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# Inverse Acoustic and Electromagnetic Scattering Theory

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Abstract. This paper is a survey of the inverse scattering problem for time-harmonic acoustic and electromagnetic waves at fixed frequency. We begin by a discussion of "weak scattering" and Newton-type methods for solving the inverse scattering problem for acoustic waves, including a brief discussion of Tikhonov's method for the numerical solution of ill-posed problems. We then proceed to prove a uniqueness theorem for the inverse obstacle problems for acoustic waves and the linear sampling method for reconstructing the shape of a scattering obstacle from far field data. Included in our discussion is a description of Kirsch's factorization method for solving this problem. We then turn our attention to uniqueness and reconstruction algorithms for determining the support of an inhomogeneous, anisotropic media from acoustic far field data. Our survey is concluded by a brief discussion of the inverse scattering problem for time-harmonic electromagnetic waves.

## 1. Introduction

The field of inverse scattering, at least for acoustic and electromagnetic waves, can be viewed as originating with the invention of radar and sonar during the Second World War. Indeed, as every viewer of World War II movies knows, the ability to use acoustic and electromagnetic waves to determine the location of hostile objects through sea water and clouds played a decisive role in the outcome of that war. Inspired by the success of radar and sonar, the prospect was raised of the possibility of not only determining the range of an object from the transmitter, but to also image the object and thereby identify it, i.e. to distinguish between a whale and a submarine or a goose and an airplane. However, it was soon realized that the problem of identification was considerably more difficult than that of simply determining the location of a target. In particular, not only was the identification problem computationally extremely expensive, and indeed beyond the capabilities of post-war computing facilities, but the problem was also ill-posed in the sense that the solution did not depend continuously on the measured data. It was not until the 1970's with the development of the mathematical theory of ill-posed problems by Tikhonov and his school in the Soviet Union and Keith Miller and others in the United States, together with the rise of high speed computing facilities, that the possibility of imaging began to appear as a realistic possibility. Since that time, the mathematical basis of the acoustic and electromagnetic inverse scattering problem has reached a level of maturity that the imaging hopes expressed in the early post-war years have to a certain extent been realized, at least in the case of electromagnetic waves with the invention of synthetic aperature radar [3], [8]. However, as the imaging demands have increased so have the mathematical and computational expectations and hence at this time it seems appropriate to make an attempt at describing the state of the art in the mathematical foundations of acoustic and electromagnetic inverse scattering theory. This article is directed towards that goal.

Before proceeding, a few caveats are perhaps in order. The first one is obvious: we are not proposing to survey the entire field of inverse scattering theory in a few pages. In particular, we will restrict our attention to a specific area, that is inverse scattering in the frequency domain and deterministic models. This means such important topics as time-reversal and scattering by random media are ignored. Even within this restrictive framework we will be selective and hence opinionated. In particular, our view is that the mathematical field of inverse scattering theory should remain close to the applications and in particular should have the numerical solution of practical imaging problems in "real time" as a high priority. Uniqueness theorems are important since they indicate what is possible to image in an ideal noise-free world but not all reconstruction algorithms are equally valuable from this point of view. Proceeding with such judgments, since the inverse scattering problem is ill-posed, restoring stability is clearly of central importance, but again not all stability results are of equal value. In particular, in order to restore stability some type of a priori information is needed and an estimate on the noise level is in general more realistic than the knowledge of, for example, an a priori bound on the curvature of the scattering object. It is freely acknowledged that points of view other than our own are both reasonable and legitimate and we are only emphasizing our own view here in order to warn the reader of what to expect in the pages that follow.

Before proceeding to a discussion of the inverse scattering problem and methods for its numerical solution we need to be clear on what inverse scattering problem we are talking about since depending on what a priori information is available there are many inverse scattering problems! For example, in using ultrasound to image the human body it is not unreasonable to assume that the density is known and equal to the density of water. In this case incident waves at a single fixed frequency are sufficient for imaging purposes whereas this is not the case if the density is not known a priori. On the other hand in imaging a target that has been (partially) coated by an unknown material, it is not reasonable to assume that the boundary condition on the surface of the scatterer is known.

Indeed, in my opinion, this last example is more typical in the sense that one usually knows neither the shape nor the material properties of a scattering object and hence neither the shape nor boundary conditions are known. Of course if a priori information on the material properties of the scatterer are known (as, for example, in the case of ultrasound imaging of the human body) it is usually beneficial to use an algorithm which makes use of this information.

The plan of this paper is as follows. Except for the final section, we shall concentrate on the scattering of time-harmonic acoustic waves at a fixed frequency. Hence in Section 2 we shall formulate the acoustic scattering problem and discuss various inverse scattering problems and their solution by either "weak-scattering" or Newton-type methods. These two methods are the work horses of inverse scattering and typically lead to the problem of solving linear integral equations of the first kind arising in either the weak-scattering approximation or in the computation of the Fréchet derivative of a nonlinear operator. With this as motivation we shall give a brief introduction to Tikhonov's method for the numerical solution of ill-posed problems.

The methods presented in Section 2 for solving acoustic inverse scattering problems rely on rather strong a priori information on the scattering object. In Section 3 we shall turn to more recent methods which avoid such strong assumptions but at the expense of needing more data. In particular, we shall concentrate on the case of obstacle scattering and prove the Kirsch–Kress uniqueness theorem [46] which in turn serves as motivation for the linear sampling method for determining the shape of the scatterer [12],[42]. We shall in addition present a recent optimization scheme of Kirsch which has certain attractive characteristics and is closely related to the linear sampling method [44].

In Section 4 we will first consider acoustic inverse scattering problems associated with an isotropic inhomogeneous medium and begin with the uniqueness theorems of Nachman [53], Novikov [57] and Ramm [66]. In this case special problems occur in the case of scattering in $R^2$. We will again discuss the linear sampling method for determining the support of the inhomogeneous scattering object, leading to an investigation of the existence, uniqueness and spectral properties of the interior transmission problem [13], [14], [19], [67]. We will then proceed to an extension of these results to the case of anisotropic media. In contrast to the case of isotropic media, variational methods rather than integral equation techniques are a more convenient tool in this case [6], [7], [31].

Finally, in Section 5, we consider the inverse scattering problem for Maxwell's equations and extend some of the results in the previous sections to this situation. However, much of what is known for the scalar case of acoustic waves remains unknown in the vector case and hence is a rich area for future study. To this end, we conclude our survey with a list of open problems for the electromagnetic inverse scattering problem.

## 2. The Inverse Scattering Problem for Acoustic Waves

We now consider the scattering of a time harmonic acoustic wave of frequency $\omega$ by an inhomogeneous medium of compact support $D$ having density $\rho_D(x)$ and sound speed $c_D(x), x \in D \subset R^3$. We assume that the boundary $\partial D$ is of class $C^2$ having unit outward normal $\nu$ (although much of the analysis which follows is also valid for Lipschitz domains-see [4],[6]) and that $\rho_D, c_D \in C^2(\bar{D})$. Then if the host medium is homogeneous with density $\rho$ and sound speed $c$, the wave number $k$ is defined by $k = \omega/c$,

$$n(x) = c/c(x), \; x \in D$$

and the pressure $p(x,t)$ is given by

$$p(x,t) = \begin{cases} \mathrm{Re}(u(x)e^{-i\omega t}) & \text{if } x \in R^3 \setminus D, \\ \mathrm{Re}(v(x)e^{-i\omega t}) & \text{if } x \in D, \end{cases}$$

then $u \in C^2(R^3 \setminus \bar{D}) \bigcap C^1(R^3 \setminus D)$ and $v \in C^2(D) \bigcap C^1(\bar{D})$ satisfy the acoustic transmission problem

$$\Delta u + k^2 u = 0 \qquad \text{in } R^3 \setminus \bar{D}, \tag{2–1a}$$

$$\rho_D(x)\nabla\left(\frac{1}{\rho_D(x)}\nabla v\right) + k^2 n(x) v = 0 \qquad \text{in } D, \tag{2–1b}$$

$$u = u^i + u^s, \tag{2–1c}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2–1d}$$

$$\begin{aligned} u &= v && \text{on } \partial D, \\ \frac{1}{\rho}\frac{\partial u}{\partial \nu} &= \frac{1}{\rho_D}\frac{\partial v}{\partial \nu} && \text{on } \partial D, \end{aligned} \tag{2–1e}$$

where $u^i$ is the incident field, which we assume is given by

$$u^i(x) = e^{ikx\cdot d}, \quad |d| = 1,$$

and the Sommerfeld radiation condition (2–1d) holds uniformly in $\hat{x} = x/|x|$, $r = |x|$.

To allow the possibility of absorption in $D$ we allow $n$ to possibly have a positive imaginary part; that is, in addition to $\mathrm{Re}\, n(x) > 0$ for $x \in D$ we require that

$$\mathrm{Im}\, n(x) \geq 0$$

for $x \in D$. The existence of a unique solution to (2–1) has been established by Werner [73].

For the sake of simplicity, we shall be concerned in this and the next two sections with certain special cases of the above transmission problem. In particular, if $\rho_D \to \infty$ we are led to the exterior Neumann problem for $u \in$

$C^2(R^3 \setminus \bar{D}) \cap C^1(R^3 \setminus D)$,

$$\Delta u + k^2 u = 0 \qquad \text{in } R^3 \setminus \bar{D}, \tag{2–2a}$$

$$u = u^i + u^s, \tag{2–2b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2–2c}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D; \tag{2–2d}$$

if $\rho_D \to 0$ we are led to the exterior Dirichlet problem for $u \in C^2(R^3 \setminus \bar{D}) \cap C(R^3 \setminus D)$,

$$\Delta u + k^2 u = 0 \qquad \text{in } R^3 \setminus \bar{D}, \tag{2–3a}$$

$$u = u^i + u^s, \tag{2–3b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2–3c}$$

$$u = 0 \qquad \text{on } \partial D; \tag{2–3d}$$

and if $\rho = \rho_D$ we are led to the inhomogeneous medium problem for $u \in C^1(R^3) \bigcap C^2(R^3 \setminus \partial D)$,

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } R^3, \tag{2–4a}$$

$$u = u^i + u^s, \tag{2–4b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2–4c}$$

where $n(x) = 1$ in $R^3 \setminus \bar{D}$.

For the purpose of exposition, in the sequel we shall restrict our attention to the exterior Dirichlet problem and inhomogeneous medium problem (the exterior Neumann problem can be treated in essentially the same way as the exterior Dirichlet problem).

We can now be more explicit about what we mean by the acoustic inverse scattering problem. In particular, using Green's theorem and the radiation condition it is easy to show that the scattered field $u^s$ has the representation

$$u^s(x) = \int_{\partial D} \left( u^s(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y) \tag{2–5}$$

for $x \in R^3 \setminus \bar{D}$ where $\Phi$ is the radiating fundamental solution to the Helmholtz equation (2–1a) defined by

$$\Phi(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, x \neq y. \tag{2–6}$$

From (2–5) and (2–6) we see that $u^s$ has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + O\left(\frac{1}{r^2}\right) \tag{2–7}$$

as $r \to \infty$ where $u_\infty$ is the *far field pattern* of the scattered field $u^s$. In the case of the exterior Dirichlet problem, the inverse scattering problem we will be concerned with is to determine $D$ from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}$ and $d$ on the unit sphere $\Omega := \{x : |x| = 1\}$ and fixed wave number $k$. For the inhomogeneous medium problem we will consider two inverse scattering problems, that of either determining $D$ from $u_\infty(\hat{x}, d)$ or $n(x)$ from $u_\infty(\hat{x}, d)$, again assuming that $k$ is fixed. In all cases, we will always assume (except in discussing uniqueness) that $u_\infty$ is not known exactly but is determined by measurements that by definition are inexact.

The inverse scattering problems defined above are particularly difficult to solve for two reasons: they are 1) nonlinear and 2) ill-posed. Of these two reasons, it is the latter that creates the most difficulty. In particular, it is easily verified that $u_\infty$ is an analytic function of both $\hat{x}$ and $d$ on the unit sphere and hence, for a given measured far field pattern (i.e. "noisy data"), in general no solution exists to the inverse scattering problem under consideration. On the other hand, if a solution does exist it does not depend continuously on the measured data in any reasonable norm. Hence, before we can begin to construct a solution to an inverse scattering problem we must explain what we mean by a solution. In order to do this it is necessary to introduce "nonstandard" information that reflects the physical situation we are trying to model. Various ideas for doing this have been introduced, ranging from a priori bounds on the curvature of $\partial D$ or the derivatives of $n(x)$ to having an a priori estimate of the noise level. The latter approach leads to what is called the Morozov discrepancy principle and will be discussed at the end of this section.

The two most popular methods for solving inverse scattering problems such as those described above are based on either what is called the "weak-scattering" approximation or on nonlinear optimization techniques. For a comprehensive discussion of such methods we refer the reader to Langenberg [51] and Biegler, et.al. [2] respectively. Here we shall content ourselves with only a brief description of these two approaches. We begin with the weak-scattering approximation, in particular the physical optics approximation for the case of the exterior Dirichlet problem.

The physical optics approximation is valid for a convex obstacle and large values of the wave number $k$. In particular, it is assumed that in a first approximation a convex object $D$ may locally be considered at each point $x$ of $\partial D$ as a plane with normal $\nu(x)$. For the exterior Dirichlet problem, this means that not only does the total field $u$ satisfy $u = 0$ on $\partial D$ but also

$$\frac{\partial u}{\partial \nu} = 2\frac{\partial u^i}{\partial \nu} \tag{2–8}$$

in the illuminated region $\partial D_- := \{x \in \partial D : \nu(x) \cdot d < 0\}$ and

$$\frac{\partial u}{\partial \nu} = 0 \tag{2–9}$$

in the shadow region $\partial D_+ := \{x \in \partial D : \nu(x) \cdot d > 0\}$. Hence, using the identity

$$0 = \int_{\partial D} \left( u^i(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^i}{\partial \nu}(y) \Phi(x,y) \right) ds(y)$$

for $x \in R^3 \setminus \bar{D}$ we see from (2–5) that under the *physical optics approximation* (2–8), (2–9)

$$u_\infty(\hat{x}, d) = -\frac{1}{2\pi} \int_{\partial D_-} \frac{\partial}{\partial \nu} e^{iky\cdot d} e^{-ik\hat{x}\cdot y}\, ds(y) = \frac{-ik}{4\pi} \int_{\partial D_1} \nu(y) \cdot d e^{ik(d-\hat{x})\cdot y}\, ds(y).$$

Hence, setting $\hat{x} = -d$, replacing $d$ by $-d$ and adding yields the *Bojarski identity*

$$\begin{aligned} u_\infty(-d,d) + \overline{u_\infty(d,-d)} &= -\frac{1}{4\pi} \int_{\partial D} \frac{\partial}{\partial \nu(y)} e^{2ikd\cdot y}\, ds(y) \\ &= \frac{k^2}{4\pi} \int_{R^3} \chi(y) e^{2ikd\cdot y}\, dy, \qquad (2\text{–}10) \end{aligned}$$

where $\chi$ is the characteristic function of $D$. Hence, under the assumption that $k$ is large, $D$ is convex and $u = 0$ on $\partial D$, (2–10) is a linear integral equation which in principle yields $D$ from a knowledge of $u_\infty$. However, the kernel of this integral equation is analytic and hence solving (2–10) is a severely ill-posed problem! We shall indicate possible methods for solving such problems at the end of this section. Note that in order to ensure injectivity we must assume that (2–10) holds for an interval of $k$ values.

An analogous procedure to the above method for attempting to solve the inverse obstacle problem can also be carried out for the inverse inhomogeneous medium problem, this time under the assumption that the wave number $k$ is small. To derive the desired integral equation we reformulate the inhomogeneous medium problem (2–4) as the *Lippmann–Schwinger integral equation*

$$u(x) = u^i(x) - k^2 \int_{R^3} \Phi(x,y) m(y) u(y)\, dy, \quad x \in R^3, \qquad (2\text{–}11)$$

where $m := 1-n$. If $k$ is small, we can solve (2–11) by successive approximations and, if we replace $u$ by the first term in this iterative process and let $r = |x| \to \infty$, we obtain the *Born approximation*

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{R^3} e^{-ik\hat{x}\cdot y} m(y) u^i(y)\, dy. \qquad (2\text{–}12)$$

(2–12) is again a linear integral equation of the first kind for the determination of $m$ from $u_\infty$ under the assumption that $k$ is sufficiently small and $\rho = \rho_D$ in (2–1). In order to ensure injectivity we must again assume that (2–12) is valid for an interval of $k$ values. For further developments in this direction see [22].

Although the weak scattering models discussed above have had considerable success, particularly in their extensions to the electromagnetic case and use in the development of synthetic aperature radar, they suffer in more complicated imaging problems where multiple scattering effects can no longer be ignored. In order to treat such problems, a considerable effort has been put into the derivation of robust nonlinear optimization schemes. The advantage of such an approach is that $u_\infty$ need only be known for a single fixed value of $k$ and multiple scattering effects are no longer ignored, although it is still necessary to have some a priori knowledge of the physical properties of the scattering object (e.g. $u = 0$ on $\partial D$ or $\rho = \rho_D$ as in the above examples). A difficulty with nonlinear optimization techniques is that they are often computationally very expensive.

We begin our discussion of nonlinear optimization methods for solving the inverse scattering problem by considering the exterior Dirichlet problem (2–3). To this end we note the solution to the direct scattering problem with a fixed incident plane wave $u^i$ defines an operator $\mathcal{F} : \partial D \to U_\infty$ which maps the boundary $\partial D$ onto the far field pattern $u_\infty$ of the scattered field. In terms of this operator, the inverse problem consists in solving the nonlinear equation $\mathcal{F}(\partial D) = u_\infty$. Having in mind that for ill-posed problems the norm in the data space has to be suitable for describing the measurement error, we make the assumption that $u_\infty$ is in the Hilbert space $L^2(\Omega)$. For $\partial D$ we need to choose a class of admissible surfaces described by some suitable parameterization and equipped with an appropriate norm. For the sake of simplicity, we restrict ourselves to the class of domains $D$ that are star-like with respect to the origin with $C^2$ boundary $\partial D$, i.e. we assume that $\partial D$ is represented in its parametric form

$$x = r(\hat{x})\hat{x}, \quad \hat{x} \in \Omega,$$

for a positive function $r \in C^2(\Omega)$. We now view the operator $\mathcal{F}$ as a mapping from $C^2(\Omega)$ into $L^2(\Omega)$ and write $\mathcal{F}(\partial D) = u_\infty$ as

$$\mathcal{F}(r) = u_\infty. \tag{2–13}$$

The following basic theorem was first proved by Kirsch [41] using variational methods and subsequently by Potthast [63] using a boundary integral equation approach (see also [34] and [48]). We note that the validity of the following theorem for the case of the exterior Neumann problem remains an open question [50].

THEOREM 2.1. *The boundary to far field map* $\mathcal{F} : C^2(\Omega) \to L^2(\Omega)$ *has a Fréchet derivative* $\mathcal{F}'$. *The linear operator* $\mathcal{F}'$ *is compact and injective with dense range.*

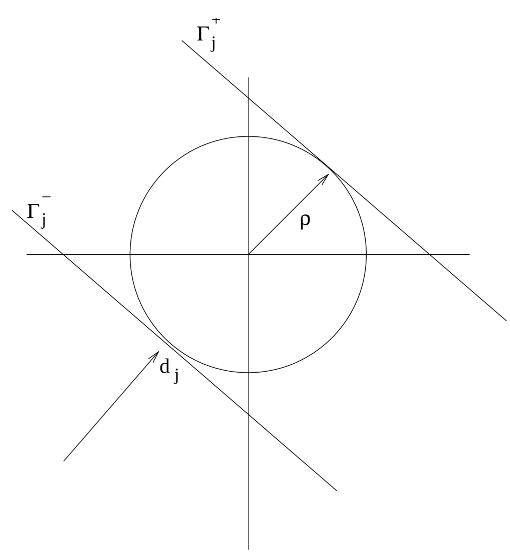
Theorem 2.1 now allows us to apply Newton's method to solve (2–13). In particular, given a far field pattern $u_\infty$ and initial guess $r_0$ to $r$, the nonlinear equation (2–13) is replaced by the linearized equation

$$\mathcal{F}(r_0) + \mathcal{F}'q = u_\infty,$$

which is then solved for $q$ to yield the new approximation $r$, given by $r_1 = r_0 + q$. Newton's method than consists in iterating this procedure. Note that since $\mathcal{F}'$ is compact each step of the iteration procedure is ill-posed. Alternate optimization strategies for determining $D$ have been proposed by numerous people in particular Kirsch and Kress [45], Angell, Kleinman and Roach [1] and Maponi et al. [52].

Newton's method can also be used to determine the coefficient $n(x)$ in the inverse inhomogeneous medium problem [32] In this case the nonlinear operator $\mathcal{F}$ is defined by means of the Lippmann–Schwinger integral equation (2–11). Other methods for determining $n(x)$ have been proposed by Colton and Monk (see Chapter 10 of [14])who use an averaging procedure to reduce the number of unknowns, Gutman and Klibanov [26], who confine themselves to reconstructing a fixed number of Fourier coefficients of $n$ where the number depends on the wave number $k$, Kleinman and Van den Berg [72], who use a modified gradient method for an output least squares formulation of the problem, and Natterer and Wübbeling [54], [55] who employ an algebraic reconstruction technique (ART) to determine $n(x)$. We shall conclude our brief discussion of nonlinear optimization schemes to solve inverse scattering problems by describing the method of Natterer and Wübbeling.

Our aim is to reconstruct the coefficient $n(x)$ from a knowledge of the far field pattern corresponding to the inhomogeneous medium scattering problem (2–4). We assume that $D \subset \{x : |x| < \rho\}$ and that our data is $p$ far field patterns $u_\infty(\hat{x}, d_j)$, $j = 1, \ldots, p$, corresponding to $p$ distinct incident plane waves. From each $u_\infty(\hat{x}, d_j)$ we can determine the Cauchy data on the planes $\Gamma_j^\pm$ perpendicular to $d_j$ for the solution $u_j$ of (2–4) corresponding to $u^i(x) = \exp(ikx \cdot d_j)$ (see figure). Assuming to begin with that $n(x)$ is known, we want to determine $u_j$



on $\Gamma_j^+$ from the ill-posed Cauchy problem

$$\Delta u_j + k^2 n(x) u_j = 0 \qquad \text{in } R^3, \tag{2–14a}$$

$$\begin{aligned} u &= f \qquad \text{on } \Gamma_j^-, \\ \nabla u \cdot d_j &= g \qquad \text{on } \Gamma_j^- \end{aligned} \tag{2–14b}$$

This can be done in a stable fashion by finite difference methods if we first filter out frequencies greater than $\kappa$ where $\kappa < k$. We now define the nonlinear operator $R_j : L^2(|x| < \rho) \to L^2(\Gamma_j^+)$ by

$$R_j(n) = u_\kappa^j\big|_{\Gamma_j^+}, \tag{2–15}$$

where $u_\kappa^j$ is the filtered solution of (2–14). Our aim is to now solve the inverse scattering problem by using an ART-type procedure to solve (2–15) for $j = 1, \ldots, p$.

To solve (2–15) for $n$ we set $g_j = u_\kappa^j\big|_{\Gamma_j^+}$ and solve this equation iteratively by first determining $n_p$ from

$$\begin{aligned} &n_0 = n^0, \\ &n_j = n_{j-1} + \omega R_j'(n_{j-1})^* C_j^{-1}(g_j - R_j(n_{j-1})), \end{aligned}$$

where $n^0$ is an initial guess, $\omega$ is a relaxation parameter, $R_j'$ is the Frechet derivative of $R_j, C_j = R_j'(0)(R_j'(0))^*$ where $*$ denotes the adjoint operator and the operator $C_j^{-1}$ can be applied through the use of Fourier transforms (see [54]). The first approximation is now defined to be $n_p$ and the procedure is repeated. For details we refer the reader to Natterer and Wübbeling [54] [55] where the computational advantages of using such an approach are discussed. An extension of this method (which is sometimes called the "adjoint field method") to the case of time-harmonic electromagnetic waves has been done by Dorn, et.al.[23].

In both the weak scattering and Newton-type methods for solving the inverse scattering problem we are faced with the problem of solving a linear operator equation of the form

$$A\varphi = f$$

where $A : X \to Y$ is compact and $X$ and $Y$ are infinite dimensional normed spaces. We shall also encounter such equations in the sequel when we consider linear sampling methods for solving the inverse scattering problem. Hence it is appropriate to conclude this section of our paper by giving some idea of how such equations can be solved numerically. The problem in doing this is that since $A$ is compact solving $A\varphi = f$ is an ill-posed problem in the sense that $A^{-1}$, if it exists, is unbounded. This follows immediately from the fact that if $A^{-1}$ were bounded then $I = A^{-1}A$ is compact, a contradiction since $X$ is infinite dimensional. Our discussion will purposefully be brief and for more information on the solution of ill-posed problems we refer the reader to Engl, Hanke and Neubauer [24], Kirsch [40] and Kress[47].

We restrict our attention to the case when $X$ and $Y$ are infinite dimensional Hilbert spaces. We denote by $(\sigma_n, \varphi_n, g_n)$ a singular system for the compact operator $A : X \to Y$, so that

$$A\varphi_n = \sigma_n g_n, \qquad A^* g_n = \sigma_n \varphi_n,$$

and we denote the null space of $A$ by $N(A)$.

Picard's Theorem. *$A\varphi = f$ is solvable if and only if $f \in N(A^*)^\perp$ and*

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} \left|(f, g_n)\right|^2 < \infty.$$

*In this case a solution is given by*

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} (f, g_n) \varphi_n.$$

Definition 2.2. The equation $A\varphi = f$ is *mildly ill-posed* if $\sigma_n = O(n^{-\beta})$, for $\beta \in R^+$, and *severely ill-posed* if the $\sigma_n$ decay faster than this.

We note that the equations appearing in inverse scattering theory are typically severely ill-posed.

For severely ill-posed problems we must use regularization methods to arrive at a solution.

Definition 2.3. Let $A : X \to Y$ be an injective compact linear operator. Then a family of bounded linear operators $R_\alpha : Y \to X, \alpha > 0$, such that

$$\lim_{\alpha \to 0} R_\alpha A \varphi = \varphi$$

for all $\varphi \in X$ is called a *regularization scheme* for $A$ with *regularization parameter* $\alpha$.

Suppose the solution $\varphi$ of $A\varphi = f$ is approximated by

$$\varphi_\alpha^\delta := R_\alpha f^\delta$$

where $\|f - f^\delta\| \le \delta$. Then

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A \varphi - \varphi$$

and hence

$$\|\varphi_\alpha^\delta - \varphi\| \le \delta \|R_\alpha\| + \|R_\alpha A \varphi - \varphi\|.$$

The first term in the above equation increases as $\alpha$ tends to zero (since $A^{-1}$ is not bounded) whereas the second term is only small when $\alpha$ tends to zero. How should $\alpha = \alpha(\delta)$ be chosen?

DEFINITION 2.4. The choice of $\alpha = \alpha(\delta)$ is called *regular* if for all $f \in A(X)$ and all $f^\delta \in Y$ with $\|f - f^\delta\| \le \delta$ we have

$$R_{\alpha(\delta)} f^\delta \to A^{-1} f$$

as $\delta$ tends to zero.

We shall now describe a regular regularization scheme due to Tikhonov and Morozov for solving the ill-posed equation $A\varphi = f$.

Assume once again that $A : X \to Y$ is compact. Then, since $A^*A \ge 0$, for every $\alpha > 0$ the operator $\alpha I + A^*A : X \to X$ is bijective with bounded inverse. The *Tikhonov regularization method* for solving $A\varphi = f$ is to set

$$R_\alpha := (\alpha I + A^*A)^{-1} A^*.$$

Then the regularized solution $\varphi_\alpha$ of $A\varphi = f$ is the unique solution of

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^* f.$$

In particular, if $A$ is injective, then

$$\varphi_\alpha = \sum_{n=l}^{\infty} \frac{\sigma_n}{\alpha + \sigma_n^2} (f, g_n)\varphi_n = R_\alpha f$$

and hence as $\alpha$ tends to zero we have $R_\alpha A\varphi \to \varphi$ for all $\varphi \in X$. The function $\varphi_\alpha$ can also be obtained by minimizing the *Tikhonov functional*

$$\|A\varphi - f\|^2 + \alpha\|\varphi\|^2.$$

Note that if $A$ is injective with dense range then $\|\varphi_\alpha\| \to \infty$ as $\alpha$ tends to zero if and only if $f \notin A(X)$.

We now turn to the choice of the regularization parameter $\alpha$. If $A$ is injective with dense range then a regular method for choosing $\alpha$ is the *Morozov discrepancy principle*. In particular, assume that we want to solve

$$A_\delta \varphi = f_\varepsilon$$

where $\|A - A_\delta\| \le \delta$ and $\|f - f_\varepsilon\| \le \varepsilon$ with known $\delta$ and $\varepsilon$, i.e. an estimate of the noise level is known a priori. We require that the residual be commensurate with the accuracy of the measurements of $A$ and $f$, i.e.

$$\|A_\delta\varphi_\alpha - f_\varepsilon\| \approx \varepsilon + \delta\|\varphi_\alpha\|.$$

In applications to the linear sampling method described in the sequel we have $\varepsilon \ll \delta$. Hence, in this case, the Morozov discrepancy principle is to choose $\alpha = \alpha(\delta)$ such that $\mu(\alpha) = 0$, where

$$\mu(\alpha) := \|A_\delta\varphi_\alpha - f_\varepsilon\|^2 - \delta^2\|\alpha_\alpha\|^2 = \sum_{n=1}^{\infty} \frac{\alpha^2 - \delta^2\sigma_n^2}{(\sigma_n^2 + \alpha)^2} \big|(f_\varepsilon, g_n)\big|^2$$

and $(\sigma_n, \varphi_n, g_n)$ is now a singular system for the operator $A_\delta$. We note that $\mu(\alpha)$ is monotonously increasing and in practice only a rough approximation to the root of $\mu(\alpha) = 0$ is necessary.

In closing this section, we make a few comments on the use of Tikhonov regularization and the Morozov discrepancy principle in solving ill-posed problems arising in inverse scattering theory. In particular, we note that in general one has no idea if the noise level is small enough so that the regularized solution of the equation with noisy data is in fact a good approximation to the solution of the noise free equation. Without further a priori information the only statement that can be made is what happens if the noise tends to zero. However, since the noise is fixed and nonzero, in general all that can be said is that there is a "nearby" equation (i.e. noisy $A$ and $f$) whose solution can be obtained and if this nearby equation is "close enough" to the noise free equation then one expects the regularized solution to behave like the true solution, assuming it exists. In particular, since without severe a priori assumptions, which are in general not available, error estimates are not known for the dependency of the regularized solution on the noise level, and the remark of Lanczos is valid: "A lack of information cannot be remedied by any mathematical trickery." Nevertheless, a regularized solution based on Tikhonov regularization and the Morozov discrepancy principle provides a rational approach for arriving at a candidate for a solution to an ill-posed problem when an a priori estimate of the noise level is available.

## 3. The Inverse Dirichlet Problem for Acoustic Waves

In the previous section we discussed two of the most popular methods for solving the inverse scattering problem, i.e. the weak scattering approximation and nonlinear optimization techniques, as well as regularization methods that can be used for their numerical solution. However, as previously mentioned, both of these methods rely on some a priori knowledge of the physical properties of the scattering object $D$ in order to know the boundary conditions on $\partial D$. Furthermore, uniqueness theorems were not discussed, in particular how much information is needed in principle to determine $D$ or, more importantly, can $D$ be determined from the far field data if the boundary conditions are not known? We view these issues as particularly important since in many practical inverse scattering problems both the material properties of the scatterer as well as its shape are unknown. In this and the sections that follow we will be paying particular attention to inverse scattering problems such as these.

In this section we will consider the inverse scattering problem associated with the exterior Dirichlet problem (2–3) and, when relevant, point out what results are in fact independent of the boundary condition on $\partial D$. We will always assume the existence of a solution $u \in C^2(R^3 \setminus \bar{D}) \cap C(R^3 \setminus D)$ to (2–3) as well as the fact that since $\partial D$ is in class $C^2$ we have $u \in C^1(R^3 \setminus D)$ [14]. We begin with

Rellich's lemma which forms the basis of the entire field of acoustic scattering theory.

THEOREM 3.1 (RELLICH'S LEMMA). *Let* $u^s$ *be a solution of the Helmholtz equation in the exterior of* $D$ *satisfying the Sommerfeld radiation condition such that the far field pattern* $u_\infty$ *of* $u^s$ *vanishes. Then* $u^s = 0$ *in* $R^3 \setminus \bar{D}$.

PROOF. For sufficiently large $|x|$ we have a Fourier expansion

$$u^s(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(r) Y_n^m(\hat{x})$$

with respect to the spherical harmonics $Y_n^m$ where the coefficients are given by

$$a_n^m(r) = \int_\Omega u^s(r\hat{x}) \overline{Y_n^m(\hat{x})}\, ds(\hat{x}).$$

Since $u^s \in C^2(R^3 \setminus \bar{D})$ and the radiation condition holds uniformly in $\hat{x}$, we can differentiate under the integral sign and integrate by parts to conclude that $a_n^m$ is a solution of the spherical Bessel equation

$$\frac{d^2 a_n^m}{dr^2} + \frac{2}{r}\frac{da_n^m}{dr} + (k^2 - \frac{n(n+1)}{r^2}) a_n^m = 0$$

satisfying the radiation condition, i.e.

$$a_n^m(r) = \alpha_n^m h_n^{(1)}(kr)$$

where $h_n^{(1)}$ is a spherical Hankel function of the first kind of order $n$ and the $\alpha_n^m$ are constants depending only on $n$ and $m$. From (2–7) we have that, since $u_\infty = 0$,

$$\lim_{r \to \infty} \int_{|x|=r} \left|u^s(x)\right|^2 ds = \int_\Omega \left|u_\infty(\hat{x})\right|^2 ds = 0.$$

But by Parseval's equality

$$\int_{|x|=r} \left|u^s(x)\right|^2 ds = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m(r)\right|^2.$$

Substituting the above expression for $a_n^m$ into this identity, letting $r$ tend to infinity, and using the asymptotic behavior of the spherical Hankel functions now yields $\alpha_n^m = 0$ for all $n$ and $m$. Hence $u^s = 0$ outside a sufficiently large sphere. By the representation formula (2–5) we see that $u^s$ is an analytic function of $x$, and hence we can now conclude that $u^s = 0$ in $R^3 \setminus \bar{D}$ by analyticity. □

In the sequel we will need two *reciprocity relations* which can be proved by a straightforward application of Green's theorem. The first of these is for scattering by a plane wave, i.e. $u^i(x) = e^{ik\hat{x}\cdot d}$ in (2–3), and is given by [14]

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}) \tag{3–1}$$

and the second of these is for scattering due to the point source $u^i(x) = \Phi(x, z)$ having far field pattern $u_\infty(\hat{x}, z)$ and is given by [62]

$$4\pi u_\infty(-d, z) = u^s(z, d) \tag{3–2}$$

for $z \in R^3 \setminus \bar{D}, d \in \Omega$, where $u^s$ is the scattered field in (2–3) corresponding to $u^i(x) = e^{ikx\cdot d}$.

We now consider the scattering problem (2–3) and let $u_\infty$ be the far field pattern of the scattered field. The *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ for this problem is defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) \tag{3–3}$$

and is easily seen to be the scattered field $v_g^s$ corresponding to the *Herglotz wave function*

$$v_g^i(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d), \quad x \in R^3$$

as incident field. The function $g \in L^2(\Omega)$ is known as the kernel of the Herglotz wave function. Of basic importance to us is the following theorem [14].

THEOREM 3.2. *The far field operator $F$ is injective with dense range if and only if there does not exist a Dirichlet eigenfunction for $D$ which is a Herglotz wave function.*

PROOF. For the $L^2$ adjoint $F^* : L^2(\Omega) \to L^2(\Omega)$ the reciprocity relation (3–1) implies that

$$F^* g = \overline{RFR\bar{g}} \tag{3–4}$$

where $R : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Rg)(d) := g(-d).$$

Hence, the operator $F$ is injective if and only if its adjoint $F^*$ is injective. Recalling that the denseness of the range of $F$ is equivalent to the injectivity of $F^*$ we therefore must only show the injectivity of $F$. To this end, we note that $Fg = 0$ with $g \neq 0$ is equivalent to the existence of a nontrivial Herglotz wave function $v_g^i$ with kernel $g$ for which the far field pattern of the corresponding scattered field $v^s$ is $v_\infty = 0$. By Rellich's lemma this implies $v^s = 0$ in $R^3 \setminus D$ and the boundary condition $v_g^i + v^s = 0$ on $\partial D$ now shows that $v_g^i = 0$ on $\partial D$. The proof is finished. □

We now want to show that the far field $F$ is normal. To this end, we need the following lemma [15].

LEMMA 3.3. *The far field operator $F$ satisfies*

$$2\pi\big((Fg, h) - (g, Fh)\big) = ik(Fg, Fh).$$

PROOF. If $v^s$ and $w^s$ are radiating solutions to the Helmholtz equation with far field patterns $v_\infty$ and $w_\infty$ then from the radiation condition and Green's theorem we obtain

$$\int_{\partial D} \left( v^s \frac{\partial \overline{w^s}}{\partial \nu} - \overline{w^s} \frac{\partial v^s}{\partial \nu} \right) ds = -2ik \int_{\Omega} v_\infty \overline{w_\infty} \, ds.$$

From the representation formula (2–5) and letting $x \to \infty$ we see that, if $w_h^i$ is a Herglotz wave function with kernel $h$, then

$$\begin{aligned} \int_{\partial D} \left( v^s(x) \frac{\partial \overline{w_h^i}}{\partial \nu}(x) - \overline{w_h^i(x)} \frac{\partial v^s}{\partial \nu}(x) \right) ds(x) \\ &= \int_{\Omega} \overline{h(d)} \int_{\partial D} \left( v^s(x) \frac{\partial e^{-ikx\cdot d}}{\partial \nu} - e^{-ikx\cdot d} \frac{\partial v^s}{\partial \nu}(x) \right) ds(x) \, ds(d) \\ &= 4\pi \int_{\Omega} \overline{h(d)} v_\infty(d) \, ds(d). \end{aligned}$$

Now let $v_g^i$ and $v_h^i$ be Herglotz wave functions with kernels $g, h \in L^2(\Omega)$, respectively, and let $v_g, v_h$ be the solutions of (2–3) with $u^i$ replaced by $v_g^i$ and $v_h^i$, respectively. Let $v_{g,\infty}$ and $v_{h,\infty}$ be the corresponding far field patterns. Then we can combine the two previous equations to obtain

$$\begin{aligned} -2ik(Fg, Fh) + 4\pi(Fg, h) - 4\pi(g, Fh) \\ &= -2ik \int_{\Omega} v_{g,\infty} \overline{v_{h,\infty}} \, ds + 4\pi \int_{\Omega} v_{g,\infty} \bar{h} \, ds - 4\pi \int_{\Omega} g \overline{v_{h,\infty}} \, ds \\ &= \int_{\partial D} \left( v_g \frac{\partial \overline{v_h}}{\partial \nu} - \overline{v_h} \frac{\partial v_g}{\partial \nu} \right) ds \end{aligned}$$

and the lemma follows from the Dirichlet boundary condition satisfied by $v_g$ and $v_h$. □

THEOREM 3.4. *The far field operator $F$ is compact and normal.*

PROOF. Since $F$ is an integral operator on $\Omega$ with a continuous kernel, it is compact. From Lemma 3.3 we have

$$(g, ikF^*Fh) = 2\pi\big((g, Fh) - (g, F^*h)\big)$$

for all $g, h \in L^2(\Omega)$ and hence

$$ikF^*F = 2\pi(F - F^*). \tag{3–5}$$

Using (3–4) we can deduce that

$$(F^*g, F^*h) = (FR\bar{h}, FR\bar{g})$$

and hence from Lemma 3.3 again it follows that

$$ik(F^*g, F^*h) = 2\pi\big((g, F^*h) - (F^*g, h)\big)$$

for all $g, h \in L^2(\Omega)$. If we now proceed as in the derivation of (3–6) we find that

$$ikFF^* = 2\pi(F - F^*) \tag{3–6}$$

and the proof is finished. □

The proof of Theorem 3.4 carries over to the case of Neumann boundary data. However for the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0$$

where $\lambda > 0$ the operator $F$ is no longer normal since Lemma 3.3 is not valid, i.e. absorption destroys normality. Finally, returning to the case of Dirichlet boundary data, if we define the *scattering operator* $S : L^2(\Omega) \to L^2(\Omega)$ by

$$S = I + \frac{ik}{2\pi}F$$

then from (3–5) and (3–6) we see that $SS^* = S^*S = I$, i.e., $S$ is unitary.

Having established the basic properties of the far field pattern and far field operator, we now turn our attention to the uniqueness of a solution to the inverse scattering problem, basing our analysis on the approach of Kirsch and Kress [46] with a subsequent simplification of the proof by Potthast [62].

THEOREM 3.5. *Assume that $D_1$ and $D_2$ are two obstacles such that the far field patterns corresponding to the exterior Dirichlet problem* (2–3) *for $D_1$ and $D_2$ coincide for all incident directions $d$. Then $D_1 = D_2$.*

PROOF. By analyticity and Rellich's lemma the scattered fields $u_1^s(\,\cdot\,, d) = u_2^s(\,\cdot\,, d)$ for the incident fields $u^i(x, d) = e^{ikx\cdot d}$ coincide in the unbounded component $G$ of the complement of $\bar{D}_1 \cup \bar{D}_2$ for all $d \in \Omega$. Then from the reciprocity relation (3–2) we can conclude that the far field patterns $u_{1,\infty}(\,\cdot\,, z) = u_{2,\infty}(\,\cdot\,, z)$ for the scattering of point sources $\Phi(\,\cdot\,, z)$ coincide for all point sources located at $z \in G$. Again by Rellich's lemma, this implies that the corresponding scattered fields satisfy $u_1^s(x, z) = u_2^s(x, z)$ for all $x, z \in G$.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. In particular, we have

$$z_n := x^* + \frac{1}{n}\nu(x^*)$$

is in $G$ for integers $n$ sufficiently large. Then, on the one hand, we have

$$\lim_{n\to\infty} u_2^s(x^*, z_n) = u_2^s(x^*, x^*)$$

since $u_2^s(x^*, \cdot)$ is continuous in a neighborhood of $x^* \notin \bar{D}_2$ due to the well-posedness of the direct scattering problem. On the other hand, we have

$$\lim_{n \to \infty} u_1^s(x^*, z_n) = \infty$$

because of the boundary condition $u_1^s(x^*, z_n) + \Phi(x^*, z_n) = 0$. This contradicts the fact that $u_1^s(x^*, z_n) = u_2^s(x^*, z_n)$ for $n$ sufficiently large and the proof in complete. □

A closer examination of the proof of Theorem 3.5 shows that the boundary condition $u = 0$ is not used explicitly but rather only the well-posedness of the direct scattering problem. Hence it is not necessary to know the boundary condition (2–3d) a priori in order to conclude that the far field pattern uniquely determines the scatterer. In fact it is not even necessary to know if $u_\infty$ is the far field pattern of (2–2), (2–3) or (2–4) in order to conclude that $D$ is uniquely determined [41], [42]. In a related direction, Potthast [62], [64] has considered the important case of finite data. In particular, if $\Omega_n \subset \Omega$ is a set of $n$ uniformly distributed unit vectors such that if

$$d(\hat{x}, \Omega_n) := \inf_{d \in \Omega_n} |\hat{x} - d|$$

then (1) $d(\hat{x}, \Omega_n) \to 0$ as $n \to \infty$; (2) $d \in \Omega_n \implies -d \in \Omega_n$ if $n$ is even; and (3) $\Omega_{n'}, \subset \Omega_n$ for $n > n'$ then the following theorem is valid.

THEOREM 3.6. *Let $u_1^\infty$ and $u_2^\infty$ be the far field patterns corresponding to one of* (2–2), (2–3) *or* (2–4). *Given $\varepsilon > 0$ there exists integers $n_0$ and $n_i$ such that if $u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d)$ for $\hat{x} \in \Omega_{n_0}, d \in \Omega_{n_i}$ then*

$$d(D_1, D_2) \le \varepsilon,$$

*where $d(D_1, D_2)$ denotes the Hausdorff distance between $D_1$ and $D_2$.*

An open problem is to determine if one incident plane wave at a fixed wave number $k$ is sufficient to uniquely determine the scatterer $D$. If it is known a priori that the boundary condition (2–3d) is satisfied and that furthermore $D$ is contained in a ball of radius $R$ such that $kR < \pi$ then it was shown by Colton and Sleeman ([20] and Corollary 5.3 of [14]) that $D$ is uniquely determined by its far field pattern for a single incident direction $d$ and fixed wave number $k$.

We now turn our attention to a method for reconstructing $D$ from an inexact knowledge of the far field pattern $u_\infty$ of the scattering problem (2–3) that is closely related to the ideas of the proof of the uniqueness Theorem 3.5. Indeed, as with Theorem 3.5, this method can be implemented without knowing a priori which of the scattering problems (2–2), (2–3) or (2–4) is associated with $u_\infty$ and in this sense has a clear advantage over the reconstruction methods discussed in the previous section. On the other hand, the implementation requires a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d$ on open subsets of $\Omega$ whereas for obstacle scattering Newton's method only requires a single incident direction $d$. Furthermore, for the case of the scattering problem (2–4), only the support $D$ is obtained rather than the coefficient $n(x)$. The method we have on mind was first introduced by Colton and Kirsch in [12], with a subsequent second version being given by Kirsch in [42] and [43] and has become known as the *linear sampling method* (related methods have been considered by Ikehata [35], Norris [56] and Potthast [65]). Here, for the sake of simplicity, we will assume that $u_\infty(\hat{x}, d)$ is known for all $\hat{x}, d \in \Omega$ rather than only on a subset of $\Omega$. For the case of the first version of the linear sampling method, this latter case can be easily handled by appealing to the result that a Herglotz wave function and its first derivatives can be approximated on compact subsets of a ball $B$ by another Herglotz wave function having a kernel that is compactly supported on $\Omega$. (This can easily be shown by assuming without loss of generality that $k^2$ is not a Dirichlet eigenvalue for $B$ and then using the ideas of the proof of Theorem 5.5 of [14] to show that Herglotz wave functions with compactly supported kernels are dense in $L^2(\partial B)$).

To describe the basic idea behind the linear sampling method, assume that for every $z \in D$ there exists a unique solution $g = g(\,\cdot\,, z) \in L^2(\Omega)$ to the *far field equation*

$$\int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) = \Phi_\infty(\hat{x}, z), \tag{3–7}$$

where

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}$$

and $u_\infty$ is the far field pattern corresponding to the scattering problem (2–3). Then, since the right hand side of (3–7) is the far field pattern of the fundamental solution $\Phi(x, z)$, it follows from Rellich's lemma that

$$\int_\Omega u^s(x, d) g(d)\, ds(d) = \Phi(x, z)$$

for $x \in R^3 \setminus D$. From the boundary condition $u = 0$ on $\partial D$ it now follows that

$$v_g^i(x) + \Phi(x, z) = 0 \quad \text{for } x \in \partial D, \tag{3–8}$$

where $v_g^i$ is the Herglotz wave function with kernel $g$. We now see from (3–8) that $v_g^i$ becomes unbounded as $z \to x \in \partial D$ and hence

$$\lim_{\substack{z \to \partial D \\ z \in D}} \big\| g(\,\cdot\,, z) \big\| = \infty,$$

that is, $\partial D$ is characterized by points $z$ where the solution of (3–7) becomes unbounded.

Unfortunately, in general the far field equation

$$Fg = \Phi_\infty(\,\cdot\,, z)$$

does not have a unique solution, nor does the above analysis say anything about what happens when $z \in R^3 \setminus D$. However, using on the one hand the fact that

Herglotz wave functions are dense in the space of solutions to the Helmholtz equation in $D$ with respect to the norm in the Sobolev space $H^1(D)$ (see [16], [21]) and on the other the factorization of the far field operator $F$ as

$$(Fg) = -\frac{1}{4\pi}\mathcal{F}S^{-1}(Hg),$$

where $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is the single layer potential

$$(S\varphi)(x) := \int\limits_{\partial D} \varphi(y)\Phi(x,y)\, ds(y), \tag{3–9}$$

$Hg$ is the trace on $\partial D$ of the Herglotz wave function, and $\mathcal{F} : H^{-1/2}(\partial D) \to L^2(\Omega)$ is defined by

$$(\mathcal{F}\varphi)(\hat{x}) := \int\limits_{\partial D} \varphi(y)\varphi^{-ik\hat{x}\cdot y}\, ds(y),$$

we can prove the following result [4].

THEOREM 3.7. *Assume that $k^2$ is not a Dirichlet eigenvalue for $D$. Then*

(1) *if $z \in D$ for every $\varepsilon > 0$ there exists a solution $g(\,\cdot\,, z) \in L^2(\Omega)$ of the inequality*

$$\left\|Fg(\,\cdot\,,z) - \Phi_\infty(\,\cdot\,,z)\right\| < \varepsilon$$

*such that*

$$\lim_{z\to\partial D}\left\|g(\,\cdot\,,z)\right\|_{L^2(\Omega)} = \infty, \qquad \lim_{z\to\partial D}\left\|v_g^i(\,\cdot\,,z)\right\|_{H^1(D)} = \infty,$$

(2) *if $z \in R^3\backslash\bar{D}$ for every $\varepsilon > 0$ and $\gamma > 0$ there exists a solution $g(\,\cdot\,,z) \in L^2(\Omega)$ of the inequality*

$$\left\|Fg(\,\cdot\,,z) - \Phi_\infty(\,\cdot\,,z)\right\| < \varepsilon + \gamma$$

*such that*

$$\lim_{\gamma\to 0}\|g(\,\cdot\,,z)\|_{L^2(\Omega)} = \infty, \qquad \lim_{\gamma\to 0}\left\|v_g^i(\,\cdot\,,z)\right\|_{H^1(D)} = \infty.$$

We note that the difference between cases (1) and (2) of this theorem is that, for $z \in D$, $\Phi_\infty(\,\cdot\,,z)$ is in the range of $\mathcal{F}$ whereas for $z \in R^3 \setminus \bar{D}$ this is no longer true.

The above theorem now suggests a numerical procedure for determining $\partial D$ from noisy far field data. In particular, let $u_\infty^\delta$ be the measured far field data, i.e. $\|u_\infty^\delta - u_\infty\| < \delta$ and assume $g$ is such that $\left\|Fg - \Phi_\infty(\,\cdot\,,z)\right\| < \varepsilon$. If $F_\delta$ is the operator $F$ with kernel $u_\infty$ replaced by $u_\infty^\delta$, then we want to find an approximation to $g$ by solving $F_\delta\varphi = \Phi_\infty(\,\cdot\,,z)$; that is, we view both the operator and the right hand side as being inexact. This equation is now solved using Tikhonov regularization and the Morozov discrepancy principle. The unknown boundary $\partial D$ is now determined by looking for those points $z$ where $\left\|\varphi(\,\cdot\,,z)\right\|$ begins to sharply increase. Numerical examples using this procedure can be found in [9], [10] and [74].

The analogue of Theorem 3.7 for the exterior Neumann problem is established in exactly the same way as Theorem 3.7 where now it is assumed that $k^2$ is not a Neumann eigenvalue for $D$. It is also possible to treat mixed boundary value problems [4], [6]. As will be seen in the next section, a similar result also holds for the inhomogeneous medium problem (2–4) as well as the more general problem (2–1) provided $k^2$ is not a transmission eigenvalue (to be defined in the following section of this paper). In particular, as with Theorems 3.5 and 3.6, it is not necessary to know the material properties of the scatterer (e.g., the boundary condition) in order to determine $D$ from a knowledge of the regularized solution of the far field equation. it is also possible to treat the case when the background medium is piecewise homogeneous by appropriately modifying the far field equation [9]. The possibility of doing this is particularly important in numerous applications, e.g. the detection of buried objects or structures under foliage.

Theorem 3.7 is complicated by the fact that in general $\Phi_\infty(\,\cdot\,,z)$ is not in the range of $F$ for neither $z \in D$ nor $z \in R^3 \setminus \bar{D}$. For the case when $F$ is normal (say nonabsorbing media and data on all of $\Omega$ rather than some subset of $\Omega$), this problem was resolved by Kirsch [42], who proposed replacing the equation $Fg = \Phi_\infty(\,\cdot\,,z)$ by $(F^*F)^{\frac{1}{4}}g = \Phi_\infty(\,\cdot\,,z)$ where $F^*$ is again the adjoint of $F$ in $L^2(\Omega)$. He was then able to show that $\Phi(\,\cdot\,,z)$ is in the range of $(F^*F)^{\frac{1}{4}}$ if and only if $z \in D$. We will now outline the main ideas of Kirsch's proof of this result. In what follows, $S : L^2(\partial D) \to L^2(\partial D)$ is the single layer potential defined by (3–9) and $G : L^2(\partial D) \to L^2(\Omega)$ is defined by $Gh = v_\infty$ where $v_\infty$ is the far field pattern of the solution to the radiating exterior Dirichlet problem with boundary data $h \in L^2(\partial D)$. The relation among the operators $F, G$ and $S$ is given by the following lemma.

Lemma 3.8. *The relation*

$$F = -4\pi G S^* G^*$$

*is valid where* $G^* : L^2(\Omega) \to L^2(\partial D)$ *and* $S^* : L^2(\partial D) \to L^2(\partial D)$ *are the* $L^2$ *adjuncts of* $G$ *and* $S$ *respectively.*

Proof. Define the operator $H : L^2(\Omega) \to L^2(\partial D)$ by

$$(Hg)(x) := \int_\Omega g(d)e^{ikx\cdot d}\,ds(d).$$

Note that $Hg$ is the Herglotz wave function with density $g$. The adjoint operator $H^* : L^2(\partial D) \to L^2(\Omega)$ is given by

$$(H^*\varphi)(\hat{x}) = \int_{\partial D} \varphi(y)e^{-ik\hat{x}\cdot y}\,ds(y)$$

and we note that $\frac{1}{4\pi}H^*\varphi$ is the far field pattern of the single layer potential (3–9). The single layer potential with continuous density $\varphi$ is continuous in $R^3$ and thus $\frac{1}{4\pi}H^*\varphi = GS\varphi$ , i.e. by a denseness argument

$$H = 4\pi S^* G^* \tag{3–10}$$

on $L^2(\partial D)$. We now observe that $Fg$ is the far field pattern of the solution to the radiating exterior Dirichlet problem with boundary data $-(Hg)(x), x \in \partial D$, and hence

$$Fg = -GHg. \tag{3–11}$$

Substituting (3–10) into (3–11) now yields the lemma. □

We now assume that $k^2$ is not a Dirichlet eigenvalue for $D$. Then by Theorems 3.2 and 3.4 the far field operator $F$ is normal and injective. In particular, there exists eigenvalues $\lambda_j \in \mathbb{C}$ of $F, j = 1, 2, \ldots$, with $\lambda_j \neq 0$, and the corresponding eigenfunctions $\psi_j \in L^2(\omega)$ form a complete orthonormal system in $L^2(\Omega)$. From Lemma 3.3 we can deduce the fact that the $\lambda_j$ all lie on the circle of radius $\frac{2\pi}{k}$ and center $\frac{2\pi i}{k}$. We also note that $\{|\lambda_j|, \psi_j, \text{sign } (\lambda_j)\psi_j\}$ is a singular system for $F$. By the preceding lemma,

$$-4\pi GS^*G^*\lambda_j = \lambda_j\psi_j.$$

If we define the functions $\varphi_j \in L^2(\partial D)$ by

$$G^*\psi_j = -\sqrt{\lambda_j}\varphi_j,$$

where we choose the branch of $\sqrt{\lambda_j}$ such that Im $\sqrt{\lambda_j} > 0$, we see that

$$GS^*\varphi_j = \frac{\sqrt{\lambda_j}}{4\pi}\psi_j. \tag{3–12}$$

A central result of Kirsch is that the functions $\varphi_j$ form a Riesz basis in the Sobolev space $H^{-1/2}(\partial D)$, i.e. $H^{-1/2}(\partial D)$ consists exactly of functions $\varphi$ of the form

$$\varphi = \sum_{j=1}^{\infty} \alpha_j\varphi_j \quad \text{with} \quad \sum_{j=1}^{\infty} |\alpha_j|^2 < \infty.$$

The proof of this result relies in a fundamental way on the normality of $F$. Using these results we can now prove the main result of [42] where in the proof of the theorem $R(A)$ denotes the range of the operator $A$.

THEOREM 3.9. *Assume that $k^2$ is not a Dirichlet eigenvalue for $D$. Then the ranges of $G : H^{1/2}(\partial D) \to L^2(\Omega)$ and $(F^*F)^{\frac{1}{4}} : L^2(\Omega) \to L^2(\Omega)$ coincide.*

PROOF. We use the fact that $S^* : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is an isomorphism. Suppose $G\varphi = \psi$ for some $\varphi \in H^{1/2}(\partial D)$. Then $(S^*)^{-1}\varphi \in H^{-1/2}(\partial D)$ and thus $(S^*)^{-1}\varphi = \sum_{j=1}^{\infty} \alpha_j\varphi_j$ with $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$. Therefore, by (3–12), we have

$$\psi = G\varphi = GS^*(S^*)^{-1}\varphi = \frac{1}{4\pi}\sum_{j=1}^{\infty} \alpha_j\sqrt{\lambda_j}\psi_j = \sum_{j=1}^{\infty} \rho_j\psi_j$$

with $\rho_j = \frac{1}{4\pi}\alpha_j\sqrt{\lambda_j}$ and thus

$$\sum_{j=1}^{\infty} \frac{|\rho_j|^2}{|\lambda_j|} = \frac{1}{(4\pi)^2} \sum_{j=1}^{\infty} |\alpha_j|^2 < \infty. \tag{3–13}$$

On the other hand, let $\psi = \sum_{j=1}^{\infty} \rho_j \psi_j$ with the $\rho_j$ satisfying (3–13) and define $\varphi := \sum_{j=1}^{\infty} \alpha_j \varphi_j$ with $\alpha_j = 4\pi\rho_j/\sqrt{\lambda_j}$. Then $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$ and hence $\varphi \in H^{-1/2}(\partial D), S^*\varphi \in H^{1/2}(\partial D)$, and

$$G(S^*\varphi) = \frac{1}{4\pi} \sum_{j=1}^{\infty} \alpha_j \sqrt{\lambda_j} \psi_j = \sum_{j=1}^{\infty} \rho_j \psi_j = \psi.$$

Since $\sqrt{|\lambda_j|}$ and $\psi_j$ are the eigenvalues and eigenfunctions, respectively, of the self-adjoint operator $(F^*F)^{\frac{1}{4}}$, we have

$$R(F^*F)^{\frac{1}{4}} = \left( \sum_{j=1}^{\infty} \rho_j \psi_j : \sum_{j=1}^{\infty} \frac{|\rho_j|^2}{|\lambda_j|} < \infty \right),$$

and, as we have shown above, this is precisely $R(G)$. □

Since $\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x, z)$, it is easy to verify that $\Phi_\infty(\,\cdot\,, z)$ is in the range of $G$ and if and only if $z \in D$, i.e. $(F^*F)^{\frac{1}{4}} g = \Phi_\infty(\,\cdot\,, z)$ is solvable if and only if $z \in D$. In particular, if regularization methods are used to solve $(F^*F)^{\frac{1}{4}} g = \Phi_\infty(\,\cdot\,, z)$ then from Section 2 of this paper we see that as the noise level on $u_\infty$ tends to zero the norm of the regularized solution remains bounded if and only if $z \in D$. Numerical examples using this procedure can be found in [10] and [42].

Under the assumption that $k^2$ is not a Neumann eigenvalue, the equation $(F^*F)^{\frac{1}{4}} g = \Phi_\infty(\,\cdot\,, z)$ can also be derived for the determination of $D$, i.e. it is not necessary to know a priori whether or not the boundary data is of Dirichlet or Neumann type. However, in both cases the derivation of this equation depends on $F$ being a normal operator. In particular this excludes the limited aperature case when $\Omega$ is replaced by a subset $\Omega_0$ of $\Omega$ as well as the case of impedance boundary data. In an effort to avoid this problem, Kirsch has introduced a simple nonlinear optimization scheme which preserves some of the advantages of the second version of the linear sampling method while at the same time avoiding the assumption of normality of $F$ [44]. We will conclude this section of our paper by describing Kirsch's optimization scheme.

The optimization scheme of Kirsch is based on the following theorem.

THEOREM 3.10. *Let $X_1$ be a (complex) reflexive Banach space with dual $X_1^*$ and dual form $\langle\,\cdot\,,\cdot\,\rangle_1$. Let $X_2$ be a (complex) Hilbert space with inner product $\langle\,\cdot\,,\cdot\,\rangle_2$ and $F : X_2 \to X_2, B : X_1 \to X_2$, compact linear operators such that $B$ is injective. Suppose there exists a bounded linear operator $A : X_1^* \to X_1$ such that $F = BAB^*$ and*

$$c_1 \|A\varphi\|_1^2 \le \left|\langle \varphi, A\varphi \rangle_1\right| \le c_2 \|A\varphi\|_1^2 \tag{3–14}$$

*for all* $\varphi \in X_1^*$ *where* $c_1$ *and* $c_2$ *are positive constants. Then for any* $\varphi \in X_2, \varphi \neq 0, \varphi \in R(BA^*)$ *if and only if*

$$\inf\left\{\left|\langle \psi, F\psi\rangle_2\right| \colon \psi \in X_2,\ \langle \psi, \varphi\rangle_2 = 1\right\} > 0.$$

PROOF. From (3–14), we have

$$\left|\langle \psi, F\psi\rangle_2\right| = \left|\langle B^*\psi,\ AB^*\psi\rangle_1\right| \geq c_1 \|AB^*\psi\|^2,$$

for all $\psi \in X_2$. Let $\varphi = BA^*\varphi_0$ for some $\varphi_0 \in X_1^*$. Then for $\psi \in X_2$ such that $\langle \psi, \varphi\rangle_2 = 1$ we have

$$\begin{aligned}\left|\langle \psi, F\psi\rangle_2\right| &\geq c_1 \|AB^*\psi\|_1^2 = \frac{c_1}{\|\varphi_0\|_1^2}\|AB^*\psi\|_1^2\,\|\varphi_0\|_1^2 \\ &\geq \frac{c_1}{\|\varphi_0\|_1^2}|\langle \varphi_0, AB^*\psi\rangle_1|^2 = \frac{c_1}{\|\varphi_0\|\|_1^2}|\langle BA^*\varphi_0, \psi\rangle_2|^2 = \frac{c_1}{\|\varphi_0|_1^2} > 0.\end{aligned}$$

Now assume that $\varphi \neq R(BA^*)$ and define the closed subspace $V := \{\psi \in X_2 : \langle \psi, \varphi\rangle_2 = 0\}$. We will show that $AB^*(V)$ is dense in $\overline{R(A)} = N(A^*)^\perp$. To see this, let $\varphi \in X_1^*$ such that $\langle \varphi, AB^*\psi\rangle_1 = 0$ for all $\psi \in V$. Then $\langle BA^*\varphi, \psi\rangle_2 = 0$ for all $\psi \in V$, i.e., $BA^*\varphi \in V^\perp = \operatorname{span}\{\varphi\}$. Since $\varphi \in R(BA^*)$ this implies that $BA^*\varphi = 0$ and hence $A^*\varphi = 0$ by the injectivity of $B$. Therefore $\varphi \in N(A^*) = \overline{R(A)}^\perp$ and hence $AB^*(V)$ is dense in $\overline{R(A)}$. We can therefore find a sequence $\{\hat{\psi}_n\}$ in $V$ such that

$$AB^*\hat{\psi}_n \to -\frac{1}{\|\varphi\|_2^2}AB^*\varphi$$

as $n \to \infty$. We now set $\psi_n = \hat{\psi}_n + \varphi/\|\varphi\|_2^2$. Then $\langle \psi_n, \varphi\rangle_2 = 1$ and $AB^*\psi_n \to 0$. From (3–14) we have

$$\left|\langle \psi_n, F\psi_n\rangle_2\right| = \left|\langle B^*\psi, AB^*\psi\rangle_1\right| \leq c_2\|AB^*\psi_n\|_1^2$$

and hence $\langle \psi_n, F\psi_n\rangle_2 \to 0$ as $n \to \infty$, i.e.,

$$\inf\left\{|\langle \psi, F\psi\rangle_2| : \psi \in X_2, \langle \psi, \varphi\rangle_2 = 1\right\} = 0. \qquad \square$$

In order to make use of Theorem 3.10, Kirsch defines $G : H^{1/2}(\partial D) \to L^2(\Omega)$ and $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ as in Lemma 3.8 (with the indicated changes in ranges and domains) and proves that if $F : L^2(\Omega) \to L^2(\Omega)$ is the far field operator corresponding to the exterior Dirichlet problem (2–3) then we again have the factorization $F = 4\pi G S^* G^*$. After showing that $S^*$ satisfies the coercivity condition (3–14) if $k^2$ is not a Dirichlet eigenvalue, and using the fact that in this case $S$ is an isomorphism, it is then possible to use Theorem 3.10 to conclude that if $k^2$ is not a Dirichlet eigenvalue we have

$$\varphi \in R(G) \iff \inf\left\{\left|\langle \psi, F\psi\rangle_{L^2(\Omega)}\right| : \psi \in L^2(\Omega), \langle \varphi, \psi\rangle_{L^2(\Omega)} = 1\right\} > 0.$$

Since $z \in D$ if and only if $\Phi_\infty(\,\cdot\,, z) \in R(G)$ we now have the following theorem [44].

THEOREM 3.11. *Assume that $k^2$ is not a Dirichlet eigenvalue. Then*

$$z \in D \iff \inf\big\{|\langle\psi, F\psi\rangle| : \psi \in L^2(\Omega),\ \langle\Phi_\infty(\,\cdot\,,z),\ \psi\rangle_{L^2(\Omega)} = 1\big\} > 0.$$

Theorem 3.11 leads in an obvious manner to a constrained optimization scheme for determining when a point $z \in R^3$ is in $D$ [44]. Note that the proof of Theorem 3.11 does not rely on the normality of $F$.

## 4. The Inverse Medium Problem for Acoustic Waves

We will now turn our attention to inverse scattering problems associated with the inhomogeneous medium problem (2–4) and ultimately the acoustic transmission problem (2–1). As in Section 3, we will focus our attention on the situation where neither the material properties of the scatterer nor its shape are known. Then it follows from the uniqueness theorems of Nachman [53] and Isakov [36], [37]that for a single frequency the best that we can hope for is to determine the shape $D$ of the scatterer. In particular, in order to determine the coefficients in (2–1b), either multi-frequency data is needed or an a priori knowledge of either $\rho_D(x)$ or $n(x)$ is required. If such information is available then nonlinear optimization techniques such as those described in Section 2 of this paper can be used to determine the coefficients. Here we will restrict ourselves to a fixed frequency and prove uniqueness theorems associated with the direct scattering problems (2–1) and (2–4) as well as reconstruction algorithms for determining $D$ from the far field pattern $u_\infty$.

In the previous section we presented three different methods for determining the shape $D$ of the scatterer from a knowledge of the far field pattern associated with the exterior Dirichlet problem (2–3), in particular the two versions of the linear sampling method based on $F$ and $(F^*F)^{\frac{1}{4}}$ respectively and the constrained optimization method based on Theorem 3.11. Each of these methods, under appropriate assumptions, can be extended to the inverse scattering problem associated with the inhomogeneous medium problems (2–4) [12], [19], [43], [44]. However, at the time of writing, only the linear sampling method associated with the far field operator $F$ has been extended to the general acoustic transmission problem (2–1) [6], [7] and the case of Maxwell's equations [11], [29], [49]. Hence, in the interest of developing a unifying theme to our paper, we will restrict our attention to the first version of the linear sampling method in order to determine $D$.

We begin our discussion by considering the inhomogeneous medium problem (2–4) and again define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d), \tag{4–1}$$

where $u_\infty$ is the far field pattern of the scattered field $u^s$ defined in (2–4). It is again possible to establish the reciprocity relations (3–1) and (3–2). However, since $\operatorname{Im} n(x) \geq 0$, we cannot expect normality of $F$ except in the case when $\operatorname{Im} n(x) = 0$. The question of when $F$ is injective with dense range is addressed by the following theorem, where the role of the interior Dirichlet problem in Theorem 3.2 is now replaced by a new type of boundary value problem called the homogeneous *interior transmission problem.*

THEOREM 4.1. *The far field operator $F$ defined by* (4–1) *is injective with dense range if and only if there does not exist $w \in C^2(D) \cap C^2(\bar{D})$ and a Herglotz wave function $v$ with kernel $g \neq 0$ such that $v, w$ is a solution to the homogeneous interior transmission problem*

$$\begin{aligned} \Delta v + k^2 v = 0 &\qquad \text{in } D, \\ \Delta w + k^2 n(x) w = 0 &\qquad \text{in } D, \end{aligned} \tag{4–2a}$$

$$\begin{aligned} v = w &\qquad \text{on } \partial D, \\ \frac{\partial w}{\partial \nu} &\qquad \text{on } \partial D. \end{aligned} \tag{4–2b}$$

PROOF. As in the case of Theorem 3.2, it suffices to establish conditions for when the far field operator is injective. To this end, we note that $Fg = 0$ with $g \neq 0$ is equivalent to the vanishing of the far field pattern of $w^s$ where $w$ is the solution of (2–4) with $u^i$ a Herglotz wave function $v$ with kernel $g$. By Rellich's lemma, $w^s = 0$ in $R^3 \setminus D$, and hence if $w = v + w^s$ we have

$$w = v \quad \text{on } \partial D, \qquad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \qquad \square$$

An elementary application of Green's theorem and the unique continuation principle for elliptic equations shows (Theorem 8.12 of [14]) that if $\operatorname{Im} n(x_0) > 0$ for some $x_0 \in D$ then the only solution of (4–2) is $v = w = 0$, i.e., in this case $F$ is injective with dense range. Knowing that the values of $k$ for which the far field operator is not injective form a discrete set is of considerable importance in the inverse scattering problem associated with (2–4), just as it is in the case of the obstacle problem (2–3) where it is known that the set of Dirichlet eigenvalues forms a discrete set. In the case of the linear sampling method, for example, this enables us to conclude that the method can fail only for a discrete set of values of $k$. From Theorem 4.1 we see that $F$ is injective if there does not exist a nontrivial solution to the homogeneous interior transmission problem. Values of $k$ for which there exists a nontrivial solution to the homogeneous interior transmission problem are called *transmission eigenvalues.* It was shown by Colton, Kirsch and Päivärinta ([13] and Section 8.6 of [14]) and by Rynne and Sleeman [67] that if there exists $\varepsilon > 0$ such that either $n(x) \geq 1 + \varepsilon$ for $x \in \bar{D}$ or $0 < n(x) \leq 1 - \varepsilon$ for $x \in \bar{D}$ then the set of transmission eigenvalues is discrete.

We now turn to the problem of the unique determination of $n(x)$ in (2–4) from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$. The original proof of this result is due to Nachman [53], Novikov [57] and Ramm [66] and is based on the fundamental paper of Sylvester and Uhlmann [71]. Here we follow a modification of the original proof due to Hähner [30] which is based on the following two lemmas, where $H^2(B)$ denotes a Sobolev space.

LEMMA 4.2. *Let $B$ be an open ball centered at the origin and containing the support of $m := 1 - n$. Then there exists a positive constant $C$ such that for each $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|\operatorname{Re} z| \geq 2k^2\|n\|_\infty$ there exists a solution $w \in H^2(B)$ to $\Delta w + k^2 nw = 0$ in $B$ of the form*

$$w(x) = e^{iz\cdot x}\big(1 + r(x)\big),$$

*where*

$$\|r\|_{L^2(B)} \leq \frac{C}{|\operatorname{Re} z|}.$$

LEMMA 4.3. *Let $B_1$ and $B_2$ be two open balls entered at the origin and containing the support of $m := 1 - n$ such that $\bar{B}_1 \subset B_2$. Then the set of total fields $\{u(\,\cdot\,, d), d \in \Omega\}$ satisfying (2–4) is complete in the closure of*

$$H := \{w \in C^2(\bar{B}_2) : \Delta w + k^2 nw = 0 \text{ in } B_2\}$$

*with respect to the norm in $L^2(B_1)$.*

We are now ready to prove the following uniqueness result for the inverse inhomogeneous medium problem (2–4).

THEOREM 4.4. *The coefficient $n(x)$ in (2–4) is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$.*

PROOF. Assume that $n_1$ and $n_2$ are such that $u_{1,\infty}(\,\cdot\,, d) = u_{2,\infty}(\,\cdot\,, d), d \in \Omega$, and let $B_1$ and $B_2$ be two open balls centered at the origin and containing the supports of $1 - n_1$ and $1 - n_2$ such that $\bar{B}_1 \subset B_2$. Then by Rellich's lemma we have $u_1(\,\cdot\,, d) = u_2(\,\cdot\,, d)$ in $R^3 \setminus \bar{B}_1$ for all $d \in \Omega$. Hence $u = u_1 - u_2$ satisfies $u = \partial u/\partial\nu$ on $\partial B_1$ and the differential equation

$$\Delta u + k^2 n_1 u = k^2(n_2 - n_1)u_2$$

in $B_1$. From this and the differential equation for $\tilde{u}_1 = u_1(\,\cdot\,, \tilde{d}), \tilde{d} \in \Omega$, we obtain

$$k^2 \tilde{u}_1 u_2(n_2 - n_1) = \tilde{u}_1(\Delta u + k^2 n_1 u) = \tilde{u}_1 \Delta u - u \Delta \tilde{u}_1.$$

From Green's theorem and the fact that the Cauchy data for $u$ vanishes on $\partial B_1$ we now have

$$\iint_{B_1} u_1(\,\cdot\,, \tilde{d}) u_2(\,\cdot\,, d)(n_1 - n_2)\, dx = 0$$

for all $d, \tilde{d} \in \Omega$. It now follows from Lemma 4.3 that

$$\iint_{B_1} w_1 w_2 (n_1 - n_2)\, dx = 0 \tag{4–3}$$

for all solutions $w_1, w_2 \in C^2(\bar{B}_2)$ of $\Delta w_1 + k^2 n_1 w_1 = 0$ and $\Delta w_2 + k^2 n_2 w_2 = 0$ in $B_2$.

Given $y \in R^3 \setminus \{0\}$ and $\rho > 0$, we now choose vectors $a, b \in R^3$ such that $\{y, a, b\}$ is an orthogonal basis in $R^3$ with the properties that $|a| = 1$ and $|b|^2 = |y|^2 + \rho^2$. Then for $z_1 := y + \rho a + ib, z_2 := y - \rho a - ib$ we have

$$z_j \cdot z_j = |\mathrm{Re}\, z_j|^2 - |\mathrm{Im}\, z_j|^2 + 2i\,\mathrm{Re}\, z_j \cdot \mathrm{Im}\, z_j = |y|^2 + \rho^2 - |b|^2 = 0$$

and $|\mathrm{Re}\, z_j|^2 = |y|^2 + \rho^2 \geq \rho^2$. In (4–3) we now substitute the solutions $w_1$ and $w_2$ from Lemma 4.2 for the coefficients $n_1$ and $n_2$ and vectors $z_1$ and $z_2$, respectively. Since $z_1 + z_2 = 2y$ this gives

$$\iint_{B_1} e^{2iy\cdot x}\big(1 + r_1(x)\big)\big(1 + r_2(x)\big)\big(n_1(x) - n_2(x)\big)\, dx = 0$$

and, passing to the limit as $\rho \to \infty$, gives

$$\iint_{B_1} e^{2iy\cdot x}\big(n_1(x) - n_2(x)\big)\, dx = 0.$$

Since this equation is true for arbitrary $y \in R^3$, by the Fourier integral theorem we have $n_1(x) = n_2(x)$ in $B_1$ and the proof is finished. □

Before proceeding to reconstruction algorithms for determining the support of $m = 1 - n$, we note that at the time of writing a uniqueness theorem for the inverse inhomogeneous medium problem (2–4) in $R^2$ analogous to Theorem 4.4 for the case of $R^3$ is unknown. The problem in $R^2$ is more difficult than the case in $R^3$ due to the fact that the inverse scattering problem for fixed frequency in $R^2$ is not overdetermined as in the $R^3$ case. i.e., in $R^2, u_\infty(\hat{x}, d)$ is a function of two variables and $n(x)$ is also a function of two variables. Nevertheless, there have been numerous partial results in this case due to Novikov [58], Sun and Uhlmann [68], Isakov and Nachman [38], Isakov and Sun [39] and Eskin [25] among others. We content ourselves here by stating a single result in this direction due to Sun and Uhlmann [70] (see also [64]) which shows that the discontinuities of $n$ are uniquely determined from the far field pattern $u_\infty$. We note that in the $R^2$ case the radiation condition (2–4c) is replaced by

$$\lim_{r\to\infty} r^{1/2}\Big(\frac{\partial u^s}{\partial r} - iku^s\Big) = 0$$

and the asymptotic behavior (2–7) is replaced by

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}).$$

THEOREM 4.5. *Let $n_1$ and $n_2$ be in $L^\infty(R^2)$ and suppose $m_1 := 1 - n_1$ and $m_2 := 1 - n_2$ have compact support. Then if $u^j_\infty$ is the far field pattern corresponding to $n_j$ for $j = 1, 2$ and $u^1_\infty(\hat{x}, d) = u^2_\infty(\hat{x}, d)$ for all $\hat{x}, d$ on the unit circle $\Omega$, then $n_1 - n_2 \in C^{0,\alpha}(R^2)$ for every $\alpha, 0 \leq \alpha < 1$.*

We now return to the three dimensional inverse scattering problem associated with (2–4). Given the fact that $n(x)$ is uniquely determined from $u_\infty$, we can now attempt to reconstruct $n(x)$ by using nonlinear optimization techniques as discussed in Section 2 of this paper. (A reconstruction procedure for determining $n$ based on the techniques used in the uniqueness Theorem 4.4 has been given by Nachman [53] and Novikov [57] although it is not clear whether or not this leads to a viable numerical procedure.) However, a reconstruction of $n(x)$ is often more than is necessary. Indeed, it is frequently sufficient to determine the support of $m = 1-n$ and, as mentioned at the beginning of this section, for fixed frequency and the more general acoustic transmission problem this is essentially all the information that can be extracted from the far field data $u_\infty$. We will now proceed to show how the linear sampling method can be used to determine the support $D$ of $m := 1 - n$ basing our analysis on the ideas of Colton and Kirsch [12] and Colton, Piana and Potthast [19]. In order to avoid the problem of transmission eigenvalues, we will limit our attention to the case when there exists a positive constant $c$ such that

$$\operatorname{Im} n(x) \geq c \tag{4–4}$$

for $x \in D$ where $\bar{D}$ is the support of $m = 1 - n$. If instead of (4–4) we have $\operatorname{Im} n(x) = 0$ for $x \in D$, the analysis that follows remains valid if we assume that $k$ is not a transmission eigenvalue.

The derivation of the linear sampling method for the inverse scattering problem associated with (2–4) is based on a projection theorem for Hilbert spaces where the inner product is replaced by a bounded sesquilinear form together with an analysis of a special inhomogeneous interior transmission problem. We begin with the projection theorem. Let $X$ be a Hilbert space with the scalar product $(\,\cdot\,,\cdot)$ and norm $\|\cdot\|$ induced by $(\,\cdot\,,\cdot)$ and let $\langle\,\cdot\,,\cdot\rangle$ be a bounded sesquilinear form on $X$ such that

$$\big|\langle\varphi,\varphi\rangle\big| \geq C\|\varphi\|^2$$

for all $\varphi \in X$ where $C$ is a positive constant. Then, using the Lax–Milgram theorem, it is easy to prove the following theorem ([19], Theorem 10.22 in [14]) where $\oplus_s$ is the orthogonal decomposition with respect to the sesquilinear form $\langle\,\cdot\,,\cdot\rangle$ and $H^{\perp_s}$ is the orthogonal complement of $H$ with respect to $\langle\,\cdot\,,\cdot\rangle$.

THEOREM 4.6. *For every closed subspace $\bar{H} \subset X$ we have the orthogonal decomposition*

$$X = H^{\perp_s} \oplus_s \bar{H}.$$

*The projection operator $P : X \to H^{\perp_s}$ defined by this decomposition is bounded in $X$.*

We now turn our attention to the problem of showing the existence of a unique weak solution $v, w$ of the interior transmission problem

$$\Delta v + k^2 v = 0 \qquad \text{in } D, \tag{4–5}$$

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D,$$
$$w - v = \Phi(\,\cdot\,, z) \qquad \text{on } \partial D,$$
$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\Phi\,(\,\cdot\,, z) \qquad \text{on } \partial D,$$

where $z \in D$, $n$ is assumed to satisfy (4–4) and $\Phi$ as usual is defined by (2–6). To motivate the following definition of a weak solution of (4–5), we note that if a solution $v, w \in C^2(D) \cap C^1(\bar{D})$ to (4–5) exists, then from Green's formula and Rellich's lemma we have

$$w(x) + k^2 \iint_D \Phi(x, y) m(y) w(y)\, dy = v(x) \qquad \text{for } x \in D,$$
$$-k^2 \iint_D \Phi(x, y) m(y) w(y)\, dy = \Phi(x, z) \qquad \text{for } x \in \partial B,$$

where $B$ is a ball centered at the origin with $\bar{D} \subset B$.

Definition 4.7. Let $H$ be the linear space of all Herglotz wave functions and $\bar{H}$ the closure of $H$ in $L^2(D)$. For $\varphi \in L^2(D)$ define the volume potential by

$$(T\varphi)(x) := \iint_D \Phi(x, y) m(y) \varphi(y)\, dy, \ x \in R^3.$$

Then a pair $v, w$ with $v \in \bar{H}$ and $w \in L^2(D)$ is said to be a weak solution of the interior transmission problem (4–5) with source point $z \in D$ if $v$ and $w$ satisfy the integral equation

$$w + k^2 T w = v$$

and the boundary condition

$$-k^2 T w = \Phi(\,\cdot\,, z) \qquad \text{on } \partial B.$$

The uniqueness of a weak solution to the interior transmission problem follows from a limiting argument using (4–4) and a simple application of Green's theorem [19], [14, Theorem 10.24]. To prove existence we will use Theorem 4.6 applied to the sesquilinear form in $L^2(D)$ defined by

$$\langle \varphi, \psi \rangle := \iint_D m(y) \varphi(y)\, \overline{\psi(y)}\, dy$$

and $H$ as defined in the above definition.

Theorem 4.8. *For every source point $z \in D$ there exists a weak solution to the interior transmission problem.*

Proof. By a translation we can assume without loss of generality that $z = 0$. We consider the space

$$H_1^0 = \operatorname{span}\big\{ j_p(k|x|) Y_p^q(\hat{x}),\ p = 1, 2, \ldots,\ -p \le q \le p \big\}$$

and the closure $H_1$ of $H_1^0$ in $L^2(D)$, where $j_p$ is a spherical Bessel function and $Y_p^q$ is a spherical harmonic. It can be shown that there exists a nontrivial $\psi \in H_1^{\perp_s} \cap \bar{H}$ such that $\langle j_0, \psi \rangle \neq 0$.

Now let $P$ be the projection operator from $L^2(D)$ onto $H^{\perp_s}$ as defined by Theorem 4.6. We first consider the integral equation

$$u + k^2 PTu = k^2 PT\psi \tag{4–6}$$

in $L^2(D)$. Since $T$ is compact and $P$ is bounded, the operator $PT$ is compact in $L^2(D)$. Hence to establish the existence of a solution to (4–6) we must prove uniqueness for the homogeneous equation. To this end, assume that $w \in L^2(D)$ satisfies

$$w + k^2 Tw = v.$$

Since $\langle w, \varphi \rangle = 0$ for all $\varphi \in H$, from the addition formula for Bessel functions we conclude that $Tw = 0$ on $\partial B$. Hence, by uniqueness for the weak interior transmission problem we have $v = w = 0$, and we obtain the continuous invertibility of $I + k^2 PT$ in $L^2(D)$.

Now let $u$ be the unique solution to (4–6) and note that $u \in H^{\perp_s}$. We define the constant $c$ and function $w \in L^2(D)$ by

$$c := -\frac{1}{k^2 \langle j_0, \psi \rangle}, \qquad w := c(u - \psi).$$

Then we compute

$$w + k^2 PTw = -c\psi$$

and hence

$$w + k^2 Tw = v$$

where $v := k^2(I - P)Tw - c\psi \in \bar{H}$. Since

$$\langle h, w \rangle = c \, \langle h, u - \psi \rangle = 0$$

for all $h \in H_1$ and

$$\langle j_0, w \rangle = c \, \langle j_0, u - \psi \rangle = -\frac{1}{k^2},$$

we have from the addition formula for Bessel functions that

$$-k^2 (Tw)(x) = ikh_0^{(1)}(k|x|) = \Phi(x, 0), \qquad x \in \partial B,$$

where $h_0^{(1)}$ is a spherical Hankel function of the first kind of order zero, and the proof is complete. □

Having Theorem 4.8 at our disposal, we can now establish the linear sampling method for determining $D$. In particular, we again consider the far field equation $Fg = \Phi_\infty(\,\cdot\,, z)$, that is,

$$\int_\Omega u_\infty(\hat{x}, d) g(d) \, ds(d) = \Phi_\infty(\hat{x}, z), \tag{4–7}$$

where $\Phi_\infty(\,\cdot\,,z)$ is the far field pattern of the fundamental solution $\Phi(\,\cdot\,,z)$. Following the proof of Theorem 4.1 we see that (4–7) has a solution if and only if $z \in D$ and the interior transmission problem (4–5) has a solution $v, w \in C^2(D) \cap C^1(\bar{D})$ such that $v$ is a Herglotz wave function with kernel $g$. This is only true in very special cases. However, by Theorem 4.8 we know there exists a (unique)weak solution $v, w$ to the interior transmission problem and that $v$ can be approximated in $L^2(D)$ by a Herglotz wave function. This fact then enables us to establish a result for the far field equation (4–7) that is analogous to Theorem 3.7 for the far field equation (3–7) corresponding to the exterior Dirichlet problem [4] (Later on in this section we shall outline how this is done for the general case of an anisotropic medium). Note that the far field equations (3–7) and (4–7) are exactly the same except of course that the far field pattern $u_\infty$ appearing in the kernel of $F$ come from different scattering problems. This means that in order to determine the support $D$ of the scatterer it is not necessary to know a priori whether the direct scattering problem is (2–3) or (2–4) or, as previously noted, (2–2). In particular, one can determine the support of the scatterer without a priori knowledge on whether or not the scattering object is penetrable or impenetrable, at least in the context of the three scattering problems (2–2), (2–3) and (2–4). In the remaining part of this section we will extend this observation to include the general acoustic transmission problem (2–1) and in fact consider the even more general case of anisotropic media. As with the case of the inhomogeneous medium problem (2–4), the basic ingredient will again be an analysis of an interior transmission problem, this time for anisotropic media.

Let $D \subset R^3$ be a bounded domain having $C^2$ boundary $\partial D$ with unit outward normal $\nu$. Let $A$ be a $3 \times 3$ matrix-valued function whose entries $a_{jk}$ ($j = 1,2,3,\ k = 1,2,3$) are continuously differentiable functions in $\bar{D}$, such that $A$ is symmetric and satisfies

$$\bar{\xi} \cdot (\operatorname{Im} A)\xi \le 0, \qquad \bar{\xi} \cdot (\operatorname{Re} A)\xi \ge \gamma|\xi|^2 \tag{4–8}$$

for all $\xi \in \mathbb{C}^3$ and $x \in \bar{D}$, where $\gamma$ is a positive constant. For a function $u \in C^1(\bar{D})$ we define the conormal derivative by

$$\frac{\partial u}{\partial \nu_A}(x) := \nu(x) \cdot A(x)\nabla u(x) \quad \text{for } x \in \partial D$$

and let $k > 0$ again be the wave number and let $n \in C(\bar{D})$ satisfy $\operatorname{Re} n > 0$ and $\operatorname{Im} n \ge 0$. The anisotropic acoustic transmission problem, for which (2–1) is the special case of an isotropic medium, is to find $u \in C^2(R^3 \setminus \bar{D}) \cap C^1(R^3 \setminus D)$ and $v \in C^2(D) \cap C^1(\bar{D})$ such that

$$\Delta u + k^2 u = 0 \qquad \text{in } R^3 \setminus \bar{D}, \tag{4–9a}$$

$$\nabla \cdot A\nabla v + k^2 n(x) v = 0 \qquad \text{in } D, \tag{4–9b}$$

$$u = u^i + u^s, \tag{4–9c}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{4–9d}$$

$$u = v \qquad \text{on } \partial D, \tag{4–9e}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu_A} \qquad \text{on } \partial D, \tag{4–9f}$$

where again $u^i(x) = e^{ikx\cdot d}$. The existence of a unique solution to (4–9) has been established by Hähner [31].

Since $u$ satisfies the radiation condition, we can again conclude that $u^s$ has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + O(r^{-2}).$$

The inverse scattering problem we are concerned with is to determine $D$ from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$. We note that the matrix $A$ is not uniquely determined by $u_\infty$ (see [27], [61]) and hence determinning $D$ is the most that can be hoped for. To this end we have the following theorem due to Hähner [31] (see also [17] and [61]).

Theorem 4.9. *Assume $\gamma > 1$. Then $D$ is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$.*

The proof of this theorem uses the ideas of Theorem 3.5 together with a continuous dependence result for an associated interior transmission problem. It follows from the results of Cakoni and Haddar [7] that Theorem 4.9 remains valid if the condition $\gamma > 1$ is replaced by the condition

$$\bar{\xi} \cdot (\operatorname{Re} A^{-1})\xi \geq \mu |\xi|^2 \tag{4–10}$$

for all $\xi \in \mathbb{C}^3$ and $x \in \bar{D}$ where $\mu$ is a positive constant such that $\mu > 1$. The isotropic case when $A = I$ is handled by Theorem 4.4.

Given the uniqueness Theorem 4.9 (and the variations on this theorem indicated above)we now want to establish the linear sampling method for determining $D$. In particular, we look for a (regularized) solution $g \in L^2(\Omega)$ of the far field equation

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) = \Phi_\infty(\hat{x}, z) \tag{4–11}$$

where $z \in R^3$ is an artificially introduced parameter point and $u_\infty$ is the far field pattern of the scattered field defined by (4–9). Following the proof of Theorem 4.1 it is easily verified that (4–11) is solvable if and only if $z \in D$ and $v, w \in C^2(D) \cap C^1(\bar{D})$ is a solution of the interior transmission problem

$$\begin{aligned} \Delta v + k^2 v &= 0 \qquad \text{in } D, \\ \nabla \cdot A\nabla w + k^2 n w &= 0 \qquad \text{in } D, \end{aligned} \tag{4–12a}$$

$$w - v = \Phi(\,\cdot\,, z) \qquad \text{on } \partial D,$$
$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\Phi(\,\cdot\,, z) \qquad \text{on } \partial D, \tag{4–12b}$$

such that $v$ is a Herglotz wave function. Values of $k$ for which a nontrivial solution to the homogeneous interior transmission problem ($\Phi = 0$) exists are again called transmission eigenvalues. As in the case of an isotropic medium, our aim is to now study the interior transmission problem (4–12) with the aim of showing that, roughly speaking, $D$ can be characterized as the set of points $z \in R^3$ where an arbitrarily good approximation of the solution to the far field equation (4–4) remains bounded (see Theorem 3.7).

We begin with uniqueness. The following theorem follows easily by an application of Green's theorem.

THEOREM 4.10. *If either* $\operatorname{Im} n > 0$ *or* $\bar{\xi} \cdot (\operatorname{Im} A)\xi < 0$ *in a neighborhood of a point* $x_0 \in D$, *then* (4–12) *has at most one solution.*

In order to study the solvability of (4–12) we first consider a modified interior transmission problem which is a compact perturbation of (4–12). In particular, let $m \in C(\bar{D})$ satisfy $m(x) > 0$ for $x \in \bar{D}$ and for $l_1, l_2 \in L^2(D)$, $f \in H^{1/2}(\partial D)$, $h \in H^{-1/2}(\partial D)$ we want to find $v, w \in H^1(D)$ such that

$$\Delta v + k^2 v = l_1 \qquad \text{in } D,$$
$$\nabla \cdot A \nabla w - m w = l_2 \qquad \text{in } D, \tag{4–13a}$$

$$w - v = f \qquad \text{on } \partial D,$$
$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \qquad \text{on } \partial D. \tag{4–13b}$$

In [6], (4–13) is reformulated as a variational problem for $(w, \mathbf{v}) \in H^1(D) \times W(D)$ where $\mathbf{v} = \nabla v$ and

$$W(D) := \{\mathbf{v} \in (L^2(D))^3 \;:\; \nabla \cdot \mathbf{v} \in L^2(D) \quad \text{and curl } \mathbf{v} = 0\}.$$

The variational problem is then solved by appealing to the Lax–Milgram theorem under the assumption that in (4–8) we have $\gamma > 1$ and $m > \gamma$. Having established the existence of a unique solution to (4–13), and using the fact that (4–13) is a compact perturbation of the interior transmission problem (4–12), we can now appeal to Theorem 4.10 to deduce the following theorem [5].

THEOREM 4.11. *Assume that either* $\operatorname{Im} n > 0$ *or* $\bar{\xi} \cdot (\operatorname{Im} A)\xi < 0$ *in a neighborhood of a point* $x_0 \in D$ *and that* $\gamma > 1$. *Then* (4–12) *has a unique solution* $v, w \in H^1(D) \times H^1(D)$ *where the boundary data* (4–12b) *is assumed in the sense of the trace operator*.

It was shown in [7] that Theorem 4.11 remains valid if the condition $\gamma > 1$ is replaced by the condition (4–10) where $\mu > 1$ (In this case $m$ is a constant restricted to satisfy $\mu^{-1} \le m < 1$).

If $A$ and $n$ do not satisfy one of the above assumptions ($\gamma > 1$ in (4–8) or $\mu > 1$ in (4–10)) then in general we cannot conclude the solvability of the interior transmission problem (4–12). In particular, (4–12) is uniquely solvable if and only if $k$ is not a transmission eigenvalue and if $\operatorname{Im} n = 0$ and $\operatorname{Im} A = 0$ it is not possible to exclude this possibility. However, from the point of view of applying regularization techniques to the far field equation (4–11), it is important to have $F$ injective with dense range and this is true if $k$ is not a transmission eigenvalue. To this end the following theorem is important [5], [7].

THEOREM 4.12. *Assume that* $\operatorname{Im} n = 0$ *and* $\operatorname{Im} A = 0$ *in* $D$ *and that one of the following conditions is satisfied*:

(1) $\gamma > 1$ *in* (4–8) *and* $n(x) \geq \gamma$ *for* $x \in D$, *or*
(2) $\mu > 1$ *in* (4–10) *and* $n$ *is a constant such that* $\mu^{-1} \leq n < 1$.

*Then the set of transmission eigenvalues forms a discrete set.*

The proof of Theorem 4.12 is based on the uniqueness of a solution to the modified interior transmission problem (4–13) together with the fact that the spectrum of a compact operator is discrete. At the time of writing it is not known whether or not transmission eigenvalues exist except for the special case when $A = I$ (isotropic media) and $n(x) = n(r)$ is spherically symmetric; see [14, Theorem 8.13].

We now turn our attention to showing how Theorems 4.11 and 4.12 lead to a justification of the linear sampling method for anisotropic media that is analogous to Theorem 3.7 for the case of the exterior Dirichlet problem. To this end we let $B$ be the bounded linear operator from $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ into $L^2(\Omega)$ which maps $(f,h) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ onto the far field data $u_\infty \in L^2(\Omega)$ of the solution of the transmission problem

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } R^3 \setminus \bar{D}, \tag{4–14a}$$

$$\nabla \cdot A \nabla v + k^2 n(x) v = 0 \qquad \text{in } D, \tag{4–14b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{4–14c}$$

$$v - u^s = f \qquad \text{on } \partial D, \tag{4–14d}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = h \qquad \text{on } \partial D, \tag{4–14e}$$

where $u^s \in H^1_{\text{loc}}(R^3 \setminus \bar{D})$ and $v \in H^1(D)$. It is shown in [5] that the range of $B$ is dense in $L^2(\Omega)$. However, $B$ is not injective. To remedy this problem, we define the subset $H(\partial D)$ of $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ by

$$H(\partial D) := \left\{ \left( v|_{\partial D}, \frac{\partial v}{\partial \nu}|_{\partial D} \right) : v \in H \right\},$$

where $H := \{v \in H^1(D) : \Delta v + k^2 v = 0 \text{ in } D\}$. Then $H(\partial D)$ equipped with the induced norm from $H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ is a Banach space and if $B_0$ is the restriction of $B$ to $H(\partial D)$ we conclude that for $k$ not a transmission eigenvalue $B_0 : H(\partial D) \to L^2(\Omega)$ is injective, compact and has dense range [5].

We now write the far field equation (4–11) in the form

$$-B_0 Hg = \Phi_\infty(\,\cdot\,, z)$$

where $Hg$ denotes the traces $(v_g|_{\partial D}, \partial v_g/\partial\nu|_{\partial D})$ for $v_g$ a Herglotz wave function with kernel $g$. Using the facts that 1) Herglotz wave functions with kernels $g \in L^2(\Omega)$ are dense in $H$ with respect to the norm in $H^1(D)$ and 2) $\Phi(\,\cdot\,, z)$ is in the range of $B_0$ if and only if $z \in D$, we can now deduce the analogue of Theorem 3.7 for anisotropic media under the assumption that either $\gamma > 1$ in (4–8) or $\mu > 1$ in (4–10) and $k$ is not a transmission eigenvalue [5], [7]. As in the case of Theorem 3.7 for the exterior Dirichlet problem,this result now yields a numerical procedure for determining the support $D$ of an anisotropic object from noisy far field data.

Partial results related to the above for the case when $A = I$ on $\partial D$ but $A \neq I$ in $D$ can be found in Chapter 7 of [62].

## 5. The Inverse Scattering Problem for Electromagnetic Waves

In this final section of our survey paper on inverse scattering problems for acoustic and electromagnetic waves we consider the scattering of a time harmonic electromagnetic wave by either a perfectly conducting obstacle or an isotropic inhomogeneous medium of compact support. We begin with the scattering of a time harmonic electromagnetic plane wave by a perfectly conducting obstacle. Let $D$ be a bounded domain in $R^3$ with connected complement such that $\partial D$ is in class $C^2$ and $\nu$ is the unit outward normal to $\partial D$. Then the direct scattering problem we are concerned with can be formulated as the problem of finding an electric field $E$ and a magnetic field $H$ such that $E, H \in C^1(R^3 \setminus \bar{D}) \cap C(R^3 \setminus D)$ and

$$\begin{aligned} \operatorname{curl} E - ikH = 0 \qquad &\text{in } R^3 \setminus \bar{D}, \\ \operatorname{curl} H + ikE = 0 \qquad &\text{in } R^3 \setminus \bar{D}, \end{aligned} \tag{5–1a}$$

with

$$\nu \times E = 0 \qquad \text{on } \partial D \tag{5–1b}$$

and

$$E = E^i + E^s, \qquad H = H^i + H^s, \tag{5–1c}$$

where $E^s, H^s$ represent the scattered field satisfying the Silver–Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{5–1d}$$

uniformly in $\hat{x} = x/|x|$ (with $r = |x|$), and the incident field $E^i, H^i$ is given by

$$E^i(x) = \frac{i}{k} \operatorname{curl} \operatorname{curl} p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d}, \tag{5–1e}$$

$$H^i(x) = \operatorname{curl} p e^{ikx\cdot d} = ikd \times p e^{ikx\cdot d}, \tag{5–1f}$$

where the wave number $k$ is positive, $d$ is a unit vector giving the direction of propagation and $p$ is the polarization vector. The existence and uniqueness of a solution to (5–1) is well known [14]. From the Stratton–Chu formula [14] it follows from (5–1) that $E^s$ has the asymptotic behavior

$$E^s(x) = \frac{e^{ikr}}{r} E_\infty(\hat{x}, d, p) + O(r^{-2}), \tag{5–2}$$

where $E_\infty$ is the *electric far field pattern* of the scattered electric field $E^s$. Note that since we are always assuming that $k$ is fixed we have suppressed the dependence of $E_\infty$ on $k$. It can easily be verified [14] that $E_\infty$ is infinitely differentiable as a function of $\hat{x}$ and $d$, linear with respect to $p$ and as a function of $\hat{x}$ is tangential to the unit sphere $\Omega$.

We now turn our attention to the scattering of the electromagnetic plane wave (5–1e), (5–1f) by an inhomogeneous medium of compact support. In this case, under appropriate assumptions [14], our problem is to find $E, H \in C^1(R^3)$ satisfying

$$\left.\begin{aligned} \operatorname{curl} E - ikH &= 0 \\ \operatorname{curl} H + ikn(x)E &= 0 \end{aligned}\right. \quad \text{in } R^3, \tag{5–3a}$$

where $n$ satisfies $C^{2,\alpha}(R^3)$ for some $0 < \alpha < 1$, $\operatorname{Re} n > 0$, $\operatorname{Im} n \geq 0$, $1 - n$ has compact support $\bar{D}$; and where

$$E = E^i + E^s, \qquad H = H^i + H^s \tag{5–3b}$$

such that $E^i, H^i$ is given by (5–1e), (5–1f) and $E^s, H^s$ again satisfies the Silver–Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{5–3c}$$

uniformly in $\hat{x}$. The existence and uniqueness of a solution to (5–3) is again well known [14] and $E^s$ can be shown to have the asymptotic behavior (5–2).

We are now in a position to define the electric far field operator and its connection to what are called electromagnetic Herglotz pairs. To this end, we define the Hilbert space $T^2(\Omega)$ by

$$T^2(\Omega) := \{a : \Omega \to \mathbb{C}^3 : a \in L^2(\Omega),\ a \cdot \hat{x} = 0 \text{ for } \hat{x} \in \Omega\}.$$

The *electric far field operator* $F : T^2(\Omega) \to T^2(\Omega)$ is then defined by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, ds(d),\ \hat{x} \in \Omega \tag{5–4}$$

where $g \in T^2(\Omega)$. We note that $F$ is a compact linear operator on $T^2(\Omega)$. An *electromagnetic Herglotz pair* is a pair of vector fields of the form

$$E(x) = \int_\Omega e^{ikx\cdot d} a(d)\, ds(d), \qquad H(x) = \frac{1}{ik}\ \mathrm{curl}\, E(x), \tag{5–5}$$

for $x \in R^3$ where $a \in T^2(\Omega)$ is the *kernel* of $E, H$. In particular, $Fg$ is the electric far field pattern for (5–1) or (5–3) respectively corresponding to the electromagnetic Herglotz pair with kernel $ikg$ as incident field.

As in the case of the inverse scattering problem for acoustic waves, of basic importance is the fact that $F$ is injective with dense range. The proof of the following two theorems and corollary follows along the same lines as previously discussed for the case of acoustic waves [14]. Recall that a Maxwell eigenfunction for $D$ is a solution of Maxwell's equations (5–1a) in $D$ satisfying (5–1b) on $\partial D$.

THEOREM 5.1. *The electric far field operator for* (5–1) *is injective with dense range if and only if there does not exist a Maxwell eigenfunction for $D$ which is an electromagnetic Herglotz pair.*

THEOREM 5.2. *The electric far field operator for* (5–3) *is injective with dense range if and only if there does not exist $E_1, H_1 \in C^1(D) \cap C(\bar{D})$ and an electromagnetic Herglotz pair $E_0, H_0$ with kernel $a \neq 0$ such that $E_0, H_0$ and $E_1, H_1$ is a solution to the homogeneous electromagnetic interior transmission problem*

$$\begin{aligned} &\mathrm{curl}\, E_0 - ikH_0 = 0, && \mathrm{curl}\, H_0 + ikE_0 = 0 && \textit{in } D,\\ &\mathrm{curl}\, E_1 - ikH_1 = 0, && \mathrm{curl}\, H_1 + ikn(x)E_1 = 0 && \textit{in } D, \end{aligned} \tag{5–6a}$$

$$\nu \times (E_1 - E_0) = 0, \qquad \nu \times (H_1 - H_0) = 0 \qquad \textit{on } \partial D. \tag{5–6b}$$

COROLLARY 5.3. *If there exists an $x_0 \in D$ such that $\mathrm{Im}\, n(x_0) > 0$ then the electric far field operator for* (5–3) *is injective with dense range.*

Proceeding as in our discussion of acoustic waves, the next topic we consider is the uniqueness of the solution to the inverse scattering problem for (5–1) and (5–3) respectively. In particular, for (5–1) we want to determine whether or not a knowledge of $E_\infty$ for fixed $k$ uniquely determines $D$ and in the case of (5–3) whether or not a knowledge of $E_\infty$for fixed $k$ uniquely determines $n(x)$. To this end, we have the following theorems due to Colton and Kress [14] and Colton and Päivärinta [18] respectively. The proofs of both results are similar to the corresponding proofs in the acoustic case already discussed. However, for the inverse scattering problem associated with (5–3), serious technical problems arise due to the fact that we must now construct a solution $E, H$ of (5–3a) such that $E$ has the form

$$E(x) = e^{i\zeta\cdot x}\big(\eta + R_\zeta(x)\big)$$

where $\zeta, \eta \in \mathbb{C}^3$, $\eta \cdot \zeta = 0$ and $\zeta \cdot \zeta = k^2$ and, in contrast to the case of acoustic waves, it is no longer true that $R_\zeta$ decays to zero as $|\zeta|$ tends to infinity. For details of how this difficulty is resolved we refer the reader to [18] and [33]. We also note that the scattering problem (5–3) corresponds to the case when the magnetic permeability $\mu$ is constant and for uniqueness results in the case when $\mu$ is no longer constant see [59], [60] and [69].

THEOREM 5.4. *Assume that $D_1$ and $D_2$ are two domains such that the electric far field patterns corresponding to the scattering problem* (5–1) *coincide for all incident directions $d \in \Omega$ and all polarizations $p \in R^3$. Then $D_1 = D_2$.*

THEOREM 5.5. *The coefficient $n(x)$ in* (5–3) *is uniquely determined by a knowledge of the electric far field pattern for all incident directions $d \in \Omega$ and all polarizations $p \in R^3$.*

Having determined the uniqueness of a solution to the inverse scattering problem, the next step is to derive reconstruction algorithms which are numerically viable. It is here that the theory for electromagnetic waves lags well behind that for acoustic waves. In particular, although methods based on the weak scattering approximation have been used extensively, particularly in problems associated with synthetic aperature radar [3], [8], the nonlinear problem has only begun to be considered. Notable accomplishments in the case of nonlinear optimization techniques to solve the inverse scattering problem associated with (5–1) have been achieved by Haas, et.al [28] and Maponi, et.al [52] whereas the case of nonlinear optimization techniques to solve the inverse scattering problem associated with (5–3) have recently been considered by Dorn, et.al [23]. Finally, based on Theorems 5.1 and 5.2 and the approximation properties of electromagnetic Herglotz pairs, Colton, Haddar and Monk [11] and Haddar and Monk [29] have used the linear sampling method to solve inverse scattering problems associated with (5–1) and (5–3) respectively. However, there is much to be done and we close this survey with a short list of open problems that await the input of new ideas for their solution:

(1) Extend the methods of Kirsch for acoustic waves discussed in Section 3 of this survey to the case of electromagnetic waves. For initial steps in this direction, see [49].
(2) Show that the set of transmission eigenvalues for the homogeneous electromagnetic interior transmission problem form a discrete set.
(3) Show that for the case of electromagnetic waves the support of an inhomogeneous anisotropic media in $R^3$ is uniquely determined by the corresponding electric far field pattern.
(4) Establish the mathematical basis of the linear sampling method for Maxwell's equations for an anisotropic medium. For partial results in a special case, see [62, Section 7.4].

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