Inside Out: Inverse Problems
MSRI Publications
Volume **47**, 2003

The Attenuated X-Ray Transform: Recent Developments

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Abstract. We survey recent work on the attenuated x-ray transform, concentrating especially on the inversion formulas found in the last few years.

1. Introduction

The attenuated x-ray transform is a variant of the classical x-ray transform in which functions are integrated over straight lines with respect to an exponential weight. It arises as a model in single photon emission computed tomography (SPECT) and in the study of the stationary linear single speed transport equation. Let a, f be continuous functions of compact support in $\mathbb{R}^n$ and let θ be a unit vector. We define the divergent beam x-ray transform of a at x in direction θ by

$$Da(x,\theta) = \int_0^\infty a(x+s\theta)\,ds,$$

where the integration is with respect to arc length. The attenuated x-ray transform of f is a function on the space of directed lines, whose value on the line l with direction θ is given by

$$P_a f(l) = \int_l f(y(\tau)) e^{-Da(y(\tau),\theta)}\,d\tau,$$

where $y(\tau)$ is an arc length parametrization of l. When the attenuation, a, is identically zero, the attenuated x-ray transform reduces to the ordinary x-ray transform. In the model of single photon emission tomography, the function f represents the spatial density of emitters which are assumed to emit photons isotropically. The function a is the linear attenuation coefficient, and so the attenuated x-ray transform is supposed to represent the photon intensity at a detector, collimated to accept only photons which have travelled along a specific line. A useful survey of the physics can be found in [9]. (The density of emitters is called the activity distribution, and so some authors denote it by a, whereas we use a for attenuation.)

In the plane, lines are often parametrized by their unit normal and directed distance from the origin, as are hyperplanes in higher dimensions. In that case, the attenuated x-ray transform is usually called the attenuated Radon transform and given by

$$R_a f(\omega, p) = \int_{x\cdot\omega=p} f(x)e^{-Da(x,\omega^\perp)}dx, \tag{1–1}$$

where $\omega^\perp = (-\sin\phi, \cos\phi)$ if $\omega = (\cos\phi, \sin\phi)$. The exponential Radon transform in $\mathbb{R}^2$ is given by

$$E_\mu f(\omega, p) = \int_{x\cdot\omega=p} f(x)e^{\mu x\cdot\omega^\perp}\,dx.$$

If the attenuation a is a constant on a convex set containing the support of f, then the attenuated Radon transform can be expressed in terms of the exponential Radon transform, and vice versa. The theory of the exponential Radon (or x-ray) transform is far more complete than that of the attenuated x-ray transform. The attenuated x-ray transform itself is a special case of the generalized x-ray transform, where the measure $e^{-Da}\,ds$ is replaced by a general measure $\mu(y,\theta)\,ds$. Even in this setting, Boman [7] has produced an example of a smooth measure on lines in the plane so that the associated generalized Radon transform has non-trivial kernel. One may further pass from lines in Euclidean space to curves on (or submanifolds of) a manifold. There has been a great deal of work done on such generalized Radon transforms, and many open questions remain, but we will only touch upon these extensions in this paper.

There are several inverse problems which can be posed for the attenuated x-ray transform. The simplest, and the one which has recently been solved, is the linear problem of recovering the activity f when the attenuation a is known. A much harder problem is to determine both f and a from $P_a f$. Given the resolution of the linear inverse problem, this amounts to determining a from $P_a f$. This is called the identification problem, and it is easy to see that there are rotationally invariant pairs of distinct a and f which give the same measurements, even when a is constant. Nonetheless, some progress has been made if one assumes that f has a special structure, e.g. a sum of delta functions, [28; 29] or a sum of point measures and an L^p function, [6]. In the special case of the exponential Radon transform in two dimensions, it has been proved that the identification problem has a unique solution if and only if f is not radial, [39; 17]. While we won't discuss the identification problem further in this article, it is worth saying that one of the tools used by Natterer is a set of consistency conditions for the range of the attenuated transform. The range is a topic discussed in Section 4, and its application to the identification problem has been an important motivation in its study.

The first uniqueness results which applied to the attenuated transform were of local nature. Local uniqueness for a generalized Radon transform R_μ means that each point x has a neighborhood U_x so that no non-trivial f supported in U_x lies in the kernel of R_μ. The size of U_x usually depends on some norms of the measure defining the generalized Radon transform, and of its derivatives. Examples of such results can be found in [23; 24]. A method yielding stronger local uniqueness results in the plane, as well as uniqueness for some problems of integral geometry, was introduced by Mukhometov [25] in the mid 70's. It was based on energy type estimates for a boundary value problem for a partial differential equation arising from a transport equation formulation of the integral transform. The method of Mukhometov was adapted by the author, [13], in the special case of the attenuated x-ray transform to prove uniqueness when the product of the diameter of the support of the activity and the supremum norm of the attenuation was not too large. Subsequently, Mukhometov's method was systematized and extended by Sharafutdinov and collaborators. An account may be found in his book [35]. Sharafutdinov also considered, see for example [36], the uniqueness problem for the attenuated x-ray transform on a class of Riemannian manifolds with boundary, where the integrals are taken over geodesics of the metric. The results are of the form that if the the integral over all geodesic segments joining boundary points of a weighted average of the attenuation and a geometric quantity depending on sectional curvatures is not too large, then the x-ray transform is injective. To our knowledge, these papers of Sharafutdinov are the only works on the attenuated x-ray transform on manifolds.

The theory for the exponential transform is fairly complete, but will not be much discussed in this paper. The first analytic inversion formulas were found in the late 1970's, [5; 42]. Recently, [31], an inversion procedure when the data is only collected for a range of 180° has been found (implementation is based on truncation of a Neumann series); this paper also has a good bibliography on inversion methods. The range was first characterized, in a complicated manner, by Kuchment and L'vin, [19], with later simplifications and extensions by Kuchment and coworkers appearing in [2; 1] and elsewhere. Their most recent contribution, [12], discusses a differential equation range characterization for a family of transforms which encompasses the exponential transform.

To the author's knowledge, most practical reconstruction in SPECT is done using iterative methods. In conventional x-ray tomography the greater speed and provable convergence properties of analytic methods have generally outweighed the benefits of iterative methods. In SPECT, where the photon flux is much smaller, the statistics of the emission process must be taken into account. The flexibility of iterative methods better allows them to account for these effects, to incorporate prior information, and to be adapted to incomplete sampling geometries. Of course, the price is that very little can be proved. The reader who wants to pursue this side of the subject might start by scanning some recent issues of *IEEE Transactions on Medical Imaging* or *Physics in Medicine and Biology*.

In the last five years, several exact inversion formulas have been found for the attenuated x-ray transform, as well as some results on characterization of the range of the transform. This paper is devoted to a survey of these results. In section 2 we introduce some standard notations and review some background results from complex analysis. In the next section, we sketch the methods of proof of the various inversion formulas. Section 4 is devoted to the range results, and the last section mentions some open problems.

2. Background and Preliminaries

Each of the inversion formulas makes use in some way of boundary values of analytic functions defined in a region in the complex plane. We recall a few results which we will need later. The first result is the Plemelj–Sokhozki formulas for the boundary values of an analytic function defined by a Cauchy integral. Suppose that L is a C^1 oriented simple path or simple closed curve in the complex plane, and that g is Hölder continuous of order α on L for some positive α. Let $G(z)$ be defined for $z \in \mathbb{C} \setminus L$ by the Cauchy integral

$$G(z) = \frac{1}{2\pi i} \int_L \frac{g(t)}{t - z}\, dt.$$

Then G is holomorphic in $\mathbb{C} \setminus L$ and the following formulas hold. For a proof we refer the reader to [26]. The existence of the principal value integrals is part of the assertion.

Proposition 2.1. *Let g, G, and L be as above. If $t_0 \in L$ and is not either endpoint in the case when L is not closed, then the limit of G from the left of L exists and is given by*

$$G_+(t_0) = \tfrac{1}{2} g(t_0) + \frac{1}{2\pi i} \int_L \frac{g(t)}{t - t_0}\, dt,$$

where the integral on L is taken in principal value sense. Similarly the limit from the right exists and is given by

$$G_-(t_0) = -\tfrac{1}{2} g(t_0) + \frac{1}{2\pi i} \int_L \frac{g(t)}{t - t_0}\, dt.$$

Corollary 2.1. *Let g be Hölder continuous of order $\alpha > 0$ on the unit circle, and let G be Cauchy integral of g, as above, for L the unit circle oriented counterclockwise. Then for ω in the unit circle,*

$$G_+(-\omega^\perp) - G_+(\omega^\perp) = \tfrac{1}{2}(g(-\omega^\perp) - g(\omega^\perp)) + \tfrac{1}{2\pi i}\, \text{p.v.} \int_{S^1} \frac{1}{\omega \cdot \theta} g(\theta)\, d\theta. \quad (2\text{–}1)$$

PROOF. Let $\zeta = e^{i\sigma}$ where $\omega = (\cos\sigma, \sin\sigma)$ and $w = e^{i\psi}$ for $\theta = (\cos\psi, \sin\psi)$. Then, by the Plemelj–Sokhozki formula,

$$G_+(-i\zeta) - G_+(i\zeta) = \tfrac{1}{2}(g(-i\zeta) - g(i\zeta)) + \frac{1}{2\pi i}\,\text{p.v.}\int_{S^1}\Big(\frac{1}{w+i\zeta} - \frac{1}{w-i\zeta}\Big) g(w)\,dw.$$

Combining terms in the integral and using $dw = iw\,d\psi$, the integral becomes

$$\text{p.v.}\int_{S^1} \frac{2w\zeta}{w^2+\zeta^2}\, g(e^{i\psi})\,d\psi.$$

But $\omega\cdot\theta = \frac{1}{2}(\bar\zeta w + \bar w \zeta)$, which in turn is equal to $\dfrac{w^2+\zeta^2}{2w\zeta}$, since both w and ζ lie on the unit circle. □

For f a smooth function with compact support on $\mathbb{R}$ the Hilbert transform of f is defined by the principal value integral

$$Hf(x) = \frac{1}{\pi}\,\text{p.v.}\int \frac{f(t)}{x-t}\,dt.$$

The Hilbert transform extends to a bounded operator on L^p, for $1 < p < \infty$.

We shall use $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz space of infinitely differentiable functions f on $\mathbb{R}^n$ which satisfy $D^\alpha f$ is $O((1+|x|)^{-k})$ for every natural number k and for every derivative D^α. The space of oriented lines in the plane can be parametrized by $\mathbb{S}^1\times\mathbb{R}$ either in x-ray coordinates where (θ, s) corresponds to the line $s\theta^\perp + \mathbb{R}\theta$ or in Radon coordinates in which (θ, s) corresponds to $s\theta + \mathbb{R}\theta^\perp$. In either case, we define the Schwartz space of the space of lines to be the infinitely differentiable functions g on $\mathbb{S}^1\times\mathbb{R}$ such that $\partial_\theta^k\partial_s^j g(\theta,s)$ is $O((1+|s|)^{-n})$ for every $n\in\mathbb{N}$.

We shall frequently use the complex differential operators $\frac{\partial}{\partial z}$, also abbreviated ∂, and $\frac{\partial}{\partial \bar z}$, abbreviated $\bar\partial$, defined as follows. In $\mathbb{R}^2$ with standard coordinates x and y,

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

With this notation, the Cauchy–Riemann equations for $f = u + iv$ are written simply as $\bar\partial f = 0$. Let D be an open set in the plane with C^1 boundary. If $g \in C_0^1(R^2)$ and $\zeta\in D$ then

$$g(\zeta) = -\frac{1}{\pi}\int_D \frac{\partial g}{\partial \bar z}(x,y)\frac{1}{z-\zeta}\,dx\,dy + \frac{1}{2\pi i}\int_{\partial D} g(x,y)\frac{1}{z-\zeta}\,dz. \tag{2–2}$$

There is an analogous formula with kernel $(\bar z - \bar\zeta)^{-1}$ involving $\partial g/\partial z$ obtained by conjugating the preceding equation after g is replaced by $\bar g$. If D is the entire plane, this shows that $(\pi z)^{-1}$ is a fundamental solution of the $\bar\partial$ operator, and $(\pi\bar z)^{-1}$ is a fundamental solution for ∂. For details see [18].

The function

$$h(\theta, s) = \tfrac{1}{2}(I + iH)Ra(\theta, s) \tag{2–3}$$

plays a role in all of the inversion formulas. In this definition, I is the identity operator, R is the Radon transform, and the Hilbert transform H is applied to Ra in the second variable. Its importance was first discovered by Natterer [27; 28] in his work on consistency conditions for the range of the attenuated transform. We have adopted his Radon parametrization rather than the x-ray parametrization for that reason. Note that $h(\theta, x\cdot\theta)$ is constant on oriented lines, but not independent of orientation since $HRa(\theta, x\cdot\theta)$ is odd in θ. Natterer proved the following lemma, with a different proof. A proof similar to the one given here was found by Boman and Strömberg [8].

LEMMA 2.1. *The coefficients of the Fourier expansion in the angular variable of the function* $h(\theta, x\cdot\theta) - Da(x, \theta^\perp)$ *are zero for negative or even index.*

PROOF. Since $Ra(\theta, x\cdot\theta) = Da(x, -\theta^\perp) + Da(x, \theta^\perp)$, it needs to be shown that

$$\tfrac{1}{2}(Da(x, -\theta^\perp) - Da(x, \theta^\perp)) + \tfrac{i}{2} HRa(\theta, x\cdot\theta)$$

has the desired property. Writing the Hilbert transform of the Radon transform as an iterated integral, and changing to polar coordinates yields

$$HRa(\theta, x\cdot\theta) = -\frac{1}{\pi}\,\text{p.v.}\int_{S^1} \frac{1}{\theta\cdot\omega} Da(x, \omega)\, d\omega.$$

By the corollary to the Plemelj–Sokhozki relations (2–1) the combination is the boundary value of an analytic function, and so has only non-negative Fourier coefficients. Since it is also an odd function, the result follows. □

In fact more can be said: if $\sum_{k\in\mathbb{Z}} m_k(x) e^{ik\phi}$ is the Fourier series expansion of $Da(x, -\theta^\perp)$, the Fourier expansion of $h(\theta, x\cdot\theta) - Da(x, \theta^\perp)$ is

$$\sum_{d>0} m_{2d+1}(x) e^{i(2d+1)\phi}.$$

This is the form (after a rotation) in which the expression enters in [4].

3. Uniqueness and Inversion

In this section we will present the inversion formulas found in the last five years. They are all formulated in two dimensions, but that is sufficient, since in higher dimensions one may restrict the full attenuated x-ray transform to lines in a family of planes whose union is the full space. (Whether in higher dimensions there exist other families of lines yielding inversion formulas is an open question.) We will present the formulas in order of discovery. Several of the authors have also stated uniqueness results for limited angle data, in which it is supposed that the attenuated x-ray transform is known only for lines with directions lying in a proper subset of the circle. The specific statements will be mentioned below.

3.1. The approach of Arbuzov, Bukhgeĭm, and Kazantsev. The first result, due to Arbuzov, Bukhgeĭm, and Kazantsev [4], is an application of the theory of A-analytic functions developed by Bukhgeĭm and collaborators. A summary of this theory may be found in [11]. The analysis begins with a formulation of the attenuated transform as a transport problem. Let f be the activity distribution and a the attenuation, and let Ω be a bounded convex set in the plane with smooth boundary. For a point $x \in \Omega$, let $\tau_{\pm}(x,\theta)$ be the point of intersection of the boundary and the ray from x in the direction $\pm\theta$. Let ∇ denote the gradient in space and consider the stationary transport equation

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = f(x). \tag{3–1}$$

This equation may be integrated along lines in direction θ to obtain

$$e^{-Da(y,\theta)}u(y,\theta)\Big|_{\tau_-(x,\theta)}^{\tau_+(x,\theta)} = \int_{\tau_-}^{\tau_+} e^{-Da(y,\theta)} f(y)\,ds(y), \tag{3–2}$$

where the integral extends over the segment of the line through x in direction θ lying in Ω. The right hand side is the attenuated x-ray transform of f and a restricted to Ω. Thus knowing the boundary values of a solution of (3–1) determines the attenuated x-ray transform (since a is assumed known). Since (3–1) is a parametrized family of ordinary differential equations, the forward problem does not have a unique solution without some specification of initial conditions. For example, if the incoming flux is given, $u(x,\theta) = u_0(x,\theta)$ for $x \in \partial\Omega$ and $\theta \cdot \nu(x) < 0$, for the $\nu(x)$ the outer normal at x, then the solution is unique. Then supposing the attenuated transform is known, (3–2) completes the specification of the boundary values of u. The questions of uniqueness and inversion of the attenuated x-ray transform are then transferred to the questions of uniqueness and inversion for f from boundary values for the transport equation.

We will now present a proof of the inversion formula of Arbuzov, Bukhgeĭm, and Kazantsev. Their proof is an application of the theory of A-analytic functions, but we have chosen to avoid them by working with Fourier series expansions directly. This loses the elegance and some of the power of the approach taken by these authors, but it shows clearly how easily the result may be attained. One word of caution: we have stayed with the conventions of Fourier analysis and write $g(\theta) \sim \sum_{k\in Z} g_k e^{ik\phi}$ whereas Arbuzov, Bukhgeĭm, and Kazantsev write $\sum g_k e^{-ik\phi}$.

Returning to the transport equation, it is clear that one may use any nonzero multiple of $e^{-Da(x,\theta)}$ as an integrating factor. Let $b(x,\theta) = h(-\theta^{\perp}, -x\cdot\theta^{\perp})$, where h is given in (2–3). Although it is not needed in what follows, a calculation similar to that in (2.1) shows that $b(x,\theta) = \frac{1}{2}(I - iH)Pa(\theta, x\cdot\theta^{\perp})$, where Pa is the parallel beam transform of a. Then b is constant on lines in direction θ and so $e^{b(x,\theta)-Da(x,\theta)}$ is also an integrating factor. Let $v(x,\theta) = e^{b(x,\theta)-Da(x,\theta)}u(x,\theta)$. Then v satisfies the equation $\theta\cdot\nabla v = f(x)e^{b(x,\theta)-Da(x,\theta)}$, which may be written in complex form, identifying $(x_1,x_2)\in\mathbb{R}^2$ with $z=x_1+ix_2$ and setting $\theta=(\cos\phi,\sin\phi)$, as

$$e^{-i\phi}\frac{\partial v}{\partial\bar z}+e^{i\phi}\frac{\partial v}{\partial z}=f(z)e^{b(z,\theta)-Da(z,\theta)}. \tag{3–3}$$

By Lemma 2.1, the Fourier coefficients of $b(x,\theta)-Da(x,\theta)$ are zero for negative (or even) index, and so the Fourier coefficients of $e^{b(x,\theta)-Da(x,\theta)}$ are zero for negative index. Thus if $v=\sum v_k e^{ik\phi}$ is substituted in (3–3) and Fourier coefficients are equated, there results the system of equations

$$\frac{\partial v_{k+1}}{\partial\bar z}+\frac{\partial v_{k-1}}{\partial z}=\begin{cases}0 & \text{for } k<0,\\ f(z)\gamma_k & \text{for } k\ge 0,\end{cases} \tag{3–4}$$

where the expansion of e^{-G}, with $G=Da(x,\theta)-b(x,\theta)$, is $\sum_{k\ge0}\gamma_k e^{ik\phi}$.

The aim of the following calculations is to show that each v_k for $k\le0$, can be expressed in terms of the boundary values of v_j for $j\le k$. Since these are given in terms of u and e^{-G} on the boundary, and since for $k\le0$, the Fourier coefficient u_k of u is the k-th Fourier coefficient of $e^G v$ which is expressible in terms of the γ_l and the Fourier coefficients v_j for $j\le k$, we can determine everywhere u_{-1} and u_0. These determine f by

$$2\operatorname{Re}\frac{\partial u_{-1}}{\partial z}+a(z)u_0=f(z),$$

which results from writing the transport equation $\theta\cdot\nabla u+au=f$ in complex form, and separating Fourier coefficients, as was worked out above for v. Here it is also used that u is real valued, and so $\partial u_{-1}+\bar\partial u_1=2\operatorname{Re}\partial u_{-1}$.

Let

$$\rho_k(x,\phi)=\sum_{j=0}^{\infty}v_{k-2j}(z)e^{i(k-2j)\phi}.$$

We assume that Ω is an open bounded convex set with C^1 boundary, and that ρ_k is C^1 in Ω and continuous on the closure. Let $\zeta\in\Omega$ and for each ϕ let $l(\phi)$ be the length of the ray from ζ to the boundary in direction $e^{i\phi}$. Denote by $w(\phi)=\zeta+l(\phi)e^{i\phi}$ the point where the ray meets the boundary. Then, for $k\le0$,

$$\begin{aligned}\rho_k(w(\phi),\phi)-\rho_k(\zeta,\phi)&=\int_0^l\frac{\partial\rho_k}{\partial s}(\zeta+se^{i\phi},\phi)\,ds\\&=\int_0^l\left(\frac{\partial\rho_k}{\partial\bar z}e^{-i\phi}+\frac{\partial\rho_k}{\partial z}e^{i\phi}\right)ds\\&=\int_0^l\sum_{j=0}^{\infty}\frac{\partial v_{k-2j}}{\partial\bar z}e^{i(k-2j-1)\phi}+\sum_{j=0}^{\infty}\frac{\partial v_{k-2j}}{\partial z}e^{i(k-2j+1)\phi}\,ds\end{aligned}$$

$$= \int_0^l \frac{\partial v_k}{\partial z} e^{i(k+1)\phi} + \sum_{j=0}^{\infty} \left(\frac{\partial v_{k-2j}}{\partial \bar{z}} + \frac{\partial v_{k-2j-2}}{\partial z} \right) e^{i(k-2j-1)\phi}\, ds$$

$$= \int_0^{l(\phi)} \frac{\partial v_k}{\partial z} e^{i(k+1)\phi}\, ds,$$

where the series in the second to last integral is zero by (3–4). From this $v_k(\zeta)$ is found:

$$\begin{aligned} v_k(\zeta) &= \frac{1}{2\pi} \int_0^{2\pi} \rho_k(\zeta,\phi) e^{-ik\phi}\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\rho_k(w(\phi),\phi) - \int_0^{l(\phi)} \frac{\partial v_k}{\partial z} e^{i(k+1)\phi}\, ds \right) e^{-ik\phi}\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \rho_k(w(\phi),\phi) e^{-ik\phi}\, d\phi - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{l(\phi)} \frac{\partial v_k}{\partial z} \frac{1}{s e^{-i\phi}} s\, ds\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=0}^{\infty} v_{k-2j}(w(\phi),\phi) e^{-2ij\phi}\, d\phi - \frac{1}{2\pi} \int_\Omega \frac{\partial v_k}{\partial z} \frac{1}{\bar{z} - \bar{\zeta}}\, dA. \end{aligned} \tag{3–5}$$

By the conjugate form of (2–2),

$$-\frac{1}{2\pi} \int_\Omega \frac{\partial v_k}{\partial z} \frac{1}{\overline{z - \zeta}}\, dA = \tfrac{1}{2} v_k(\zeta) + \frac{1}{4\pi i} \int_{\partial\Omega} v_k(w) \frac{1}{\bar{w} - \bar{\zeta}}\, d\bar{w}.$$

From $w(\phi) = \zeta + l(\phi) e^{i\phi}$,

$$e^{-i2\phi} = \frac{\overline{w - \zeta}}{w - \zeta}, \qquad d\phi = \frac{1}{2i} \left(\frac{1}{w - \zeta}\, dw - \frac{1}{\bar{w} - \bar{\zeta}}\, d\bar{w} \right).$$

Substituting these into (3–5) and gathering terms gives

$$\begin{aligned} v_k(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \Biggl(dw \frac{1}{w - \zeta} \sum_{j=0}^{\infty} v_{k-2j}(w) \left(\frac{\overline{w - \zeta}}{w - \zeta} \right)^j \\ - d\bar{w} \frac{1}{\bar{w} - \bar{\zeta}} \sum_{j=1}^{\infty} v_{k-2j}(w) \left(\frac{\overline{w - \zeta}}{w - \zeta} \right)^j . \Biggr) \end{aligned} \tag{3–6}$$

Recalling that $v = e^{-G} u$ and that we have reversed the indexing of [4], this is the k-th component of the equation in Theorem 4.3 of [4].

3.2. Novikov's inversion formula. In the late spring of 2000, Novikov circulated a manuscript with an inversion formula for the attenuated transform. A revised version with some additional results was written in the fall and has now appeared in [33]. A published announcement with an outline of the proof appears in [32]. The paper makes heavy use of the boundary value distributions $(x \pm i0)^{-1}$ and becomes notationally dense when operators are dressed with $\pm$ and direction subscripts. We will first present Novikov's formula in his own notation, and then modify the notation to accord with that used in this paper. Suppose that a and f are Hölder continuous of order α, for some $\alpha \in (0,1)$, that there is an $\varepsilon > 0$ such that they are $O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$ and that $\sup_{0<|y|\le 1} |y|^{-\alpha}\bigl|f(x+y)-f(x)\bigr|$ is also $O(|x|^{-1-\varepsilon})$, and similarly for a. Then, using the notations $D_\theta u(x) := Du(x,\theta)$, $P_\theta^\perp u(s) := Pu(\theta, s\theta^\perp)$, $P_{a,\theta}^\perp u(s) := P_a u(\theta, s\theta^\perp)$ and

$$H_\pm v(s) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(t)}{s \pm i0 - t}\,dt$$

and

$$\exp\bigl(\pm(2i)^{-1}H_\mp P_\theta^\perp a\bigr)v(s) = \exp\bigl(\pm(2i)^{-1}H_\mp P_\theta^\perp a(s)\bigr)v(s),$$

Novikov's formula reads

$$f(x) = -\frac{1}{4\pi}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\int_{S^1}\varphi(x,\theta)(\theta_1 + i\theta_2)\,d\theta,$$

where

$$\varphi(x,\theta) = \exp\bigl(-D_{-\theta}a(x)\bigr)m(x\cdot\theta^\perp,\theta),$$

with m given by

$$\begin{aligned} m(s,\theta) &= m_+(s,\theta) - m_-(s,\theta)\\ &= (2i)^{-1}\exp\bigl(-(2i)^{-1}H_+P_\theta^\perp a(s)\bigr)H_+\bigl(\exp\bigl((2i)^{-1}H_-P_\theta^\perp a\bigr)P_{a,\theta}^\perp f\bigr)(s)\\ &\quad -(-2i)^{-1}\exp\bigl((2i)^{-1}H_-P_\theta^\perp a(s)\bigr)H_-\bigl(\exp\bigl(-(2i)^{-1}H_+P_\theta^\perp a\bigr)P_{a,\theta}^\perp f\bigr)(s). \end{aligned} \tag{3–7}$$

It may be easily shown that

$$-\frac{1}{2i}H_+ = \tfrac12(I + iH), \qquad \frac{1}{2i}H_- = \tfrac12(I - iH),$$

which is useful when trying to compare Novikov's development with the work of others. Further since a and f are real, m_+ and m_- are conjugate, so m is pure imaginary.

Novikov's proof is also based on reformulation of the problem as a scattering problem for the transport equation

$$\theta\cdot\nabla u + au = f. \tag{3–8}$$

Let $\psi^+ = \psi^+(x,\theta)$ be the solution of the transport equation satisfying

$$\lim_{s\to-\infty}\psi^+(x + s\theta,\theta) = 0.$$

Then $\lim_{s\to\infty}\psi^+(x+s\theta,\theta) = P_a f(\theta, x\cdot\theta^\perp)$ which identifies the attenuated transform as scattering data. Let Σ be the complex quadric in $\mathbb{C}^2$ given by $\Sigma = \{(\theta_1,\theta_2)\in\mathbb{C}^2 : \theta_1^2 + \theta_2^2 = 1\}$. (This intersects the real space in the unit circle.) Novikov shows that for $\theta = (\theta_1,\theta_2)\in\Sigma\setminus\mathbb{S}^1$ there is a unique solution ψ of the complex transport equation

$$\theta\cdot\nabla\psi(x,\theta) + a(x)\psi(x,\theta) = f(x), \qquad x\in\mathbb{R}^2,$$

which satisfies $\psi(x,\theta) \to 0$ as $|x| \to \infty$. An explicit formula is given (see (3–11) below). It is not remarked by Novikov, but simplifies some of his later results, to observe that

$$\bar{\psi}(x,\theta) = \psi(x,\bar{\theta}), \tag{3–9}$$

which follows directly from conjugating the differential equation and using the uniqueness assertion. The next step is to study the limit of these solutions as θ tends to the real space. (This is the most delicate part of the analysis.) It is proved that for $\theta \in \mathbb{S}^1$, the limits

$$\psi_{\pm}(x,\theta) = \lim_{0<\tau\to 0} \psi(x,\omega(\pm\tau))$$

for $\omega(\tau) = \sqrt{1+\tau^2}\theta + i\tau\theta^{\perp}$ with τ real and the positive square root, exist and are continuous, and satisfy the real transport equation (3–8), with initial conditions

$$\lim_{s\to-\infty} \psi_{\pm}(x+s\theta,\theta) = m_{\pm}(x\cdot\theta^{\perp},\theta),$$

where $m_{\pm}$ were defined in (3–7). It follows from (3–9) that ψ_+ and ψ_- are conjugate. The difference $\psi_+ - \psi_-$ is then a solution of the homogeneous transport equation with value at $-\infty$ equal to $m_+ - m_-$. Since

$$\varphi(x,\theta) = e^{-Da(x,-\theta)}(m_+ - m_-)$$

is the unique such solution, it must hold that

$$\varphi(x,\theta) = \psi_+ - \psi_-.$$

(This is the φ of the inversion formula.) Next the quadric Σ is seen to have the holomorphic parametrization $\theta(\lambda)$, given by

$$\theta_1 = \frac{\lambda+\lambda^{-1}}{2}, \qquad \theta_2 = \frac{\lambda-\lambda^{-1}}{2i},$$

for $\lambda \in \mathbb{C}\setminus\{0\}$. Moreover, the unit circle $T \subset \mathbb{C}$ corresponds to the unit circle $S^1 \subset \mathbb{R}^2$, with the interior of the unit circle in $\mathbb{C}$ mapping to the the subset of Σ parametrized above by $\omega(\tau)$, for $\tau > 0$ (and all θ). It is easy to check that $\theta(\lambda)$ also satisfies

$$\theta(1/\bar{\lambda}) = \bar{\theta}(\lambda). \tag{3–10}$$

It is then shown that for each $x \in \mathbb{R}^2$, $\psi(x,\theta(\lambda))$ is holomorphic in λ for $\lambda \in \mathbb{C}\setminus\{0\cup T\}$, and that the limit from inside the circle (resp. from outside the circle) correspond to the boundary values $\psi_{\pm}$, respectively. The analyticity results from the specific form of the solution of the complex transport equation, referred to above. Here it is appropriate to indicate the form: for $\theta = \theta(\lambda)$ one has

$$\psi(x,\theta(\lambda)) = \int_{\mathbb{R}^2} e^{-G_{\theta(\lambda)}a(x)} G(x-y,\theta(\lambda)) e^{G_{\theta(\lambda)}a(y)} f(y)\,dy, \tag{3–11}$$

where $(x_1, x_2) \in \mathbb{R}^2$ is identified with $z = x_1 + ix_2 \in \mathbb{C}$,

$$G(z, \theta(\lambda)) = \frac{\operatorname{sgn}(1 - |\lambda|)}{2\pi i(i/2)(\lambda\bar{z} - z/\lambda)},$$

for $\lambda \neq 0, \lambda \notin T$, and $G_\theta a(x)$ is the convolution $\int G(x-w, \theta) a(w)\, dw$. Moreover, from (3–9) and (3–10),

$$\psi(x, \theta(\lambda)) = \bar{\psi}(x, \theta(\bar{\lambda}^{-1})),$$

so the Laurent expansion in the punctured circle determines that in the exterior of the circle, and vice versa. Looking at the kernel, $G(x, \theta(\lambda))$, it is clear that the expansion around $\lambda = 0$ has the form, with $\zeta = y_1 + iy_2$ and $z = x_1 + ix_2$

$$\psi(x, \theta(\lambda)) = \lambda \int \frac{f(\zeta)}{\pi(z - \zeta)}\, dy + O(\lambda^2),$$

and so the expansion at infinity is

$$\psi(x, \theta(\lambda)) = \bar{\psi}(x, \theta(\bar{\lambda}^{-1})) = \lambda^{-1} \int \frac{f(\zeta)}{\pi(\overline{z - \zeta})}\, dy + O(\lambda^{-2}).$$

Using this last relation and taking the limit as the contour shrinks to the unit circle we get

$$\begin{aligned}\int \frac{f(\zeta)}{\pi(\overline{z - \zeta})}\, dy &= \frac{1}{2\pi i} \int_T \psi_-(x, \theta(\lambda))\, d\lambda \\ &= -\frac{1}{2\pi i} \int_T (\psi_+(x, \theta(\lambda)) - \psi_-(x, \theta(\lambda)))\, d\lambda \\ &= -\frac{1}{2\pi} \int_{S^1} (\psi_+(x, \theta) - \psi_-(x, \theta))(\theta_1 + i\theta_2)\, d\theta,\end{aligned}$$

which finishes the proof of the formula, since $(\pi\bar{z})^{-1}$ is a fundamental solution for ∂.

Since the constant terms of the Laurent expansions inside and outside the circle are both zero, a similar chain of equalities shows that

$$0 = \frac{1}{2\pi i} \int_T (\psi_+ - \psi_-) \frac{d\lambda}{\lambda} = \frac{1}{2\pi} \int_{S^1} (\psi_+(x, \theta) - \psi_-(x, \theta))\, d\theta. \tag{3–12}$$

This is a consistency condition which will be used later.

Novikov proves a limited angle theorem assuming that f is continuous with compact support. If $P_a f(\theta, s)$ is known for all $s \in \mathbb{R}$ and θ in a set of positive length, then f is uniquely determined.

The inversion formula of Novikov has been implemented by Kunyansky, [22], with further particulars in [21]. Kunyansky also shows that in the case of constant attenuation Novikov's formula reduces to the inversion formula for the exponential transform given by Tretiak and Metz, [42].

3.3. The results of Natterer and of Boman and Strömberg. Natterer, [30], works with the attenuated Radon transform, (1–1), and proves the inversion formula

$$f(x) = \frac{1}{4\pi} \operatorname{Re} \operatorname{div} \int_{S^1} \theta e^{Da(x,\theta^\perp)} e^{-h(\theta, x\cdot\theta)} H(e^h R_a f)(\theta, x \cdot \theta)\, d\theta,$$

where h is given by (2–3) and the Hilbert transform H is applied to $e^h R_a f$ in the second variable. This formula can be shown to be equivalent to Novikov's formula after changing from x-ray to Radon coordinates and taking account of the parities of the constituent functions under change of sign in the argument (these occur since $R_a f(\theta, s) = P_a f(\theta^\perp, -s)$). Natterer's proof is very economical: our prose description will be nearly as long as his full exposition.

It begins with a few lemmas. The first is the result given above as (2.1) on the Fourier coefficients of $u(x, \theta) = h(\theta, x\cdot\theta) - Da(x, \theta^\perp)$, and the second, which evaluates the integrals $\int_0^{2\pi} \frac{\theta}{x\cdot\theta} e^{il\phi}\, d\phi$ (with $\theta = (\cos\phi, \sin\phi)$), is easily derivable from (2–1). The proof of the main theorem has two steps. The first expresses the integrand in the inversion formula as

$$\theta e^{-u(x,\theta)} H(e^h R_a f)(\theta, x\cdot\theta) = \frac{1}{\pi} \int_{R^2} f(y) \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, dy$$

(there are two minor misprints in his equation (2.4)) and then shows that

$$\operatorname{Re} \int_{S^1} \frac{\theta}{(x-y)\cdot\theta} e^{u(y,\theta)-u(x,\theta)}\, d\theta = 2\pi \frac{x-y}{|x-y|^2}$$

which is a multiple of a fundamental solution of the divergence operator. The proof of this formula follows by expanding $e^{u(y,\cdot\,)-u(x,\cdot\,)}$ in Fourier series, applying the lemma on the evaluation of the integrals $\int \frac{\theta}{x\cdot\theta} e^{il\phi}\, d\phi$, and observing that only the $l = 0$ term of the Fourier expansion contributes to the real part. Natterer does not formulate a precise theorem on the necessary regularity of f, but notes that the formula does hold pointwise for $f \in C_0^1(\mathbb{R}^2)$. The conditions on a are even less specific: only that it is sufficiently smooth and of sufficiently rapid decay at infinity. Natterer has implemented the formula and presents an example reconstruction.

The work of Boman and Strömberg, [8], does not yet have its final form, so we can only indicate the preliminary results given by Boman in lecture. First they prove an inversion formula for continuous functions with compact support in an open set Ω for the generalized Radon transform

$$R_\rho f(\theta, p) = \int_{x\cdot\theta=p} f(x)\rho(\theta, x)\, ds, \qquad (\theta, p) \in \mathbb{S}^1 \times \mathbb{R},$$

for complex measures $\rho(x, \theta)$ such that for each $x \in \Omega$, $\rho(x, \cdot\,)$ extends to a continuous nowhere zero function on the unit disk which is analytic on the open disk, ρ is Hölder continuous on $\Omega \times T$, and such that $\arg \rho(x, \theta)$ is constant on each oriented line $x \cdot \theta = p$ for $x \in \Omega$. With $m(x) = \frac{1}{2\pi} \int \rho(x, \theta)\, d\theta$ the (angular) mean value of ρ, assumed to be real, the inversion formula takes the form

$$f(x) = \frac{1}{4\pi m(x)} \operatorname{div} \left(m(x) \operatorname{Re} \int_{S^1} \theta (H R_\rho f)(\theta, x \cdot \theta) \frac{1}{\rho(x, \theta)}\, d\theta \right).$$

They then observe that their argument can be applied to any measure ρ_0 for which there is a nowhere zero function τ, constant on oriented lines, such that $\rho = \tau \rho_0$ satisfies the conditions above. They prove that if $\rho_0 = e^q$ is real, and $q(x, \theta) = w(x, \theta) + u(x, \theta)$ for real u, w where u and the conjugate function $\tilde{w}$ are constant on all oriented lines $x \cdot \theta = p$ in Ω, then the trick applies with $\tau = e^{-u + i\tilde{w}}$. It is further shown that for the attenuated transform, where $q = -Da(x, \theta^\perp)$, this kind of decomposition holds, so their inversion formula applies. They also prove a limited angle theorem which states that for the generalized Radon transform R_ρ, with ρ satisfying the conditions above, if $R_\rho f(\theta, p) = 0$ for all $p \in \mathbb{R}$ and θ in an non-empty open subset of the circle for some compactly supported continuous function f, then f must be identically zero.

3.4. Additional remarks. We add a few remarks on the preceding formulas and cite a new paper that has just come to our attention.

REMARK 3.1. The analysis of the stationary transport equation (3–1) is an important ingredient in the study of inverse problems for the stationary transport equation with scattering. This seems to have the motivation for much of the work by the Novosibirsk group. Indeed, it was in this context that (to our knowledge) it was first observed, see [3], that local uniqueness for the attenuated transform in dimension two implies global uniqueness for compactly supported activity distributions in higher dimensions. We refer the reader to the recent thesis and paper by Tamasan, [41; 40], for applications of the recent inversion formulas for the attenuated transform to inverse problems for the more general transport equations, and for further references.

REMARK 3.2. The results of Natterer and of Boman and Strömberg provide more direct proofs of inversions formulas of Novikov type, but they clearly owe their formulation to the work of Novikov. It is not hard to modify Natterer's Fourier series expansion to arrive at the fundamental solution of ∂ instead of working with the real part of the vector adjoint to arrive at a fundamental solution for the divergence operator.

REMARK 3.3. The inversion procedure given by Arbuzov, Bukhgeĭm, and Kazantsev is fairly complicated, because it requires evaluating u_0 and u_{-1}, both of which require the full sequence of $\{v_k\}$ for $k \leq 0$. One explanation may be that the method is underspecified. The derivation leading up to (3–6) holds no matter what choices are made in (3–2) to complete the boundary values (e.g. zero incoming or some other), and it would be no surprise if some choice might lead to a more efficient inversion formula.

Remark 3.4. In a recent work, [10], which came to the author too late for a full treatment in this paper, A. A. Bukhgeim and Kazantsev have found an inversion formula adapted to divergent beam geometry, where it is supposed that the attenuation and activity are supported in the unit disk, and lines are parametrized by direction and point of intersection with the unit circle. The formula makes use of the circular Hilbert transform Γ, given by

$$\Gamma q(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cot(\phi - \alpha)}{2} q(\alpha)\, d\alpha,$$

with respect to both direction at points on the measurement circle, and with respect to position on the measurement circle. Assuming that f is square integrable on the unit disk and that a is C^2 on the closed unit disk, they state and outline the proof of the following:

$$f(z) = \frac{\partial}{\partial \bar{z}} \frac{i}{\pi} \int_0^{2\pi} e^{-i\phi} e^{-m(z,\phi)} \operatorname{Im}\bigl(e^{-2(Da)^+(\gamma(z,\phi),\phi)} v^*(\gamma(z,\phi),\phi)\bigr)\, d\phi,$$

where for t in the unit circle,

$$v^*(t,\phi) = (I + i\Gamma)(v(\,\cdot\,,\phi) - \tfrac{1}{2} v(\gamma(\,\cdot\,,\phi+\pi),\phi))(t)$$

with $v(t,\phi) = e^{-2(Da)^+(t,\phi)} D_a f(t,\phi)$. Here the notations are $\theta = (\cos\phi, \sin\phi)$, $m(z,\phi) = Da(z,-\theta)$, $\gamma(z,\phi)$ is the point of intersection of the unit circle with the ray from z in the direction $-\theta$, $D_a f(z,\phi) = \int_{-\infty}^0 f(z+s\theta) \exp(-Da(z+s\theta,-\theta))\, ds$, and $(Da)^+(z,\phi) = \frac{1}{2}(I - i\Gamma) m^{\text{odd}}(z,\cdot)(\phi)$, where m^{odd} is the odd part of $m(z,\phi)$ with respect to the angular variable.

4. Range Characterization

Prior to presenting what is known about range characterization for the attenuated x-ray transform, it is valuable to recall what is known for the classical x-ray and Radon transforms. The results are most easily stated for the Radon transform. The chief theorem, which was first proved fully by Helgason, [16], gives a characterization of the range of the Radon transform on $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}^n$.

Theorem 4.1. *A function $g \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ is in the range of the Radon transform on $\mathcal{S}(\mathbb{R}^n)$ if and only if*

(i) *g is even, that is, $g(\theta,p) = g(-\theta,-p)$ for all $(\theta,p) \in \mathbb{S}^{n-1} \times \mathbb{R}$, and*
(ii) *for each natural number m the function $p_m(\theta) = \int_{\mathbb{R}} g(\theta,p) p^m\, dp$ is the restriction to $\mathbb{S}^{n-1}$ of homogeneous polynomial on $\mathbb{R}^n$.*

The last set of conditions are called the moment conditions. They also enter what are called the Paley–Wiener type theorems for the Radon transform, which characterize the range of the transform on various spaces of compactly supported functions. There are analogous results on the range of the classical x-ray transform in higher dimensions, [37], but also characterizations of the range as the solution space of certain differential equations.

In the mid-80's, Solmon, [38], gave a very nice extension of the range theorem, which said that if g is an even smooth function of rapid decay on $\mathbb{S}^{n-1} \times \mathbb{R}$, then g is the Radon transform of a function f which is $O(|x|^{-n})$, moreover f is $O(|x|^{-n-m-1})$, $m \geq 0$, as $|x| \to \infty$ if and only if g satisfies the moment conditions through order m. Furthermore, any k-th order derivative of f is $O(|x|^{-n-1-m-k})$ as $|x| \to \infty$.

For the attenuated Radon transform, the first result found was an analogue of the moment conditions. Natterer, [27; 28], showed the following,

THEOREM 4.2. *Let $k > m \geq 0$ be integers. If $g = R_a f$ for $a, f \in \mathcal{S}(\mathbb{R}^2)$ then*

$$\int_{\mathbb{R}} \int_0^{2\pi} p^m e^{ik\phi + 1/2(I+iH)Ra} g(\omega, p)\, dp\, d\omega = 0,$$

where as usual, $\omega = (\cos\phi, \sin\phi)$. (Additional conditions result from taking the conjugate, since g is real.)

The proof follows from writing out the integral defining the attenuated Radon transform, applying Fubini's theorem, and (2.1) on the vanishing of the negative Fourier coefficients of $e^{h(x, x\cdot\theta) - Da(x, \theta^\perp)}$. About ten years later, Kuchment and L'vin gave a characterization of the range of the exponential Radon transform, [19]. One consequence was that Natterer's conditions are not sufficient to characterize the range, even for the case of constant attenuation. Something which was obviously lacking was the analogue of the evenness condition in the classical range theorem. Novikov, [33; 32] found such a condition, and has proved, [34], the following theorem.

THEOREM 4.3. *Let a, g be in the Schwartz space $\mathcal{S}(\mathbb{S}^1 \times \mathbb{R})$ and let g satisfy*

$$\mathrm{Re} \int_{S^1} e^{-Da(x, -\theta)} e^{1/2(I+iH)Pa(\theta, x\cdot\theta^\perp)} H_+ (e^{1/2(I-iH)Pa(\theta, \cdot)} g(\theta, \cdot))\, (\theta^\perp \cdot x)\, d\theta = 0 \quad \text{for } x \in \mathbb{R}^2. \tag{4–1}$$

Then there is a C^∞ function f such that f and all its derivatives are $O(|x|^{-2})$ as $|x| \to \infty$ with $P_a f = g$.

The necessity of (4–1) was established in (3–12). The scheme of the proof is to define f by Novikov's inversion formula, using g in place of $P_a f$, and then to prove that the resulting function has the required decay and that its image under the attenuated transform is g. In the case of the classical x-ray transform, (4–1) takes the form

$$0 = \int_{S^1} Hg(\theta, x \cdot \theta^\perp)\, d\theta \text{ for all } x \in \mathbb{R}^2.$$

It is obvious that if $g = Pf$ then the conditions are satisfied, for then Pf is even and so $HPf(\theta, x \cdot \theta^{\perp})$ is odd in θ, but the converse takes some work. It suffices to prove that under the hypothesis $0 = \int_{S^1} Hg(\theta, x \cdot \theta^{\perp})\, d\theta$ the function g must be even, for one can then apply Solmon's theorem, but the author has not found a simple proof of this.

Arbuzov, Bukhgeĭm, and Kazantsev give a different range result, whose consequence and interpretation in the context of the attenuated transform have not, to the author's knowledge, been fully worked out. (Kuchment has some ideas on the matter. A discussion, along with many other matters related to the present paper, can be found in [20].) Recalling the Plemelj–Sokhozki relations from Section 2, one sees that a Hölder continuous function g on the boundary of a simply connected domain with smooth boundary is the boundary value of a function analytic in the domain (its Cauchy integral) if and only if the principal value integral over the boundary is one-half the value of the function on the boundary. The same result holds for Hölder continuous functions on the boundary, taking values in X^{m+2} of the scale of Banach space defining A-analyticity. (See [4] for further explanation of the following.) Since the transmuted function $e^{-G}u$ is A-analytic, if the boundary values are Hölder continous, they must be equal to twice the principal value integral over the boundary.

5. Open Problems

In the author's opinion, the "most wanted" of the open problems is to find an explicit set of consistency conditions which characterize the range of the attenuated transform on functions of compact support or rapid decay. As mentioned above, Novikov has proved the necessity of (4–1), and its sufficiency for a rapidly decaying smooth function on the space of lines to be the attenuated transform of a function with quadratic decay. Natterer's conditions, (4.2), are also necessary range conditions for functions of rapid decay. It is unknown whether the union of these conditions is sufficient. There are some implicit conditions, such as the function produced by the inversion formula has the desired property, but these do not seem useful for application elsewhere, such as to the identification problem. An allied question would be the existence of a support theorem for the attenuated transform. If f is continuous and decays faster than any reciprocal power of $|x|$ and $P_a f(l) = 0$ for every directed line l disjoint from some compact convex set K, is f supported in K?

Inversion from partial data. In the plane, Novikov and Boman–Strömberg proved uniqueness for compactly supported activity distributions when the line directions are restricted to a non-empty open subset of the circle. Natterer has posed the question of whether there exists a stable inversion formula when the directions comprise half the circle. If such could be found, it could help make analytic methods more competitive with iterative methods in clinical applications.

The analogous problem for the exponential transform has been treated by Noo and Wagner in [31].

In higher dimensions, for what submanifolds of the manifold of lines can inversion formulas be derived? Gel'fand, Gindikin, and Shapiro [14] have given conditions on curve families with densities in the plane for admissibility, and for curve families in higher dimensions this was extended in [15]. These papers are concerned with a geometric condition (admissibility) which corresponds to a certain differential form being closed. In complex space this can lead to local inversion formulae, though not in real space. For the attenuated transform one might start by making a similar analysis on submanifolds of oriented curves.

As mentioned in the introduction, there has been a lot of work on generalized Radon transforms on manifolds, but very little specifically about the attenuated x-ray transform. In particular, the uniqueness problem remains open. One might hope also for an inversion formula, perhaps under more stringent hypotheses.

Acknowledgments

The author would like to thank Jan Boman, Alexander Bukhgeĭm, Frank Natterer, Alex Tamasan and Gunther Uhlmann for helpful discussions at MSRI. He also thanks the Mathematical Sciences Research Institute, MSRI, of Berkeley, CA for enabling his participation in the program in Inverse Problems during fall 2001. He would also like to thank Professor R. Novikov for sending him preprints of the papers [34; 33] and R. Clackdoyle for stimulating his interest in these problems.

References

[1] V. Aguilar, L. Ehrenpreis, and P. Kuchment, *Range conditions for the exponetial Radon transform*, J. d'Anal. Math. **68** (1996), 1–13.

[2] V. Aguilar and P. Kuchment, *Range conditions for the multidimensional exponential X-ray transform*, Inverse Problems **11** (1995), 977–982.

[3] D. S. Anikonov, *Multidimensional inverse problems for the transport equation*, Differentsial′nye Uravneniya **20** (1984), no. 5, 817–824, English trans. Differential Equations, 20, 608–614. MR 86f:35171

[4] E. V. Arbuzov, A. L. Bukhgeĭm, and S. G. Kazantsev, *Two-dimensional tomography problems and the theory of A-analytic functions*, Siberian Advances in Mathematics **8** (1998), 1–20.

[5] S. Bellini, M. Pacentini, C. Cafforio, and F. Rocca, *Compensation of tissue absorbtion in emission tomography*, IEEE Trans. Acoust. Speech Signal Process. **27** (1979), 213–218.

[6] J. Boman, *On generalized Radon transforms with unknown measures*, Integral Geometry and Tomography (Arcata, CA) (E. Grinberg and E. T. Quinto, eds.), Contemporary Mathematics, vol. 113, AMS, Providence R.I., 1990, pp. 5–15.

[7] ———, *An example of nonuniqueness for a generalized Radon transform*, J. Anal. Math. **61** (1993), 395–401. MR 94j:44004

[8] J. Boman and J.-O. Strömberg, *Novikov's inversion formula for the attenuated Radon transform – a new approach*, In preparation.

[9] T. F. Budinger, G. T. Gullberg, and R. H. Huesman, *Emission computed tomography*, Image Reconstruction from Projections (G. T. Herman, ed.), Topics in Applied Physics, vol. 32, Springer, Berlin - New York, 1979, pp. 147–246.

[10] A. A. Bukhgeim and S. G. Kazantsev, *The attenuated Radon transform inversion formula for divergent beam geometry*, preprint, 6 pp., 2002.

[11] A. L. Bukhgeĭm, *Supplement: Inversion formulas in inverse problems*, pp. 323–376, in M. M. Lavrent'ev and L. Ya. Savel'ev, *Linear Operators and Ill-Posed Problems*, Consultants Bureau, N.Y., 1995.

[12] L. Ehrenpreis, P. Kuchment, and A. Panchenko, *Attenuated Radon transform and F. John's equation I: Range conditions*, Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis (E. L. Grinberg, S. Berhanu, M. Knopp, G. Mendoza, and E.T. Quinto, eds.), Contemporary Mathematics, vol. 251, Amer. Math. Soc., 2000, pp. 173–188.

[13] D. V. Finch, *Uniqueness for the attenuated x-ray transform in the physical range*, Inverse Problems (1986), 197–203.

[14] I. M. Gel'fand, S. G. Gindikin, and Z. Ya. Shapiro, *A local problem of integral geometry*, Functional Analysis and its Applications **13** (1979), 87–102.

[15] I. M. Gel'fand and M. I. Graev, *Admissible n-dimensional complexes of curves in* $\mathbf{R}^n$, Funct. Anal. Appl. **14** (1980), 274–281.

[16] S. Helgason, *The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds*, Acta Math. **113** (1965), 153–180.

[17] A. Hertle, *The identification problem for the constantly attenuated Radon transfrom*, Math. Z. **197** (1988), 13–19.

[18] L. Hörmander, *The analysis of linear partial differential operators I*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[19] P. A. Kuchment and S. Ya. L'vin, *Paley-Wiener theorem for the exponential Radon transform*, Acta Appl. Math. **18** (1990), 251–260.

[20] P. A. Kuchment and E. T. Quinto, *Some problems of integral geometry arising in tomography*, chapter 11 in L. Ehrenpreis' forthcoming book on the Radon Transform.

[21] L. A. Kunyanksy, *Analytic reconstruction algorithms in emission tomography with variable attenuation*, J. Comp. Math. in Sci. and Engr., to appear.

[22] L. A. Kunynasky, *A new SPECT reconstruction algorithm based on the Novikov explict inversion formula*, Inverse Problems **17** (2001), 293–306.

[23] M. M. Lavrent'ev and A. L. Bukhgeĭm, *On a class of problems of integral geometry*, Dokl. Akad. Nauk SSSR **211** (1973), 38–39, Engl. trans. in Soviet Math Dokl., **14** (1973).

[24] A. Markoe and E. T. Quinto, *An elementary proof of local invertibility for generalized and attenuated Radon transforms*, SIAM J. Math. Anal. **16** (1985), 1114–1119.

[25] R. G. Mukhometov, *The problem of recovery of a two-dimensional Riemannian metric and integral geometry*, Dokl. Akad. Nuak SSSR **232** (1977), 32–35, English trans. in Soviet Math. Dokl. **18** (1977), pp. 27–31.

[26] N. I. Muskhelishvili, *Singular integral equations*, Noordhoff, Groningen, 1953.

[27] F. Natterer, *The ill-posedness of Radon's integral equation*, Symposium on Ill-Posed Problems: Theory and Practice, Newark, DE, 1979.

[28] ______, *The identification problem in emission computed tomography*, Mathematical aspects of computerized tomography (Oberwolfach, 1980), Springer, Berlin, 1981, pp. 45–56. MR 84m:92008

[29] ______, *Computerized tomography with unknown sources*, SIAM J. Appl. Math. **43** (1983), no. 5, 1201–1212. MR 86i:92011

[30] ______, *Inversion of the attenuated Radon transform*, Inverse Problems **17** (2001), 113–119.

[31] F. Noo and J.-M. Wagner, *Image reconstruction in 2D SPECT with* 180° *acquisition*, Inverse Problems **17** (2001), 1357–1371.

[32] R. G. Novikov, *Une formule d'inversion pour la transformation d'un rayonnement X atténué*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 12, 1059–1063. MR 2002e:44003

[33] ______, *An inversion formula for the attenuated X-ray transformation*, Ark. Mat. **40** (2002), 145–167.

[34] ______, *On the range characterization for the two-dimensional attenuated X-ray transformation*, Inverse Problems **18** (2002), 677–700.

[35] V. A. Sharafutdinov, *Integral geometry of tensor fields*, VSP, Utrecht, 1994. MR 97h:53077

[36] ______, *On emission tomography of inhomogeneous media*, SIAM J. Appl. Math. **55** (1995), no. 3, 707–718. MR 96e:44004

[37] D. C. Solmon, *The x-ray transform*, J. Math. Anal. Appl. **56** (1976), 61–83.

[38] ______, *Asymptotic formulas for the dual Radon transform and applications*, Math. Z. **195** (1987), 321–343.

[39] ______, *The identification problem for the exponential Radon transform*, Math. Meth. Appl. Sci. **18** (1995), 687–695.

[40] A. Tamasan, *In inverse boundary value problem in two-dimensional transport*, Inverse Problems **18** (2002), 209–219.

[41] ______, *A two dimensional inverse boundary value problem in radiation transport*, Ph.D. thesis, U. Washington, 2002.

[42] O. J. Tretiak and C. Metz, *The exponential Radon transform*, SIAM J. Appl. Math. **39** (1980), 341–354.

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