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On the Pair-Crossing Number

PAVEL VALTR

Abstract. By a *drawing* of a graph G, we mean a drawing in the plane such that vertices are represented by distinct points and edges by arcs. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum possible number of crossings in a drawing of G. The *pair-crossing number* $\text{pair-cr}(G)$ of G is the minimum possible number of (unordered) crossing pairs in a drawing of G. Clearly, $\text{pair-cr}(G) \le \mathrm{cr}(G)$ holds for any graph G. Let $f(k)$ be the maximum $\mathrm{cr}(G)$, taken over all graphs G with $\text{pair-cr}(G) = k$. Obviously, $f(k) \ge k$. Pach and Tóth [2000] proved that $f(k) \le 2k^2$. Here we give a slightly better asymptotic upper bound $f(k) = O(k^2/\log k)$. In case of x-monotone drawings (where each vertical line intersects any edge at most once) we get a better upper bound $f^{\mathrm{mon}}(k) \le 4k^{4/3}$.

1. Introduction

By a *drawing* of a graph G, we mean a drawing in the plane such that vertices are represented by distinct points and edges by arcs. The arcs are allowed to cross, but they may not pass through vertices (except for their endpoints) and no point is an internal point of three or more arcs. Two arcs may have only finitely many common points. A *crossing* is a common internal point of two arcs. A *crossing pair* is a pair of edges which cross each other at least once. A drawing is *planar*, if there are no crossings in it. A *subdrawing* of a drawing is defined analogously as a subgraph of a graph.

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum possible number of crossings in a drawing of G. The *pair-crossing number* $\text{pair-cr}(G)$ of G is the minimum possible number of (unordered) crossing pairs in a drawing of G.

In this paper we investigate the relation between the crossing number and the pair-crossing number. Clearly, $\text{pair-cr}(G) \le \mathrm{cr}(G)$ holds for any graph G. The problem of deciding whether $\mathrm{cr}(G) = \text{pair-cr}(G)$ holds for every G appears quite challenging. Let $f(k)$ be the maximum $\mathrm{cr}(G)$, taken over all graphs G with $\text{pair-cr}(G) = k$. Obviously, $f(k) \ge k$. Pach and Tóth [2000] proved that

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$f(k) \le 2k^2$. In fact, they proved this bound in a stronger version when the pair-crossing number is replaced by the so-called odd-crossing number, which is the minimum number of pairs of edges in a drawing that cross each other an odd number of times. Here we find a slightly better asymptotic upper bound on $f(k)$:

Theorem 1. $f(k) = O(k^2/\log k)$.

The improvement is small but its proof gives some insight to the structure of possible counterexamples to $f(k) = k$.

We get a significantly subquadratic upper bound in the case of *(x-)monotone* drawings. A drawing D is *monotone* if every edge is drawn as an x-monotone curve, meaning that no vertical line intersects it more than once. The *monotone crossing number* $\mathrm{cr}^{\mathrm{mon}}(G)$ is the minimum possible number of crossings in a monotone drawing of G. The *monotone pair-crossing number* $\text{pair-cr}^{\mathrm{mon}}(G)$ is defined analogously — it is the minimum possible number of (unordered) crossing pairs in a monotone drawing of G. Let $f^{\mathrm{mon}}(k)$ be the maximum $\mathrm{cr}^{\mathrm{mon}}(G)$, taken over all graphs G with $\text{pair-cr}^{\mathrm{mon}}(G) = k$. Obviously, $f^{\mathrm{mon}}(k) \ge k$.

Theorem 2. $f^{\mathrm{mon}}(k) \le 4k^{4/3}$.

Theorem 1 is proved in Section 2 and Theorem 2 in Section 3.

Remarks. 1. It is possible that our results hold also if the (monotone) pair-crossing number is replaced by the so-called (monotone) odd-crossing number (see [Pach and Tóth 2000] for the definition of the odd-crossing number and for a similar result). We did not investigate this question.

2. Some related results can be found in [Kolman and Matoušek 2004]. In particular, these authors prove that

$$\mathrm{cr}(G) = O\left(\log^3 |V| \left(\text{pair-cr}(G) + \sum_{v \in V} (\deg v)^2\right)\right)$$

for any graph $G = (V, E)$.

3. One could hope to prove $f(k) = k$ by a contradiction, considering local modifications of a drawing witnessing $f(k) > k$. We tried this approach but it does not seem to work in some straightforward way. Our difficulties with this approach might have an explanation in an example [Kratochvíl and Matoušek 1994] of a drawing in which it is not possible to eliminate multiple crossings of edge pairs without introducing new crossing pairs.

2. A Logarithmic Improvement over the Quadratic Bound

Here we give a simple proof of $f(k) \le 2k^2$ and then refine the method, thereby obtaining $f(k) = O(k^2/\log k)$.

A simple proof of the quadratic bound. The bound $f(k) \le 2k^2$ can be proved easily; this was probably known to experts but, as far as we know, hasn't appeared in print. Let G be a graph with pair-cr$(G) = k$. Consider a drawing D_0 of G witnessing pair-cr$(G) = k$. At most $2k$ edges, the *bad edges*, are involved in at least one crossing in D_0. The remaining edges, the *good edges*, form a planar subdrawing D_{pl} of D_0. Each of the bad edges is drawn in a single face of D_{pl}. Let us choose a drawing D of G that extends D_{pl} such that each bad edge is drawn within a single face of D_{pl}, and the number of crossings is minimized among all such drawings.

We now show that every two edges cross at most once in the drawing D. Suppose on the contrary that x_1, x_2 are common crossings of two edges e, f. We swap the portions of e and f between x_1 and x_2, thereby eliminating x_1, x_2 and introducing no new crossings (see Figure 1). If the swap creates selfintersections of e or f, we easily eliminate them without introducing any new crossings (see Figure 2). We get a contradiction with the minimum number of crossings in D.

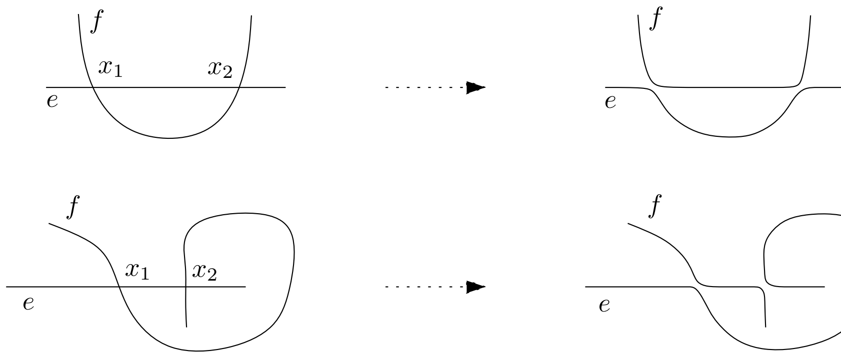

Figure 1. Swapping e, f between x_1, x_2 (two cases).

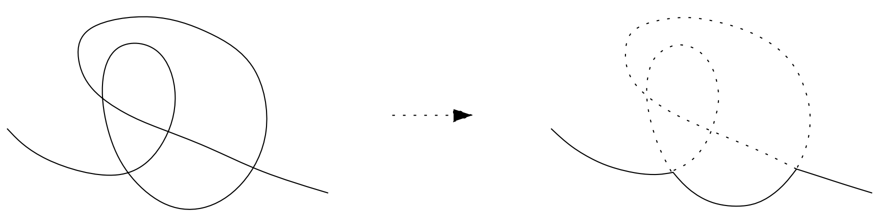

Figure 2. Eliminating selfintersections of an edge.

Thus, any two edges in D cross each other at most once.

It follows that there are at most $\binom{2k}{2} \le 2k^2$ crossings in D.

The logarithmic improvement. Here we prove Theorem 1. Let G be a graph with pair-cr$(G) = k$. Let us consider a drawing D_0 of G witnessing pair-cr$(G) = k$. Let t be a suitable parameter to be fixed later (it will be of order $\log k$). Let us call an edge of G *good* if it crosses no edge in the drawing D_0, *light* if it crosses at least one and at most t edges in D_0, and *heavy* if it crosses more than t edges in D_0. Although we later redraw light and heavy edges several times, the notation "good", "light", or "heavy" is fixed for each edge of G by the above definition. Let l be the number of light edges and h the number of heavy edges.

Let D_1 be the subdrawing of D_0 formed by the good and light edges, and let D_{pl} be its planar subdrawing formed by the good edges only.

Consider a cell of D_{pl}. Suppose that some light edge in this cell crosses at least 2^t other light edges. Then we can decrease the number of crossings in D_1 without introducing any new crossing pair of edges, as can be seen from the following result of Schaefer and Štefankovič [2004] (implicitly contained in the proof of their Theorem 3.2): *Let D be a drawing of a graph G, and let e be an edge of G that crosses at most t other edges in D. Suppose that e has at least 2^t crossings in D. Then the edge e and the edges crossing it can be redrawn* (*within a small neighborhood of e*) *in such a way that the obtained drawing D' of G has fewer crossings than D and that there are no new crossing pairs of edges in D'* (*compared to D*).

Applying the result of Schaefer and Štefankovič finitely many times, we obtain a redrawing D_2 of D_1 with the same or smaller number of crossing pairs, such that each light edge is redrawn within the same face of D_{pl} and is involved in at most $2^t - 1$ crossings. Thus, there are at most $l \cdot (2^t - 1)/2$ crossings in D_2 (recall that l is the number of light edges).

Now, let D_3 be a redrawing of D_2 such that each light edge is redrawn within the same face of D_{pl} and that the number of crossings in D_3 is minimized. D_3 has at most as many crossings as D_2, i.e., at most $l \cdot (2^t - 1)/2$ crossings. Moreover, every two edges in D_3 cross each other at most once (otherwise we could argue analogously as in Figs. 1 and 2).

Finally, we add the heavy edges to the drawing D_3, in such a way that each heavy edge is drawn in the same face of D_{pl} as in D_0, the number of heavy-light[1] crossings is minimized, and subject to this, the number of heavy-heavy crossings is minimized. Let D_4 be the obtained drawing of G.

We claim that each heavy edge crosses any other edge at most once. To see this, first suppose that a heavy edge e crosses a light edge f at least twice, and let x_1 and x_2 be two crossings of e and f. Let z_e be the number of crossings of the portion of e between x_1 and x_2 with light edges, and similarly for z_f. If $z_f \leq z_e$, then e can be routed along f between x_1 and x_2, thereby decreasing

[1] A crossing is *heavy-light*, if it is a crossing of a heavy edge with a light edge. *Heavy-heavy* and *light-light* crossings are defined analogously.

the number of heavy-light crossings. See Figure 3. Possible selfintersections of e are eliminated as in Figure 2. If $z_f > z_e$, then the drawing D_3 did not have the

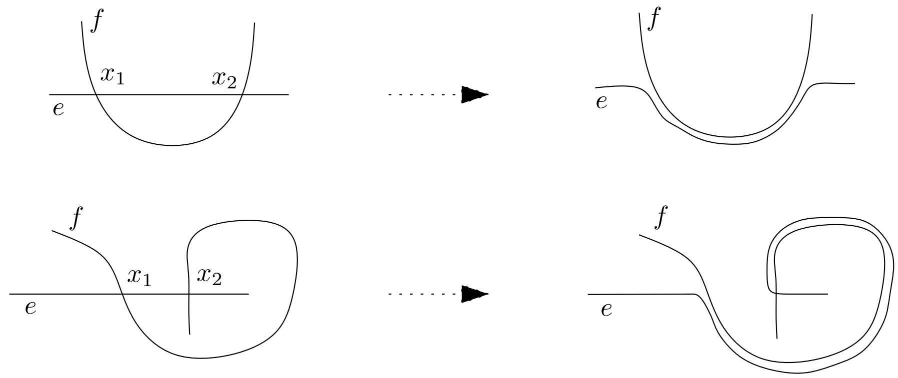

Figure 3. Rerouting e along f between x_1 and x_2 (two cases).

minimum number of crossings, as the number of crossings in it could be decreased by routing f along e. Again, possible selfintersections of f are eliminated as in Figure 2.

Similarly, suppose that two heavy edges e and f cross at least twice, and let x_1, x_2 be two of their common crossings. Then swapping the portions of e and f between x_1 and x_2 eliminates x_1 and x_2; see Figure 1. (As above, possible selfintersections of e or f are eliminated as in Figure 2.)

Thus, the heavy edges are involved in at most $\binom{h}{2} + h \cdot l \leq h(h+l) \leq h \cdot 2k$ crossings. The good edges are involved in no crossings and the number of light-light crossings is at most $l \cdot (2^t - 1)/2$. Thus, the total number of crossings in D_4 is at most $h \cdot 2k + l \cdot (2^t - 1)/2$. Using the obvious inequalities $l \leq 2k$ and $h \leq 2k/t$, this is at most $O(k^2/t + k2^t)$. Setting $t = \frac{1}{2}\log_2 k$, say, gives the claimed bound. The proof of Theorem 1 is complete.

3. Monotone Drawings

In this section we prove Theorem 2. Let G be a graph with pair-$\text{cr}^{\text{mon}}(G) = k$. Among all monotone drawings of G witnessing pair-$\text{cr}^{\text{mon}}(G) = k$, we choose a drawing D with the minimum number of crossings. We define *good, light*, and *heavy* edges in D in the same way as in the proof of Theorem 1 (now, the parameter t will be equal to $k^{1/3}$).

Lemma 1. Let e be a light edge in D. Then e intersects each edge at most $2t-1$ times.

Proof. Consider an edge $f \in E(D), f \neq e$. Since D is monotone, each pair of consecutive common crossings of e, f determines a *lens* bounded by one of the edges e, f from above and by the other one from below. Let L be such a lens.

We claim that at least one edge intersecting e has an endpoint inside L. Suppose that this is not true. A *sling* in L is a continuous portion of an edge such that it is contained in L and its endpoints lie on e (see Figure 4). If there were some slings in L, we could reroute them along e (and outside L) in such a way that no new crossing pairs are introduced and the number of crossings is decreased (see Figure 4). Thus, there are no slings in L. It follows that rerouting

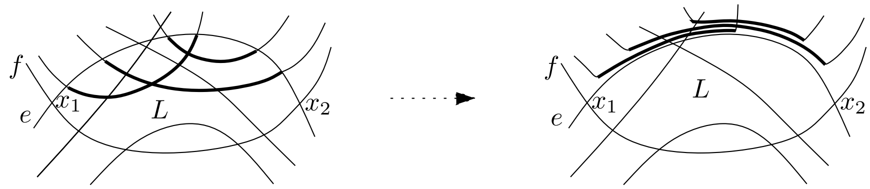

Figure 4. Three slings (bold) in a lens L determined by e, f and rerouting these slings along e.

f along e at the lens L (see Figure 5) decreases the number of crossings and introduces no new crossing pairs — a contradiction with the choice of D. Thus, there had to be an edge intersecting e and having an endpoint inside L.

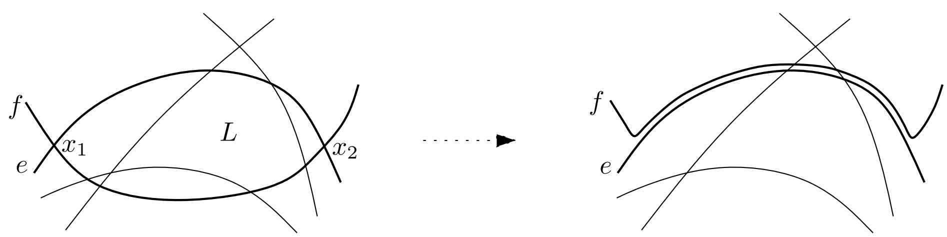

Figure 5. Rerouting f along e at the lens L.

Since at most t edges intersect e (e is light), it follows that there are at most $2(t-1)$ lenses determined by e, f. Thus, e, f cross each other at most $2t-1$ times. □

There are k crossing pairs in D. By Lemma 1, each crossing pair involving at least one light edge has at most $2t-1$ common crossings. Thus, there are at most $k(2t-1)$ crossings involving at least one light edge.

We redraw the heavy edges so that there are no crossings with good edges, the number of heavy-light crossings is minimized, and subject to this, the number of heavy-heavy crossings is minimized.

The obtained drawing has at most $k(2t-1)$ crossings involving at least one light edge. Moreover, any two heavy edges cross at most once, for otherwise we could get a better drawing by swapping these two edges as in Figure 1 (top). Since there are at most $\lfloor 2k/t \rfloor$ heavy edges, the total number of crossings is at most $k(2t-1)+\binom{\lfloor 2k/t \rfloor}{2}$. Choosing $t=k^{1/3}$, this is at most $4k^{4/3}$. This finishes the proof of Theorem 2.

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Pavel Valtr
Department of Applied Mathematics
and
Institute for Theoretical Computer Science (ITI)
Charles University
Malostranské nám. 25, 118 00
Praha 1
Czech Republic