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Engineering Applications of the Motion-Group Fourier Transform

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Abstract. We review a number of engineering problems that can be posed or solved using Fourier transforms for the groups of rigid-body motions of the plane or three-dimensional space. Mathematically and computationally these problems can be divided into two classes: (1) physical problems that are described as degenerate diffusions on motion groups; (2) enumeration problems in which fast Fourier transforms are used to efficiently compute motion-group convolutions. We examine engineering problems including the analysis of noise in optical communication systems, the allowable positions and orientations reachable with a robot arm, and the statistical mechanics of polymer chains. In all of these cases, concepts from noncommutative harmonic analysis are put to use in addressing real-world problems, thus rendering them tractable.

1. Introduction

Noncommutative harmonic analysis is a beautiful and powerful area of pure mathematics that has connections to analysis, algebra, geometry, and the theory of algorithms. Unfortunately, it is also an area that is almost unknown to engineers. In our research group, we have addressed a number of seemingly intractable "real-world" engineering problems that are easily modeled and/or solved using techniques of noncommutative harmonic analysis. In particular, we have addressed physical/mechanical problems that are described well as functions or processes on the rotation and rigid-body-motion groups. The interactions and evolution of these functions are described using group-theoretic convolutions and diffusion equations, respectively. In this paper we provide a survey of some of these applications and show how computational harmonic analysis on motion groups is used.

The group of rigid-body motions, denoted as $\mathrm{SE}(N)$ (shorthand for "special Euclidean" group in N-dimensional space), is a unimodular semidirect product group, and general methods for constructing unitary representations of such Lie groups have been known for some time (see [1; 25; 35], for example). In the past 40 years, the representation theory and harmonic analysis for the Euclidean groups have been developed in the pure mathematics and mathematical physics literature. The study of matrix elements of irreducible unitary representation of SE(3) was initiated by N. Vilenkin [39; 40] in 1957 (some particular matrix elements are also given in [41]). The most complete study of $\widetilde{\mathrm{SE}}(3)$ (the universal covering group of SE(3)) with application to the harmonic analysis was given by W. Miller in [28]. The representations of SE(3) were also studied in [16; 36; 37]. In recent works, fast Fourier transforms for SE(2) and SE(3) have been proposed [24], and an operational calculus has been constructed [5].

However, despite the considerable progress in mathematical developments of the representation theory of SE(3), these achievements have not yet been widely incorporated in engineering and applied fields. In work summarized here we try to fill this gap. A more detailed treatment of numerous applications can be found in [6].

In Section 2 we review the representation theory of SE(2), give the matrix elements of the irreducible unitary representations and review the definition of the Fourier transform for SE(2). We also review operational properties of the Fourier transform. We do not go into the intricate details of the Fourier transform for SE(3), as those are provided in the references described above and they add little to the understanding of how to apply noncommutative harmonic analysis to real-world problems. Sections 3, 4 and 5 are devoted to application areas: coherent optical communications, robotics, and polymer statistical mechanics, respectively.

2. Fourier Analysis of Motion

In this section we review the basic definitions and properties of the Euclidean motion groups. Our emphasis is on the motion group of the plane, but most of the concepts extend in a natural way to three-dimensional space. See [6] for a complete treatment.

2.1. Euclidean motion group. The Euclidean motion group, SE(N), is the semidirect product of $\mathbb{R}^N$ with the special orthogonal group, SO(N). We denote elements of SE(N) as $g = (\boldsymbol{a}, A) \in \mathrm{SE}(N)$ where $A \in \mathrm{SO}(N)$ and $\boldsymbol{a} \in \mathbb{R}^N$. The identity element is $e = (0, I)$ where I is the $N \times N$ identity matrix. For any $g = (\boldsymbol{a}, A)$ and $h = (\boldsymbol{r}, R) \in \mathrm{SE}(N)$, the group law is written as $g \circ h = (\boldsymbol{a} + A\boldsymbol{r}, AR)$, and $g^{-1} = (-A^T\boldsymbol{a}, A^T)$. Any $g = (\boldsymbol{a}, A) \in \mathrm{SE}(N)$ acts transitively on a position $\boldsymbol{x} \in \mathbb{R}^N$ as

$$g \cdot \boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{a}.$$

That is, position vector $\boldsymbol{x}$ is rigidly moved by rotation followed by a translation.

Often in the engineering literature, no distinction is made between a *motion*, g, and the result of that motion acting on the identity element (called a *pose* or *reference frame*). Hence, we interchangeably use the words "motion" and "frame" when referring to elements of SE(N).

It is convenient to think of an element of SE(N) as an $(N+1) \times (N+1)$ matrix of the form:

$$g = \begin{pmatrix} A & \boldsymbol{a} \\ 0^T & 1 \end{pmatrix}.$$

In the engineering literature, matrices with this kind of structure are called *homogeneous transforms*.

For example, each element of SE(2) can be parameterized using polar coordinates as:

$$g(r, \theta, \phi) = \begin{pmatrix} \cos\phi & -\sin\phi & r\cos\theta \\ \sin\phi & \cos\phi & r\sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

where $r \geq 0$ is the magnitude of translation. SE(2) is a 3-dimensional manifold much like $\mathbb{R}^3$. We can integrate over SE(2) using the volume element $d(g(r,\theta,\phi)) = (4\pi^2)^{-1} r\, dr\, d\theta\, d\phi$. This volume element is bi-invariant in the sense that it does not change under left and right shifts by any fixed element $h \in \mathrm{SE}(2)$:

$$d(h \circ g) = d(g \circ h) = d(g).$$

Bi-invariant volume elements exist for SE(N) for $N = 2, 3, 4, \ldots$. A group with bi-invariant volume element is called a *unimodular* group.

The Lie group SE(2) has an associated Lie algebra $se(2)$. Physically, elements of SE(2) describe finite motions in the plane, whereas elements of $se(2)$ represent infinitesimal motions. Since SE(2) is a three-dimensional Lie group, there are three independent directions along which any infinitesimal motion can be decomposed. The vector space of all such motions relative to the identity element $e \in \mathrm{SE}(2)$ together with the matrix commutator operation defines $se(2)$. As with any vector space, we can choose an appropriate basis. One such basis for the Lie algebra $se(2)$ consists of the following three matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following one-parameter motions are obtained by exponentiating the above basis elements of $se(2)$:

$$g_1(t) = \exp(tX_1) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$g_2(t) = \exp(tX_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix};$$

$$g_3(t) = \exp(tX_3) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the purposes of the current discussion, we can take as a definition of $se(2)$ the vector space spanned by any linear combination of X_1, X_2, and X_3. The exponential mapping

$$\exp : se(2) \to \mathrm{SE}(2)$$

is well-defined for every element of $se(2)$ and is invertible except at a set of measure zero in SE(2).

Any rigid-body motion in the plane can be expressed as an appropriate combination of these three basic motions. For example, $g = g_1(x)g_2(y)g_3(\phi)$.

2.2. Differential operators on SE(2)**.** The way to take partial derivatives of a function of motion is to evaluate

$$\tilde{X}_i^R f \triangleq \frac{d}{dt} f(g \circ \exp(tX_i))|_{t=0}, \qquad \tilde{X}_i^L f \triangleq \frac{d}{dt} f(\exp(tX_i) \circ g)|_{t=0}.$$

(In our notation, R means that the exponential appears on the right, and L means that it appears on the left. This means that $\tilde{X}_i^R$ is invariant under left shifts, while $\tilde{X}_i^L$ is invariant under right shifts. Our notation is different than others in the mathematics literature where the superscript denotes the invariance of the vector field formed by the concatenation of these derivatives.) Explicitly, we find the differential operators $\tilde{X}_i^R$ in polar coordinates to be [6]

$$\begin{aligned} \tilde{X}_1^R &= \cos(\phi - \theta)\frac{\partial}{\partial r} + \frac{\sin(\phi-\theta)}{r}\frac{\partial}{\partial \theta}, \\ \tilde{X}_2^R &= -\sin(\phi - \theta)\frac{\partial}{\partial r} + \frac{\cos(\phi-\theta)}{r}\frac{\partial}{\partial \theta}, \\ \tilde{X}_3^R &= \frac{\partial}{\partial \phi}, \end{aligned}$$

and in Cartesian coordinates to be

$$\tilde{X}_1^R = \cos\phi\frac{\partial}{\partial x} - \sin\phi\frac{\partial}{\partial y}, \quad \tilde{X}_2^R = \sin\phi\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}, \quad \tilde{X}_3^R = \frac{\partial}{\partial \phi}.$$

The differential operators $\tilde{X}_i^L$ in polar coordinates are

$$\tilde{X}_1^L = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \quad \tilde{X}_2^L = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}, \quad \tilde{X}_3^L = \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \theta}.$$

2.3. Fourier analysis on SE(2)**.** The Fourier transform, $\mathcal{F}$, of a function of motion, $f(g)$ where $g \in \mathrm{SE}(N)$, is an infinite-dimensional matrix defined as [6]:

$$\mathcal{F}(f) = \hat{f}(p) = \int_G f(g)U(g^{-1}, p)\, d(g)$$

where $U(g,p)$ is an infinite dimensional matrix function with the property that $U(g_1 \circ g_2, p) = U(g_1,p)U(g_2,p)$. This kind of matrix is called a *matrix representation* of SE(N). It has the property that it converts convolutions on SE(N) into matrix products:

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\mathcal{F}(f_1).$$

In the case when $N = 2$, the original function is reconstructed as

$$\mathcal{F}^{-1}(\hat{f}) = f(g) = \int_0^\infty \operatorname{trace}(\hat{f}(p)U(g,p))p\,dp,$$

and the matrix elements of $U(g,p)$ are expressed explicitly as [6]:

$$u_{mn}(g(r,\theta,\phi),p) = j^{n-m} e^{-j[n\phi+(m-n)\theta]} J_{n-m}(p\,r)$$

where $J_\nu(x)$ is the ν^{th} order Bessel function and $j = \sqrt{-1}$. This inverse transform can be written in terms of elements as

$$f(g) = \sum_{m,n\in\mathbb{Z}} \int_0^\infty \hat{f}_{mn} u_{nm}(g,p)p\,dp. \tag{2–1}$$

In analogy with the classical Fourier transform, which converts derivatives of functions of position into algebraic operations in Fourier space, there are operational properties for the motion-group Fourier transform.

By the definition of the SE(2)-Fourier transform $\mathcal{F}$ and operators $\tilde{X}_i^R$ and $\tilde{X}_i^L$, we can write the Fourier transform of the derivatives of a function of motion as

$$\mathcal{F}[\tilde{X}_i^R f] = \tilde{u}(X_i,p)\hat{f}(p), \qquad \mathcal{F}[\tilde{X}_i^L f] = -\hat{f}(p)\tilde{u}(X_i,p),$$

where

$$\tilde{u}(X_i,p) \triangleq \frac{d}{dt} U(\exp(tX_i),p)\bigg|_{t=0}.$$

Explicitly,

$$u_{mn}(\exp(tX_1),p) = j^{n-m} J_{m-n}(pt).$$

We know that

$$\frac{d}{dx} J_m(x) = \tfrac{1}{2}[J_{m-1}(x) - J_{m+1}(x)]$$

and

$$J_{m-n}(0) = \begin{cases} 1 & \text{for } m-n=0, \\ 0 & \text{for } m-n\neq 0. \end{cases}$$

Hence,

$$\tilde{u}_{mn}(X_1,p) = \frac{d}{dt} u_{mn}(\exp(tX_1),p)\bigg|_{t=0} = -\frac{jp}{2}(\delta_{m,n+1} + \delta_{m,n-1}).$$

Likewise,

$$u_{mn}(\exp(tX_2),p) = j^{n-m} e^{-j(n-m)\pi/2} J_{m-n}(pt) = J_{m-n}(pt),$$

and so

$$\tilde{u}_{mn}(X_2,p) = \frac{d}{dt}u_{mn}(\exp(tX_2),p)\bigg|_{t=0} = \frac{p}{2}(J_{m-n-1}(0) - J_{m-n+1}(0)) = \frac{p}{2}(\delta_{m,n+1} - \delta_{m,n-1}).$$

Similarly, we find

$$u_{mn}(\exp(tX_3),p) = e^{-jmt}\delta_{m,n}$$

and

$$\tilde{u}_{mn}(X_3,p) = \frac{d}{dt}u_{mn}(\exp(tX_3),p)\bigg|_{t=0} = -jm\delta_{m,n}.$$

Fast Fourier transforms for SE(2) and SE(3) have been outlined in [6; 24]. Operational properties for SE(3) which are analogous to those presented here for SE(2) can be found in [5; 6]. Subsequent sections in this paper describe various applications of motion-group Fourier analysis to problems in engineering.

3. Phase Noise in Coherent Optical Communications

In optical communications, laser light is used to transmit information along fiber optic cables. There are several methods that are used to transmit and detect information within the light. Coherent detection (in contrast to direct detection) is a method that has the ability to detect the phase, frequency, amplitude and polarization of the incident light signal . Therefore, information can be transmitted via phase, frequency, amplitude, or polarization modulation. However, the phase of the light emitted from a semiconductor laser exhibits random fluctuations due to spontaneous emissions in the laser cavity [19]. This phenomenon is commonly referred to as *phase noise*. Phase noise puts strong limitations on the performance of coherent communication systems. Evaluating the influence of phase noise is essential in system design and optimization and has been studied extensively in the literature [10; 12]. Analytical models that describe the relationship between phase noise and the filtered signal are found in [2; 11]. In particular, the Fokker–Planck approach represents the most rigorous description of phase noise effects [13; 14]. To better apply this approach to system design and optimization, an efficient and powerful computational tool is necessary. In this section, we describe one such tool that is based on the motion-group Fourier transform. Readers unfamiliar with the technical terms used below are referred to [21]. The discussion in the following paragraph provides a context for this particular engineering application, but the value of noncommutative harmonic analysis in this context is solely due to its ability to solve equation (3–1).

Let $s(t)$ be the input signal to a bandpass filter which is corrupted by phase noise. Using the equivalent baseband representation and normalizing it to unit amplitude, this signal can be written as [14]

$$s(t) = e^{j\phi(t)}$$

where $\phi(t)$ is the phase noise, usually modeled as a Brownian motion process. The function $h(t)$ is the impulse response of the bandpass filter. The output of the bandpass filter is denoted $z(t)$. Let us represent $z(t)$ through its real and imaginary parts:

$$z(t) = x(t) + jy(t) = r(t)e^{j\theta(t)}.$$

The 3-D Fokker–Planck equation defining the probability density function (pdf) of $z(t)$ is derived as [2; 45]:

$$\frac{\partial f}{\partial t} = -h(t)\cos\phi \frac{\partial f}{\partial x} - h(t)\sin\phi \frac{\partial f}{\partial y} + \frac{D}{2}\frac{\partial^2 f}{\partial \phi^2} \tag{3–1}$$

with initial condition $f(x, y, \phi; 0) = \delta(x)\delta(y)\delta(\phi)$, where δ being the Dirac delta function. The parameter D is related to the laser line width Δv by $D = 2\pi\Delta v$. Having an efficient method for solving equation (3–1) is of great importance in the design of filters.

A number of papers have attempted to solve the above equations using a variety of techniques including series expansions, numerical methods based on discretizing the domain, and analytical methods [42; 45]. However, all of them are based on classical partial differential equation solution techniques.

In our work, we present a new method for solving these methods using harmonic analysis on groups. These techniques reduce the above Fokker–Planck equations to systems of linear ordinary differential equations with constant or time-varying coefficients in a generalized Fourier space. The solution to this system of equations in generalized Fourier space is simply a matrix exponential for the case of constant coefficients. A usable solution is then generated via the generalized Fourier inversion formula.

Using the differential operators defined on the motion group, the 3-D Fokker–Planck equation in (3–1) can be rewritten as

$$\frac{\partial f}{\partial t} = \Big(-h(t)\tilde{X}_2^R + \frac{D}{2}(\tilde{X}_3^R)^2\Big) f. \tag{3–2}$$

This equation describes a kind of process that evolves on the group of rigid-body motions SE(2). Applying the motion-group Fourier transform to (3–2), we can convert it to an infinite system of linear ordinary differential equations:

$$\frac{d\hat{f}}{dt} = A(t)\hat{f}. \tag{3–3}$$

For equation (3–2), the matrix is

$$A(t) = -h(t)\tilde{u}(X_2, p) + \frac{D}{2}\big(\tilde{u}(X_3, p)\big)^2$$

and its elements are

$$A(t)_{mn} = -h(t)\frac{p}{2}(\delta_{m,n+1} - \delta_{m,n-1}) - \frac{D}{2}m^2\delta_{m,n}.$$

Numerical methods such as Runge–Kutta integration can be applied to easily solve the truncated version of this system. In the case when $h(t)$ is a constant, then A is a constant matrix and the solution to the resulting linear time-invariant system can be written in closed form as

$$\hat{f}(p;t) = \exp(At)$$

with the initial condition that $\hat{f}(p;0)$ is the infinite-dimensional identity matrix. In practice we truncate A at finite dimension, then exponentiate.

Once we get the solution to (3–3), we can then substitute it into the Fourier inversion formula for the motion group in (2–1) to recover the pdf $f(g;t)$ of $z(t)$. To get the pdf $f(r,\theta;t)$ is just an integration with respect to ϕ as

$$f(r,\theta;t) = \frac{1}{2\pi}\int_0^{2\pi} f(g;t)d\phi = \sum_{n\in\mathbb{Z}} j^{-n}e^{-jn\theta}\int_0^{\infty}\hat{f}_{0,n}J_{-n}(p\,r)p\,dp. \qquad (3–4)$$

Integrating equation (3–4) over θ will give us the marginal pdf of $|z(t)|$ as:

$$f(r;t) = \int_0^{\infty}\hat{f}_{0,0}(p)J_0(p\,r)p\,dp. \qquad (3–5)$$

Using our method, we can get a simple and compact expression for the marginal pdf for the output of the bandpass filter given in (3–5).

For details and numerical results generated using this approach, see [43].

4. Robotics

A robotic manipulator arm is a device used to position and orient objects in space. The set of all reachable positions and orientations is called the workspace of the arm. A robot arm that can attain only a finite number of different states is called a discretely-actuated manipulator. For such manipulators, it is a combinatorially explosive problem to enumerate by brute force all possible states for arms that have a high degree of articulation. The function that describes the relative density of reachable positions and orientations in the workspace (called a *workspace density function*) has been shown to be an important quantity in planning the motions of these manipulator arms [4]. This function is denoted as $f(g;L)$ where $g\in \mathrm{SE}(N)$, and L is the length of the arm.

Noncommutative harmonic analysis enters in this problem as a way to reduce this complexity. It was shown in [4] that the workspace density function $f(g;L_1+L_2)$ for two concatenated manipulator segments with length L_1 and L_2 is the motion-group convolution

$$f(g;L_1+L_2) = f(g;L_1) * f(g;L_2) = \int_G f(h;L_1)f(h^{-1}\circ g;L_2)\,dh, \qquad (4–1)$$

where h is a dummy variable of integration and dh is the bi-invariant (Haar) measure for $\mathrm{SE}(N)$. That is, given two short arms with known workspace densities, we can generate the workspace density of the long arm generated by stacking one short arm on the other using equation (4–1). In order to perform these convolutions efficiently, the concept of FFTs for the motion groups was studied in [6].

In the rest of this section, we discuss an alternative method for generating manipulator workspace density functions that does not explicitly compute convolutions. Instead, it relies on the same kinds of degenerate diffusions we have seen already in the context of phase noise.

4.1. Inspiration of the algorithm. Consider a discretely-actuated serial manipulator which consists of concatenated segments called modules. Suppose that each module can reach 16 different states. The workspace of this manipulator with 2 modules, 3 modules and 4 modules can be generated by brute force enumeration because 16^2, 16^3, and 16^4 are not terribly huge numbers. It is easy to imagine that the size of the workspace will spread out with the increment of modules. This enlargement of the workspace is just like the diffusion produced by a drop of ink spreading in a cup of water. Inspired by this observation, we view the workspace of a manipulator as something that grows/evolves from a single point source at the base as the length of the manipulator increases from zero. The workspace is generated after the manipulator grows to full length.

4.2. Implementation of the algorithm. With this analogy, we then need to determine what kind of diffusion equation is suitable to model this process. We get such an equation by realizing that some characteristics of manipulators are similar to those of polymer chains like DNA.

During our study of conformational statistics in polymer science, we derived a diffusion-type equation defined on the motion group [7]. This equation describes the probability density function of the position and orientation of the distal end of a stiff macromolecule chain relative to its proximal end. By involving parameters which indicate the kinematic properties of a manipulator into this equation, we can modify it to the diffusion-type equation describing the evolution of the workspace density function. It is written explicitly as

$$\frac{\partial f}{\partial L} = \big(\alpha \tilde{X}_1^R + \beta (\tilde{X}_1^R)^2 + \tilde{X}_3^R + \varepsilon (\tilde{X}_3^R)^2\big) f. \tag{4–2}$$

Here f stands for the workspace density function, and L is the manipulator length. The differential operators $\tilde{X}_1^R$ and $\tilde{X}_3^R$ are those defined on SE(2) given earlier. Parameters β, ε and α describe the kinematic properties of manipulators. We define these kinematic properties as flexibility, extensibility and the degree of asymmetry. The parameter β describes the flexibility of a manipulator in the sense of how much a segment of the manipulator can bend per unit length. A larger value of β means that the manipulator can bend a lot. The parameter ε describes the extensibility of a manipulator in the sense of how much a manipulator can extend along its backbone direction. A larger value of ε means that the manipulator can extend a lot. The parameter α describes the asymmetry in how the manipulator bends. When $\alpha = 0$, the manipulator can reach left and right with equal ease. When $\alpha < 0$, there is a preference for bending to the left, and when $\alpha > 0$ there is a preference for bending to the right. Since α, β, and ε are qualitative descriptions of the kinematic properties of a manipulator, they are not directly measurable.

This simple three-parameter model qualitatively captures the behavior that has been observed in numerical simulations of workspace densities of discretely-actuated variable-geometry truss manipulators [23]. Clearly, equation (4–2) can be solved in the same way as the phase-noise equation. We have done this in [43].

5. Statistical Mechanics of Macromolecules

In this section, we show how certain quantities of interest in polymer physics can be generated numerically using Euclidean-group convolutions. We also show how for wormlike polymer chains, a partial differential equation governs a process that evolves on the motion group and describes the diffusion of end-to-end position and orientation. This equation can be solved using the SE(3)-Fourier transform in a manner very similar to the way the phase-noise Fokker–Planck was addressed in Section 3. This builds on classical works in polymer theory such as [8; 15; 20; 22; 34; 44].

5.1. Mass density, frame density, and Euclidean group convolutions. In statistical mechanical theories of polymer physics, it is essential to compute ensemble properties of polymer chains averaged over all of their possible conformations [9; 27]. Noncommutative harmonic analysis provides a tool for computing probability densities used in these averages.

In this subsection we review three statistical properties of macromolecular ensembles. These are: (1) The ensemble mass density for the whole chain $\rho(\boldsymbol{x})$, which is generated by imagining that one end of the chain is held fixed and a cloud is generated by all possible conformations of the chain superimposed on each other; (2) The ensemble tip frame density $f(g)$ (where g is the frame of reference of the distal end of the chain relative to the fixed proximal end); (3) The function $\mu(g, \boldsymbol{x})$, which is the ensemble mass density of all configurations which grow from the identity frame fixed to one end of the chain and terminate at the relative frame g at the other end. Figures that describe these quantities can be found in [3].

The functions ρ, f, and μ are related to each other. Given $\mu(g, \boldsymbol{x})$, the ensemble mass density is calculated by adding the contribution of each μ for each different end position and orientation:

$$\rho(\boldsymbol{x}) = \int_G \mu(g, \boldsymbol{x})\, dg. \tag{5–1}$$

This integration is written as being over all motions of the end of the chain, but only frames g in the support of μ contribute to the integral. Here G is shorthand for SE(3) and dg denotes the invariant integration measure for SE(3).

In an analogous way, it is not difficult to see that integrating the $\boldsymbol{x}$-dependence out of μ provides the total mass of configurations of the chain starting at frame e and terminating at frame g. Since each chain has mass M, this means that the frame density $f(g)$ is related to $\mu(g, \boldsymbol{x})$ as:

$$f(g) = \frac{1}{M} \int_{\mathbb{R}^3} \mu(g, \boldsymbol{x}) d\boldsymbol{x}. \tag{5–2}$$

We note the total number of frames attained by one end of the chain relative to the other is

$$F = \int_G f(g)\, dg.$$

It then follows that

$$\int_{\mathbb{R}^3} \rho(\boldsymbol{x}) d\boldsymbol{x} = F \cdot M.$$

If the functions $\rho(\boldsymbol{x})$ and $f(g)$ are known for the whole chain then a number of important thermodynamic and mechanical properties of the polymer can be determined [6].

We can divide the chain into P segments that are short enough to allow brute force enumeration calculation of $\rho_i(\boldsymbol{x})$ and $f_i(g)$ for $i = 1, \ldots, P$, where g is the *relative* frame of reference of the distal end of the segment with respect to the proximal one. For a homogeneous chain, such as polyethylene, these functions are the same for each value of $i = 1, \ldots, P$.

In the general case of a heterogeneous chain, we can calculate the functions $\rho_{i,i+1}(\boldsymbol{x})$, $f_{i,i+1}(g)$, and $\mu_{i,i+1}(g, \vec{x})$ for the concatenation of segments i and $i+1$ from those of segments i and $i+1$ separately in the following way:

$$\rho_{i,i+1}(\boldsymbol{x}) = F_{i+1}\rho_i(\boldsymbol{x}) + \int_G f_i(h)\rho_{i+1}(h^{-1} \circ \boldsymbol{x})\, dh, \tag{5–3}$$

$$f_{i,i+1}(g) = (f_i * f_{i+1})(g) = \int_G f_i(h) f_{i+1}(h^{-1} \circ g)\, dh. \tag{5–4}$$

and

$$\mu_{i,i+1}(g, \vec{x}) = \int_G \big(\mu_i(h, \vec{x}) f_{i+1}(h^{-1} \circ g) + f_i(h)\mu_{i+1}(h^{-1} \circ g, h^{-1} \circ \vec{x})\big)\, dh. \tag{5–5}$$

In these expressions $h \in G = \mathrm{SE}(3)$ is a dummy variable of integration. The meaning of equation (5–3) is that the mass density of the ensemble of all conformations of two concatenated chain segments results from two contributions. The first is the mass density of all the conformations of the lower segment (weighted by the number of different upper segments it can carry, which is $F_{i+1} = \int_G f_{i+1}\, dg$). The second contribution results from rotating and translating the mass density of the ensemble of the upper segment, and adding the contribution at each of these poses (positions and orientations). This contribution is weighted by the number of frames that the distal end of the lower segment can attain relative to its base. Mathematically $L(h)\rho_{i+1}(\boldsymbol{x}) = \rho_{i+1}(h^{-1} \circ \boldsymbol{x})$ is a left-shift operation which geometrically has the significance of rigidly translating and rotating the function $\rho_{i+1}(\boldsymbol{x})$ by the transformation h. The weight $f_i(h)\,dh$ is the number of configurations of the i^{th} segment terminating at frame of reference h.

The meaning of equation (5–4) is that the distribution of frames of reference at the terminal end of the concatenation of segments i and $i+1$ is the group-theoretical *convolution* of the frame densities of the terminal ends of each of the two segments relative to their respective bases. This equation holds for exactly the same reason why equation (4–1) does in the context of robot arms.

Equation (5–5) says that there are two contributions to $\mu_{i,i+1}(g,\vec{x})$. The first comes from adding up all the contributions due to each $\mu_i(h,\vec{x})$. This is weighted by the number of upper segment conformations with distal ends that reach the frame g given that their base is at frame h. The second comes from adding up all shifted (translated and rotated) copies of $\mu_{i+1}(g,\vec{x})$, where the shifting is performed by the lower distribution, and the sum is weighted by the number of distinct configurations of the lower segment that terminate at h. This number is $f_i(h)\,dh$.

Equations (5–3), (5–4) and (5–5) can be iterated as described in [3; 6].

5.2. Statistics of stiff molecules as solutions to PDEs on SO(3) **and** SE(3)**.** Experimental measurements of the stiffness constants of DNA and other stiff (or semi-flexible) macromolecules have been reported in a number of papers, as well as the statistical mechanics of such molecules. See [17; 26; 29; 30; 31; 32; 33; 38], for example.

The stiffness and chirality (how helical the molecule is) can be described with parameters D_{lk} and d_l for $l,k = 1,2,3$. In particular, D_{lk} are the elements of the inverse of the stiffness matrix. When a force is applied, these constants determine how easily one end of the molecule deflects from the helical shape that it assumes when no forces act on it. The parameters d_l describe the helical shape of an undeformed molecule with flexibility described by D_{lk}. These parameters are described in detail in [7].

Degenerate diffusion equations describing the evolution of position and orientation of frames of reference attached to points on the chain at different values of length, L, have been derived [6; 43]. These equations incorporate stiffness and chirality information and are written in terms of SE(3) differential operators as

$$\left(\frac{\partial}{\partial L} - \frac{1}{2}\sum_{k,l=1}^{3} D_{lk}\tilde{X}_l^R\tilde{X}_k^R - \sum_{l=1}^{3} d_l\tilde{X}_l^R + \tilde{X}_6^R\right) f = 0. \qquad (5\text{–}6)$$

The initial conditions are $f(\boldsymbol{a},A;0) = \delta(\boldsymbol{a})\delta(A)$ where $g = (\boldsymbol{a},A)$.

This equation has been solved using the operational properties of the SE(3) Fourier transform in [5; 6; 43].

6. Conclusions

This paper has reviewed a number of applications of harmonic analysis on the motion groups. This illustrates the power of noncommutative harmonic analysis, and its potential as a computational and analytical tool for solving real-world problems. We hope that this review will stimulate interest among others working in the field of noncommutative harmonic analysis to apply these methods to problems in engineering, and we hope that those in the engineering sciences will appreciate noncommutative harmonic analysis for the powerful tool that it is.

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