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A Mathematical and Deterministic Analysis of the Time-Reversal Mirror

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Abstract. We give a mathematical analysis of the "time-reversal mirror", in what concerns phenomena described by the genuine acoustic equation with Dirichlet or impedance boundary conditions. An ideal situation is first considered, followed by the boundary-data, impedance and internal time-reversal methods. We explore the relationship between local decay of energy and accuracy of the method, and explain the positive effect of ergodicity.

1. Introduction: Principle of the Method

In all time reversal experiments, a finite time $0 < T < \infty$ is chosen. At time $t = 0$ waves are emitted from a localized source, recorded in time (for $0 < t < T$) by an array of receivers-transducers, time-reversed and retransmitted in the media during the time $(T < t < 2T)$; for instance the first signal to arrive is reemitted last and the last to arrive is remitted first. In this second step $(t > T)$ one can introduce amplification. The process is possibly repeated several times, leading in some cases to an automatic focusing on the most reflective target in a multiple target media. This has several applications in nondestructive testing, medical techniques such as lithotripsy and hyperthermia, underwater acoustics, etc. See [13].

The intuitive reasons why such a process may work are:

(1) The wave equation is invariant with respect to the symmetry $t \in (0,T) \mapsto 2T-t \in (T,2T)$.
(2) At high frequencies waves propagate as rays.
(3) Inhomogeneities, randomness and ergodicity contribute to much better refocusing.

Point (1) is really at the origin of the method; the same type of technique in diffusive media (phenomena governed by the diffusion equation) seems more difficult to analyse.

Point (2) gives a good intuitive description of the phenomena. Also it can be used to understand point (3) through multipathing, observing that the number of rays which connect the source with the transducers is greatly enhanced by the complexity of the media. For the case of random or ergodic media, it is necessary to study and derive some high frequency asymptotics, and this study is the backbone of the present contribution. Using classical analysis as our general tool, we start from the basic properties of the wave equation and continue with a mathematical version of the high frequency asymptotics called microlocal analysis. We depart from other points of view where randomness is of crucial importance (see for instance [4] and [1]). We do not assume any randomness but we consider the effect of ergodicity which is the deterministic counterpart of randomness.

We will consider three basic examples of time-reversal methods: the boundary-data time-reversal method, or BDTRM, the impedance time reversal mirror, or IMTRM, and the internal time-reversal method, or INTRM. In all three cases the phenomena are described by solutions to the acoustic equation in a homogeneous medium of dimension d:

$$\partial_t^2 u - \Delta u = 0. \tag{1–1}$$

In the BDTRM the solution is defined in the complement $\Omega \subset \mathbb{R}^d$ of a bounded obstacle $K \subset \mathbb{R}^d$ that forms a cavity $\mathcal{C} \subset \Omega$ with an aperture Γ. The boundary of $\mathcal{C}$ is therefore the union of Γ and $\Gamma_c = \bar{\mathcal{C}} \cap \partial\Omega$ (Figure 1). Given an arbitrary but possibly large time T, the solution of (1–1) is assumed to solve a homogeneous boundary condition on $\partial\Omega$ — say, for simplicity, the Dirichlet boundary condition

$$u_i(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T).$$

In the mean time the value of $u_i(x,t)$ is observed on Γ. For $t > T$ one considers the reversed solution u_r defined for $T < t < 2T$ by the equations

$$\partial_t^2 u_r - \Delta u_r = 0,$$

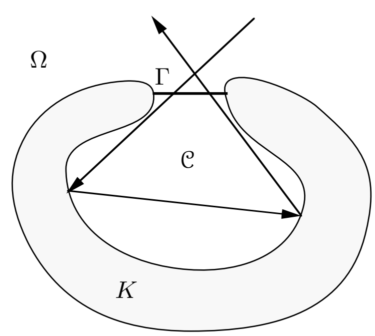

Figure 1. Cavity $\mathcal{C}$ with an aperture Γ contained in the complement Ω of a bounded obstacle K. A broken ray is shown.

with initial conditions

$$u_r(x,T) = u_i(x,T), \qquad \partial_t u_r(x,T) = \partial_t u_i(x,T)$$

and boundary conditions

$$u_r(x,t) = \begin{cases} 0 & \text{on } \Gamma_c = \bar{\mathcal{C}} \cap \partial\Omega, \\ u(x, 2T-t) & \text{on } \Gamma. \end{cases}$$

In the impedance time-reversal problem, IMTRM (Figure 2), one considers the same wave equation

$$\partial_t^2 u_i - \Delta u_i = 0$$

in a (closed) bounded cavity $\mathcal{C}$ whose boundary has a region Γ thought of as being covered with transducers (sensors). Away from Γ a homogeneous Dirichlet boundary condition holds, while on Γ an *impedance boundary condition* holds:

$$\begin{aligned} \partial_t u_i + Z(x)\partial_n u_i(x,t) &= 0 \quad \text{on } \Gamma \times (0,T), \\ u_i(x,t) &= 0 \quad \text{on } \partial\mathcal{C} \setminus \Gamma \times (0,T), \end{aligned} \tag{1–2}$$

where $Z(x)$ is a strictly positive function representing the impedance of the transducers that cover the region Γ. Here and below ∂_n denotes the outward normal to the boundary. For $0 \le t \le T$, the value of $\partial_t u_i$ on $\Gamma \times (0,T)$ is recorded and $\partial_n u_i$ is computed using (1–2). Then for $T < t < 2T$ one considers the reversed solution u_r defined by the equations

$$\partial_t^2 u_r - \Delta u_r = 0,$$

with initial conditions

$$\begin{aligned} u_r(x,T) &= u_i(x,T), \\ \partial_t u_r(x,T) &= \partial_t u_i(x,T). \end{aligned}$$

and the Neumann–Dirichlet boundary conditions

$$\begin{aligned} \partial_n u_r(x,t) &= \partial_n u_r(x, 2t-T) && \text{on } \Gamma \times (0,T), \\ u_r(x,t) &= 0 && \text{on } \partial\mathcal{C} \setminus \Gamma \times (0,T). \end{aligned}$$

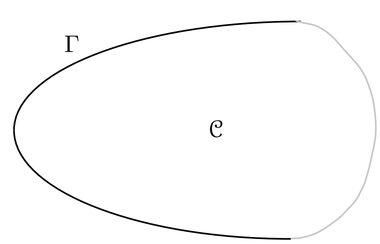

Figure 2. Cavity $\mathcal{C}$ with an impedance time-reversal mirror on a subset Γ of the boundary (in black).

In the internal time-reversal problem, INTRM, one considers for $0 < t < T$ the solution of a homogeneous boundary value problem (for instance with the Dirichlet boundary condition) in a bounded set $\mathcal{C}$,

$$\begin{aligned} \partial_t^2 u_i - \Delta u_i = 0 \quad & \text{in } \mathcal{C}, \\ u_i(x,t) = 0 \quad & \text{on } \partial\mathcal{C} \times (0,T), \end{aligned}$$

with initial conditions

$$u(\,\cdot\,,0) \equiv 0, \quad \partial_t u(x,0) = \phi(x),$$

and one introduces a bounded function $\Xi(x)$ with support in a subset σ of $\mathcal{C}$. The support of $\Xi(x)$ represents the domain of action of the transducer. This is where the signal is recorded and reemitted. For $0 < t < T$, record the value of $\partial_t u_i(x,t)$ on σ and for $T < t < 2T$ consider the solution of the problem

$$\begin{aligned} \partial_t^2 u_r - \Delta u_r = \Xi(x)\partial_t u(x, 2T{-}t) \quad & \text{in } \Omega, \\ u(x,t) = 0 \quad & \text{on } \partial\mathcal{C} \times (0,T), \end{aligned} \tag{1–3}$$

with initial conditions as above:

$$\begin{aligned} u_r(x,T) &= u_i(x,T), \\ \partial_t u_r(x,T) &= \partial_t u_i(x,T). \end{aligned} \tag{1–4}$$

It is mainly the IMTRM and the INTRM that correspond to real physical experiments. In the laboratory the impedance time-reversal mirror is usually performed by using a setup that measures and records the field on Γ, and after a time T transmits the time-reversed field in $\mathcal{C}$. Such a time-reversal mirror setup is made of an array of reversible piezoelectric transducers on Γ, which can be used now as microphones to record the field, now as loudspeakers to retransmit the time-reversed field ([11], [12] and [13]). When the transducers are used as microphones, due to their elastic properties, the boundary condition is usually an absorbing condition relating the normal derivative of the field to its time derivative through a local impedance condition of type

$$\partial_t u_i + Z(x)\partial_n u_i = 0.$$

In the first step, for $t < T$, the microphones measure the incident acoustic pressure field which is directly proportional to the time derivative of the acoustic potential $\partial_t u$.

In the second step, for $T < t < 2T$, the loudspeakers impose on Γ the normal velocity field which results from the time reversal of the component measured in the first step according to the formula:

$$\partial_n u_r(x,t) = \partial_n u_i(x, 2T{-}t) = -\frac{1}{Z(x)}\partial_t u_i(x, 2T{-}t).$$

The INTRM has been the object of several practical and numerical experiments trying to evaluate how the ergodicity of the cavity would contribute to the refocusing of the wave. An experiment due to C. Draeger is shown in Figure 5; other experimental or numerical results can be found in [9] and [10].

Even if it is not so close to applications, the BDTRM is studied because in very special cases an exact reversal is obtained. This elucidates how the difference between real and ideal time reversal method is related to the question of local decay of energy.

Therefore this chapter is organized as follows.

(i) Section 2 gives an ideal example of exact time reversal, based only on the strong form of Huygens' principle.
(ii) Section 3 analyzes the BDTRM, mainly in relation with the problem of the local decay, well known in the mathematical community.
(iii) Section 4 is devoted to the IMTRM, which appears closely related to the question of stabilization.
(iv) Section 5 concerns the refocusing by the INTRM in an ergodic cavity. It is shown how such phenomena can be explained in term of recent theorems about quantum mixing.

The present chapter follows with some improvements the ideas of an earlier article [2], which included a comparison with the classical theory of control. The experimental and numerical results were performed by Casten Draeger, who pionnered the study of the ergodic cavity.

2. Example of an Exact Time-Reversal Method

One can fully understand why the method works and what its limitations are by starting with an "academic case" as described below. Consider in $\mathbb{R}^3$ the acoustic equation

$$\partial_t^2 u_i - \Delta u_i = 0, \tag{2–1}$$

with prescribed initial conditions $u_i(x,0)$ and $\partial_t u_i(x,0)$ having their support in a ball $B_{\rho_1} = \{x : |x| < \rho_1\}$ of radius $0 < \rho_1 < \infty$.

Assume that the observation region Γ is the boundary of a bounded open set $\mathcal{C}$ containing the ball B_{ρ_1} and contained in a bigger ball $B_{\rho_2} = \{x : |x| < \rho_2\}$:

$$\text{supp } u_i(x,0) \cup \text{ supp } \partial_t u_i(x,0) \subset B_{\rho_1} \subset \mathcal{C} \subset B_{\rho_2}$$

(see Figure 3). Observe this solution (defined in $\mathbb{R}^3 \times (0,T)$) on $(\partial\mathcal{C} = \Gamma) \times (0,T)$. For $0 < T < t \le 2T$ introduce the solution of the reversed problem:

$$\partial_t^2 u_r - \Delta u_r = 0 \quad \text{in } \mathcal{C} \times \{T < t < 2T\} \tag{2–2}$$

with initial conditions

$$\begin{aligned} u_r(x,T) &= u_i(x,T), \\ \partial_t u_r(x,T) &= \partial_t u_i(x,T) \end{aligned} \tag{2–3}$$

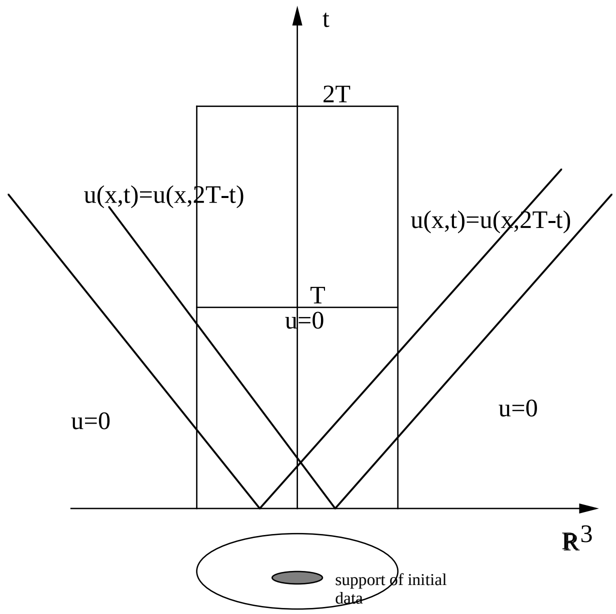

Figure 3. Finite speed of propagation and Huygens' principle in three space variables.

and boundary conditions

$$u_r(x,t) = u_i(x,2T-t) \quad \text{on } \Gamma \times (T,2T), \tag{2–4}$$

where as usual $\Gamma = \partial\mathcal{C}$. Then the following easy theorem, a direct consequence of Huygens' principle, precisely indicates the validity of the method:

THEOREM 2.1. *Consider the solution u_r defined in $\mathcal{C} \times (T,2T) \subset \mathbb{R}^3 \times (T,2T)$ by equations* (2–1), (2–2), (2–3) *and* (2–4). *Then, under the hypothesis*

$$T > \rho_1 + \rho_2$$

one has in $\mathcal{C}$

$$u_r(x,2T) = u_i(x,0), \qquad \partial_t u_r(x,2T) = -\partial_t u_i(x,0).$$

PROOF. Consider $U(x,t)$ defined in $\mathcal{C} \times (0,2T)$ by the formulas

$$U(x,t) = \begin{cases} u_i(x,t) & \text{for } 0 < t < T, \\ u_r(x,t) & \text{for } T < t < 2T. \end{cases}$$

Such a function is a solution of a mixed time-dependent boundary value problem.

As a consequence of the strong form of Huygens' principle ([17, Theorem 1.3, p. 96 and figure 3]) the initial solution $u_i(x,t)$ is zero in the cone

$$\{(x,t) : |x| \le t - \rho_1\}$$

and one has, for $t = T$:

$$U(x,T) \equiv \partial_t U(x,T) \equiv 0 \text{ in } \Omega.$$

Furthermore for the boundary condition one has, for $T < t < 2T$, by construction:

$$U(x,t) = U(x,\, 2T{-}t) \quad \text{ on } \partial\mathcal{C} \times (T, 2T). \tag{2–5}$$

The function obtained by time symmetry around T is a solution of the same problem. Both the data and the equation are therefore invariant with respect to the time symmetry around $t = T$. the uniqueness of the mixed boundary-value problem for the wave equation [6] on $\mathcal{C} \times (0, 2T)$ implies the relation

$$U(x,t) = U(x,\, 2T{-}t) \quad \text{on } \Omega \times (T, 2T), \tag{2–6}$$

and the result follows. □

3. The Boundary-Data Time-Reversal Method (BDTRM)

The preceding example, together with recent results on the decay of the solution of the exterior problem, leads to an understanding of the possibilities and limitations of the method in a cavity. Once again for simplicity the problem is considered in $\mathbb{R}^3$ or $\mathbb{R}^d$ with d odd. (The case d even introduces some algebraic decay of the local energy.) As mentioned in the introduction for the exterior problem, the solution is defined in the complement Ω of a bounded obstacle $K \subset \mathbb{R}^d$ which forms a cavity $\mathcal{C} \subset \Omega$ with an aperture Γ and the boundary of $\mathcal{C}$ is therefore the union of Γ and $\Gamma_c = \bar{\mathcal{C}} \cap \partial\Omega$ (Figure 2).

Given an arbitrary but possibly large time T, the solution $u_i(x,t)$ of (1–1) is assumed to evolve with homogeneous Dirichlet boundary condition on $\partial\Omega$. In the mean time the value of $u_i(x,t)$ is observed on Γ and for $T < t < 2T$ one considers the reversed solution u_r defined for $T < t < 2T$ by the equations

$$\partial_t^2 u_r - \Delta u_r = 0, \tag{3–1}$$

with initial conditions

$$\begin{aligned} u_r(x,T) &= u_i(x,T), \\ \partial_t u_r(x,T) &= \partial_t u_i(x,T), \end{aligned} \tag{3–2}$$

and boundary conditions

$$u_r(x,t) = \begin{cases} 0 & \text{on } \Gamma_c = \bar{\mathcal{C}} \cap \partial\Omega, \\ u_i(x,\, 2T{-}t) & \text{on } \Gamma. \end{cases} \tag{3–3}$$

Observe that $u_r(x,t)$ decomposes in $\mathcal{C} \times (T, 2T)$ into the sum of two functions

$$u_r(x,t) = u_D(x,t) + u_R(x,t),$$

which are solutions of

$$\begin{aligned} \partial_t^2 u_D - \Delta u_D = 0 &\quad \text{in } \mathcal{C}\times(0,2T),\\ u_D(x,t) = 0 &\quad \text{on } \big(\partial\mathcal{C} = (\bar{\mathcal{C}}\cap\partial\Omega)\cup\Gamma\big)\times(0,2T),\\ u_D(x,T)=0,\quad \partial_t u_D(x,T) = \partial_t u(x,T) \end{aligned}$$

and

$$\begin{aligned} \partial_t^2 u_R - \Delta u_R = 0 &\quad \text{in } \mathcal{C}\times(0,2T),\\ u_R(x,t) = 0 &\quad \text{on } (\bar{\mathcal{C}}\cap\partial\Omega)\times(T,2T),\\ u_R(x,t) = u_i(x,t) &\quad \text{on } \Gamma\times(0,T),\\ u_R(x,t) = u_i(x,2T-t) &\quad \text{on } \Gamma\times(T,2T),\\ u_R(x,T)=u_i(x,T),\quad \partial_t u_R(x,T) = 0 &\quad \text{on } \mathcal{C}. \end{aligned}$$

We have:

(i) $u_R(x,t)$ is time symmetric with respect to T,
(ii) $u_D(x,t)+u_R(x,t)$ coincide with $u_i(x,t)$ for $0<t<T$ and with $u_r(x,t)$ for $T<t<2T$.

Thus the difference between $\big(u_i(x,0),\partial_t u_i(x,0)\big)$ and $\big(u_r(x,2T),-\partial_t u_r(x,2T)\big)$ is bounded in the energy norm

$$E_{\mathcal{C}}(u) = \frac{1}{2}\int_{\mathcal{C}} \big(|\nabla u|^2 + |\partial_t u|^2\big)\,dx$$

by twice the energy norm of $u_D(x,t)$, which is time invariant and equal for $t=T$ to

$$\int_{\mathcal{C}} |\partial_t u_i(x,T)|^2 dx.$$

Using the standard notation concerning the energy norm and the Sobolev space $H_0^1(\mathcal{C})$ one obtains:

PROPOSITION 3.1. *Assume that u_r is constructed with the algorithm* (3–1), (3–2) *and* (3–3). *Then*

$$\big\|(u_r(x,2T),-\partial_t u_r(x,T)) - (u(x,0),\partial_t u(x,0))\big\|^2_{H_0^1(\mathcal{C})\times L^2(\mathcal{C})} \le \int_{\mathcal{C}} |\partial_t u(x,T)|^2 dx.$$

A consequence of this proposition is that the validity of the time-reversal method can be estimated in terms of the local energy decay of the solution of the wave equation in an exterior problem. Such problems have been studied in detail; some historical information can be found in the revised version of Lax and Phillips [17]. Since the first edition of this book it has become known that with initial data of compact support and finite energy the local energy decays to zero as $t\to\infty$. However it is also known that this decay depends on the geometry of the classical Hamiltonian flow, which describes the evolution of rays of geometrical optics in $\bar\Omega\times\mathbb{R}_t$. In the present case, where the coefficients are constant, and with the canonical identification between tangent and cotangent space, these rays are defined as continuous maps $t \mapsto \gamma(t) = (x(t), \xi(t))$ from $\mathbb{R}_t$ with values in $\bar{\Omega} \times S^{d-1}$, according to the following prescription:

In $\Omega \times S^{d-1}$ rays propagate with constant velocity

$$\dot{x}(t) = \xi, \quad \dot{\xi}(t) = 0.$$

Then the interaction with the boundary is described as follow.

First consider only the rays that, coming from the interior, intersect transversally $\partial\Omega$ at a point x_b and at a time t_b. Extend them for further time by specular reflection according to the formula

$$\dot{x}_b^+ = \xi_b^+ = \xi_b^- - 2(\xi_b^-, n_b)n_b. \tag{3–4}$$

With several reflections one obtains broken rays which are continuous maps from $\mathbb{R}_t$ with value in $\bar{\Omega} \times S^{d-1}$. The compressed broken hamiltonian flow $t \mapsto (x(t), \xi(t))$ is defined as the closure of these broken rays in $\bar{\Omega} \times S^{d-1}$, for the C^0 topology (Figure 1). Under very general hypothesis (but not always; see [14, vol. 3, p. 438]) the curves of the compressed broken hamiltonian flow (which are called bicharacteristics) realize a "foliation" of $\bar{\Omega} \times S^{d-1}$ and the singularities of the solutions propagate along these bicharacteristics. As a consequence, the following definition and theorem are now classic in microlocal analysis:

DEFINITION 3.1. A bounded obstacle $K \subset \mathbb{R}^n$ is *nontrapping* if there exists a ball B_ρ with $\Omega = \mathbb{R}^n \setminus K$, $K \subset B_\rho$, and a time $T > 0$ such that for any compressed broken ray $t \mapsto (x(t), \xi(t))$ with initial data satisfying

$$x(0) \in \Omega \cap B_\rho,$$

one has

$$x(t) \notin B_\rho \quad \text{for } t > T.$$

When this is not the case, the obstacle is *trapping*.

THEOREM 3.1. [17] *Consider the exterior problem with homogeneous boundary conditions* (*say Dirichlet or Neumann boundary conditions*) *and initial data with compact support and finite energy the local energy decays always to zero. When the obstacle is nontrapping this decay is uniform* (*and exponential when the dimension of the space is odd*). *When the obstacle is trapping this decay may be arbitrarily slow. More precisely*, *in odd dimensions*, *the solutions of*

$$\begin{aligned} \partial_t^2 u - \Delta u = 0 \quad &\textit{in } \Omega, \\ u(x,t) = 0 \quad &\textit{on } \partial\Omega \end{aligned} \tag{3–5}$$

with initial conditions $(u(x,0), \partial_t u(x,0))$ *of finite energy and compact support satisfy the following assertions*:

(i) *If the obstacle* $K = \mathbb{R}^n \setminus \Omega$ *is not trapping, there exists a constant* β *such that*

$$E_\rho(u)(t) = \frac{1}{2}\int_{\Omega\cap B_\rho} \big(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\big)\,dx \le \tfrac{1}{2}e^{-\beta t}\int_\Omega \big(|\nabla u(x,0)|^2 + |\partial_t u(x,0)|^2\big)\,dx.$$

(ii) *If the obstacle is trapping, for any pair* ε, T *there exists a solution* u_ε *of* (3–5) *such that*

$$E_\rho(u_\varepsilon)(t) \ge \tfrac{1}{2}(1-\varepsilon)\int_\Omega \big(|\nabla u_\varepsilon(x,0)|^2 + |\partial_t u_\varepsilon(x,0)|^2\big)\,dx. \tag{3–6}$$

for all $t \in (0,T)$.

The proof of (3–6) is constructed with the concentration of high frequency solutions along a trapped ray for which higher norms would blow up with ε.

On the other hand, if the solution is uniformly bounded for all time in a subspace of higher regularity, the Rellich and Banach Steinhaus theorems imply the existence of a uniform rate of decay.

For a precise statement it is convenient to write the wave equation as a group of transformations in the energy space $\mathcal{E}$ introduced by Lax and Phillips [17]. This space is the closure for the norm

$$\|(u,v)\|^2 = \frac{1}{2}\int_\Omega \big(|\nabla u|^2 + |v|^2\big)\,dx \tag{3–7}$$

of the set of smooth functions (u,v) with compact support in Ω. The generator $\mathcal{A}$ of this wave group and its domain $D(\mathcal{A})$ are defined by the formulas

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

and

$$D(\mathcal{A}) = \{U = (u,v) \in \mathcal{E} : \mathcal{A}U \in \mathcal{E}\}. \tag{3–8}$$

The quantity $\|\mathcal{A}^s U\|_{\mathcal{E}} + \|U\|_{\mathcal{E}} = \|U\|_{D(\mathcal{A}^s)}$ is invariant under the action of the wave group and the conjunction of the Rellich and Banach–Steinhaus Theorems implies, for $s > 0$, the existence of a positive continuous function $\phi(t,s)$ going to zero with $t \to \infty$ such that, for any solution with initial data having support in B_ρ,

$$E_\rho(u)(t) = \frac{1}{2}\int_{\Omega\cap B_\rho} \big(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\big)\,dx \le \phi(t,s)\|(u(x,0),\partial_t u(x,0))\|^2_{D(\mathcal{A}^s)}.$$

The optimal result (involving no hypotheses on the geometry) on the decay of $\phi(t,s)$ has been obtained by Burq, using Carleman estimates:

Proposition 3.2. [5] *For any solution of the wave equation with Dirichlet boundary data* (3–5) *and initial data supported in* $B_\rho \cap \Omega$ *one has*:

$$E_\rho(u)(t) = \frac{1}{2}\int_{\Omega\cap B_\rho} \left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\right) dx \\ \leq \frac{C}{\log(2+t)^{2s}} \|(u(x,0), \partial_t u(x,0))\|^2_{D(\mathcal{A}^s)},$$

where the constant C *depends only on the domain* Ω *and the number* ρ.

Proposition 3.1, Theorem 3.1 and Proposition 3.2 together have the following consequence for the analysis of the boundary time-reversal method in a cavity:

Theorem 3.2. *Assume that* u_r *is constructed with the algorithm* (3–1), (3–2), (3–3), *with* Γ, *where the time symmetry is done*, *being the aperture of the cavity*. *Then*:

(i) *For a nontrapping obstacle in odd dimensions*, *there is a constant* β *for which*

$$\left\|(u_r(x,2T),\, -\partial_t u_r(x,T)) - (u(x,0),\, \partial_t u(x,0))\right\|_{H^1_0(\mathcal{C})\times L^2(\mathcal{C})} \\ \leq C e^{-\beta T}\int_\Omega \left(|\nabla u(x,0)|^2 + |\partial_t u(x,0)|^2\right) dx$$

(ii) *For either trapping or nontrapping obstacles*, *the following estimate holds if the initial data are smooth*:

$$\left\|(u_r(x,2T), -\partial_t u_r(x,T)) - (u(x,0), \partial_t u(x,0))\right\|_{H^1_0(\mathcal{C})\times L^2(\mathcal{C})} \\ \leq C\frac{C}{\log(2+T)^{2s}} \left\|(u(x,0), \partial_t u(x,0))\right\|^2_{D(\mathcal{A}^s)},$$

and this estimate is optimal. *It is saturated when a stable periodic orbit is contained in* $\mathcal{C}$ *and does not meet* Γ [5].

Remark 3.1. This theorem gives qualitative results on intuitive facts. It shows that the BDTRM always works at least for smooth solutions and large time T. The larger T and the bigger Γ, the better the reconstruction. The reconstruction is obtained with an accuracy $e^{-\beta T}$ when the dimension is odd and the aperture is large enough to capture all the rays of geometric optic. In the worst case when an essential part of the initial signal propagates near a closed stable geodesic which does not meet the aperture the accuracy of the reconstruction is in $O((\log T)^k)$ with k depending on the smoothness of the initial data. In some situations where the set of geodesics which do not meet the aperture in a finite time is "unstable and small", the reconstruction of a smooth signal is obtained with an error of the order of T^{-k}; see [15].

4. The Impedance Time-Reversal Mirror (IMTRM)

In the IMTRM the initial wave u_i evolves for $0 \le t \le T$ in a bounded cavity $\mathcal{C}$:

$$\partial_t^2 u_i - \Delta u_i = 0 \quad \text{in } \mathcal{C}, \tag{4–1}$$

with boundary conditions

$$\begin{aligned} u_i(x,t) &= 0 \quad \text{on } (\partial\mathcal{C} \setminus \Gamma), \\ \partial_t u_i(x,t) + Z(x)\partial_n u_i(x,t) &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{4–2}$$

Then, for $T \le t \le 2T$, one considers the solution u_r of the reversed problem:

$$\partial_t^2 u_r - \Delta u_r = 0 \quad \text{in } \mathcal{C}, \tag{4–3}$$

with boundary conditions

$$\begin{aligned} &\partial_n u_R(x,t) = \partial_n u_R(x,\, 2t-T) && \text{on } \Gamma \times (0,T), \\ &u_R(x,t) = 0 && \text{on } \partial\mathcal{C} \setminus \Gamma \times (0,T). \end{aligned} \tag{4–4}$$

and initial data

$$u_r(x,T) = u_i(x,T), \quad \partial_t u_r(x,T) = \partial_t u_i(x,T). \tag{4–5}$$

Proposition 4.1. *Assume that u_r is constructed with the algorithm* (4–3), (4–4) *and* (4–5). *Then*

$$\big\|(u_r(x,2T), -\partial_t u_r(x,T)) - (u(x,0), \partial_t u(x,0))\big\|^2_{H^1(\mathcal{C})\times L^2(\mathcal{C})} \le \int_{\mathcal{C}} |\partial_t u(x,T)|^2 dx.$$

Proof. One introduces the T-symmetric solution u_R of of the Neumann–Dirichlet boundary-value problem:

$$\partial_t^2 u_R - \Delta u_R = 0 \quad \text{in } \mathcal{C} \times (0,2T)$$

with boundary conditions

$$\begin{aligned} &u_R(x,t) = 0 && \text{on } (\partial\mathcal{C} \setminus \Gamma) \times (0,2T), \\ &\partial_n u_R(x,t) = \partial_n u_i(x,t) && \text{on } \Gamma \times (0,T), \\ &\partial_n u_R(x,t) = \partial_n u_i(x,\, 2T-t) && \text{on } \Gamma \times (T,2T) \end{aligned}$$

and initial data (at time $t = T$):

$$u_R(x,T) = u_i(x,T), \qquad \partial_t u_R(x,T) = 0. \tag{4–6}$$

With this symmetric function the proof is completed along the lines of the Proposition 3.1. □

Proposition 4.1 implies that the accuracy of the method relies on the decay of the energy norm

$$\frac{1}{2}\int_{\mathcal{C}} \big(|\nabla u_i(x,t)|^2 + |\partial_t u_i(x,t)|^2\big)\,dx.$$

This decay has been extensively studied in connection with the problem of the stabilization by boundary feedback ([3], [5], [18]) and it appears that the properties are exactly of the same nature as for the local decay of the exterior problem. It is convenient to introduce a function space $\mathcal{E}$ and an unbounded operator $\mathcal{A}$ adapted as above to the introduction of the variable $v = \partial_t u$ and $U = (u, v)$, according to the formulas

$$\begin{aligned} \mathcal{E} &= \big\{U = (u,v) \in H^1(\mathcal{C}) \times L^2(\mathcal{C}) : u = 0 \text{ on } \partial\mathcal{C} \setminus \Gamma\big\},\\ \|U\|^2_{\mathcal{E}} = \|(u,v)\|^2_{\mathcal{E}} &= \frac{1}{2}\int_{\mathcal{C}} \big\{|\nabla u(x,t)|^2 + |v(x,t)|^2\big\}\,dx,\\ D(\mathcal{A}) &= \{U = (u,v) \in \mathcal{E} : \Delta u \in L^2(\mathcal{C}),\ v \in H^1(\mathcal{C}),\ v + \partial_n u = 0 \text{ on } \Gamma\big\}. \end{aligned}$$

Multiplication of the equation

$$\partial_t^2 u - \Delta u = 0$$

by $\partial_t u$ and integration over $\mathcal{C}$ gives, with the boundary condition (4–2), the energy identity

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{C}} \big(|\nabla u_i(x,t)|^2 + |\partial_t u_i(x,t)|^2\big)\,dx\right) + \int_\Gamma Z(x)|\partial_n u|^2\,d\sigma_x = 0,$$

which leads through classical functional analysis to the following statement.

The operator $\mathcal{A}$ is, in $\mathcal{E}$, the generator of a strongly continuous contraction semigroup, and

$$\lim_{t\to\infty} e^{t\mathcal{A}} U_0 = 0$$

for any initial data $U_0 = (u(x,0), \partial_t u(x,0) = v(x,0))$.

Once again the rate of decay depends on the geometry. Following [3] one says that Γ *geometrically stabilizes* the cavity $\mathcal{C}$, if there exists a time T such that any generalized ray $t \in [0,T] \mapsto x(t) \in \bar{\mathcal{C}}$ intersects Γ at least once in a nondiffractive point. The following results are now well known; see [3], [18], [5].

Theorem 4.1. (i) *If Γ geometrically stabilizes $\mathcal{C}$, there exists a constant $\beta > 0$ such that*

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{C}} \big(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\big)\,dx = \|U(t)\|^2_{\mathcal{E}} \le e^{-\beta t}\|U(0)\|^2_{\mathcal{E}}\\ = \frac{1}{2}\int_{\mathcal{C}} \big(|\nabla u(x,0)|^2 + |\partial_t u(x,0)|^2\big)\,dx. \end{aligned}$$

(ii) *If Γ does not geometrically stabilize $\mathcal{C}$, the decay may be arbitrary slow in the sense of Theorem* 3.1(*ii*).

(iii) *However, for any sufficiently smooth initial data the following estimate is always valid* (*and optimal in the absence of hypotheses on the geometry*)

$$\frac{1}{2}\int_{\Omega\cap B_\rho}\big(|\nabla u(x,t)|^2+|\partial_t u(x,t)|^2\big)\,dx \le \frac{C}{\log(2+t)^{2s}}\big\|(u(x,0),\partial_t u(x,0))\big\|^2_{D(\mathcal{A}^s)}.$$

With Proposition 4.1 and Theorem 4.1 one concludes as in the previous section how the accuracy of the method depends on the size of Γ and on the time of observation.

5. Internal Time-Reversal Method in an Ergodic Cavity (INTRM)

Intuition suggests that the domain where the time-reversal process occurs can be much smaller if the compressed hamiltonian flow is ergodic and if the time of "action" is large enough. This has been corroborated by numerical simulation and ultrasonic experiments made on a silicium wafer by C. Draeger and M. Fink ([9], [10]). What is observed with a time-reversal experiment conducted on one single point is a very good refocusing of a localized initial signal. The mathematical explanation, as described below, relies on (1) an asymptotic formula which in [9] is called the "cavity formula", and (2) the notion of quantum ergodicity, which is closely related to classical ergodicity ([21], [8], [23], [24]).

As in the previous section, $\mathcal{C}$ denotes a bounded open set and Δ is the Laplace operator with Dirichlet boundary condition on $\partial\mathcal{C}$. It is convenient to introduce the operators

$$\exp\big(it(-\Delta)^{1/2}\big),\quad \sin\big(t(-\Delta)^{1/2}\big),\quad \cos\big(t(-\Delta)^{1/2}\big).$$

The solution of the initial value problem

$$\begin{aligned}\partial_t^2 u-\Delta u=0 &\quad \text{in } \mathcal{C},\\ u(x,t)\equiv 0 &\quad \text{on } \partial\mathcal{C}\end{aligned} \tag{5–1}$$

with the initial condition

$$u(x,0)=0 \qquad \partial_t u(x,0)=\psi(x) \tag{5–2}$$

is given by

$$u(x,t)=(-\Delta)^{-1/2}\sin(t(-\Delta)^{1/2})\psi. \tag{5–3}$$

The solution of the problem

$$\begin{aligned}\partial_t^2 u-\Delta u=f(x,t) &\quad \text{in } \mathcal{C},\\ u(x,t)\equiv 0 &\quad \text{on } \partial\mathcal{C}\end{aligned}$$

with initial conditions

$$u(x,0)=0,\qquad \partial_t u(x,0)=0,$$

is given by

$$u(x,t) = \int_0^t (-\Delta)^{-1/2} \sin\big((t-s)(-\Delta)^{1/2}\big) f(s)\, ds. \tag{5–4}$$

Observe that (5–3) and (5–4) are well defined (this can be done by duality) not only for L^2 functions but also for distributions in $\mathcal{D}'(\mathcal{C})$ and that, with the introduction of the eigenvalues and eigenvectors of $-\Delta$, namely

$$-\Delta\phi_k = \omega_k^2 \phi_k, \quad \phi_k(x) = 0 \quad \text{on } \partial\mathcal{C}, \quad 1 \le k \le \infty$$

the kernel of the operator

$$(-\Delta)^{-1/2} \sin(t(-\Delta)^{1/2})$$

is the distribution

$$k(x,y,t) = \sum_{1 \le k \le \infty} \frac{\sin t\omega_k}{\omega_k} \phi_k(x) \otimes \phi_k(y),$$

which turns out to be the (fundamental) solution of the problem

$$\partial_t^2 k(x,y,t) - \Delta_x k(x,y,t) = \delta_t \otimes \delta_y.$$

For the INTRM one observes the solution u_i of (5–1) and (5–2) on a subset $\sigma \subset \mathcal{C}$ (which may be arbitrary small), introduces an L^∞ function $\Xi(x)$ with support contained in σ and eventually introduces for $T < t < 2T$ the solution of the problem

$$\begin{aligned} \partial_t^2 u_r - \Delta u_i &= K\Xi(x)\partial_t u_i(x,\, 2T-t) && \text{in } \mathcal{C}, \\ u_i(x,t) &= 0 && \text{on } \partial\mathcal{C}, \end{aligned} \tag{5–5}$$

with initial conditions

$$u_r(x,T) = u_r(x,T), \qquad \partial_t u_r(x,T) = \partial_t u_r k(x,T).$$

In (5–5) K represents an amplification factor which may be large. Therefore

$$\begin{aligned} \partial_t u_r(x,2T) &= \cos\big(2T(-\Delta)^{1/2}\big)\psi \\ &\quad + K \int_T^{2T} \cos\big((2T-t)(-\Delta)^{1/2}\big) \Xi \cos\big((2T-t)(-\Delta)^{1/2}\big)\psi\, dt. \end{aligned}$$

To use the ergodicity property T will be taken large enough. This also reinforces the influence of the reemitted signal which is also amplified by the factor Ampl. Accordingly one writes

$$\begin{aligned} \partial_t u_r(x,2T) = T\bigg(&\frac{1}{T} \cos\big(2T(-\Delta)^{1/2}\big)\psi \\ &+ \frac{\text{Ampl}}{T} \int_T^{2T} \cos\big((2T-t)(-\Delta)^{1/2}\big) \Xi \cos\big((2T-t)(-\Delta)^{1/2}\big)\psi dt\bigg). \end{aligned}$$

In any convenient sense, and in particular for the energy norm (with initial data of finite energy), one has

$$\lim_{T\to\infty}\frac{1}{T}\cos(2T(-\Delta)^{1/2})\psi=0.$$

Therefore

$$u_r(x,2T)\simeq\frac{\operatorname{Ampl}T}{T}\int_T^{2T}\cos((2T-t)(-\Delta)^{1/2}))\Xi\cos\bigl((2T-t)(-\Delta)^{1/2}\bigr)\psi\,dt$$

whenever this limit exists.

This result follows from the cavity equation and the quantum chaos principle. One has

$$\frac{1}{T}\int_T^{2T}\cos\bigl((2T-t)(-\Delta)^{1/2}\bigr)\Xi\cos((2T-t)(-\Delta)^{1/2})\psi\,dt$$
$$=\frac{1}{4T}\int_0^T(e^{it(-\Delta)^{1/2}}+e^{-it(-\Delta)^{1/2}})\Xi(e^{it(-\Delta)^{1/2}}+e^{it(-\Delta)^{1/2}})dt\psi.$$

which is written as the sum of two terms:

$$M(T)\psi=\frac{1}{4T}\int_{-T}^{T}e^{it(-\Delta)^{1/2}}\Xi e^{it(-\Delta)^{1/2}}\psi\,dt,$$
$$N(T)\psi=\frac{1}{4T}\int_{-T}^{T}e^{it(-\Delta)^{1/2}}\Xi e^{-it(-\Delta)^{1/2}}\psi\,dt.$$

For $M(T)$ we have:

Proposition 5.1 (Cavity formula). *The family of operators $T\mapsto M(T)$ is uniformly equibounded in $L^2(\Omega)$ and for $T\to\infty$ it converges weakly to* 0.

Proof. Observe that one has

$$\|M(T)\|\le\tfrac{1}{2}|\Xi\|_{L^\infty(\mathcal{C})}. \tag{5–6}$$

Then for any pair of eigenvectors $(\phi_k(x),\,\phi_l(x))$,

$$\lim_{T\to0}(M(T)\phi_k,\phi_l)=\lim_{T\to0}\frac{1}{4T}\int_{-T}^{T}(\Xi e^{it(-\Delta)^{1/2}}\phi_k,e^{-it(-\Delta)^{1/2}}\phi_l)\,dt$$
$$=\lim_{T\to0}\frac{\sin\bigl(T(\omega_k+\omega_l)\bigr)}{2T(\omega_k+\omega_l)}(\Xi\phi_k,\phi_l)=0,$$

and the result follows by density. □

Remark 5.1. By Rellich's theorem, it follows from the above proposition that

$$\lim\|M(T)\psi\|_{L^2(\mathcal{C})}=0.$$

for any $\psi\in H^s(\mathcal{C})$ with $s>0$.

From the cavity formula one deduces the relation

$$u_R(x,2T) \simeq TN(T) \simeq \tfrac{1}{2}\mathrm{Ampl}\left(\frac{K}{2T}\int_{-T}^{T} e^{it(-\Delta)^{1/2}}\Xi(e^{-it(-\Delta)^{1/2}})dt\right)\psi.$$

Start from the Hamiltonian system

$$\dot{x}(t) = \xi, \qquad \dot{\xi}(t) = 0.$$

For any function f write, whenever that makes sense (i.e., when the trajectory $x(s)$ for $s \in [0,t]$ remains in $\mathcal{C}$)

$$V(t)f = f\big(x(t), \xi(t)\big).$$

With the introduction of the broken hamiltonian flow, extend this operator on the functions defined on $S^*(\bar{\mathcal{C}}) = \big\{(x,\xi) : x \in \mathcal{C},\, |\xi| = 1\big\}$ and denote by

$$|S^*(\bar{\mathcal{C}})| = \iint_{\mathcal{C}\times\{|\xi|=1\}} dx\, d\xi$$

the volume of this cosphere bundle. Further, denote by σ_P the principal symbol of any zero order pseudodifferential operator P.

Definition 5.1. (i) The flow is *classically ergodic* if

$$\lim_{t\to\infty} V(t)f = \bar{f} = \frac{1}{|S^*(\bar{\mathcal{C}})|}\int_{\mathcal{C}\times\{|\xi|=1\}} f(x,\xi)\, dx\, d\xi.$$

in the weak L^* topology, for any continuous function $f \in C^0(S^*(\bar{\mathcal{C}}))$.

(ii) Let $\Pi_l = \sum_{1\le k\le l} \phi_k \otimes \phi_k^*$ be the projection onto the space spanned by the l first eigenvectors of $-\Delta$. An operator $K \in \mathcal{L}(L^2(\mathcal{C}))$ is *spectrally regularizing* if it satisfies the bound

$$\|\Pi_l K \Pi_l\|_{HS}^2 = o(l).$$

(iii) The flow is *quantum ergodic* if, for any zero-order pseudodifferential operator P and in the weak operator limit, one has

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{it(-\Delta)^{1/2}} P e^{-it(-\Delta)^{1/2}} dt = \langle P\rangle I + K$$

with K spectrally regularizing and

$$\langle P\rangle = \frac{1}{|S^*(\bar{\mathcal{C}})|}\int_{\mathcal{C}\times\{|\xi|=1\}} \sigma_P(x,\xi)\, dx\, d\xi = \lim_{l\to\infty} \sum_{1\le k\le l} \frac{1}{l}(P\phi_k, \phi_k).$$

It has been proved that classical ergodicity implies quantum ergodicity. See [8], [21], [23], [24].

Therefore it follows from Proposition 5.1 that:

THEOREM 5.1. *The INTRM solution constructed* (*see* (5–5) *and* (1–3)) *satisfies*, *as* $T \to \infty$:

$$u_R(x, 2T) \simeq \tfrac{1}{2} T \operatorname{Ampl}\big(\langle \Xi \rangle \psi + K\psi\big)$$

with K *spectrally regularizing and*

$$\langle \Xi \rangle = \lim_{l \to \infty} \sum_{1 \le k \le l} \frac{1}{l} (\Xi \phi_k, \phi_k) = \sum_{1 \le k \le l} \frac{1}{l} \int_{\mathcal{C}} \big(\Xi(x)\phi_k(x), \phi_k(x)\big)\, dx.$$

REMARK 5.2. The notion of spectrally regularizing is not very explicit in its present form. However, for any pseudodifferential operator P of zero order,

$$\lim_{l \to \infty} \frac{1}{l} \|\Pi_l P \Pi_l\|_{HS}^2 = \langle P \rangle = \frac{1}{|S^*(\bar{\mathcal{C}})|} \int_{\mathcal{C} \times \{|\xi|=1\}} |\sigma_P(x, \xi)|^2 \, dx \, d\xi;$$

see [25, Prop. 1.1(ii)]. Therefore any spectrally regularizing pseudodifferential operator P has its principal symbol equal to zero and has a regularizing effect. Similarly one shows [23, p. 921] that in general (at least when the spectra of Δ has bounded multiplicity) K is compact. This is why the preceding theorem carries pertinent information when the initial data ψ is a distribution with a single singularity located at one point, say A, then the reversed solution is a sum of a more regular term $\frac{1}{2} T K \psi$ and of a leading term proportional to

$$\tfrac{1}{2} T \operatorname{Ampl} \langle \Xi \rangle \psi,$$

which also has a singularity at the point A. In this sense for large time the refocusing is perfect and this is in agreement with the experiment of C. Draeger (see Figure 5) and the numerical simulations of [9] and [10].

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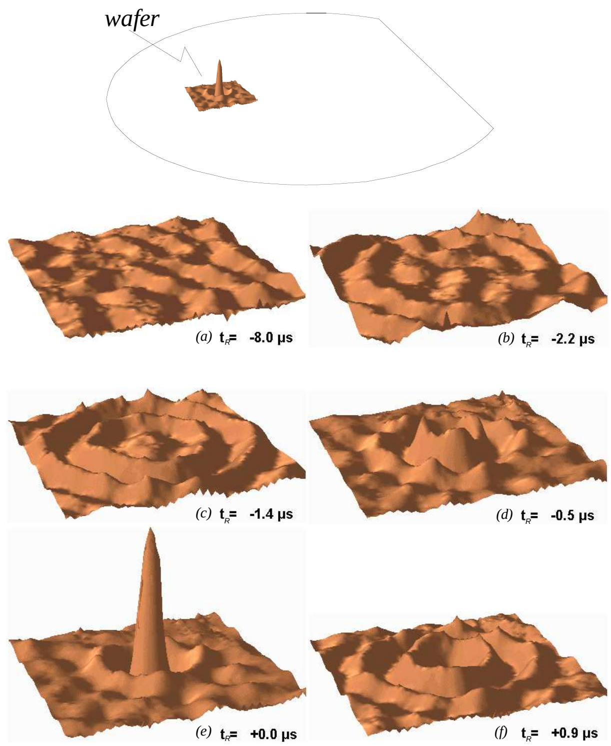

Figure 4. Linear representation of the acoustic field around point A during the refocusing after a time reversal of $T = 2\,\text{ms}$. Measurements have been made ([9], [10]) on a square of side 15 mm, with a spatial step of 0.25 mm.

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