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Betti Number Bounds, Applications and Algorithms

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Abstract. Topological complexity of semialgebraic sets in $\mathbb{R}^k$ has been studied by many researchers over the past fifty years. An important measure of the topological complexity are the Betti numbers. Quantitative bounds on the Betti numbers of a semialgebraic set in terms of various parameters (such as the number and the degrees of the polynomials defining it, the dimension of the set etc.) have proved useful in several applications in theoretical computer science and discrete geometry. The main goal of this survey paper is to provide an up to date account of the known bounds on the Betti numbers of semialgebraic sets in terms of various parameters, sketch briefly some of the applications, and also survey what is known about the complexity of algorithms for computing them.

1. Introduction

Let R be a real closed field and S a semialgebraic subset of R^k, defined by a Boolean formula, whose atoms are of the form $P = 0$, $P > 0$, $P < 0$, where $P \in \mathcal{P}$ for some finite family of polynomials $\mathcal{P} \subset \mathrm{R}[X_1, \dots, X_k]$. It is well known [Bochnak et al. 1987] that such sets are finitely triangulable. Moreover, if the cardinality of $\mathcal{P}$ and the degrees of the polynomials in $\mathcal{P}$ are bounded, then the number of topological types possible for S is finite [Bochnak et al. 1987]. (Here, two sets have the same topological type if they are semialgebraically homeomorphic). A natural problem then is to bound the topological complexity of S in terms of the various parameters of the formula defining S.

One measure of topological complexity are the various Betti numbers of S. The i-th Betti number of S (which we will denote by $b_i(S)$) is the rank of $H_i(S, \mathbb{Z})$. In case, R happens to be $\mathbb{R}$ then $H_i(S, \mathbb{Z})$ denotes the i-th singular homology group of S with integer coefficients. For semialgebraic sets defined

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over general real closed fields the definition of homology groups requires more care and several possibilities exists. For instance, if S is closed and bounded, then using the fact that S is finitely triangulable, $H_i(S,\mathbb{Z})$ can be taken to be the i-th simplicial homology group of S, and this definition agrees with the previous definition in case $R = \mathbb{R}$. For a general locally closed semialgebraic set, one can take for $H_i(S,\mathbb{Z})$ the i-th Borel–Moore homology groups, which are defined in terms of the simplicial homology groups of the one-point compactification of S, and which are known to be invariants under semialgebraic homeomorphisms [Bochnak et al. 1987]. Note that, even though some of the early results on bounding the Betti numbers of semialgebraic sets were stated only over $\mathbb{R}$, the bounds can be shown to hold over any real closed field by judicious applications of the Tarski–Seidenberg transfer principle. We refer the reader to [Basu et al. 2003] (Chapter 7) for more details.

2. Early Bounds

For a polynomial $P \in R[X_1,\ldots,X_k]$, we denote by $\mathrm{Z}(P,\mathrm{R}^k)$ the set of zeros of P in R^k. The first results on bounding the Betti numbers of algebraic sets are due to Oleinik and Petrovsky [1949; 1951; 1949a; 1949b]. They considered the problem of bounding the Betti numbers of a nonsingular real algebraic hypersurface in $\mathbb{R}^k$ defined by a single polynomial equation of degree d. More precisely, they prove that the sum of the even Betti numbers, as well as the sum of the odd Betti numbers, of a nonsingular real algebraic hypersurface in R^k defined by a polynomial of degree d are each bounded by $\frac{1}{2}d^k +$ lower order terms. Independently, Thom [1965] proved a similar bound of $\frac{1}{2}d(2d-1)^{k-1}$ on the sum of all the Betti numbers of $\mathrm{Z}(P,\mathbb{R}^k)$, where P is only assumed to be nonnegative over $\mathbb{R}^k$ without the assumption that $\mathrm{Z}(P,\mathbb{R}^k)$ is a nonsingular hypersurface. Milnor [1964] also proved the same bound in the case $\mathrm{Z}(P,\mathbb{R}^k)$ is an arbitrary real algebraic subset. Moreover, he proved a bound of $(sd)(2sd-1)^{k-1}$ on the sum of the Betti numbers of a basic semialgebraic set defined by the conjunction of s weak inequalities $P_1 \geq 0,\ldots,P_s \geq 0$, with $P_i \in \mathbb{R}[X_1,\ldots,X_k], \deg(P_i) \leq d$. Note that there is a cost for generality: the bounds of Thom and Milnor are slightly weaker (in the leading constant) than those proved by Oleinik and Petrovsky. Note also that these bounds on the sum of the Betti numbers of an algebraic set are tight, since the solutions to the system of equations,

$$(X_1-1)(X_1-2)\cdots(X_1-d) = \cdots = (X_k-1)(X_k-2)\cdots(X_k-d) = 0,$$

or equivalently of the single equation

$$\big((X_1-1)(X_1-2)\cdots(X_1-d)\big)^2 + \cdots + \big((X_k-1)(X_k-2)\cdots(X_k-d)\big)^2 = 0,$$

consist of d^k isolated points and the only nonzero Betti number of this set is $b_0 = d^k$.

The method used to obtain these bounds is based on a basic fact from Morse theory – that the sum of the Betti numbers of a compact, nonsingular, hypersurface in $\mathbb{R}^k$ is at most the number of critical points of a well chosen projection. In case of a nonsingular real algebraic variety, the critical points of a projection map satisfy a simple system of algebraic equations obtaining by setting the polynomial defining the hypersurface, as well as $k-1$ different partial derivatives to zero. The number of solutions to such a system can be bounded from above by Bezout's theorem. The case of an arbitrary real algebraic variety (not necessarily compact and nonsingular) is reduced to the compact, nonsingular case by carefully using perturbation arguments.

Even though the bounds mentioned above are bounds on the sum of all the Betti numbers, in different combinatorial applications it suffices to have bounds only on the zero-th Betti number (that is the number of connected components). For instance, given a finite set of polynomials $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, a natural question is how many of the 3^s sign conditions in $\{0, 1, -1\}^{\mathcal{P}}$ are actually realized at points in R^k. We define

$$\begin{cases} \operatorname{sign} x = 0 & \text{if and only if } x = 0, \\ \operatorname{sign} x = 1 & \text{if and only if } x > 0, \\ \operatorname{sign} x = -1 & \text{if and only if } x < 0. \end{cases}$$

Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$. A sign condition on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. A strict sign condition on $\mathcal{P}$ is an element of $\{1, -1\}^{\mathcal{P}}$. We say that $\mathcal{P}$ realizes the sign condition σ at $x \in \mathrm{R}^k$ if

$$\bigwedge_{P \in \mathcal{P}} \operatorname{sign} P(x) = \sigma(P).$$

The realization of the sign condition σ is

$$\mathcal{R}(\sigma) = \left\{ x \in \mathrm{R}^k \;\middle|\; \bigwedge_{P \in \mathcal{P}} \operatorname{sign} P(x) = \sigma(P) \right\}.$$

The sign condition σ is realizable if $\mathcal{R}(\sigma)$ is nonempty.

Warren [1968] proved a bound of $(4esd/k)^k$ on the number of strict sign conditions realized by a set of s polynomials in $\mathbb{R}^k$ whose degrees are bounded by d. Alon [1995] extended this result to all sign conditions by proving a bound of $(8esd/k)^k$. The fact that these bounds are polynomial in s (for fixed values of k) is important in many applications. Note that this bound is tight since it is an easy exercise to prove that the number of sign conditions realized by a family of linear polynomials in general position is

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{s}{i} \binom{s-i}{j}; \tag{2–1}$$

see for example [Basu et al. 2003].

3. Early Applications

One of the first applications of the bounds of Oleinik–Petrovsky, Thom and Milnor, was in proving lower bounds in theoretical computer science. The model for computation was taken to be algebraic decision trees. Given an input $x \in \mathbb{R}^k$, an algebraic decision tree decides membership of x in a certain fixed semialgebraic set $S \subset \mathbb{R}^k$. Starting from the root of the tree, at each internal node, v, of the tree, it evaluates a polynomial $f_v \in \mathbb{R}[X_1, \ldots, X_k]$ (where $\deg(f_v) \leq d$, for some fixed constant d), at the point $(x_1, \ldots, x_k)$ and branches according to the sign of the result. The leaf nodes of the tree are labelled as accepting or rejecting. On an input $x \in \mathbb{R}^k$, the algebraic decision tree accepts x if and only if the computation terminates at an accepting leaf node. Moreover, an algebraic decision tree tests membership in S, if it accepts x if and only if $x \in S$. The main idea behind using the Oleinik–Petrovsky, Thom and Milnor bounds in proving lower bounds for the problem of testing membership in a certain semialgebraic set $S \subset \mathbb{R}^k$ is that if the set S is topologically complicated, then an algebraic decision tree testing membership in it has to have large depth.

Ben-Or [1983] proved that the depth of an algebraic computation tree testing membership in S must be $\Omega(\log b_0(S))$. Several extensions of this result were proved by Yao [1995; 1997]. He proved that instead of $b_0(S)$ one could use in fact the Euler characteristic of S (which is the alternating sum of the Betti numbers), as well as the sum of the Betti numbers of S. This made the theorem useful for proving lower bounds for a wider class of problems by including sets with a single connected component but complicated topology [Montaña et al. 1991]. Another early application of the Oleinik–Petrovsky, Thom and Milnor bounds was in proving upper bounds on the number of order types of simple configurations of points in $\mathbb{R}^k$. Given an ordered set, S, of s points in $\mathbb{R}^k$, the order type of S is determined by the $\binom{s}{k+1}$ orientations of the $\binom{s}{k+1}$ oriented simplices spanned by $(k+1)$-tuples of points. A point configuration is simple if no $k+1$ of them are affinely dependent. Using Milnor's bound on the Betti numbers of basic semialgebraic sets Goodman and Pollack [1986b] proved an upper bound of s^{k^2} on the number of realizable simple order types of s points in $\mathbb{R}^k$ [Goodman and Pollack 1986a] rather than the trivial bound of 2^s. as well as on the number of combinatorial types of simple polytopes with s vertices in $\mathbb{R}^k$ [Goodman and Pollack 1986a]. In fact, Milnor's bound actually yields a bound on the number of isotopy classes of simple configurations of s points in $\mathbb{R}^k$. The isotopy class of a point configuration in $\mathbb{R}^k$ consists of all point configurations in $\mathbb{R}^k$ having the same order type which are reachable by continuous order type preserving deformations of the original point configuration. Alon [1995] extended these bounds to all configurations – not necessarily simple ones.

All of these applications are based on the simple observation that different strict sign conditions must belong to different connected components. Any situation where *geometric types* can be characterized by a sign condition gives an application of this type. Two other application in this spirit are bounds on the number of *weaving patterns* of lines [Pach et al. 1993] and the size of a grid which will support all order types of s points in the plane [Goodman et al. 1989; 1990].

4. Modern Bounds

Pollack and Roy [1993] proved a bound of $\binom{s}{k}O(d)^k$ on the number of connected components of the realizations of all realizable sign conditions of a family of s polynomials of degrees bounded by d. The proof was based on Oleinik–Petrovsky, Thom and Milnor's results for algebraic sets, as well as with deformation techniques and general position arguments.

From this bound one can deduce a tight bound on the number of isotopy classes of *all* point configurations in $\mathbb{R}^k$ (not just the simple ones). Note that Warren's bound mentioned before is a bound on the number of realizable strict sign conditions (extended by Alon to all sign conditions) but not on the number of connected components of their realizations. Thus, Warren's (or Alon's) bounds cannot be used to bound the number of isotopy classes (of simple or nonsimple configurations).

In some applications, notably in geometric transversal theory as well in bounding the complexity of the configuration space in robotics, it is useful to study the realizations of sign conditions of a family of s polynomials in $\mathrm{R}[X_1, \ldots, X_k]$ restricted to a real variety $\mathrm{Z}(Q, \mathrm{R}^k)$ where the real dimension of the variety $\mathrm{Z}(Q, \mathrm{R}^k)$ can be much smaller than k. In [Basu et al. 1996] it was shown that the number of connected components of the realizations of all realizable sign condition of a family, $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ of s polynomials, restricted to a real variety of dimension k', where the degrees of the polynomials in $\mathcal{P} \cup \{Q\}$ are all bounded by d, is bounded by $\binom{s}{k'}O(d)^k$.

There are also results bounding the sum of the Betti numbers of semialgebraic sets defined by a conjunction of weak inequalities. Milnor [1964] proved a bound of $(sd)(2sd-1)^{k-1}$ on the sum of the Betti numbers of a basic semialgebraic set defined by the conjunction of s weak inequalities $P_1 \geq 0, \ldots, P_s \geq 0$, with $P_i \in \mathbb{R}[X_1, \ldots, X_k]$ such that $\deg(P_i) \leq d$. In another direction, Barvinok [1997] proved a bound of $k^{O(s)}$ on the sum of the Betti numbers of a basic, closed semialgebraic set defined by polynomials of degree at most 2. Unlike all previous bounds, this bound is polynomial in k for fixed values of s.

Extending such bounds to arbitrary semialgebraic sets is not trivial, because Betti numbers are not additive and the union of two topologically trivial semialgebraic sets can clearly have arbitrarily large higher Betti numbers. Basu [1999] proved a bound on the sum of the Betti numbers of a $\mathcal{P}$-closed semialgebraic set on a variety. A $\mathcal{P}$-closed semialgebraic set is one defined by a Boolean formula without negations whose atoms are of the form $P \geq 0$ or $P \leq 0$ with $P \in \mathcal{P}$. The bound is $s^{k'}O(d)^k$. Very recently Gabrielov and Vorobjov [$\geq$ 2005], succeeded in removing even the $\mathcal{P}$-closed assumption at the cost of a slightly worse bound.

They showed that the sum of the Betti numbers of an arbitrary semialgebraic set defined by a Boolean formula whose atoms are of the form $P = 0$, $P > 0$ or $P < 0$ with $P \in \mathcal{P}$, is bounded by $O(s^2d)^k$.

There have been recent refinements of the bounds on the Betti numbers of semialgebraic sets in another direction. All the bounds mentioned above are either bounds on the number of connected components or on the sums of all (or even or odd) Betti numbers. Basu [2003] proved different bounds (for each i) on the i-th Betti number of a basic, closed semialgebraic set on a variety. If S is a basic closed semialgebraic set defined by s polynomials in $\mathrm{R}[X_1, \ldots, X_k]$ of degree d, restricted to a real variety of dimension k' and defined by a polynomial of degree bounded by d, then $b_i(S)$ is bounded by $\binom{s}{k'-i}O(d)^k$. In the same paper, a bound of $s^\ell k^{O(\ell)}$ on the $(k-\ell)$-th Betti number of a basic, closed semialgebraic set defined by polynomials of degree at most 2 is proved. For fixed ℓ this bound is polynomial in both s and k. More recently, in [Basu et al. 2005] the authors bound (for each i) the sum of the i-th Betti number over all realizations of realizable sign conditions of a family of polynomials restricted to a variety of dimension k' by

$$\sum_{1 \le j \le k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

This generalizes and makes more precise the bound in [Basu et al. 1996] which is the special case with $i = 0$. The technique of the proof uses a generalization of the Mayer–Vietoris exact sequence.

All the bounds on the Betti numbers of semialgebraic sets described above, depend on the degrees of the polynomials used in describing the semialgebraic set. However, it is well known that in the case of real polynomials of one variable, the number of real zeros can be bounded in terms of the number of monomials appearing in the polynomial (independent of the degree). This is an easy consequence of Descartes' law of signs [Basu et al. 2003]. Hence, it is natural to hope for a similar result in higher dimensions. Khovansky [1991] proved a bound of $2^{m^2}(mk)^k$ on the number of isolated real solutions of a system of k polynomial equations in k variables in which the number of monomials appearing with nonzero coefficients is bounded by m. Using this, one can obtain similar bounds on the sum of the Betti numbers of an algebraic set defined by a polynomial with at most m monomials in its support. The semialgebraic case requires some additional technique and it was shown in [Basu 1999] that the sum of the Betti numbers of a $\mathcal{P}$-closed semialgebraic set on a variety, is bounded by $s^{k'}2^{O(km^2)}$, where m is a bound on the number of monomials.

5. Modern Applications

Using [Pollack and Roy 1993] one immediately obtains reasonably tight bounds on the number of isotopy classes of not necessarily simple geometric objects such as the number of isotopy classes (with respect to order type) of configurations of n points in R^k or the number of isotopy classes (with respect to combinatorial type) of $k-$polytopes with n vertices.

Using [Basu et al. 1996], Goodman, Pollack, and Wenger [Goodman et al. 1996] were able to extend the known bounds on the number of *geometric permutations* (1-order types) induced by line transversals ($\ell = 1$) to the number of ℓ-order types induced by ℓ-flat transversals to n convex sets in R^3. As is the case for line transversals in $\mathbb{R}^3$, the lower bounds are about the square root of the upper bounds (in the plane, the corresponding result is tight [Edelsbrunner and Sharir 1990]). A much fuller discussion of Geometric Transversal Theory can be found in [Goodman et al. 1993].

6. Algorithms

A natural algorithmic problem is to design efficient algorithms for computing the Betti numbers of a given semialgebraic set. Clearly the problem of deciding whether a given semialgebraic set is empty is NP-hard, and counting its number of connected component is #P-hard. However, in view of the bounds described above we could hope for an algorithm having complexity polynomial in the number of polynomials and their degrees and singly exponential in the number of variables. This seems to be a very difficult problem in general and only partial results exist in this direction.

The cylindrical algebraic decomposition [Collins 1975] makes it possible to compute triangulations, and thus the number of connected components [Schwartz and Sharir 1983] as well as the higher Betti numbers in time polynomial in the number of polynomials and their degrees and doubly exponential in the number of variables (see [Basu et al. 2003]).

Various singly exponential time algorithms have been obtained for finding a point in every connected component of an algebraic set [Canny 1988b; Renegar 1992], of a semialgebraic set [Grigor'ev and Vorobjov 1988; Canny 1988b; Heintz et al. 1989; Renegar 1992], in every connected component of the sign conditions defined by a family of polynomials on a variety [Basu et al. 1997].

Computing the exact number of connected components in singly exponential time is a more difficult problem. The notion of a roadmap introduced by Canny [1988a] is the key to the solution. The basic algorithm has since been generalized and refined in several papers [Canny 1988a; 1993; Grigor'ev and Vorobjov 1992; Heintz et al. 1994; Gournay and Risler 1993; Basu et al. 2000] (see [Basu et al. 2003] for more details). Single exponential algorithms for computing the Euler–Poincaré characteristic (which is the alternating sum of the Betti numbers) of algebraic (as well as $\mathcal{P}$-closed semialgebraic) sets are described in [Basu 1999]. However, the problem of computing all the Betti numbers in single exponential time remains open.

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