the time (7% false positives).

If a mammogram is positive, which of the following is nearest to the odds that cancer is present? Assume that all women over 50 get tested.

- A. 9 out of 10
- B. 8 out of 10
- C. 7 out of 10
- D. 1 out of 6
- E. 1 out of 14
- F. 1 out of 22

The correct choice of F is surprising to many; hence, the term paradox. In fact, the 380 cancer cases would generate roughly 325 positive tests, while the 99,620 noncancer cases would generate nearly 7000 false positives, so the odds that a positive test indicates cancer is about 1 in 22.

**Example 4.** The graphs in Figure 1 on the next page appeared in the *New York Times* on August 11, 2002 (p. 25).

*Sample tasks*

1. Explain why these graphs are titled "The Rise in Spending."
2. From 1990 to 1995, how did the cost of health insurance premiums change?
3. Explain why inflation is included in each of the graphs.
4. In what year was the rate of increase in prescription drug costs the greatest?

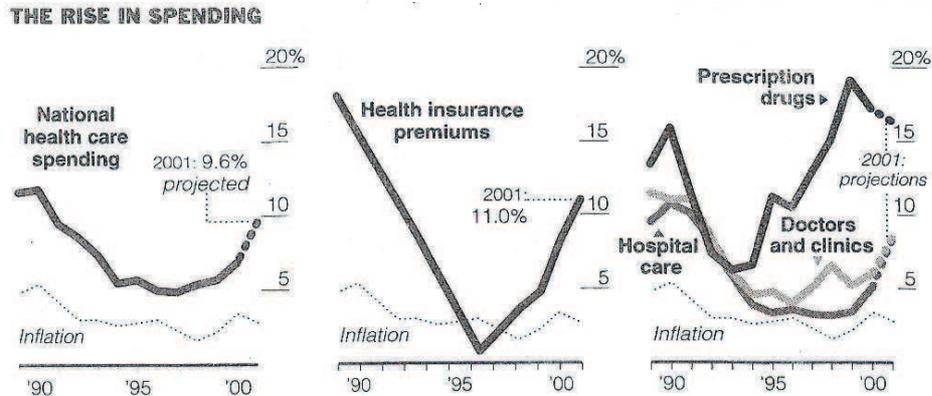


Figure 1. Used with permission, © 2002, *New York Times*.

**Example 5.** The graphs in Figure 2 appeared in the *New York Times* on April 7, 1995 (p. A29).

Sample tasks

1. Can both of these views be correct? Explain.
2. In each bar graph there is a “bar” over \$20,000 to \$30,000. Do these two bars represent the same quantity? Explain.

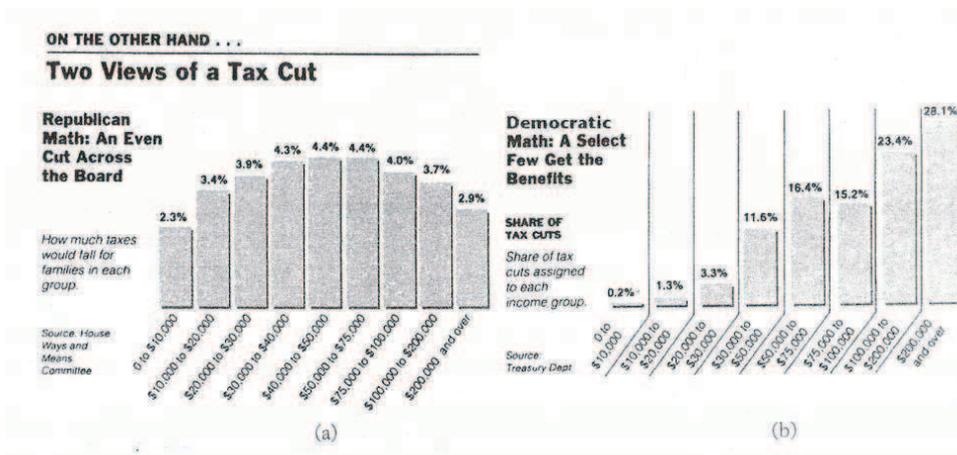


Figure 2. Used with permission. © 1995, *New York Times*

**What the examples imply.** What do the above examples imply about critical proficiencies in mathematics needed for citizenship? The following seem clear:

- Calculate and estimate with decimals and fractions.
- Recognize and articulate mathematics.
- Generalize and abstract specific mathematical situations.
- Understand and use functions as process, especially to describe linear and exponential growth.
- Have facility with algebra.
- Use calculators to explore and compute.
- Formulate and analyze situations using geometry and measurement.
- Analyze and describe data using statistics and probability.

College mathematics courses that cover the mathematical topics in the list above have been proposed and developed over the past century. For example, Allendoerfer [1947] described such a course in an article entitled “Mathematics for Liberal Arts Students.” However, the bulk of the enrollments in college mathematics, even those for general education, remain in traditional courses in college algebra, trigonometry, and calculus.

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