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Finite Quantum Groupoids and Their Applications

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Abstract. We give a survey of the theory of finite quantum groupoids (weak Hopf algebras), including foundations of the theory and applications to finite depth subfactors, dynamical deformations of quantum groups, and invariants of knots and 3-manifolds.

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1. Introduction

By *quantum groupoids* we understand weak Hopf algebras introduced in [BNSz], [BSz1], [N]. These objects generalize Hopf algebras (in fact, many Hopf-algebraic concepts can be extended to the quantum groupoid case) and usual finite groupoids (see [NV1] for discussion). Every quantum groupoid has two canonical subalgebras that play the same role as the space of units in a groupoid and projections on these subalgebras generalizing the source and target maps in a groupoid. We use the term "quantum groupoid" instead of "weak Hopf algebra" to stress this similarity that leads to many interesting constructions and examples.

Our initial motivation for studying quantum groupoids in [NV1], [NV2],[N1] was their connection with depth 2 von Neumann subfactors, first mentioned in [O2], which was also one of the main topics of [BNSz], [BSz1], [BSz2], [NSzW]. It was shown in [NV2] that quantum groupoids naturally arise as non-commutative symmetries of subfactors, namely if $N \subset M \subset M_1 \subset M_2 \subset \cdots$ is the Jones tower constructed from a finite index depth 2 inclusion $N \subset M$ of II_1 factors, then $H = M' \cap M_2$ has a canonical structure of a quantum groupoid acting outerly on M_1 such that $M = M_1^H$ and $M_2 = M_1 \rtimes H$; moreover $\widehat{H} = N' \cap M_1$ is a quantum groupoid dual to H.

In [NV3] we extended this result to show that quantum groupoids give a description of arbitrary finite index and finite depth II_1 subfactors via a Galois correspondence and explained how to express subfactor invariants such as bimodule categories and principal graphs in quantum groupoid terms. Thus, in this respect quantum groupoids play the same role as Ocneanu's paragroups [O1], [D].

Quantum groupoids also appear naturally in the theory of dynamical deformations of quantum groups [EV], [Xu]. It was shown in [EN] that for every simple Lie algebra $\mathfrak{g}$ the corresponding Drinfeld–Jimbo quantum group $U_q(\mathfrak{g})$ for q a primitive root of unity has a family of dynamical twists that give rise to self-dual finite quantum groupoids. To every such a twist one can associate a solution of the quantum dynamical Yang–Baxter equation [ES].

As in the case of Hopf algebras, the representation category $\mathrm{Rep}(H)$ of a quantum groupoid H is a monoidal category with duality. Existence of additional (quasitriangular, ribbon, or modular) structures on H makes $\mathrm{Rep}(H)$, respectively, braided, ribbon, or modular [NTV]. It is well known [RT], [T], that such categories give rise to invariants of knots and 3-manifolds.

The survey is organized as follows.

In Section 2 we give basic definitions and examples of finite quantum groupoids, discuss their fundamental properties (following [BNSz] and [NV1]) and relations with other versions of quantum groupoids.

The exposition of the theory of integrals in Section 3 follows [BNSz] and is similar to the Hopf algebra case [M]. The main results here are extensions of

the fundamental theorem for Hopf modules and the Maschke theorem. The latter gives an equivalence between existence of normalized integrals and semisimplicity of a quantum groupoid.

In Section 4 we define the notions of action and smash product and extend the well known Blattner–Montgomery duality theorem for Hopf algebra actions to quantum groupoids (see [N2]).

In Section 5 we describe monoidal structure and duality on the representation category of a quantum groupoid ([BSz2] and [NTV]). Then we distinguish special cases of quantum groupoids leading to braided, ribbon, and modular categories of representations.

We also develop the Drinfeld double construction and emphasize the role of factorizable quantum groupoids in producing modular categories (see [NTV]). Section 6 contains an extension of the Drinfeld twisting procedure to quantum groupoids and construction of a quantum groupoid via a *dynamical twisting* of a Hopf algebra. Important concrete examples of such twistings are *dynamical quantum groups* at roots of 1 [EN].

Semisimple and C^*-quantum groupoids are analyzed in Section 7. Following [BNSz], our discussion is concentrated around the existence of the Haar integral in connection with a special implementation of the square of the antipode.

Sections 8 and 9 are devoted to the description of finite index an finite depth subfactors via crossed products of II_1 factors with C^*-quantum groupoids. These results were obtained in [NV2], [NV3] and partially in [N1], [NSzW].

2. Definitions and Examples

In this section we give definitions and discuss basic properties of finite quantum groupoids. Most of the material presented here can be found in [BNSz] and [NV1].

Throughout this paper we use Sweedler's notation for comultiplication, writing $\Delta(b) = b_{(1)} \otimes b_{(2)}$. Let k be a field.

2.1. Definition of the quantum groupoid. By *quantum groupoids* or *weak Hopf algebras* we understand the objects introduced in [BNSz], [BSz1] as a generalization of ordinary Hopf algebras.

Definition 2.1.1. A *(finite) quantum groupoid* over k is a finite dimensional k-vector space H with the structures of an associative algebra $(H, m, 1)$ with multiplication $m : H \otimes_k H \to H$ and unit $1 \in H$ and a coassociative coalgebra (H, Δ, ε) with comultiplication $\Delta : H \to H \otimes_k H$ and counit $\varepsilon : H \to k$ such that:

(i) The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$(\Delta \otimes \mathrm{id})\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1), \tag{2–1}$$

(ii) The counit is a k-linear map satisfying the identity:

$$\varepsilon(fgh) = \varepsilon(fg_{(1)})\,\varepsilon(g_{(2)}h) = \varepsilon(fg_{(2)})\,\varepsilon(g_{(1)}h), \tag{2–2}$$

for all $f, g, h \in H$.

(iii) There is a linear map $S : H \to H$, called an *antipode*, such that, for all $h \in H$,

$$m(\mathrm{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \tag{2–3}$$

$$m(S \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)), \tag{2–4}$$

$$m(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes S)(\Delta \otimes \mathrm{id})\Delta(h) = S(h). \tag{2–5}$$

A quantum groupoid is a Hopf algebra if and only if the comultiplication is unit-preserving if and only if the counit is a homomorphism of algebras.

A *morphism* between quantum groupoids H_1 and H_2 is a map $\alpha : H_1 \to H_2$ which is both algebra and coalgebra homomorphism preserving unit and counit and which intertwines the antipodes of H_1 and H_2, i.e., $\alpha \circ S_1 = S_2 \circ \alpha$. The image of a morphism is clearly a quantum groupoid. The tensor product of two quantum groupoids is defined in an obvious way.

2.2. Counital maps and subalgebras. The linear maps defined in (2–3) and (2–4) are called *target* and *source counital maps* (see examples below for explanation of the terminology) and denoted ε_t and ε_s respectively:

$$\varepsilon_t(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \qquad \varepsilon_s(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)). \tag{2–6}$$

In the next proposition we collect several useful properties of the counital maps.

Proposition 2.2.1. *For all $h, g \in H$,*

(i) *counital maps are idempotents in* $\mathrm{End}_k(H)$:

$$\varepsilon_t(\varepsilon_t(h)) = \varepsilon_t(h), \qquad \varepsilon_s(\varepsilon_s(h)) = \varepsilon_s(h);$$

(ii) *the relations between ε_t, ε_s, and comultiplication are*

$$(\mathrm{id} \otimes \varepsilon_t)\Delta(h) = 1_{(1)}h \otimes 1_{(2)}, \qquad (\varepsilon_s \otimes \mathrm{id})\Delta(h) = 1_{(1)} \otimes h1_{(2)};$$

(iii) *the images of counital maps are characterized by*

$$h = \varepsilon_t(h) \text{ if and only if } \Delta(h) = 1_{(1)}h \otimes 1_{(2)},$$
$$h = \varepsilon_s(h) \text{ if and only if } \Delta(h) = 1_{(1)} \otimes h1_{(2)};$$

(iv) *$\varepsilon_t(H)$ and $\varepsilon_s(H)$ commute*;

(v) *one also has identities dual to* (ii):

$$h\varepsilon_t(g) = \varepsilon(h_{(1)}g)h_{(2)}, \qquad \varepsilon_s(g)h = h_{(1)}\varepsilon(gh_{(2)}).$$

Proof. We prove the identities containing the target counital map, the proofs of their source counterparts are similar. Using the axioms (2–1) and (2–2) we compute

$$\varepsilon_t(\varepsilon_t(h)) = \varepsilon(1_{(1)}h)\varepsilon(1'_{(1)}1_{(2)})1'_{(2)} = \varepsilon(1_{(1)}h)\varepsilon(1_{(2)})1_{(3)} = \varepsilon_t(h),$$

where $1'$ stands for the second copy of the unit, proving (i). For (ii) we have

$$\begin{aligned} h_{(1)} \otimes \varepsilon_t(h_{(2)}) &= h_{(1)}\varepsilon(1_{(1)}h_{(2)}) \otimes 1_{(2)} \\ &= 1_{(1)}h_{(1)}\varepsilon(1_{(2)}h_{(2)}) \otimes 1_{(3)} = 1_{(1)}h \otimes 1_{(2)}. \end{aligned}$$

To prove (iii) we observe that

$$\Delta(\varepsilon_t(h)) = \varepsilon(1_{(1)}h)1_{(2)} \otimes 1_{(3)} = \varepsilon(1_{(1)}h)1'_{(1)}1_{(2)} \otimes 1'_{(2)} = 1'_{(1)}\varepsilon_t(h) \otimes 1'_{(2)},$$

on the other hand, applying $(\varepsilon \otimes \mathrm{id})$ to both sides of $\Delta(h) = 1_{(1)}h \otimes 1_{(2)}$, we get $h = \varepsilon_t(h)$. (iv) is immediate in view of the identity $1_{(1)} \otimes 1'_{(1)}1_{(2)} \otimes 1'_{(2)} = 1_{(1)} \otimes 1_{(2)}1'_{(1)} \otimes 1'_{(2)}$. Finally, we show (v):

$$\begin{aligned} \varepsilon(h_{(1)}g)h_{(2)} &= \varepsilon(h_{(1)}g_{(1)})h_{(2)}\varepsilon_t(g_{(2)}) = \varepsilon(h_{(1)}g_{(1)})h_{(2)}g_{(2)}S(g_{(3)}) \\ &= hg_{(1)}S(g_{(2)}) = h\varepsilon_t(g), \end{aligned}$$

where the antipode axiom (2–3) is used. □

The images of the counital maps

$$H_t = \varepsilon_t(H) = \{h \in H \mid \Delta(h) = 1_{(1)}h \otimes 1_{(2)}\}, \tag{2–7}$$

$$H_s = \varepsilon_s(H) = \{h \in H \mid \Delta(h) = 1_{(1)} \otimes h1_{(2)}\} \tag{2–8}$$

play the role of bases of H. The next proposition summarizes their properties.

Proposition 2.2.2. *H_t (resp. H_s) is a left (resp. right) coideal subalgebra of H. These subalgebras commute with each other; moreover*

$$H_t = \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in \widehat{H}\}, \qquad H_s = \{(\mathrm{id} \otimes \phi)\Delta(1) \mid \phi \in \widehat{H}\},$$

i.e., H_t (resp. H_s) is generated by the right (resp. left) tensorands of $\Delta(1)$.

Proof. H_t and H_s are coideals by Proposition 2.2.1(iii), they commute by 2.2.1(iv). We have $H_t = \varepsilon(1_{(1)}H)1_{(2)} \subset \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in \widehat{H}\}$, conversely $\phi(1_{(1)})1_{(2)} = \phi(1_{(1)})\varepsilon_t(1_{(2)}) \subset H_t$, therefore $H_t = \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in \widehat{H}\}$. To see that it is an algebra we note that $1 = \varepsilon_t(1) \in H_t$ and for all $h, g \in H$ compute, using Proposition 2.2.1(ii) and (v):

$$\begin{aligned} \varepsilon_t(h)\varepsilon_t(g) &= \varepsilon(\varepsilon_t(h)_{(1)}g)\varepsilon_t(h)_{(2)} \\ &= \varepsilon(1_{(1)}\varepsilon_t(h)g)1_{(2)} = \varepsilon_t(\varepsilon_t(h)g) \in H_t. \end{aligned}$$

The statements about H_s are proven similarly. □

Definition 2.2.3. We call H_t (resp. H_s) a *target* (resp. *source*) *counital subalgebra* of H.

2.3. Properties of the antipode. The properties of the antipode of a quantum groupoid are similar to those of a finite-dimensional Hopf algebra.

PROPOSITION 2.3.1 ([BNSz], 2.10). *The antipode S is unique and bijective. Also, it is both algebra and coalgebra anti-homomorphism.*

PROOF. Let $f * g = m(f \otimes g)\Delta$ be the convolution of $f, g \in \mathrm{End}_k(H)$. Then $S * \mathrm{id} = \varepsilon_s$, $\mathrm{id} * S = \varepsilon_t$, and $S * \mathrm{id} * S = S$. If S' is another antipode of H then

$$S' = S' * \mathrm{id} * S' = S' * \mathrm{id} * S = S * \mathrm{id} * S = S.$$

To check that S is an algebra anti-homomorphism, we compute

$$\begin{aligned} S(1) &= S(1_{(1)})1_{(2)}S(1_{(3)}) = S(1_{(1)})\varepsilon_t(1_{(2)}) = \varepsilon_s(1) = 1, \\ S(hg) &= S(h_{(1)}g_{(1)})\varepsilon_t(h_{(2)}g_{(2)}) = S(h_{(1)}g_{(1)})h_{(2)}\varepsilon_t(g_{(2)})S(h_{(3)}) \\ &= \varepsilon_s(h_{(1)}g_{(1)})S(g_{(2)})S(h_{(2)}) = S(g_{(1)})\varepsilon_s(h_{(1)})\varepsilon_t(g_{(2)})S(h_{(2)}) = S(g)S(h), \end{aligned}$$

for all $h, g \in H$, where we used Proposition 2.2.1(iv) and easy identities $\varepsilon_t(hg) = \varepsilon_t(h\varepsilon_t(g))$ and $\varepsilon_s(hg) = \varepsilon_t(\varepsilon_s(h)g)$. Dualizing the above arguments we show that S is also a coalgebra anti-homomorphism:

$$\begin{aligned} \varepsilon(S(h)) &= \varepsilon(S(h_{(1)})\varepsilon_t(h_{(2)})) = \varepsilon(S(h_{(1)})h_{(2)}) = \varepsilon(\varepsilon_t(h)) = \varepsilon(h), \\ \Delta(S(h)) &= \Delta(S(h_{(1)})\varepsilon_t(h_{(2)})) \\ &= \Delta(S(h_{(1)}))(\varepsilon_t(h_{(2)}) \otimes 1) \\ &= \Delta(S(h_{(1)}))(h_{(2)}S(h_{(4)}) \otimes \varepsilon_t(h_{(3)})) \\ &= \Delta(\varepsilon_s(h_{(1)}))(S(h_{(3)}) \otimes S(h_{(2)})) \\ &= S(h_{(3)}) \otimes \varepsilon_s(h_{(1)})S(h_{(2)}) = S(h_{(2)}) \otimes S(h_{(1)}). \end{aligned}$$

The proof of the bijectivity of S can be found in ([BNSz], 2.10). □

Next, we investigate the relations between the antipode and counital maps.

PROPOSITION 2.3.2. *We have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $\varepsilon_s \circ S = S \circ \varepsilon_t$. The restriction of S defines an algebra anti-isomorphism between counital subalgebras H_t and H_s.*

PROOF. Using results of Proposition 2.3.1 we compute

$$S(\varepsilon_s(h)) = S(1_{(1)})\varepsilon(h1_{(2)}) = \varepsilon(1_{(1)}S(h))1_{(2)} = \varepsilon_t(S(h)),$$

for all $h \in H$. The second identity is proven similarly. Clearly, S maps H_t to H_s and vice versa. Since S is bijective, and $\dim H_t = \dim H_s$ by Proposition 2.2.2, therefore $S|_{H_t}$ and $S|_{H_s}$ are anti-isomorphisms. □

PROPOSITION 2.3.3 ([NV1], 2.1.12). *Any nonzero morphism $\alpha : H \to K$ of quantum groupoids preserves counital subalgebras, i.e. $H_t \cong K_t$ and $H_s \cong K_s$. Thus, quantum groupoids with a given target (source) counital subalgebra form a full subcategory of the category of all finite quantum groupoids.*

PROOF. It is clear that $\alpha|_{H_t} : H_t \to K_t$ is a homomorphism. If we write

$$\Delta(1_H) = \sum_{i=1}^{n} w_i \otimes z_i$$

with $\{w_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ linearly independent, then

$$\Delta(1_K) = \sum_{i=1}^{n} \alpha(w_i) \otimes \alpha(z_i).$$

By Proposition 2.2.2, $K_t = \operatorname{span}\{\alpha(z_i)\}$, i.e., $\alpha|_{H_t}$ is surjective. Since

$$z_j = \varepsilon_t(z_j) = \sum_{i=1}^{n} \varepsilon(w_i z_j) z_i,$$

then $\varepsilon(w_i z_j) = \delta_{ij}$, therefore,

$$\begin{aligned}\dim H_t = n = \sum_{i=1}^{n} \varepsilon_H(w_i z_i) &= \sum_{i=1}^{n} \varepsilon_H(w_i S(z_i)) \\ &= \varepsilon_H(\varepsilon_t(1_H)) = \varepsilon_H(1_H) = \varepsilon_K(1_K) = \dim K_t,\end{aligned}$$

so $\alpha|_{H_t}$ is bijective. The proof for source subalgebras is similar. □

Let us recall that a k-algebra A is said to be *separable* [P] if the multiplication epimorphism $m : A \otimes_k A \to A$ has a right inverse as an $A-A$ bimodule homomorphism. This is equivalent to the existence of a *separability element* $e \in A \otimes_k A$ such that $m(e) = 1$ and $(a \otimes 1)e = e(1 \otimes a)$ for all $a \in A$.

PROPOSITION 2.3.4. *The counital subalgebras H_t and H_s are separable, with separability elements $e_t = (S \otimes \mathrm{id})\Delta(1)$ and $e_s = (\mathrm{id} \otimes S)\Delta(1)$, respectively.*

PROOF. For all $z \in H_t$ we compute, using Propositions 2.2.1 and 2.3.2:

$$\begin{aligned}1_{(1)} S^{-1}(z) \otimes 1_{(2)} &= S^{-1}(z)_{(1)} \otimes \varepsilon_t(S^{-1}(z)_{(2)}) \\ &= 1_{(1)} \otimes \varepsilon_t(1_{(2)} S^{-1}(z)) = 1_{(1)} \otimes \varepsilon_t(1_{(2)} z) = 1_{(1)} \otimes 1_{(2)} z,\end{aligned}$$

applying $(S \otimes \mathrm{id})$ to this identity we get $e_t z = z e_t$. Clearly, $m(e_t) = 1$, whence e_t is a separability element. The second statement follows similarly. □

2.4. The dual quantum groupoid. The set of axioms of Definition 2.1.1 is self-dual. This allows to define a natural quantum groupoid structure on the dual vector space $\widehat{H} = \mathrm{Hom}_k(H, k)$ by "reversing the arrows":

$$\langle\, h,\, \phi\psi \,\rangle = \langle\, \Delta(h),\, \phi \otimes \psi \,\rangle, \tag{2–9}$$

$$\langle\, h \otimes g,\, \widehat{\Delta}(\phi) \,\rangle = \langle\, hg,\, \phi \,\rangle, \tag{2–10}$$

$$\langle\, h,\, \widehat{S}(\phi) \,\rangle = \langle\, S(h),\, \phi \,\rangle, \tag{2–11}$$

for all $\phi, \psi \in \widehat{H}$, $h, g \in H$. The unit $\widehat{1}$ of $\widehat{H}$ is ε and counit $\widehat{\varepsilon}$ is $\phi \mapsto \langle\, \phi,\, 1 \,\rangle$.

In what follows we will use the Sweedler arrows, writing

$$h \rightharpoonup \phi = \phi_{(1)}\langle\, h,\, \phi_{(2)}\,\rangle, \qquad \phi \leftharpoonup h = \langle\, h,\, \phi_{(1)}\,\rangle\phi_{(2)}, \tag{2–12}$$

for all $h \in H, \phi \in \widehat{H}$.

The counital subalgebras of $\widehat{H}$ are canonically anti-isomorphic to those of H. More precisely, the map $H_t \ni z \mapsto (z \rightharpoonup \varepsilon) \in \widehat{H}_s$ is an algebra isomorphism with the inverse given by $\chi \mapsto (1 \leftharpoonup \chi)$. Similarly, the map $H_s \ni z \mapsto (\varepsilon \leftharpoonup z) \in \widehat{H}_t$ is an algebra isomorphism ([BNSz], 2.6).

Remark 2.4.1. The opposite algebra H^{op} is also a quantum groupoid with the same coalgebra structure and the antipode S^{-1}. Indeed,

$$\begin{aligned} S^{-1}(h_{(2)})h_{(1)} &= S^{-1}(\varepsilon_s(h)) = S^{-1}(1_{(1)})\varepsilon(h1_{(2)}) \\ &= S^{-1}(1_{(1)})\varepsilon(hS^{-1}(1_{(2)})) = \varepsilon(h1_{(1)})1_{(2)}, \\ h_{(2)}S^{-1}(h_{(1)}) &= S^{-1}(\varepsilon_t(h)) = \varepsilon(1_{(1)}h)S^{-1}(1_{(2)}) \\ &= \varepsilon(S^{-1}(1_{(1)})h)S^{-1}(1_{(2)}) = 1_{(1)}\varepsilon(1_{(2)}h), \\ S^{-1}(h_{(3)})h_{(2)}S^{-1}(h_{(1)}) &= S^{-1}(h_{(1)}S(h_{(2)})h_{(3)}) = S^{-1}(h). \end{aligned}$$

Similarly, the co-opposite coalgebra H^{cop} (with the same algebra structure as H and the antipode S^{-1}) and $(H^{\mathrm{op/cop}}, S)$ are quantum groupoids.

2.5. Examples: groupoid algebras and their duals. As group algebras and their duals are the easiest examples of Hopf algebras, groupoid algebras and their duals provide examples of quantum groupoids ([NV1], 2.1.4).

Let G be a finite *groupoid* (a category with finitely many morphisms, such that each morphism is invertible), then the groupoid algebra kG (generated by morphisms $g \in G$ with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise) is a quantum groupoid via:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G. \tag{2–13}$$

The counital subalgebras of kG are equal to each other and coincide with the abelian algebra spanned by the identity morphisms:

$$(kG)_t = (kG)_s = \mathrm{span}\{gg^{-1} \mid g \in G\}.$$

The target and source counital maps are given by the operations of taking the target (resp. source) object of a morphism:

$$\varepsilon_t(g) = gg^{-1} = \mathrm{id}_{target(g)} \quad \text{and} \quad \varepsilon_s(g) = g^{-1}g = \mathrm{id}_{source(g)}.$$

The dual quantum groupoid $\widehat{kG}$ is isomorphic to the algebra of functions on G, i.e., it is generated by idempotents p_g, $g \in G$ such that $p_g p_h = \delta_{g,h} p_g$, with the following structure operations

$$\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,gg^{-1}}, \quad S(p_g) = p_{g^{-1}}. \tag{2–14}$$

The target (resp. source) counital subalgebra is precisely the algebra of functions constant on each set of morphisms of G having the same target (resp. source) object. The target and source maps are

$$\varepsilon_t(p_g) = \sum_{vv^{-1}=g} p_v \quad \text{and} \quad \varepsilon_s(p_g) = \sum_{v^{-1}v=g} p_v.$$

2.6. Examples: quantum transformation groupoids. It is known that any group action on a set (i.e., on a commutative algebra of functions) gives rise to a groupoid [R]. Extending this construction, we associate a quantum groupoid with any action of a Hopf algebra on a separable algebra ("finite quantum space").

Namely, let H be a Hopf algebra and B be a separable (and, therefore, finite dimensional and semisimple [P]) right H-module algebra with the action $b \otimes h \mapsto b \cdot h$, where $b \in B, h \in H$. Then B^{op}, the algebra opposite to B, becomes a left H-module algebra via $h \otimes a \mapsto h \cdot a = a \cdot S_H(h)$. One can form a *double crossed product algebra* $B^{\mathrm{op}} \rtimes H \ltimes B$ on the vector space $B^{\mathrm{op}} \otimes H \otimes B$ with the multiplication

$$(a \otimes h \otimes b)(a' \otimes h' \otimes b') = (h_{(1)} \cdot a')a \otimes h_{(2)}{h'}_{(1)} \otimes (b \cdot {h'}_{(2)})b',$$

for all $a, a' \in B^{\mathrm{op}}$, $b, b' \in B$, and $h, h' \in H$.

Assume that k is an algebraically closed field of characteristic zero and let e be the symmetric separability element of B (note that e is an idempotent when considered in $B \otimes B^{\mathrm{op}}$). Let $\omega \in B^*$ be uniquely determined by $(\omega \otimes \mathrm{id})e = (\mathrm{id} \otimes \omega)e = 1$.

One can check that ω is the trace of the left regular representation of B and verifies the following identities:

$$\omega((h \cdot a)b) = \omega(a(b \cdot h)), \qquad e^{(1)} \otimes (h \cdot e^{(2)}) = (e^{(1)} \cdot h) \otimes e^{(2)},$$

where $a \in B^{\mathrm{op}}, b \in B$, and $e = e^{(1)} \otimes e^{(2)}$.

The structure of a quantum groupoid on $B^{\mathrm{op}} \rtimes H \ltimes B$ is given by

$$\Delta(a \otimes h \otimes b) = (a \otimes h_{(1)} \otimes e^{(1)}) \otimes ((h_{(2)} \cdot e^{(2)}) \otimes h_{(3)} \otimes b), \tag{2–15}$$

$$\varepsilon(a \otimes h \otimes b) = \omega(a(h \cdot b)) = \omega(a(b \cdot S(h))), \tag{2–16}$$

$$S(a \otimes h \otimes b) = b \otimes S(h) \otimes a. \tag{2–17}$$

2.7. Examples: Temperley–Lieb algebras. It was shown in [NV3] (see Section 9 for details) that any inclusion of type II_1 factors with finite index and depth ([GHJ], 4.1) gives rise to a quantum groupoid describing the symmetry of this inclusion. In the case of *Temperley–Lieb algebras* ([GHJ], 2.1) we have this way the following example.

Let $k = \mathbb{C}$ be the field of complex numbers, $\lambda^{-1} = 4\cos^2 \frac{\pi}{n+3}$ $(n \geq 2)$, and $e_1, e_2, \ldots$ be a sequence of idempotents satisfying, for all i and j, the braid-like relations

$$e_i e_{i\pm 1} e_i = \lambda e_i, \qquad e_i e_j = e_j e_i \text{ if } |i-j| \geq 2.$$

Let $A_{k,l}$ be the algebra generated by $1, e_k, e_{k+1}, \dots e_l$ $(k \le l)$, σ be the algebra anti-isomorphism of $H = A_{1,2n-1}$ determined by $\sigma(e_i) = e_{2n-i}$ and $P_k \in A_{2n-k,2n-1} \otimes A_{1,k}$ be the image of the symmetric separability idempotent of $A_{1,k}$ under $(\sigma \otimes \mathrm{id})$.

Finally, we denote by τ the non-degenerate Markov trace ([GHJ], 2.1) on H and by w the index of the restriction of τ on $A_{n+1,2n-1}$ [W], i.e., the unique central element in $A_{n+1,2n-1}$ such that $\tau(w\,\cdot)$ is equal to the trace of the left regular representation of $A_{n+1,2n-1}$.

Then the following operations give a quantum groupoid structure on H:

$$\begin{aligned}
\Delta(yz) &= (z \otimes y)P_{n-1}, \qquad y \in A_{n+1,2n-1}, \quad z \in A_{1,n-1},\\
\Delta(e_n) &= (1 \otimes w)P_n(1 \otimes w^{-1}),\\
S(h) &= w^{-1}\sigma(h)w,\\
\varepsilon(h) &= \lambda^{-n}\tau(hfw), \quad h \in A,
\end{aligned}$$

where in the last line

$$f = \lambda^{n(n-1)/2}(e_n e_{n-1} \cdots e_1)(e_{n+1} e_n \cdots e_2) \cdots (e_{2n-1} e_{2n-2} \cdots e_n)$$

is the Jones projection corresponding to the n-step basic construction.

The source and target counital subalgebras of $H = A_{1,2n-1}$ are

$$H_s = A_{n+1,2n-1} \quad \text{and} \quad H_t = A_{1,n-1},$$

respectively. The example corresponding to $n = 2$ is a C^*-quantum groupoid of dimension 13 with the antipode having an infinite order (it was studied in detail in [NV2], 7.3).

See Sections 8 and 9 of this survey and the Appendix of [NV3] for the explanation of how quantum groupoids can be constructed from subfactors.

2.8. Other versions of a quantum groupoid. Here we briefly discuss several notions of quantum groupoids that appeared in the literature and relations between them. All these objects generalize both usual groupoid algebras and their duals and Hopf algebras. We apologize for possible non-intentional omissions in the list below.

Face algebras of Hayashi [H1] were defined as Hopf-like objects containing an abelian subalgebra generated by "bases". Non-trivial examples of such objects and applications to monoidal categories and II_1 subfactors were considered in [H2] and [H3]. It was shown in ([N], 5.2) that face algebras are precisely quantum groupoids whose counital subalgebras are abelian.

Generalized Kac algebras of Yamanouchi [Y] were used to characterize C^*-algebras arising from finite groupoids; they are exactly C^*-quantum groupoids with $S^2 = \mathrm{id}$, see ([NV1], 2.5) and ([N], 8.7).

The idea of a quantum transformation groupoid (see 2.6) has been explored in a different form in [Mal1] (resp., [V]), where it was shown that an action of a Hopf algebra on a commutative (resp., noncommutative) algebra gives rise to

a specific quantum groupoid structure on the tensor product of these algebras. This construction served as a strong motivation for [Mal2], where a quite general approach to quantum groupoids has been developed.

The definition of *quantum groupoids* in [Lu] and [Xu] is more general than the one we use. In their approach the existence of the bases is a part of the axioms, and they do not have to be finite-dimensional. These objects give rise to Lie bialgebroids as classical limits. It was shown in [EN] that every quantum groupoid in our sense is a quantum groupoid in the sense of [Lu] and [Xu], but not vice versa.

A suitable functional analytic framework for studying quantum groupoids (not necessarily finite) in the spirit of the multiplicative unitaries of S. Baaj and G. Skandalis [BS] was developed in [EVal], [E2] in connection with depth 2 inclusions of von Neumann algebras. A closely related notion of a *Hopf bimodule* was introduced in [V1] and then studied extensively in [EVal]. In the finite-dimensional case the equivalence of these notions to that of a C^*-quantum groupoid was shown in [V2] and [BSz3] and, respectively, in [V2], [NV1].

3. Integrals and Semisimplicity

3.1. Integrals in quantum groupoids

Definition 3.1.1 ([BNSz], 3.1). A left (right) *integral* in H is an element $l \in H$ ($r \in H$) such that

$$hl = \varepsilon_t(h)l, \qquad (rh = r\varepsilon_s(h)) \qquad \text{for all } h \in H. \tag{3–1}$$

These notions clearly generalize the corresponding notions for Hopf algebras ([M], 2.1.1). We denote $\int_H^l$ (respectively, $\int_H^r$) the space of left (right) integrals in H and by $\int_H = \int_H^l \cap \int_H^r$ the space of two-sided integrals.

An integral in H (left or right) is called *non-degenerate* if it defines a non-degenerate functional on $\widehat{H}$. A left integral l is called *normalized* if $\varepsilon_t(l) = 1$. Similarly, $r \in \int_H^r$ is normalized if $\varepsilon_s(r) = 1$.

A dual notion to that of left (right) integral is the left (right) invariant measure. Namely, a functional $\phi \in \widehat{H}$ is said to be a left (right) *invariant measure* on H if

$$(\mathrm{id} \otimes \phi)\Delta = (\varepsilon_t \otimes \phi)\Delta, \qquad (\text{resp.}, (\phi \otimes \mathrm{id})\Delta = (\phi \otimes \varepsilon_s)\Delta). \tag{3–2}$$

A left (right) invariant measure is said to be normalized if $(\mathrm{id} \otimes \phi)\Delta(1) = 1$ (resp., $(\phi \otimes \mathrm{id})\Delta(1) = 1$).

Example 3.1.2. (i) Let G^0 be the set of units of a finite groupoid G, then the elements $l_e = \sum_{gg^{-1}=e} g\,(e \in G^0)$ span $\int_{kG}^l$ and elements $r_e = \sum_{g^{-1}g=e} g\,(e \in G^0)$ span $\int_{kG}^r$.
(ii) If $H = (kG)^*$ then $\int_H^l = \int_H^r = \mathrm{span}\{p_e,\, e \in G^0\}$.

The next proposition gives a description of the set of left integrals.

PROPOSITION 3.1.3 ([BNSz], 3.2). *The following conditions for $l \in H$ are equivalent*:

(i) $l \in \int_H^l$,
(ii) $(1 \otimes h)\Delta(l) = (S(h) \otimes 1)\Delta(l)$ *for all* $h \in H$,
(iii) $(\mathrm{id} \otimes l)\Delta(\widehat{H}) = \widehat{H}_t$,
(iv) $(\mathrm{Ker}\ \varepsilon_t)l = 0$,
(v) $S(l) \in \int_H^r$.

PROOF. The proof is a straightforward application of Definitions 2.1.1, 3.1.1, Propositions 2.2.1 and 2.3.2, and is left as an exercise for the reader. □

3.2. Hopf modules. Since a quantum groupoid H is both algebra and coalgebra, one can consider modules and comodules over H. As in the theory of Hopf algebras, an H-Hopf module is an H-module which is also an H-comodule such that these two structures are compatible (the action "commutes" with coaction):

DEFINITION 3.2.1. A right H-Hopf module is such a k-vector space M that

(i) M is a right H-module via $m \otimes h \mapsto m \cdot h$,
(ii) M is a right H-comodule via $m \mapsto \rho(m) = m^{(0)} \otimes m^{(1)}$,
(iii) $(m \cdot h)^{(0)} \otimes (m \cdot h)^{(1)} = m^{(0)} \cdot h_{(1)} \otimes m^{(1)} h_{(2)}$

for all $m \in M$ and $h \in H$.

REMARK 3.2.2. Condition (iii) above means that ρ is a right H-module map, where $(M \otimes H)\Delta(1)$ is a right H-module via $(m \otimes g)\Delta(1) \cdot h = (m \cdot h_{(1)}) \otimes g h_{(2)}$, for all $m \in M$ and $g, h \in H$.

EXAMPLE 3.2.3. H itself is an H-Hopf module via $\rho = \Delta$.

EXAMPLE 3.2.4. The dual vector space $\widehat{H}$ becomes a right H-Hopf module:

$$\phi \cdot h = S(h) \rightharpoonup \phi \qquad \phi^{(0)} \langle\, \psi,\, \phi^{(1)} \,\rangle = \psi\phi,$$

for all $\phi, \psi \in \widehat{H}$, $h \in H$. Indeed, we need to check that

$$(\phi \cdot h)^{(0)} \otimes (\phi \cdot h)^{(1)} = \phi^{(0)} \cdot h_{(1)} \otimes \phi^{(1)} h_{(2)}, \qquad \phi \in \widehat{H},\ h \in H.$$

Evaluating both sides against $\psi \in \widehat{H}$ in the first factor we have:

$$\begin{aligned}
(\phi^{(0)} \cdot h_{(1)}) \langle\, \psi,\, \phi^{(1)} h_{(2)} \,\rangle &= (\psi_{(1)}\phi) \cdot h_{(1)} \langle\, \psi_{(2)},\, h_{(2)} \,\rangle \\
&= \psi_{(1)}\phi_{(1)} \langle\, S(\psi_{(2)}\phi_{(2)}),\, h_{(1)} \,\rangle \langle\, \psi_{(3)},\, h_{(2)} \,\rangle \\
&= \psi_{(1)}\phi_{(1)} \langle\, S^{-1}\varepsilon_s(\psi_{(2)})\phi_{(2)},\, S(h) \,\rangle \\
&= \psi\phi_{(1)} \langle\, \phi_{(2)},\, S(h) \,\rangle \\
&= \psi(S(h) \rightharpoonup \phi) = \psi(\phi \cdot h) \\
&= (\phi \cdot h)^{(0)} \langle\, \psi,\, (\phi \cdot h)^{(1)} \,\rangle.
\end{aligned}$$

where we used Proposition 2.3.4 and that $S^{-1}\varepsilon_s(\psi) \in \widehat{H}_t$ for all $\psi \in \widehat{H}$.

The fundamental theorem for Hopf modules over Hopf algebras ([M], 1.9.4) generalizes to quantum groupoids as follows:

THEOREM 3.2.5 ([BNSz], 3.9). *Let M be a right H-Hopf module and*

$$N = \mathrm{Coinv} M = \{m \in M \mid m^{(0)} \otimes m^{(1)} = m1_{(1)} \otimes 1_{(2)}\} \tag{3–3}$$

be the set of its coinvariants. The H_t-module tensor product $N \otimes_{H_t} H$ (where N is a right H_t-submodule) is a right H-Hopf module via

$$(n \otimes h) \cdot g = n \otimes hg, \qquad (n \otimes h)^{(0)} \otimes (n \otimes h)^{(1)} = (n \otimes h_{(1)}) \otimes h_{(2)}, \tag{3–4}$$

for all $h, g \in H$, $n \in N$ and the map

$$\alpha : N \otimes_{H_t} H \to M : n \otimes h \mapsto n \cdot h \tag{3–5}$$

is an isomorphism of right H-Hopf modules.

PROOF. Proposition 2.2.1(iii) implies that the H-Hopf module structure on $N \otimes_{H_t} H$ is well defined, and it is easy to check that α is a well defined homomorphism of H-Hopf modules. We will show that

$$\beta : M \to N \otimes_{H_t} H : m \mapsto (m^{(0)} \cdot S(m^{(1)})) \otimes m^{(2)} \tag{3–6}$$

is an inverse of α. First, we observe that $m^{(0)} \cdot S(m^{(1)}) \in N$ for all $m \in M$, since

$$\begin{aligned}
\rho(m^{(0)} \cdot S(m^{(1)})) &= (m^{(0)} \cdot S(m^{(3)})) \otimes m^{(1)} S(m^{(2)}) \\
&= (m^{(0)} \cdot S(1_{(2)} m^{(1)}) \otimes S(1_{(1)}) \\
&= (m^{(0)} \cdot S(m^{(1)}) 1_{(1)}) \otimes 1_{(2)},
\end{aligned}$$

so β maps to $N \otimes_{H_t} H$. Next, we check that it is both module and comodule map:

$$\begin{aligned}
\beta(m \cdot h) &= m^{(0)} \cdot h_{(1)} S(m^{(1)} h_{(2)}) \otimes m^{(2)} h_{(3)} \\
&= m^{(0)} \cdot \varepsilon_t(h_{(1)}) S(m^{(1)}) \otimes m^{(2)} h_{(2)} \\
&= m^{(0)} \cdot S(m^{(1)} 1_{(1)}) \otimes m^{(2)} 1_{(2)} h = \beta(m) \cdot h, \\
\beta(m^{(0)}) \otimes m^{(1)} &= m^{(0)} \cdot S(m^{(1)}) \otimes m^{(2)} \otimes m^{(3)} \\
&= \beta(m)^{(0)} \otimes \beta(m)^{(1)},
\end{aligned}$$

Finally, we verify that $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$:

$$\begin{aligned}
\alpha \circ \beta(m) &= m^{(0)} \cdot S(m^{(1)}) m^{(2)} = m^{(0)} \cdot \varepsilon_s(m^{(1)}) \\
&= m^{(0)} \cdot 1_{(1)} \varepsilon(m^{(1)} 1_{(2)}) = m^{(0)} \varepsilon(m^{(1)}) = m, \\
\beta \circ \alpha(n \otimes h) &= \beta(n \cdot h) = \beta(n) \cdot h \\
&= n \cdot 1_{(1)} S(1_{(2)}) \otimes 1_{(3)} h = n \otimes h,
\end{aligned}$$

which completes the proof. □

Remark 3.2.6. $\int^l_{\widehat{H}} = \mathrm{Coinv}\widehat{H}$ where $\widehat{H}$ is an H-Hopf module as in Example 3.2.4. Indeed, the condition

$$\lambda^{(1)} \otimes \lambda^{(2)} = \lambda \cdot 1_{(1)} \otimes 1_{(2)}$$

is equivalent to $\phi\lambda = (S(1_{(1)}) \rightharpoonup \lambda)\langle \phi, 1_{(2)} \rangle = \varepsilon_t(\phi)\lambda$.

Corollary 3.2.7. *$\widehat{H} \cong \int^l_{\widehat{H}} \otimes_{H_t} H$ as right H-Hopf modules. In particular, $\int^l_{\widehat{H}}$ is a non-zero subspace of $\widehat{H}$.*

Proof. Take $M = \widehat{H}$ in Theorem 3.2.5 and use Remark 3.2.6. □

3.3. Maschke's Theorem. The existence of left integrals in quantum groupoids leads to the following generalization of Maschke's Theorem, well-known for Hopf algebras ([M], 2.2.1).

Theorem 3.3.1 ([BNSz], 3.13). *Let H be a finite quantum groupoid, then the following conditions are equivalent:*

(i) *H is semisimple,*
(ii) *There exists a normalized left integral l in H,*
(iii) *H is separable.*

Proof. (i) $\Rightarrow$ (ii): Suppose that H is semisimple, then since $\mathrm{Ker}\,\varepsilon_t$ is a left ideal in H we have $\mathrm{Ker}\,\varepsilon_t = Hp$ for some idempotent p. Therefore, $\mathrm{Ker}\,\varepsilon_t(1-p) = 0$ and $l = 1 - p$ is a left integral by Lemma 3.1.3(iv). It is normalized since $\varepsilon_t(l) = 1 - \varepsilon_t(p) = 1$. (ii) $\Rightarrow$ (iii): if l is normalized then $l_{(1)} \otimes S(l_{(2)})$ is a separability element of H by Lemma 3.1.3(ii). (iii) $\Rightarrow$ (i): this is a standard result [P]. □

Corollary 3.3.2. *Let G be a finite groupoid and for every $e \in G^0$, where G^0 denotes the unit space of G, let $|e| = \#\{g \in G \mid gg^{-1} = e\}$. Then kG is semisimple if and only if $|e| \neq 0$ in k for all $e \in G^0$.*

Proof. In the notation of Example 3.1.2(i) the element $l = \sum_{e \in G^0} \frac{1}{|e|} l_e$ is a normalized integral in kG. □

Given a left integral l, one can show ([BNSz], 3.18) that if there exists $\lambda \in \widehat{H}$ such that $\lambda \rightharpoonup l = 1$, then it is unique, it is a left integral in $\widehat{H}$ and $l \rightharpoonup \lambda = \widehat{1}$. Such a pair (l, λ) is called a *dual pair* of left integrals. One defines dual pairs of right integrals in a similar way.

4. Actions and Smash Products

4.1. Module and comodule algebras

Definition 4.1.1. An algebra A is a (left) *H-module algebra* if A is a left H-module via $h \otimes a \to h \cdot a$ and

(1) $h \cdot ab = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,

(2) $h \cdot 1 = \varepsilon_t(h) \cdot 1$.

If A is an H-module algebra we will also say that H acts on A.

Definition 4.1.2. An algebra A is a (right) *H-comodule algebra* if A is a right H-comodule via $\rho : a \to a^{(0)} \otimes a^{(1)}$ and

(1) $\rho(ab) = a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)}$,
(2) $\rho(1) = (\mathrm{id} \otimes \varepsilon_t)\rho(1)$.

It follows immediately that A is a left H-module algebra if and only if A is a right $\widehat{H}$-comodule algebra.

Example 4.1.3. (i) The target counital subalgebra H_t is a trivial H-module algebra via $h \cdot z = \varepsilon_t(hz)$, $h \in H$, $z \in H_t$.
(ii) H is an $\widehat{H}$-module algebra via the dual action $\phi \rightharpoonup h = h_{(1)}\langle \phi, h_{(2)} \rangle$, $\phi \in \widehat{H}$, $h \in H$.
(iii) Let $A = C_H(H_s) = \{a \in H \mid ay = ya \quad \forall y \in H_s\}$ be the centralizer of H_s in H, then A is an H-module algebra via the adjoint action $h \cdot a = h_{(1)}aS(h_{(2)})$.

4.2. Smash products. Let A be an H-module algebra, then a *smash product* algebra $A\#H$ is defined on a k-vector space $A \otimes_{H_t} H$, where H is a left H_t-module via multiplication and A is a right H_t-module via

$$a \cdot z = S^{-1}(z) \cdot a = a(z \cdot 1), \qquad a \in A, z \in H_t,$$

as follows. Let $a \# h$ be the class of $a \otimes h$ in $A \otimes_{H_t} H$, then the multiplication in $A \# H$ is given by the familiar formula

$$(a \# h)(b \# g) = a(h_{(1)} \# b) \# h_{(2)}g, \qquad a, b, \in A,\ h, g \in H,$$

and the unit of $A \# H$ is $1 \# 1$.

Example 4.2.1. H is isomorphic to the trivial smash product algebra $H_t \# H$.

4.3. Duality for actions. An analogue of the Blattner–Montgomery duality theorem for actions of quantum groupoids was proven in [N2]. Let H be a finite quantum groupoid and A be a left H-module algebra. Then the smash product $A \# H$ is a left $\widehat{H}$-module algebra via

$$\phi \cdot (a \# h) = a \# (\phi \rightharpoonup h), \quad \phi \in \widehat{H},\ h \in H,\ a \in A. \tag{4–1}$$

In the case when H is a finite dimensional Hopf algebra, there is an isomorphism $(A \# H) \# \widehat{H} \cong M_n(A)$, where $n = \dim H$ and $M_n(A)$ is an algebra of n-by-n matrices over A [BM]. This result extends to quantum groupoid action in the form $(A \# H) \# \widehat{H} \cong \mathrm{End}(A \# H)_A$, where $A \# H$ is a right A-module via multiplication (note that $A \# H$ is not necessarily a free A-module, so that we have $\mathrm{End}(A \# H)_A \ncong M_n(A)$ in general).

THEOREM 4.3.1 ([N2], 3.1, 3.2). *The map* $\alpha : (A \# H) \# \widehat{H} \to \mathrm{End}(A \# H)_A$ *defined by*

$$\alpha((x \# h) \# \phi)(y \# g) = (x \# h)(y \# (\phi \rightharpoonup g)) = x(h_{(1)} \cdot y) \# h_{(2)}(\phi \rightharpoonup g)$$

for all $x, y \in A$, $h, g \in H$, $\phi \in \widehat{H}$ *is an isomorphism of algebras.*

PROOF. A straightforward (but rather lengthly) computation shows that α is a well defined homomorphism, cf. ([N2], 3.1). Let $\{f_i\}$ be a basis of H and $\{\xi^i\}$ be the dual basis of $\widehat{H}$, i.e., such that $\langle f_i, \xi^j \rangle = \delta_{ij}$ for all i, j, then the element $\sum_i f_i \otimes \xi^i \in H \otimes_k \widehat{H}$ does not depend on the choice of $\{f_i\}$ and the following map

$$\begin{aligned} \beta : \mathrm{End}(A \# H)_A &\to (A \# H) \# \widehat{H} \\ : T &\mapsto \sum_i T(1 \# f_{i(2)})(1 \# S^{-1}(f_{i(1)})) \# \xi^i. \end{aligned}$$

is the inverse of α. □

COROLLARY 4.3.2. $H \# \widehat{H} \cong \mathrm{End}(H)_{H_t}$, *therefore*, $H \# \widehat{H}$ *is a semisimple algebra.*

PROOF. Theorem 4.3.1 for $A = H_t$ shows that H is a projective and finitely generated H_t-module such that $\mathrm{End}(H)_{H_t} \cong H \# \widehat{H}$. Therefore, H_t and $H \# \widehat{H}$ are Morita equivalent. Since H_t is semisimple (as a separable algebra), $H \# \widehat{H}$ is semisimple. □

5. Representation Category of a Quantum Groupoid

Representation categories of quantum groupoids were studied in [BSz2] and [NTV].

5.1. Definition of $\mathrm{Rep}(H)$**.** For a quantum groupoid H let $\mathrm{Rep}(H)$ be the category of representations of H, whose objects are H-modules of finite rank and whose morphisms are H-linear homomorphisms. We show that, as in the case of Hopf algebras, $\mathrm{Rep}(H)$ has a natural structure of a monoidal category with duality.

For objects V, W of $\mathrm{Rep}(H)$ set

$$V \otimes W = \{x \in V \otimes_k W \mid x = \Delta(1) \cdot x\}, \tag{5–1}$$

with the obvious action of H via the comultiplication Δ (here $\otimes_k$ denotes the usual tensor product of vector spaces).

Since $\Delta(1)$ is an idempotent, $V \otimes W = \Delta(1) \cdot (V \otimes_k W)$. The tensor product of morphisms is the restriction of usual tensor product of homomorphisms. The standard associativity isomorphisms $\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ are functorial and satisfy the pentagon condition, since Δ is coassociative. We will suppress these isomorphisms and write simply $U \otimes V \otimes W$.

The target counital subalgebra $H_t \subset H$ has an H-module structure given by $h \cdot z = \varepsilon_t(hz)$, where $h \in H$, $z \in H_t$.

LEMMA 5.1.1. *H_t is the unit object of* $\mathrm{Rep}(H)$.

PROOF. Define a left unit homomorphism $l_V : H_t \otimes V \to V$ by

$$l_V(1_{(1)} \cdot z \otimes 1_{(2)} \cdot v) = z \cdot v, \qquad z \in H_t,\, v \in V.$$

It is an invertible H-linear map with the inverse $l_V^{-1}(v) = S(1_{(1)}) \otimes 1_{(2)} \cdot v$. Moreover, the collection $\{l_V\}_V$ gives a natural equivalence between the functor $H_t \otimes (\ )$ and the identity functor. Similarly, the right unit homomorphism $r_V : V \otimes H_t \to V$ defined by

$$r_V(1_{(1)} \cdot v \otimes 1_{(2)} \cdot z) = S(z) \cdot v, \qquad z \in H_t,\, v \in V,$$

has the inverse $r_V^{-1}(v) = 1_{(1)} \cdot v \otimes 1_{(2)}$ and satisfies the necessary properties. Finally, one can check the triangle axiom, i.e., that

$$(\mathrm{id}_V \otimes l_W) = (r_V \otimes \mathrm{id}_W)$$

for all objects V, W of $\mathrm{Rep}(H)$ and $v \in V$, $w \in W$ ([NTV], 4.1). □

Using the antipode S of H, we can provide $\mathrm{Rep}(H)$ with a duality. For any object V of $\mathrm{Rep}(H)$ define the action of H on $V^* = \mathrm{Hom}_k(V, k)$ by $(h \cdot \phi)(v) = \phi(S(h) \cdot v)$, where $h \in H, v \in V, \phi \in V^*$. For any morphism $f : V \to W$ let $f^* : W^* \to V^*$ be the morphism dual to f (see [T], I.1.8).

For any V in $\mathrm{Rep}(H)$ define the duality homomorphisms

$$d_V : V^* \otimes V \to H_t, \qquad b_V : H_t \to V \otimes V^*$$

as follows. For $\sum_j \phi^j \otimes v_j \in V^* \otimes V$ set

$$d_V(\sum_j \phi^j \otimes v_j) = \sum_j \phi^j(1_{(1)} \cdot v_j)1_{(2)}. \tag{5–2}$$

Let $\{g_i\}_i$ and $\{\gamma^i\}_i$ be bases of V and V^* respectively, dual to each other. The element $\sum_i g_i \otimes \gamma^i$ does not depend on choice of these bases; moreover, for all $v \in V, \phi \in V^*$ one has $\phi = \sum_i \phi(g_i)\gamma^i$ and $v = \sum_i g_i\gamma^i(v)$. Set

$$b_V(z) = z \cdot \sum_i g_i \otimes \gamma^i. \tag{5–3}$$

PROPOSITION 5.1.2. *The category* $\mathrm{Rep}(H)$ *is a monoidal category with duality.*

PROOF. We know already that $\mathrm{Rep}(H)$ is monoidal. One can check ([NTV], 4.2) that d_V and b_V are H-linear. To show that they satisfy the identities

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \qquad (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*},$$

take $\sum_j \phi^j \otimes v_j \in V^* \otimes V, z \in H_t$. Using the isomorphisms l_V and r_V identifying $H_t \otimes V$, $V \otimes H_t$ and V, for all $v \in V$ and $\phi \in V^*$ we have:

$$\begin{aligned}
(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V)(v) &= (\mathrm{id}_V \otimes d_V)(b_V(1_{(1)} \cdot 1) \otimes 1_{(2)} \cdot v) \\
&= (\mathrm{id}_V \otimes d_V)(b_V(1_{(2)}) \otimes S^{-1}(1_{(1)}) \cdot v) \\
&= \sum_i (\mathrm{id}_V \otimes d_V)(1_{(2)} \cdot g_i \otimes 1_{(3)} \cdot \gamma^i \otimes S^{-1}(1_{(1)}) \cdot v) \\
&= \sum_i 1_{(2)} \cdot g_i \otimes (1_{(3)} \cdot \gamma^i)(1'_{(1)} S^{-1}(1_{(1)}) \cdot v) 1'_{(2)} \\
&= 1_{(2)} S(1_{(3)}) 1'_{(1)} S^{-1}(1_{(1)}) \cdot v \otimes 1'_{(2)} = v, \\
(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V)(\phi) &= (d_V \otimes \mathrm{id}_{V^*})(1_{(1)} \cdot \phi \otimes b_V(1_{(2)})) \\
&= \sum_i (d_V \otimes \mathrm{id}_{V^*})(1_{(1)} \cdot \phi \otimes 1_{(2)} \cdot g_i \otimes 1_{(3)} \cdot \gamma^i) \\
&= \sum_i (1_{(1)} \cdot \phi)(1'_{(1)} 1_{(2)} \cdot g_i) 1'_{(2)} \otimes 1_{(3)} \cdot \gamma^i \\
&= 1'_{(2)} \otimes 1_{(3)} 1_{(1)} S(1'_{(1)} 1_{(2)}) \cdot \phi = \phi,
\end{aligned}$$

which completes the proof. □

5.2. Quasitriangular quantum groupoids.

Definition 5.2.1. A quasitriangular quantum groupoid is a pair $(H, \mathcal{R})$ where H is a quantum groupoid and $\mathcal{R} \in \Delta^{\mathrm{op}}(1)(H \otimes_k H)\Delta(1)$ satisfying the following conditions:

$$\Delta^{\mathrm{op}}(h)\mathcal{R} = \mathcal{R}\Delta(h), \tag{5–4}$$

for all $h \in H$, where Δ^{op} denotes the comultiplication opposite to Δ,

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{5–5}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{5–6}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, etc. as usual, and such that there exists $\bar{\mathcal{R}} \in \Delta(1)(H \otimes_k H)\Delta^{\mathrm{op}}(1)$ with

$$\mathcal{R}\bar{\mathcal{R}} = \Delta^{\mathrm{op}}(1), \qquad \bar{\mathcal{R}}\mathcal{R} = \Delta(1). \tag{5–7}$$

Note that $\bar{\mathcal{R}}$ is uniquely determined by $\mathcal{R}$: if $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ are two elements of $\Delta(1)(H \otimes_k H)\Delta^{\mathrm{op}}(1)$ satisfying the previous equation, then

$$\bar{\mathcal{R}} = \bar{\mathcal{R}}\Delta^{\mathrm{op}}(1) = \bar{\mathcal{R}}\mathcal{R}\bar{\mathcal{R}}' = \Delta(1)\bar{\mathcal{R}}' = \bar{\mathcal{R}}'.$$

For any two objects V and W of $\mathrm{Rep}(H)$ define $c_{V,W} : V \otimes W \to W \otimes V$ as the action of R_{21}:

$$c_{V,W}(x) = R^{(2)} \cdot x^{(2)} \otimes R^{(1)} \cdot x^{(1)}, \tag{5–8}$$

where $x = x^{(1)} \otimes x^{(2)} \in V \otimes W$ and $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$.

PROPOSITION 5.2.2. *The family of homomorphisms* $\{c_{V,W}\}_{V,W}$ *defines a braiding in* $\mathrm{Rep}(H)$. *Conversely, if* $\mathrm{Rep}(H)$ *is braided, then there exists* $\mathcal{R}$, *satisfying the properties of Definition* 5.2.1 *and inducing the given braiding.*

PROOF. Note that $c_{V,W}$ is well-defined, since $\mathcal{R}_{21} = \Delta(1)\mathcal{R}_{21}$. To prove the H-linearity of $c_{V,W}$ we observe that

$$\begin{aligned} c_{V,W}(h \cdot x) &= \mathcal{R}^{(2)} h_{(2)} \cdot x^{(2)} \otimes \mathcal{R}^{(1)} h_{(1)} \cdot x^{(1)} \\ &= h_{(1)} \mathcal{R}^{(2)} \cdot x^{(2)} \otimes h_{(2)} \mathcal{R}^{(1)} \cdot x^{(1)} = h \cdot (c_{V,W}(x)). \end{aligned}$$

The inverse of $c_{V,W}$ is given by

$$c_{V,W}^{-1}(y) = \bar{\mathcal{R}}^{(1)} \cdot y^{(2)} \otimes \bar{\mathcal{R}}^{(2)} \cdot y^{(1)}, \qquad \text{where } y = y^{(1)} \otimes y^{(2)} \in W \otimes V,$$

therefore $c_{V,W}$ is is an isomorphism. Finally, we check the braiding identities. Let $x = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \in U \otimes V \otimes W$, then

$$\begin{aligned} &(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)(x) = \\ &= (\mathrm{id}_V \otimes c_{U,W})(\mathcal{R}^{(2)} \cdot x^{(2)} \otimes \mathcal{R}^{(1)} \cdot x^{(1)} \otimes x^{(3)}) \\ &= \mathcal{R}^{(2)} \cdot x^{(2)} \otimes \mathcal{R}'^{(2)} \cdot x^{(3)} \otimes \mathcal{R}'^{(1)} \mathcal{R}^{(1)} \cdot x^{(1)} \\ &= \mathcal{R}^{(2)}_{(1)} \cdot x^{(2)} \otimes \mathcal{R}^{(2)}_{(2)} \cdot x^{(3)} \otimes \mathcal{R}^{(1)} \cdot x^{(1)} = c_{U,V\otimes W}(x). \end{aligned}$$

Similarly, we have $(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = c_{U\otimes V,W}$.

The third equality of this computation shows that the relations of Definition 5.2.1 are equivalent to the braiding identities. □

LEMMA 5.2.3. *Let* $(H, \mathcal{R})$ *be a quasitriangular quantum groupoid. Then* $\mathcal{R}$ *satisfies the quantum Yang–Baxter equation*:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

PROOF. It follows from the first two relations of Definition 5.2.1 that

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = (\mathrm{id} \otimes \Delta^{\mathrm{op}})(\mathcal{R})\mathcal{R}_{23} = \mathcal{R}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \qquad \square$$

REMARK 5.2.4. Let us define two linear maps $\mathcal{R}_1, \mathcal{R}_2 : \widehat{H} \to H$ by

$$\mathcal{R}_1(\phi) = (\mathrm{id} \otimes \phi)(\mathcal{R}), \quad \mathcal{R}_2(\phi) = (\phi \otimes \mathrm{id})(\mathcal{R}), \quad \text{for all } \phi \in \widehat{H}.$$

Then condition (5–5) of Definition 5.2.1 is equivalent to $\mathcal{R}_1$ being a coalgebra homomorphism and algebra anti-homomorphism and condition (5–6) is equivalent to $\mathcal{R}_2$ being an algebra homomorphism and coalgebra anti-homomorphism. In other words, $\mathcal{R}_1 : \widehat{H} \to H^{\mathrm{op}}$ and $\mathcal{R}_2 : \widehat{H} \to H^{\mathrm{cop}}$ are homomorphisms of quantum groupoids.

PROPOSITION 5.2.5. *For any quasitriangular quantum groupoid* $(H, \mathcal{R})$, *we have*:

$$\begin{aligned} (\varepsilon_s \otimes \mathrm{id})(\mathcal{R}) &= \Delta(1), \qquad (\mathrm{id} \otimes \varepsilon_s)(\mathcal{R}) = (S \otimes \mathrm{id})\Delta^{\mathrm{op}}(1), \\ (\varepsilon_t \otimes \mathrm{id})(\mathcal{R}) &= \Delta^{\mathrm{op}}(1), \qquad (\mathrm{id} \otimes \varepsilon_t)(\mathcal{R}) = (S \otimes \mathrm{id})\Delta(1), \\ (S \otimes \mathrm{id})(\mathcal{R}) &= (\mathrm{id} \otimes S^{-1})(\mathcal{R}) = \bar{\mathcal{R}}, \qquad (S \otimes S)(\mathcal{R}) = \mathcal{R}. \end{aligned}$$

Proof. The proof is essentially the same as ([Ma], 2.1.5). □

Proposition 5.2.6. *Let $(H, \mathcal{R})$ be a quasitriangular quantum groupoid. Then*

$$S^2(h) = uhu^{-1}$$

for all $h \in H$, where $u = S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ is an invertible element of H such that

$$u^{-1} = \mathcal{R}^{(2)}S^2(\mathcal{R}^{(1)}), \quad \Delta(u) = \bar{\mathcal{R}}\bar{\mathcal{R}}_{21}(u \otimes u).$$

Likewise, $v = S(u) = \mathcal{R}^{(1)}S(\mathcal{R}^{(2)})$ obeys $S^{-2}(h) = vhv^{-1}$ and

$$v^{-1} = S^2(\mathcal{R}^{(1)})\mathcal{R}^{(2)}, \quad \Delta(v) = \bar{\mathcal{R}}\bar{\mathcal{R}}_{21}(v \otimes v).$$

Proof. Note that $S(\mathcal{R}^{(2)})y\mathcal{R}^{(1)} = S(y)u$ for all $y \in H_s$. Hence, we have

$$\begin{aligned} S(h_{(2)})uh_{(1)} &= S(h_{(2)})S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}h_{(1)} = S(\mathcal{R}^{(2)}h_{(2)})\mathcal{R}^{(1)}h_{(1)} \\ &= S(h_{(1)}\mathcal{R}^{(2)})h_{(2)}\mathcal{R}^{(1)} = S(\mathcal{R}^{(2)})\varepsilon_s(h)\mathcal{R}^{(1)} \\ &= S(\varepsilon_s(h))u, \end{aligned}$$

for all $h \in H$. Therefore,

$$\begin{aligned} uh &= S(1_{(2)})u1_{(2)}h = S(\varepsilon_t(h_{(2)})uh_{(1)} \\ &= S(h_{(2)}S(h_{(3)}))uh_{(1)} = S^2(h_{(3)})S(h_{(2)})uh_{(1)} \\ &= S^2(h_{(2)})S(\varepsilon_s(h_{(1)}))u = S(\varepsilon_s(h_{(1)})S(h_{(2)}))u = S^2(h)u. \end{aligned}$$

The remaining part of the proof follows the lines of ([Ma], 2.1.8). The results for v can be obtained by applying the results for u to the quasitriangular quantum groupoid $(H^{\mathrm{op/cop}}, \mathcal{R})$. □

Definition 5.2.7. The element u defined in Proposition 5.2.6 is called *the Drinfeld element* of H.

Corollary 5.2.8. *The element $uv = vu$ is central and obeys*

$$\Delta(uv) = (\bar{\mathcal{R}}\bar{\mathcal{R}}_{21})^2(uv \otimes uv).$$

The element $uv^{-1} = vu^{-1}$ is group-like and implements S^4 by conjugation.

Proposition 5.2.9. *Given a quasitriangular quantum groupoid $(H, \mathcal{R})$, consider a linear map $F : \widehat{H} \to H$ given by*

$$F : \phi \mapsto (\phi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R}), \qquad \phi \in \widehat{H}. \tag{5–9}$$

Then the range of F belongs to $C_H(H_s)$, the centralizer of H_s.

Proof. Take $y \in H_s$. Then we have:

$$\phi(\mathcal{R}^{(2)}\mathcal{R}'^{(1)})\mathcal{R}^{(1)}\mathcal{R}'^{(2)}y = \phi(\mathcal{R}^{(2)}y\mathcal{R}'^{(1)})\mathcal{R}^{(1)}\mathcal{R}'^{(2)} = \phi(\mathcal{R}^{(2)}\mathcal{R}'^{(1)})y\mathcal{R}^{(1)}\mathcal{R}'^{(2)},$$

therefore $F(\phi) \in C_H(H_s)$, as required. □

Definition 5.2.10 (cf. [Ma], 2.1.12). A quasitriangular quantum groupoid is *factorizable* if the above map $F : \widehat{H} \to C_H(H_s)$ is surjective.

The factorizability of H means that $\mathcal{R}$ is as non-trivial as possible, in contrast to *triangular* quantum groupoids, for which $\mathcal{R}_{21} = \bar{\mathcal{R}}$ and the range of F is equal to H_t.

5.3. The Drinfeld double. To define the *Drinfeld double* $D(H)$ of a quantum groupoid H, consider on the vector space $\widehat{H}^{\mathrm{op}} \otimes_k H$ an associative multiplication

$$(\phi \otimes h)(\psi \otimes g) = \psi_{(2)}\phi \otimes h_{(2)} g \langle\, S(h_{(1)}),\, \psi_{(1)} \,\rangle \langle\, h_{(3)},\, \psi_{(3)} \,\rangle, \tag{5–10}$$

where $\phi, \psi \in \widehat{H}^{\mathrm{op}}$ and $h, g \in H$. Then one can verify that the linear span J of the elements

$$\phi \otimes zh - (\varepsilon \leftharpoonup z)\phi \otimes h, \quad z \in H_t, \tag{5–11}$$

$$\phi \otimes yh - (y \rightharpoonup \varepsilon)\phi \otimes h, \quad y \in H_s, \tag{5–12}$$

is a two-sided ideal in $\widehat{H}^{\mathrm{op}} \otimes_k H$. Let $D(H)$ be the factor-algebra $(\widehat{H}^{\mathrm{op}} \otimes_k H)/J$ and let $[\phi \otimes h]$ denote the class of $\phi \otimes h$ in $D(H)$.

Proposition 5.3.1. *$D(H)$ is a quantum groupoid with the unit $[\varepsilon \otimes 1]$, and comultiplication, counit, and antipode given by*

$$\Delta([\phi \otimes h]) = [\phi_{(1)} \otimes h_{(1)}] \otimes [\phi_{(2)} \otimes h_{(2)}], \tag{5–13}$$

$$\varepsilon([\phi \otimes h]) = \langle\, \varepsilon_t(a),\, \phi \,\rangle, \tag{5–14}$$

$$S([\phi \otimes a]) = [S^{-1}(\phi_{(2)}) \otimes S(h_{(2)})] \langle\, h_{(1)},\, \phi_{(1)} \,\rangle \langle\, S(h_{(3)}),\, \phi_{(3)} \,\rangle. \tag{5–15}$$

Proof. The proof is a straightforward verification that all the structure maps are well-defined and satisfy the axioms of a quantum groupoid, which is carried out in full detail in ([NTV], 6.1). □

Proposition 5.3.2. *The Drinfeld double $D(H)$ has a canonical quasitriangular structure given by*

$$\mathcal{R} = \sum_i [\xi^i \otimes 1] \otimes [\varepsilon \otimes f_i], \qquad \bar{\mathcal{R}} = \sum_j [S^{-1}(\xi_j) \otimes 1] \otimes [\varepsilon \otimes f_j] \tag{5–16}$$

where $\{f_i\}$ and $\{\xi^i\}$ are bases in H and $\widehat{H}$ such that $\langle\, f_i,\, \xi^j \,\rangle = \delta_{ij}$.

Proof. The identities $(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ and $(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ can be written as (identifying $[(\widehat{H})^{\mathrm{op}} \otimes 1]$ with $(\widehat{H})^{\mathrm{op}}$ and $[\varepsilon \otimes H]$ with H):

$$\sum_i \xi^i_{(1)} \otimes \xi^i_{(1)} \otimes f_i = \sum_{ij} \xi^i \otimes \xi^j \otimes f_i f_j,$$

$$\sum_i \xi^i \otimes f_{i(1)} \otimes f_{i(2)} = \sum_{ij} \xi^j \xi^i \otimes f_j \otimes f_i.$$

The above equalities can be verified, e.g., by evaluating both sides against an element $a \in H$ in the third factor (resp., against $\phi \in (\widehat{H})^{\mathrm{op}}$ in the first factor),

see ([Ma], 7.1.1). It is also straightforward to check that $\mathcal{R}$ is an intertwiner between Δ and Δ^{op} and check that $\bar{\mathcal{R}}\mathcal{R} = \Delta(1)$ and $\mathcal{R}\bar{\mathcal{R}} = \Delta^{\mathrm{op}}(1)$, see ([NTV], 6.2). □

Remark 5.3.3. The dual quantum groupoid $\widehat{D(H)}$ consists of all $\sum_k h_k \otimes \phi_k$ in $H \otimes_k \widehat{H}^{\mathrm{op}}$ such that

$$\sum_k (h_k \otimes \phi_k)|_J = 0.$$

The structure operations of $\widehat{D(H)}$ are obtained by dualizing those of $D(H)$:

$$\begin{aligned}
\left(\sum_k h_k \otimes \phi_k\right)\left(\sum_l g_l \otimes \psi_l\right) &= \sum_{kl} h_k g_l \otimes \phi_k \psi_l,\\
1_{\widehat{D(H)}} &= 1_{(2)} \otimes (\varepsilon \leftharpoonup 1_{(1)}),\\
\Delta\left(\sum_k h_k \otimes \phi_k\right) &= \sum_{ijk} (h_{k(2)} \otimes \xi^i \phi_{k(1)} \xi^j) \otimes (S(f_i) h_{k(1)} f_j \otimes \phi_{k(2)}),\\
\varepsilon\left(\sum_k h_k \otimes \phi_k\right) &= \sum_k \varepsilon(h_k)\widehat{\varepsilon}(\phi_k),\\
S\left(\sum_k h_k \otimes \phi_k\right) &= \sum_{ijk} f_i S^{-1}(h_k) S(f_j) \otimes \xi^i S(\phi_k) \xi^j,
\end{aligned}$$

for all $\sum_k h_k \otimes \phi_k$, $\sum_l g_l \otimes \psi_l \in \widehat{D(H)}$, where $\{f_i\}$ and $\{\xi^j\}$ are dual bases.

Corollary 5.3.4 ([NTV], 6.4). *The Drinfeld double $D(H)$ is factorizable in the sense of Definition* 5.2.10.

Proof. One can use the explicit form of the R-matrix (5–16) of $D(H)$ and the description of the dual $\widehat{D(H)}$ to check that in this case the map F from (5–9) is surjective. □

5.4. Ribbon quantum groupoids.

Definition 5.4.1. A ribbon quantum groupoid is a quasitriangular quantum groupoid with an invertible central element $\nu \in H$ such that

$$\Delta(\nu) = \mathcal{R}_{21}\mathcal{R}(\nu \otimes \nu) \quad \text{and} \quad S(\nu) = \nu. \tag{5–17}$$

The element ν is called a *ribbon element* of H.

For an object V of $\mathrm{Rep}(H)$ we define the twist $\theta_V : V \to V$ to be the multiplication by ν:

$$\theta_V(v) = \nu \cdot v, \quad v \in V. \tag{5–18}$$

Proposition 5.4.2. *Let $(H, \mathcal{R}, \nu)$ be a ribbon quantum groupoid. The family of homomorphisms $\{\theta_V\}_V$ defines a twist in the braided monoidal category* $\mathrm{Rep}(H)$ *compatible with duality. Conversely, if $\theta_V(v) = \nu \cdot v$ is a twist in* $\mathrm{Rep}(H)$, *then ν is a ribbon element of H.*

PROOF. Since ν is an invertible central element of H, the homomorphism θ_V is an H-linear isomorphism. The twist identity follows from the properties of ν:

$$c_{W,V}c_{V,W}(\theta_V \otimes \theta_W)(x) = \mathcal{R}_{21}\mathcal{R}(\nu \cdot x^{(1)} \otimes \nu \cdot x^{(2)}) = \Delta(\nu) \cdot x = \theta_{V\otimes W}(x),$$

for all $x = x^{(1)} \otimes x^{(2)} \in V \otimes W$. Clearly, the identity $\mathcal{R}_{21}\mathcal{R}(\nu \otimes \nu) = \Delta(\nu)$ is equivalent to the twist property. It remains to prove that

$$(\theta_V \otimes \mathrm{id}_{V^*})b_V(z) = (\mathrm{id}_V \otimes \theta_{V^*})b_V(z),$$

for all $z \in H_t$, i.e., that

$$\sum_i \nu z_{(1)} \cdot \gamma^i \otimes z_{(2)} \cdot g_i = \sum_i z_{(1)} \cdot \gamma^i \otimes \nu z_{(2)} \cdot g_i,$$

where $\sum_i \gamma^i \otimes g_i$ is the canonical element in $V^* \otimes V$. Evaluating the first factors of the above equality on an arbitrary $v \in V$, we get the equivalent condition:

$$\sum_i (\nu z_{(1)} \cdot \gamma^i)(v) z_{(2)} \cdot g_i = \sum_i (z_{(1)} \cdot \gamma^i)(v) \nu z_{(2)} \cdot g_i,$$

which reduces to $z_{(2)}S(\nu z_{(1)}) \cdot v = S(z_{(1)})\nu z_{(2)} \cdot v$. The latter easily follows from the centrality of $\nu = S(\nu)$ and properties of H_t. □

PROPOSITION 5.4.3. *The category* $\mathrm{Rep}(H)$ *is a ribbon category if and only if* H *is a ribbon quantum groupoid.*

PROOF. Follows from Propositions 5.1.2, 5.2.2, and 5.4.2. □

For any endomorphism f of the object V, we define, following [T], I.1.5, its *quantum trace*

$$\mathrm{tr}_q(f) = d_V c_{V,V^*}(\theta_V f \otimes \mathrm{id}_{V^*})b_V \tag{5–19}$$

with values in $\mathrm{End}(H_t)$ and the *quantum dimension* of V by $\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V)$.

COROLLARY 5.4.4 ([NTV], 7.4). *Let* $(H, \mathcal{R}, \nu)$ *be a ribbon quantum groupoid,* f *be an endomorphism of an object* V *in* $\mathrm{Rep}(H)$. *Then*

$$\mathrm{tr}_q(f)(z) = \mathrm{Tr}\,(S(1_{(1)})u\nu f)z1_{(2)}, \qquad \dim_q(V)(z) = \mathrm{Tr}\,(S(1_{(1)})u\nu)z1_{(2)}, \tag{5–20}$$

where Tr *is the usual trace of an endomorphism, and* u *is the Drinfeld element.*

COROLLARY 5.4.5. *Let* k *be an algebraically closed field of characteristic zero. If the trivial* H*-module* H_t *is irreducible (which happens exactly when* $H_t \cap Z(H) = k$, *i.e., when* H *is connected* ([N1], 3.11, [BNSz], 2.4)), *then* $\mathrm{tr}_q(f)$ *and* $\dim_q(V)$ *are scalars*:

$$\mathrm{tr}_q(f) = (\dim H_t)^{-1}\ \mathrm{Tr}\,(u\nu f), \qquad \dim_q(V) = (\dim H_t)^{-1}\mathrm{Tr}\,(u\nu). \tag{5–21}$$

PROOF. Any endomorphism of an irreducible module is the multiplication by a scalar, therefore, we must have $\mathrm{Tr}\,(S(1_{(1)})u\nu f)1_{(2)} = \mathrm{tr}_q(f)1$. Applying the counit to both sides and using that $\varepsilon(1) = \dim H_t$, we get the result. □

5.5. Towards modular categories. In [RT] a general method of constructing invariants of 3-manifolds from modular Hopf algebras was introduced. Later it became clear that the technique of Hopf algebras can be replaced by a more general technique of modular categories (see [T]). In addition to quantum groups, such categories also arise from skein categories of tangles and, as it was observed by A. Ocneanu, from certain bimodule categories of type II_1 subfactors.

The representation categories of *quantum groupoids* give quite general construction of modular categories. Recall some definitions needed here. Let $\mathcal{V}$ be a ribbon *Ab*-category over k, i.e., such that all $\mathrm{Hom}(V, W)$ are k-vector spaces (for all objects $V, W \in \mathcal{V}$) and both operations $\circ$ and $\otimes$ are k-bilinear.

An object $V \in \mathcal{V}$ is said to be *simple* if any endomorphism of V is multiplication by an element of k. We say that a family $\{V_i\}_{i\in I}$ of objects of $\mathcal{V}$ dominates an object V of $\mathcal{V}$ if there exists a finite set $\{V_{i(r)}\}_r$ of objects of this family (possibly, with repetitions) and a family of morphisms $f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}$ such that $\mathrm{id}_V = \sum_r f_r g_r$.

A modular category ([T], II.1.4) is a pair consisting of a ribbon *Ab*-category $\mathcal{V}$ and a finite family $\{V_i\}_{i\in I}$ of simple objects of $\mathcal{V}$ satisfying four axioms:

(i) There exists $0 \in I$ such that V_0 is the unit object.
(ii) For any $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to V_i^*.
(iii) All objects of $\mathcal{V}$ are dominated by the family $\{V\}_{i\in I}$.
(iv) The square matrix $S = \{S_{ij}\}_{i,j\in I} = \{\mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i})\}_{i,j\in I}$ is invertible over k (here tr_q is the quantum trace in a ribbon category defined by (5–19)).

If a quantum groupoid H is connected and semisimple over an algebraically closed field, modularity of $\mathrm{Rep}(H)$ is equivalent to $\mathrm{Rep}(H)$ being ribbon and such that the matrix $S = \{S_{ij}\}_{i,j\in I} = \{\mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i})\}_{i,j\in I}$, where I is the set of all (equivalent classes of) irreducible representations, is invertible. The following proposition extends a result known for Hopf algebras ([EG], 1.1).

Proposition 5.5.1. *If H is a connected, ribbon, factorizable quantum groupoid over an algebraically closed field k of characteristic zero possessing a normalized two-sided integral, then* $\mathrm{Rep}(H)$ *is a modular category.*

Proof. Here we only need to prove the invertibility of the matrix formed by

$$\begin{aligned} S_{ij} &= \mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i}) \\ &= (\dim H_t)^{-1}\mathrm{Tr}\,((u\nu) \circ c_{V_i,V_j} \circ c_{V_j,V_i}) \\ &= (\dim H_t)^{-1}(\chi_j \otimes \chi_i)((u\nu \otimes u\nu)\mathcal{R}_{21}\mathcal{R}), \end{aligned}$$

where V_i are as above, $I = \{1, \dots n\}$, $\{\chi_j\}$ is a basis in the space $C(H)$ of characters of H (we used above the formula (5–21) for the quantum trace).

It was shown in ([NTV], 5.12, 8.1) that the map $F : \phi \mapsto (\phi \otimes \mathrm{id})(R_{21}R)$ is a linear isomorphism between $C(H) \leftharpoonup u\nu$ and the center $Z(H)$ (here $(\chi \leftharpoonup u\nu)(a) = \chi(u\nu a)\ \forall a \in H,\ \chi \in C(H)$). So, there exists an invertible matrix

$T = (T_{ij})$ representing the map F in the bases $\{\chi_j\}$ of $C(H)$ and $\{e_i\}$ of $Z(H)$, i.e., such that $F(\chi_j \leftharpoonup u\nu) = \sum_i T_{ij} e_i$. Then

$$\begin{aligned} S_{ij} &= (\dim H_t)^{-1} \chi_i(u\nu F(\chi_j \leftharpoonup u\nu)) = (\dim H_t)^{-1} \sum_k T_{kj} \chi_i(u\nu e_k) \\ &= (\dim H_t)^{-1} (\dim V_i) \chi_i(u\nu) T_{ij}. \end{aligned}$$

Therefore, $S = DT$, where $D = \operatorname{diag}\{(\dim H_t)^{-1}(\dim V_i)\chi_i(u\nu)\}$. Theorem 7.2.2 below shows that the existence of a normalized two-sided integral in H is equivalent to H being semisimple and possessing an invertible element g such that $S^2(x) = gxg^{-1}$ for all $x \in H$ and $\chi(g^{-1}) \neq O$ for all irreducible characters χ of H. Then $u^{-1}g$ is an invertible central element of H and $\chi_i(u^{-1}) \neq 0$ for all χ_i. By Corollary 5.2.8, $uS(u) = c$ is invertible central, therefore $\chi_i(u) = \chi_i(c)\chi_i(S(u^{-1}))) \neq 0$. Hence, $\chi_i(u\nu) \neq 0$ for all i and D is invertible. □

6. Twisting and Dynamical Quantum Groups

In this section we present a generalization of the Drinfeld twisting construction to quantum groupoids developed in [NV1], [Xu], and [EN]. We show that *dynamical* twists of Hopf algebras give rise to quantum groupoids. An important concrete example is given by dynamical quantum groups at roots of 1 [EN], which are dynamical deformations of the Drinfeld–Jimbo–Lusztig quantum groups $U_q(\mathfrak{g})$. The resulting quantum groupoids turn out to be selfdual, which is a fundamentally new property, not satisfied by $U_q(\mathfrak{g})$.

Most of the material of this section is taken from [EN].

6.1. Twisting of quantum groupoids. We describe the procedure of constructing new quantum groupoids by twisting a comultiplication. Twisting of Hopf algebroids without an antipode was developed in [Xu] and a special case of twisting of $*$-quantum groupoids was considered in [NV1].

Definition 6.1.1. A *twist* for a quantum groupoid H is a pair $(\Theta, \bar{\Theta})$, with

$$\Theta \in \Delta(1)(H \otimes_k H), \quad \bar{\Theta} \in (H \otimes_k H)\Delta(1), \quad \text{and} \quad \Theta\bar{\Theta} = \Delta(1) \tag{6–1}$$

satisfying the following axioms:

$$(\varepsilon \otimes \mathrm{id})\Theta = (\mathrm{id} \otimes \varepsilon)\Theta = (\varepsilon \otimes \mathrm{id})\bar{\Theta} = (\mathrm{id} \otimes \varepsilon)\bar{\Theta} = 1, \tag{6–2}$$

$$(\Delta \otimes \mathrm{id})(\Theta)(\Theta \otimes 1) = (\mathrm{id} \otimes \Delta)(\Theta)(1 \otimes \Theta), \tag{6–3}$$

$$(\bar{\Theta} \otimes 1)(\Delta \otimes \mathrm{id})(\bar{\Theta}) = (1 \otimes \bar{\Theta})(\mathrm{id} \otimes \Delta)(\bar{\Theta}), \tag{6–4}$$

$$(\Delta \otimes \mathrm{id})(\bar{\Theta})(\mathrm{id} \otimes \Delta)(\Theta) = (\Theta \otimes 1)(1 \otimes \bar{\Theta}), \tag{6–5}$$

$$(\mathrm{id} \otimes \Delta)(\bar{\Theta})(\Delta \otimes \mathrm{id})(\Theta) = (1 \otimes \Theta)(\bar{\Theta} \otimes 1). \tag{6–6}$$

Note that for ordinary Hopf algebras the above notion coincides with the usual notion of twist and each of the four conditions (6–3) – (6–6) implies the other

three. But since Θ and $\bar\Theta$ are, in general, not invertible we need to impose all of them.

The next Proposition extends Drinfeld's twisting construction to the case of quantum groupoids.

PROPOSITION 6.1.2. *Let $(\Theta,\bar\Theta)$ be a twist for a quantum groupoid H. Then there is a quantum groupoid H_Θ having the same algebra structure and counit as H with a comultiplication and antipode given by*

$$\Delta_\Theta(h)=\bar\Theta\Delta(h)\Theta,\qquad S_\Theta(h)=v^{-1}S(h)v, \tag{6–7}$$

for all $h\in H_\Theta$, where $v=m(S\otimes\mathrm{id})\Theta$ is invertible in H_Θ.

PROOF. The proof is a straightforward verification, see [EN] for details. □

REMARK 6.1.3. (a) One can check that the counital maps of the twisted quantum groupoid H_Θ are given by

$$(\varepsilon_t)_\Theta(h)=\varepsilon(\Theta^{(1)}h)\Theta^{(2)},\qquad (\varepsilon_s)_\Theta(h)=\bar\Theta^{(1)}\varepsilon(h\bar\Theta^{(2)}).$$

(b) It is possible to generalize the above twisting construction by weakening the counit condition (6–2) and requiring only the existence of $u,w\in H$ such that

$$\varepsilon(\Theta^{(1)}w)\Theta^{(2)}=\Theta^{(1)}\varepsilon(\Theta^{(2)}w)=1,\qquad \varepsilon(u\bar\Theta^{(1)})\bar\Theta^{(2)}=\bar\Theta^{(1)}\varepsilon(u\bar\Theta^{(2)})=1. \tag{6–8}$$

Then the counit of the twisted quantum groupoid H_Θ is

$$\varepsilon_\Theta(h)=\varepsilon(uhw),\qquad h\in H. \tag{6–9}$$

(c) If $(\Theta,\bar\Theta)$ is a twist for H and $x\in H$ is an invertible element such that $\varepsilon_t(x)=\varepsilon_s(x)=1$ then $(\Theta^x,\bar\Theta^x)$, where

$$\Theta^x=\Delta(x)^{-1}\Theta(x\otimes x)\quad\text{ and }\quad \bar\Theta^x=(x^{-1}\otimes x^{-1})\bar\Theta\Delta(x),$$

is also a twist for H. The twists $(\Theta,\bar\Theta)$ and $(\Theta^x,\bar\Theta^x)$ are called *gauge equivalent* and x is called a *gauge transformation*. Given such an x, the map $h\mapsto x^{-1}hx$ is an isomorphism between quantum groupoids H_Θ and H_{Θ^x}.

(d) A twisting of a quasitriangular quantum groupoid is again quasitriangular. Namely, if $(\Theta,\bar\Theta)$ is a twist and $(\mathcal{R},\bar{\mathcal{R}})$ is a quasitriangular structure for H then the quasitriangular structure for H_Θ is given by $(\bar\Theta_{21}\mathcal{R}\Theta,\bar\Theta\bar{\mathcal{R}}\Theta_{21})$. The proof of this fact is exactly the same as for Hopf algebras.

EXAMPLE 6.1.4. We will show that for every (non-commutative) separable algebra B over an algebraically closed field k of characteristic zero there is a family of quantum groupoid structures H_q on the vector space $B^{\mathrm{op}}\otimes B$ considered in ([BSz2], 5.2) that can be understood in terms of twisting. Let $H=H_{1_B}$ be a quantum groupoid with the following structure operations:

$$\Delta(b\otimes c)=(b\otimes e^{(1)})\otimes(e^{(2)}\otimes c),\qquad \varepsilon(b\otimes c)=\omega(bc),\qquad S(b\otimes c)=c\otimes b,$$

for all $b, c \in B$, where $e = e^{(1)} \otimes e^{(2)}$ is the symmetric separability idempotent of B and ω is defined by the condition $(\omega \otimes \mathrm{id})e = 1$. Note that $S^2 = \mathrm{id}$ and H, as an algebra, is generated by its counital subalgebras, i.e., it is dual to an *elementary* quantum groupoid of ([NV1], 3.2).

Observe that the pair

$$\Theta = (1 \otimes e^{(1)}q^{-1}) \otimes (e^{(2)} \otimes 1) \qquad \bar{\Theta} = (1 \otimes e^{(1)}) \otimes (e^{(2)} \otimes 1) = \Delta(1),$$

where q is an invertible element of B with $e^{(1)}q^{-1}e^{(2)} = 1$, satisfies (6–3) – (6–6) and that conditions (6–8) hold for $u = 1$ and $w = q$. According to Proposition 6.1.2 and Remark 6.1.3(b), these data define a twisting of H such that the comultiplication and antipode of the twisted quantum groupoid are given by

$$\Delta(b \otimes c) = (b \otimes e^{(1)}q^{-1}) \otimes (e^{(2)} \otimes c), \quad \varepsilon(b \otimes c) = \omega(qcb), \quad S(b \otimes c) = qcq^{-1} \otimes b,$$

for all $b, c \in B$. The target and source counital subalgebras of H_q are $B^{\mathrm{op}} \otimes 1$ and $1 \otimes B$. The square of the antipode is implemented by $g_q = q \otimes q$. Since $\mathrm{Ad}\, g_q$ is an invariant of H_q, this example shows that there can be uncountably many non-isomorphic semisimple quantum groupoids with the same underlying algebra (for noncommutative B).

6.2. Dynamical twists of Hopf algebras. We describe a method of constructing twists of quantum groupoids, which is a finite-dimensional modification of the construction proposed in [Xu] and is dual to that of [EV], cf. [EN].

Dynamical twists first appeared in the work of Babelon [Ba], see also [BBB].

Let U be a Hopf algebra over an algebraically closed field k" and let $A = \mathrm{Map}(\mathbb{T}, k)$ be a commutative and cocommutative Hopf algebra of functions on a finite Abelian group $\mathbb{T}$ which is a Hopf subalgebra of U. Let P_μ, $\mu \in \mathbb{T}$ be the minimal idempotents in A.

Definition 6.2.1. We say that an element x in $U^{\otimes n}$, $n \geq 1$, has *zero weight* if x commutes with $\Delta^n(a)$ for all $a \in A$, where $\Delta^n : A \to A^{\otimes n}$ is the iterated comultiplication.

Definition 6.2.2. An invertible, zero-weight $U^{\otimes 2}$-valued function $J(\lambda)$ on $\mathbb{T}$ is called a *dynamical twist* for U if it satisfies the following functional equations:

$$(\Delta \otimes \mathrm{id})J(\lambda)(J(\lambda + h^{(3)}) \otimes 1) = (\mathrm{id} \otimes \Delta)J(\lambda)(1 \otimes J(\lambda)), \tag{6–10}$$

$$(\varepsilon \otimes \mathrm{id})J(\lambda) = (\mathrm{id} \otimes \varepsilon)J(\lambda) = 1. \tag{6–11}$$

Here and in what follows the notation $\lambda + h^{(i)}$ means that the argument λ is shifted by the weight of the i-th component, e.g., $J(\lambda + h^{(3)}) = \sum_\mu J(\lambda + \mu) \otimes P_\mu \in U^{\otimes 2} \otimes_k A$.

Note that for every fixed $\lambda \in \mathbb{T}$ the element $J(\lambda) \in U \otimes U$ does not have to be a twist for U in the sense of Drinfeld. It turns out that an appropriate object for which J defines a twisting is a certain quantum groupoid that we describe next.

Observe that the simple algebra $\mathrm{End}_k(A)$ has a natural structure of a cocommutative quantum groupoid given as follows (cf. Section 2.5):

Let $\{E_{\lambda\mu}\}_{\lambda,\mu\in\mathbb{T}}$ be a basis of $\mathrm{End}_k(A)$ such that

$$(E_{\lambda\mu}f)(\nu) = \delta_{\mu\nu}f(\lambda), \qquad f \in A,\ \lambda,\mu,\nu \in \mathbb{T}, \tag{6–12}$$

then the comultiplication, counit, and antipode of $\mathrm{End}_k(A)$ are given by

$$\Delta(E_{\lambda\mu}) = E_{\lambda\mu} \otimes E_{\lambda\mu}, \quad \varepsilon(E_{\lambda\mu}) = 1, \quad S(E_{\lambda\mu}) = E_{\mu\lambda}. \tag{6–13}$$

Define the tensor product quantum groupoid $H = \mathrm{End}_k(A) \otimes_k U$ and observe that the elements

$$\Theta = \sum_{\lambda\mu} E_{\lambda\lambda+\mu} \otimes E_{\lambda\lambda}P_\mu \quad \text{and} \quad \bar{\Theta} = \sum_{\lambda\mu} E_{\lambda+\mu\lambda} \otimes E_{\lambda\lambda}P_\mu \tag{6–14}$$

define a twist for H. Thus, according to Proposition 6.1.2, $H_\Theta = (\mathrm{End}_k(A) \otimes_k U)_\Theta$ becomes a quantum groupoid . It is non-commutative, non-cocommutative, and not a Hopf algebra if $|\mathbb{T}| > 1$.

It was shown in [EN], following [Xu], that H_Θ can be further twisted by means of a dynamical twist $J(\lambda)$ on U. Namely, if $J(\lambda)$ is a dynamical twist on U embedded in $H \otimes_k H$ as

$$J(\lambda) = \sum_{\lambda} E_{\lambda\lambda}J^{(1)}(\lambda) \otimes E_{\lambda\lambda}J^{(2)}(\lambda), \tag{6–15}$$

then the pair $(F(\lambda), \bar{F}(\lambda))$, where

$$F(\lambda) = J(\lambda)\Theta \quad \text{and} \quad \bar{F}(\lambda) = \bar{\Theta}J^{-1}(\lambda)$$

defines a twist for $H = \mathrm{End}_k(A) \otimes_k U$. Thus, every dynamical twist $J(\lambda)$ for a Hopf algebra U gives rise to a quantum groupoid $H_J = H_{J(\lambda)\Theta}$.

Remark 6.2.3. According to Proposition 6.1.2, the antipode S_J of H_J is given by $S_J(h) = v^{-1}S(h)v$ for all $h \in H_J$, where S is the antipode of H and

$$v = \sum_{\lambda\mu} E_{\lambda+\mu\lambda}(S(J^{(1)})J^{(2)})(\lambda)P_\mu.$$

Remark 6.2.4. There is a procedure dual to the one described above. It was shown in [EV] that given a dynamical twist $J(\lambda)$ it is possible to deform the multiplication on the vector space $D = \mathrm{Map}(\mathbb{T} \times \mathbb{T}, k) \otimes_k U^*$ and obtain a *dynamical quantum group* D_J. The relation between H_J and D_J in the case when $\dim U < \infty$ was established in [EN], where it was proven that D_J is isomorphic to H_J^{*op}.

6.3. Dynamical twists for $U_q(\mathfrak{g})$ at roots of 1. Suppose that $\mathfrak{g}$ is a simple Lie algebra of type A, D or E and q is a primitive ℓth root of unity in k, where $\ell \geq 3$ is odd and coprime with the determinant of the Cartan matrix $(a_{ij})_{ij=1,\dots,m}$ of $\mathfrak{g}$.

Let $U = U_q(\mathfrak{g})$ be the corresponding quantum group which is a finite dimensional Hopf algebra with generators E_i, F_i, K_i, where $i = 1, \dots, m$ and relations as in [L].

Let $\mathbb{T} \cong (\mathbb{Z}/\ell\mathbb{Z})^m$ be the abelian group generated by K_i, $i = 1, \dots, m$. For any m-tuple of integers $\lambda = (\lambda_1, \dots, \lambda_m)$ we will write $K_\lambda = K_1^{\lambda_1} \dots K_m^{\lambda_m} \in \mathbb{T}$.

Let $\mathcal{R}$ be the universal R-matrix of $U_q(\mathfrak{g})$ and Ω be the "Cartan part" of $\mathcal{R}$ ([T]).

For arbitrary non-zero constants $\Lambda_1, \dots, \Lambda_m$ define a Hopf algebra automorphism Λ of U by setting

$$\Lambda(E_i) = \Lambda_i E_i, \quad \Lambda(F_i) = \Lambda_i^{-1} F_i, \quad \text{and} \quad \Lambda(K_i) = K_i \quad \text{for all } i = 1, \dots, m.$$

We will say that $\Lambda = (\Lambda_1, \dots, \Lambda_m)$ is *generic* if the spectrum of Λ does not contain ℓth roots of unity.

Note that the algebra U is $\mathbb{Z}$-graded with

$$\deg(E_i) = 1, \quad \deg(F_i) = -1, \quad \deg(K_i) = 0, \qquad i = 1, \dots, m, \tag{6–16}$$

and $\deg(XY) = \deg(X) + \deg(Y)$ for all X and Y. Of course, there are only finitely many non-zero components of U since it is finite dimensional.

Let U_+ be the subalgebra of U generated by the elements E_i, K_i, $i = 1, \dots, m$, U_- be the subalgebra generated by F_i, K_i, $i = 1, \dots, m$, and $I_\pm$ be the kernels of the projections from $U_\pm$ to the elements of zero degree.

The next Proposition was proven in [ABRR], [ES], [ESS] for generic values of q and in [EN] for q a root of unity:

Proposition 6.3.1. *For every generic Λ there exists a unique element $J(\lambda) \in 1 + I_+ \otimes I_-$ that satisfies the following ABRR relation* [ABRR], [ES], [ESS]:

$$(\operatorname{Ad} K_\lambda \circ \Lambda \otimes \mathrm{id})(\mathcal{R} J(\lambda) \Omega^{-1}) = J(\lambda), \quad \lambda \in \mathbb{T}. \tag{6–17}$$

The element

$$\mathcal{J}(\lambda) = J(2\lambda + h^{(1)} + h^{(2)}) \tag{6–18}$$

is a dynamical twist for $U_q(\mathfrak{g})$ in the sense of Definition 6.2.2.

Proof. The existence and uniqueness of the solution of (6–17) and the fact that $\mathcal{J}(\lambda)$ satisfies (6–10) are established by induction on the degree of the first component of $J(\lambda)$, see ([EN], 5.1). □

Remark 6.3.2. (a) The reason for introducing the "shift" automorphism Λ is to avoid singularities in equation [ABRR]. Thus, we have a family of dynamical deformations of $U_q(\mathfrak{g})$ (and, therefore, a family of quantum groupoids depending on m parameters).

(b) One can generalize the above construction and associate a dynamical twist

$\mathcal{J}_T(\lambda)$ with any *generalized Belavin–Drinfeld triple* which consists of subsets Γ_1, Γ_2 of the set $\Gamma = (\alpha_1, \dots, \alpha_m)$ of simple roots of $\mathfrak{g}$ together with an inner product preserving bijection $T : \Gamma_1 \to \Gamma_2$, see [EN], [ESS].

Example 6.3.3. Let us give an explicit expression for the twists $J(\lambda)$ and $\mathcal{J}(\lambda)$ in the case $\mathfrak{g} = sl(2)$. $U_q(\mathfrak{g})$ is then generated by E, F, K with the standard relations. The element analogous to $J(\lambda)$ for generic q was computed already in [Ba] (see also [BBB]). If we switch to our conventions, this element will take the form

$$J(\lambda) = \sum_{n=0}^{\infty} q^{-n(n+1)/2} \frac{(1-q^2)^n}{[n]_q!} (E^n \otimes F^n) \prod_{\nu=1}^{n} \frac{\Lambda q^{2\lambda}}{1 - \Lambda q^{2\lambda+2\nu}(K \otimes K^{-1})}.$$

It is obvious that the formula for q being a primitive ℓ-th root of unity is simply obtained by truncating this formula:

$$J(\lambda) = \sum_{n=0}^{\ell-1} q^{-n(n+1)/2} \frac{(1-q^2)^n}{[n]_q!} (E^n \otimes F^n) \prod_{\nu=1}^{n} \frac{\Lambda q^{2\lambda}}{1 - \Lambda q^{2\lambda+2\nu}(K \otimes K^{-1})}.$$

Therefore,

$$\mathcal{J}(\lambda) = \sum_{n=0}^{\ell-1} q^{-n(n+1)/2} \frac{(1-q^2)^n}{[n]_q!} (E^n \otimes F^n) \prod_{\nu=1}^{n} \frac{\Lambda q^{4\lambda} K \otimes K}{1 - \Lambda q^{4\lambda+2\nu}(K^2 \otimes 1)}.$$

An important new property of the resulting quantum groupoids $H_{\mathcal{J}} = U_q(\mathfrak{g})_{\mathcal{J}}$, compared with quantum groups $U_q(\mathfrak{g})$ is their selfduality. A twisting of a quasi-triangular quantum groupoid is again quasitriangular (Remark 6.1.3(d)), so the twisted R-matrix

$$\mathcal{R}(\lambda) = \bar{\Theta}_{21} \mathcal{R} \Theta = \sum_{\lambda\mu\nu} E_{\lambda\lambda+\nu} P_\mu \mathcal{R}^{\mathcal{J}(1)}(\lambda) \otimes E_{\lambda+\mu\lambda} \mathcal{R}^{\mathcal{J}(2)}(\lambda) P_\nu,$$

where $\mathcal{R}^J(\lambda) = \mathcal{J}_{21}^{-1}(\lambda) R \mathcal{J}(\lambda)$, establishes a homomorphism between $D_{\mathcal{J}} = H_{\mathcal{J}}^{*op}$ and $H_{\mathcal{J}}$. One can show that the image of $\mathcal{R}(\lambda)$ contains all the generators of $U_q(\mathfrak{g})$, i.e., that the above homomorphism is in fact an isomorphism; see ([EN], 5.3).

Remark 6.3.4. The same selfduality result holds for any generalized Belavin–Drinfeld triple for which T is an automorphism of the Dynkin diagram of $\mathfrak{g}$ ([EN], 5.4).

7. Semisimple and C^*-Quantum Groupoids

7.1. Definitions.

Definition 7.1.1. A quantum groupoid is said to be *semisimple* (resp., $*$- *or* C^**-quantum groupoid*) if its algebra H is semisimple (resp., a $*$-algebra over a field k with involution, or finite-dimensional C^*-algebra over the field $\mathbb{C}$ such that Δ is a $*$-homomorphism).

Groupoid algebras and their duals give examples of commutative and cocommutative semisimple quantum groupoids (Corollary 3.3.2), which are C^*-quantum groupoids if the ground field is $\mathbb{C}$ (in which case $g^* = g^{-1}$ for all $g \in G$).

We will describe the class of quantum groupoids possessing *Haar integrals*, i.e., normalized two-sided integrals (note that if such an integral exists then it is unique and is an S-invariant idempotent).

Definition 7.1.2 [W, 1.2.1]. Given an inclusion of unital k-algebras $N \subset M$, a *conditional expectation* $E : M \to N$ is an $N - N$ bimodule map E such that $E(1) = 1$. A *quasi-basis* for E is an element $\sum_i x_i \otimes y_i \in M \otimes_k M$ such that

$$\sum_i E(mx_i)y_i = m = \sum_i x_i E(y_i m), \qquad \text{for all } m \in M.$$

One can check that $\operatorname{Index} E = \sum_i x_i y_i \in Z(M)$ does not depend on a choice of a quasi-basis; this element is called the *index* of E.

One can apply this definition to a non-degenerate functional f on H. If (l, λ) is a dual pair of left integrals, then $l_{(2)} \otimes S^{-1}(l_{(1)})$ is a quasi-basis for λ and

$$\operatorname{Index} \lambda = S^{-1} \cdot \varepsilon_t(l) \in H_s \cap Z(H).$$

So a non-degenerate left integral is normalized if and only if its dual has index 1.

7.2. Existence of the Haar integral. Given a dual bases $\{f_i\}$ and $\{\xi^i\}$ of H and $\widehat{H}$, respectively, let us consider the following canonical element

$$\chi = \sum_i (\xi^i \leftharpoonup S^{-2}(f_i)).$$

One can show that χ is a left integral in $\widehat{H}$ such that $\chi(xy) = \chi(yS^2(x))$ for all $x, y \in H$ and $l \rightharpoonup \chi = \widehat{S}^2(\widehat{1} \leftharpoonup l)$ for any left integral l in H (see [BNSz], 3.25).

Lemma 7.2.1 ([BNSz], 3.26). (i) *The Haar integral $h \in H$ exists if and only if the above mentioned χ is non-degenerate, in which case (h, χ) is a dual pair of left integrals.*
(ii) *A left integral l is a Haar integral if and only if $\varepsilon_s(l) = 1$.*

Proof. (ii) If $\varepsilon_s(l) = 1$, then by $l \rightharpoonup \chi = \widehat{S}^2(\widehat{1} \leftharpoonup l)$ one has $l \rightharpoonup \chi = \widehat{1}$. Therefore, (h, χ) is a dual pair of non-degenerate left integrals. The property $\chi(xy) = \chi(yS^2(x))$ is equivalent to that the quasi-basis of χ satisfies

$$l_{(2)} \otimes S^{-1}(l_{(1)}) = S(l_{(1)}) \otimes l_{(2)},$$

from where $\Delta(l) = \Delta(S(l))$ and $l = S(l)$. Furthermore, $\varepsilon_t(l) = \varepsilon_t(S(l)) = S \cdot \varepsilon_s(l) = 1$. Thus, l is a Haar integral. The inverse statement is obvious.
(i) The "only if" part follows from the proof of (ii). If χ is non-degenerate and h is its dual left integral, then, as above, $S(h) = h$; so h is a two-sided integral and since $l \rightharpoonup \chi = \widehat{S}^2(\widehat{1} \leftharpoonup l)$, it is normalized. $\square$

THEOREM 7.2.2 ([BNSz], 3.27). *Let H be a finite quantum groupoid over an algebraically closed field k. Then the following conditions are equivalent*:

(i) *There exists a Haar integral*,
(ii) *H is semisimple and there exists an invertible element $g \in H$ such that $gxg^{-1} = S^2(x)$ for all $x \in H$ and $Tr(\pi_\alpha(g^{-1})) \neq 0$ for all irreducible representations π_α of H (here Tr is the usual trace on a matrix algebra).*

PROOF. The assumption on k is used only to ensure that $H = \oplus_\alpha M_{n_\alpha}(k)$, once knowing that it is semisimple, so that there is a k-basis of matrix units $\{e^\alpha_{ij}\}$ in H.

(ii) $\Longrightarrow$ (i): Recall that any trace τ on H is completely determined by its *trace vector* $\tau_\alpha = \tau(1_\alpha)$. Now let $\tau : H \to k$ be the trace with trace vector $\tau_\alpha = \mathrm{tr}\pi_\alpha(g^{-1})$. Then τ is non-degenerate and has a quasi-basis

$$\sum_i x_i \otimes y_i = \sum_\alpha \frac{1}{\tau_\alpha} \sum_{i,j=1}^{n_\alpha} e^\alpha_{ij} \otimes e^\alpha_{ji}.$$

Notice that $\sum_i x_i g^{-1} y_i = 1$. It is straightforward to verify that $\chi' = g \rightharpoonup \tau$ coincides with χ, so χ is non-degenerate and therefore its dual left integral l has $\varepsilon_s(l) = 1$ by $l \rightharpoonup \chi = \widehat{S}^2(\widehat{1} \leftharpoonup l)$. Thus, l is a Haar integral.

(i) $\Longrightarrow$ (ii): If h is a Haar integral, then H is semisimple by Theorem 3.3.1. Let τ be a non-degenerate trace on H, then there exists a unique $i \in H$ such that $i \rightharpoonup \tau = \widehat{1} = \tau \leftharpoonup i$. One can verify that i is a two-sided non-degenerate integral in H and that $S(i) = i$ (see [BNSz], I.3.21). To prove that S^2 is inner, it is enough to construct a non-degenerate functional χ on H such that $\chi(xy) = \chi(yS^2(x))$ for all $x, y \in H$. But the proof of Lemma 7.2.1 shows that this is the case for the dual left integral to i. □

REMARK 7.2.3. (a) There is a unique normalization of g from Theorem 7.2.2(ii) such that the following conditions hold ([BNSz], 4.4):

(i) $\mathrm{tr}(\pi_\alpha(g^{-1})) = \mathrm{tr}(\pi_\alpha(g))$ for all irreducible representations π_α of H (here tr is a usual trace on a matrix algebra);
(ii) $S(g) = g^{-1}$.

One can show that such a g is *group-like*, i.e.,

$$\Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g).$$

The element g implementing S^2 and satisfying normalization conditions (i) and (ii) is called the *canonical group-like element* of H.

(b) Since $\chi = g \rightharpoonup \tau$ is the dual left integral to h, its quasi-basis $\sum_i x_i g^{-1} \otimes y_i$ equals to $h_{(2)} \otimes S^{-1}(h_{(1)})$, which implies the formula

$$(S \otimes \mathrm{id})\Delta(h) = \sum_i x_i \otimes g^{-1} y_i = \sum_\alpha 1/\tau_\alpha \sum_{ij} e^\alpha_{ij} g^{-1/2} \otimes g^{-1/2} e^\alpha_{ji}.$$

A dual notion to that of the Haar integral is the Haar measure. Namely, a functional $\phi \in \widehat{H}$ is said to be a *Haar measure* on H if it is a normalized left and right invariant measure and $\phi \circ S = \phi$. Any of the equivalent conditions of Theorem 7.2.1 is also equivalent to existence (and uniqueness) of the Haar measure on $\widehat{H}$.

7.3. C^*-quantum groupoids. Definition 7.1.1 and the uniqueness of the unit, counit and the antipode (see Proposition 2.3.1) imply that

$$1^* = 1, \quad \varepsilon(x^*) = \overline{\varepsilon(x)}, \quad (S \circ *)^2 = \mathrm{id}$$

for all x in any $*$-quantum groupoid H. It is also easy to check the relations

$$\varepsilon_t(x)^* = \varepsilon_t(S(x)^*), \qquad \varepsilon_t(x)^* = \varepsilon_t(S(x)^*),$$

which imply that H_t and H_s are $*$-subalgebras, and to show that the dual, $\widehat{H}$, is also a $*$-quantum groupoid with respect to the $*$-operation

$$\langle\, x,\, \phi^* \,\rangle = \overline{\langle\, S(x)^*,\, \phi \,\rangle} \qquad \text{for all } \phi \in \widehat{H},\ x \in H. \tag{7–1}$$

The $*$-operation allows to simplify the axioms of a quantum groupoid (cf. the axioms used in [NV1], [N1]). The second parts of equalities (2–1) and (2–2) of Definition 2.1.1 follow from the rest of the axioms, also $S * \mathrm{id} = \varepsilon_s$ is equivalent to $\mathrm{id} * S = \varepsilon_t$. Alternatively, under the condition that the antipode is both algebra and coalgebra anti-homomorphism, the axioms (2–1) and (2–2) and can be replaced by the identities of Proposition 2.2.1 (ii) and (v) involving the target counital map.

Theorem 7.3.1 [BNSz, 4.5]. *In a C^*-quantum groupoid the Haar integral h exists, $h = h^*$ and*

$$(\phi, \psi) = \langle\, h,\, \phi^* \psi \,\rangle, \qquad \phi, \psi \in \widehat{H}$$

is a scalar product making $\widehat{H}$ a Hilbert space where the left regular representation of $\widehat{H}$ is faithful. Thus, $\widehat{H}$ is a C^*-quantum groupoid, too.

Proof. Clearly, H and g verify all the conditions of Theorem 7.2.2, from where the existence of Haar integral follows. Since h is non-degenerate, the scalar product $(\cdot, \cdot)$ is also non-degenerate. By the equality

$$(\phi, \phi) = \langle\, h,\, \phi^* \phi \,\rangle = \overline{\langle\, S(h_{(1)})^*,\, \phi \,\rangle} \langle\, h_{(2)},\, \phi \,\rangle,$$

positivity of $(\cdot, \cdot)$ follows from Remark 7.2.3(b). □

We will denote by $\widehat{h}$ the Haar measure of $\widehat{H}$.

Remark 7.3.2. ε is a positive functional, i.e., $\varepsilon(x^* x) \geq 0$ for all $x \in H$. Indeed, for all $x \in H$ we have

$$\varepsilon(x^* x) = \varepsilon(x^* 1_{(1)}) \varepsilon(1_{(2)} 1'_{(2)}) \varepsilon(1'_{(1)} x) = \varepsilon(\varepsilon_t(x)^* \varepsilon_t(x)) = \langle\, \widehat{h}, \varepsilon_t(x)^* \varepsilon_t(x) \,\rangle \geq 0,$$

where $\widehat{h}|_{H_t} = \varepsilon|_{H_t}$, since $\langle\, \widehat{h}, z \,\rangle = \langle\, \widehat{\varepsilon}_t(\widehat{h}), z \,\rangle = \langle\, \widehat{1}, z \,\rangle$ for all $z \in H_t$.

The Haar measure provides target and source *Haar conditional expectations* (all properties are easy to verify):

$$E_t : H \to H_t : E_t(x) = (\mathrm{id} \otimes \widehat{h})\Delta(x),$$
$$E_s : H \to H_s : E_s(x) = (\widehat{h} \otimes \mathrm{id})\Delta(x).$$

Let us introduce an element $g_t = E_t(h)^{1/2} \in H_t$.

Remark 7.3.3 ([BNSz], 4.12). The element g_t is positive and invertible, and the canonical group-like element of H can be written as $g = g_t S(g_t^{-1})$.

Proposition 7.3.4 ([NTV], 9.3). *If H is a C^*-quantum groupoid, then $D(H)$ is a quasitriangular C^*-quantum groupoid.*

Proof. The result follows from the fact that $\widehat{D(H)}$ is a C^*-quantum groupoid, which is easy to verify using the explicit formulas from Remark 5.3.3. □

Proposition 7.3.5. *A quasitriangular C^*-quantum groupoid H is automatically ribbon with the ribbon element $\nu = u^{-1}g = gu^{-1}$, where u is the Drinfeld element from Definition* 5.2.7 *and g is the canonical group-like element.*

Proof. Since u also implements S^2 (Proposition 5.2.6), $\nu = u^{-1}g$ is central, therefore $S(\nu)$ is also central. Clearly, u must commute with g. The same Proposition gives $\Delta(u^{-1}) = \mathcal{R}_{21}\mathcal{R}(u^{-1} \otimes u^{-1})$, which allows us to compute

$$\Delta(\nu) = \Delta(u^{-1})(g \otimes g) = \mathcal{R}_{21}\mathcal{R}(u^{-1}g \otimes u^{-1}g) = \mathcal{R}_{21}\mathcal{R}(\nu \otimes \nu).$$

Propositions 5.2.5 and 5.2.6 and the trace property imply that

$$\begin{aligned}\mathrm{tr}(\pi_\alpha(u^{-1})) &= \mathrm{tr}(\pi_\alpha(\mathcal{R}^{(2)}S^2(\mathcal{R}^{(1)}))) \\ &= \mathrm{tr}(\pi_\alpha(S^3(\mathcal{R}^{(1)})S(\mathcal{R}^{(2)})) = \mathrm{tr}(\pi_\alpha(S(u^{-1}))).\end{aligned}$$

Since $u^{-1} = \nu g^{-1}$ and ν is central, the above relation means that

$$\mathrm{tr}(\pi_\alpha(\nu))\mathrm{tr}(\pi_\alpha(g^{-1})) = \mathrm{tr}(\pi_\alpha(S(\nu)))\mathrm{tr}(\pi_\alpha(g)),$$

and, therefore, $\mathrm{tr}(\pi_\alpha(\nu)) = \mathrm{tr}(\pi_\alpha(S(\nu)))$ for any irreducible representation π_α, which shows that that $\nu = S(\nu)$ (cf. [EG]). □

Remark 7.3.6. For a connected ribbon C^*-quantum groupoid we have:

$$\mathrm{tr}_q(f) = (\dim H_t)^{-1}\mathrm{Tr}_V(g \circ f),\ \dim_q(V) = (\dim H_t)^{-1}\mathrm{Tr}_V(g),$$

where V is an H-module and $f \in \mathrm{End}_k(V)$.

7.4. C^*-quantum groupoids and unitary modular categories. To define the (unitary) representation category $\mathrm{URep}(H)$ of a C^*-quantum groupoid H we consider *unitary* H-modules, i.e., H-modules V equipped with a scalar product

$$(\cdot,\cdot) : V \times V \to \mathbb{C} \qquad \text{such that} \qquad (h \cdot v, w) = (v, h^* \cdot w)\ \forall h \in H, v, w \in V.$$

The notion of a morphism in this category is the same as in $\mathrm{Rep}(H)$.

The monoidal product of $V, W \in \mathrm{URep}(H)$ is defined as follows. We construct a tensor product $V \otimes_{\mathbb{C}} W$ of Hilbert spaces and note that the action of $\Delta(1)$ on this left H-module is an orthogonal projection. The image of this projection is, by definition, the monoidal product of V, W in $\mathrm{URep}(H)$. Clearly, this definition is compatible with the monoidal product of morphisms in $\mathrm{Rep}(H)$.

For any $V \in \mathrm{URep}(H)$, the dual space V^* is naturally identified $(v \to \overline{v})$ with the conjugate Hilbert space, and under this identification we have $h \cdot \overline{v} = \overline{S(h)^* \cdot v}$ $(v \in V, \overline{v} \in V^*)$. In this way V^* becomes a unitary H-module with scalar product $(\overline{v}, \overline{w}) = (w, gv)$, where g is the canonical group-like element of H.

The unit object in $\mathrm{URep}(H)$ is H_t equipped with scalar product $(z,t)_{H_t} = \varepsilon(zt^*)$ (it is known [BNSz], [NV1] that the restriction of ε to H_t is a non-degenerate positive form). One can verify that the maps l_V, r_V and their inverses are isometries and that $\mathrm{URep}(H)$ is a monoidal category with duality (see also [BNSz], Section 3). In a natural way we have a notion of a *conjugation* of morphisms ([T], II.5.1).

Remark 7.4.1. a) One can check that for a quasitriangular $*$-quantum groupoid H the braiding is an isometry in $\mathrm{URep}(H) : c_{V,W}^{-1} = c_{V,W}^*$.

b) For a ribbon C^*-quantum groupoid H, the twist is an isometry in $\mathrm{URep}(H)$. Indeed, the relation $\theta_V^* = \theta_V^{-1}$ is equivalent to the identity $S(u^{-1}) = u^*$, which follows from Proposition 5.2.5 and Remark 7.4.1a).

A *Hermitian ribbon category* is a ribbon category endowed with a conjugation of morphisms $f \mapsto \overline{f}$ satisfying natural conditions of ([T], II.5.2). A *unitary ribbon category* is a Hermitian ribbon category over the field $\mathbb{C}$ such that for any morphism f we have $\mathrm{tr}_q(f\overline{f}) \geq 0$. In a natural way we have a *conjugation* of morphisms in $\mathrm{URep}(H)$. Namely, for any morphism $f : V \to W$ we define $\overline{f} : W \to V$ as $\overline{f}(w) = \overline{f^*(\overline{w})}$ for any $w \in W$. Here $\overline{w} \in W^*, f^* : W^* \to V^*$ is the standard dual of f (see [T], I.1.8) and $\overline{f^*(\overline{w})} \in V$. For the proof of the following lemma see ([NTV], 9.7).

Lemma 7.4.2. *Given a quasitriangular C^*-quantum groupoid H,* $\mathrm{URep}(H)$ *is a unitary ribbon Ab-category with respect to the above conjugation of morphisms.*

The next proposition extends ([EG], 1.2).

Proposition 7.4.3. *If H is a connected C^*-quantum groupoid, then* $\mathrm{Rep}(D(H))$ *is a unitary modular category.*

Proof. The proof follows from Lemmas 7.4.2, 5.5.1 and Propositions 5.3.4, 7.3.4. □

8. C^*-Quantum Groupoids and Subfactors: The Depth 2 Case

In this section we characterize finite index type II_1 subfactors of depth 2 in terms of C^*-quantum groupoids. Recall that a II_1 factor is a $*$-algebra of operators on a Hilbert space that coincides with its second centralizer, or, equivalently, is weakly closed (i.e., is a von Neumann algebra), with the trivial center that admits a finite trace. An example of such a factor is the group von Neumann algebra of a discrete group whose every non-trivial conjugacy class is infinite, the corresponding trace given by its Haar measure. It is also known, that there exists a unique, up to an isomorphism, hyperfinite (i.e., generated by an increasing sequence of finite-dimensional C^*-algebras) II_1 factor.

There is a notion of index for subfactors, extending the notion of index of a subgroup in a group, though it can be non-integer. For the foundations of the subfactor theory see [GHJ], [JS].

8.1. Actions of C^*-quantum groupoids on von Neumann algebras. Let a von Neumann algebra M be a left H-module algebra in the sense of Definition 4.1.1 via a weakly continuous action of a C^*-quantum groupoid

$$H \otimes_{\mathbb{C}} M \ni x \otimes m \mapsto (x \triangleright m) \in M$$

such that $(x \triangleright m)^* = S(x)^* \triangleright m^*$ and $x \triangleright 1 = 0$ if and only if $\varepsilon_t(t) = 0$.

Then one can show ([NSzW], 3.4.2) that the smash product algebra (now we call it *crossed product algebra* and denote by $M \rtimes H$), equipped with an involution $[m \otimes x]^* = [(x^*_{(1)} \triangleright m^*) \otimes x^*_{(2)}]$, is a von Neumann algebra.

The collection $M^H = \{m \in M \mid x \triangleright m = \varepsilon_t(x) \triangleright m,\ \forall x \in H\}$ is a von Neumann subalgebra of M, called a *fixed point subalgebra*. The centralizer $M' \cap M \rtimes H$ always contains the source counital subalgebra H_s. Indeed, if $y \in H_s$, then $\Delta(y) = 1_{(1)} \otimes 1_{(2)} y$, therefore

$$\begin{aligned}[1 \otimes y][m \otimes 1] &= [(y_{(1)} \triangleright m) \otimes y_{(2)}] = [(1_{(1)} \triangleright m) \otimes 1_{(2)} y] \\ &= [m \otimes y] = [m \otimes 1][1 \otimes y],\end{aligned}$$

for any $m \in M$, and $H_s \subset M' \cap M \rtimes H$. An action of H is called *minimal* if $H_s = M' \cap M \rtimes H$.

One can define the dual action of $\widehat{H}$ on the von Neumann algebra $M \rtimes H$ as in (4–1) and construct the von Neumann algebra $(M \rtimes H) \rtimes \widehat{H}$.

Remark 8.1.1. The following results hold true if $S^2 = \mathrm{id}$, i.e., if H is involutive ([N1], 4.6, 4.7).

(a) $M \subset M \rtimes H \subset (M \rtimes H) \rtimes \widehat{H}$ is a basic construction;
(b) $M \rtimes H$ is free as a left M-module;

(c) $(M \rtimes H) \rtimes \widehat{H} \cong M \otimes_{\mathbb{C}} M_n(\mathbb{C})$ for some integer n.

Example 8.1.2. For a trivial action of H on H_t and the corresponding dual action of $\widehat{H}$ on $H \cong H_t \rtimes H$ (see Example 4.1.3 (i) and (ii)), $H_t \subset H \subset H \rtimes \widehat{H}$ is the basic construction of finite-dimensional C^*-algebras with respect to the Haar conditional expectation E_t ([BSz2], 4.2).

If H is a connected C^*-quantum groupoid having a minimal action on a II_1 factor M then one can show ([NSzW], 4.2.5, 4.3.5) that $\widehat{H}$ is also connected and that

$$N = M^H \subset M \subset M_1 = M \rtimes H \subset M_2 = (M \rtimes H) \rtimes \widehat{H} \subset \cdots$$

is the Jones tower of factors of finite index with *the derived tower*

$$N' \cap N = \mathbb{C} \subset N' \cap M = H_t \subset N' \cap M_1 = H \subset N' \cap M_2 = H \rtimes \widehat{H} \subset \cdots$$

The fact that the last triple of finite-dimensional C^*-algebras is a basic construction, means exactly that the subfactor $N \subset M$ has depth 2. Moreover, the finite-dimensional C^*-algebras

$$\begin{array}{ccc} H & \subset & H \rtimes \widehat{H} \\ \cup & & \cup \\ H_s \cong \widehat{H}_t & \subset & \widehat{H}, \end{array} \tag{8–1}$$

form *a canonical commuting square*, which completely determines the equivalence class of the initial subfactor. This implies that any biconnected C^*-quantum groupoid has at most one minimal action on a given II_1 factor and thus corresponds to no more than one (up to equivalence) finite index depth 2 subfactor.

Remark 8.1.3. It was shown in ([N1], 5) that any biconnected involutive C^*-quantum groupoid has a minimal action on the hyperfinite II_1 factor. This action is constructed by iterating the basic construction for the square (8–1) in the horizontal direction, see [N1] for details.

8.2. Construction of a C^*-quantum groupoid from a depth 2 subfactor. Let $N \subset M$ be a finite index ($[M : N] = \lambda^{-1}$) depth 2 II_1 subfactor and

$$N \subset M \subset M_1 \subset M_2 \subset \cdots \tag{8–2}$$

the corresponding Jones tower, $M_1 = \langle M, e_1 \rangle$, $M_2 = \langle M_1, e_2 \rangle, \ldots,$ where $e_1 \in N' \cap M_1$, $e_2 \in M' \cap M_2, \cdots$ are the Jones projections. The depth 2 condition means that $N' \cap M_2$ is the basic construction of the inclusion $N' \cap M \subset N' \cap M_1$. Let τ be the trace on M_2 normalized by $\tau(1) = 1$.

Let us denote

$$A = N' \cap M_1, \quad B = M' \cap M_2, \quad B_t = M' \cap M_1, \quad B_s = M_1' \cap M_2$$

and let Tr be the trace of the regular representation of B_t on itself. Since both τ and Tr are non-degenerate, there exists a positive invertible element $w \in Z(B_t)$

such that $\tau(wz) = \mathrm{Tr}\,(z)$ for all $z \in B_t$ (the index of $\tau|_{M'\cap M_1}$ in the sense of [W]).

Proposition 8.2.1. *There is a canonical non-degenerate duality form between A and B defined by*

$$\langle\, a,\, b\,\rangle = \lambda^{-2}\tau(ae_2e_1wb) = \lambda^{-2}\mathrm{Tr}\,(E_{B_t}(bae_2e_1)), \tag{8–3}$$

for all $a \in A$ and $b \in B$.

Proof. If $a \in A$ is such that $\langle\, a,\, B\,\rangle = 0$, then, using [PP1], 1.2, one has

$$\tau(ae_2e_1B) = \tau(ae_2e_1(N' \cap M_2)) = 0,$$

therefore, using the properties of τ and Jones projections, we get

$$\tau(aa^*) = \lambda^{-1}\tau(ae_2a^*) = \lambda^{-2}\tau(ae_2e_1(e_2a^*)) = 0,$$

so $a = 0$. Similarly for $b \in B$. □

Using the duality $\langle\, a,\, b\,\rangle$, one defines a coalgebra structure on B:

$$\langle\, a_1 \otimes a_2,\, \Delta(b)\,\rangle = \langle\, a_1a_2,\, b\,\rangle, \tag{8–4}$$

$$\varepsilon(b) = \langle\, 1,\, b\,\rangle, \tag{8–5}$$

for all $a, a_1, a_2 \in A$ and $b \in B$. Similarly, one defines a coalgebra structure on A.

We define a linear endomorphism $S_B : B \to B$ by

$$E_{M_1}(be_1e_2) = E_{M_1}(e_2e_1S_B(b)), \tag{8–6}$$

Note that $\tau \circ S_B = \tau$. Similarly one can define a linear endomorphism $S_A : A \to A$ such that $E_{M'}(S_A(a)e_2e_1) = E_{M'}(e_1e_2a)$ and $\tau \circ S_A = \tau$.

Proposition 8.2.2 [NV2, 4.5, 5.2]. *The following identities hold*:

(i) $S_B(B_s) = B_t$,
(ii) $S_B^2(b) = b$ *and* $S_B(b)^* = S_B(b^*)$,
(iii) $\Delta(b)(S_B(z) \otimes 1) = \Delta(b)(1 \otimes z)$, $\quad \Delta(bz) = \Delta(b)(z \otimes 1)$, $(\forall z \in B_t)$,
(iv) $S_B(bc) = S_B(c)S_B(b)$,
$\Delta(S_B(b)) = (1 \otimes w^{-1})(S_B(b_{(2)}) \otimes S_B(b_{(1)}))(S_B(w) \otimes 1)$.

Define an antipode and new involution of B by

$$S(b) = S_B(wbw^{-1}), \qquad b^\dagger = S(w)^{-1}b^*S(w). \tag{8–7}$$

Theorem 8.2.3. *With the above operations B becomes a biconnected C^*-quantum groupoid with the counital subalgebras B_s and B_t.*

PROOF. Since $\Delta(1) \in B_s \otimes B_t$ ([NV2], 4.6) and B_t commutes with B_s, we have

$$\begin{aligned}(\mathrm{id} \otimes \Delta)\Delta(1) &= 1_{(1)} \otimes \Delta(1)(1_{(2)} \otimes 1) \\ &= (1 \otimes \Delta(1))(\Delta(1) \otimes 1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))\end{aligned}$$

which is axiom (2–1) of Definition 2.1.1. The axiom (2–2) dual to it is obtained similarly, considering the comultiplication in A.

As a consequence of the "symmetric square" relations $BM_1 = M_1B = M_2$ one obtains ([NV2], 4.12) the identity

$$bx = \lambda^{-1}E_{M_1}(b_{(1)}xe_2)b_{(2)} \quad \text{for all} \quad b \in B,\, x \in M_1, \tag{8–8}$$

from where it follows that

$$E_{M_1}(bxye_2) = \lambda^{-1}E_{M_1}(b_{(1)}xe_2)E_{M_1}(b_{(2)}ye_2) \quad \text{for all} \quad b \in B,\, x, y \in M_1. \tag{8–9}$$

We use this identity to prove that Δ is a homomorphism:

$$\begin{aligned}\langle\, a_1a_2,\, bc\,\rangle &= \langle\, \lambda^{-1}E_{M_1}(ca_1a_2e_2),\, b\,\rangle \\ &= \langle\, \lambda^{-2}E_{M_1}(c_{(1)}a_1e_2)E_{M_1}(c_{(2)}a_2e_2),\, b\,\rangle \\ &= \langle\, \lambda^{-1}E_{M_1}(c_{(1)}a_1e_2), b_{(1)}\,\rangle\langle\, \lambda^{-1}E_{M_1}(c_{(2)}a_2e_2),\, b_{(2)}\,\rangle \\ &= \langle\, a_1,\, b_{(1)}c_{(1)}\,\rangle\langle\, a_2,\, b_{(2)}c_{(2)}\,\rangle,\end{aligned}$$

for all $a_1, a_2 \in A$, whence $\Delta(bc) = \Delta(b)\Delta(c)$. Next,

$$\begin{aligned}\langle\, a,\, \varepsilon(1_{(1)}b)1_{(2)}\,\rangle &= \langle\, 1,\, 1_{(1)}b\rangle\langle\, a,\, 1_{(2)}\,\rangle \\ &= \langle\, \lambda^{-1}E_{M_1}(be_2), 1_{(1)}\,\rangle\langle\, a,\, 1_{(2)}\,\rangle \\ &= \langle\, \lambda^{-1}E_{M_1}(be_2)a, 1\,\rangle = \langle\, a,\, \lambda^{-1}E_{M_1}(be_2)\,\rangle,\end{aligned}$$

therefore $\varepsilon_t(b) = \lambda^{-1}E_{M_1}(be_2)$ (similarly, $\varepsilon_s(b) = \lambda^{-1}E_{M_1'}(e_2b)$). Using Equation (8–9) we have

$$\begin{aligned}\langle\, a,\, b_{(1)}S(b_{(2)})\,\rangle &= \langle\, a,\, b_{(1)}S_B(wb_{(2)}w^{-1})\,\rangle \\ &= \lambda^{-3}\tau(E_{M_1}(S_B(wb_{(2)}w^{-1})ae_2)e_2e_1wb_{(1)}) \\ &= \lambda^{-3}\tau(E_{M_1}(e_2awb_{(2)}w^{-1})e_2e_1wb_{(1)}) \\ &= \lambda^{-3}\tau(E_{M_1}(e_2awb_{(2)}w^{-1})E_{M_1}(e_2e_1wb_{(1)})) \\ &= \lambda^{-2}\tau(E_{M_1}(e_2ae_1wb)) = \langle\, a,\, \varepsilon_t(b)\,\rangle.\end{aligned}$$

The antipode S is anti-multiplicative and anti-comultiplicative, therefore axiom (2–5) of Definition 2.1.1 follows from the other axioms. Clearly, $\varepsilon_t(B) = M' \cap M_1$ and $\varepsilon_s(B) = M_1' \cap M_2$. B is biconnected, since the inclusion $B_t = M' \cap M_1 \subset B = M' \cap M_2$ is connected ([GHJ], 4.6.3) and $B_t \cap B_s = (M' \cap M_1) \cap (M_1' \cap M_2) = \mathbb{C}$.

Finally, from the properties of Δ and S_B we have $\Delta(b^\dagger) = \Delta(b)^{\dagger\otimes\dagger}$. □

REMARK 8.2.4. (i) $S^2(b) = gbg^{-1}$, where $g = S(w)^{-1}w$.

(ii) The Haar projection of B is $e_2 w$ and the normalized Haar functional is $\phi(b) = \tau(S(w)wb)$.

(iii) The non-degenerate duality $\langle\ ,\ \rangle$ makes $A = N' \cap M_1$ the C^*-quantum groupoid dual to B.

(iv) Since the Markov trace ([GHJ], 3.2.4) of the inclusion $N \subset M$ is also the Markov trace of the finite-dimensional inclusion $A_t \subset A$, it is clear that $[M : N] = [A : A_t] = [B : B_t]$. We call last number *the index of the quantum groupoid* B.

(v) Let us mention two classification results obtained in ([NV2], 4.18, 4.19) by quantum groupoid methods:

(a) If $N \subset M$ is a depth 2 subfactor such that $[M : N]$ is a square free integer (i.e., $[M : N]$ is an integer which has no divisors of the form n^2, $n > 1$), then $N' \cap M = \mathbb{C}$, and there is a (canonical) minimal action of a Kac algebra B on M_1 such that $M_2 \cong M_1 \rtimes B$ and $M = M_1^B$.

(b) If $N \subset M$ is a depth 2 II_1 subfactor such that $[M : N] = p$ is prime, then $N' \cap M = \mathbb{C}$, and there is an outer action of the cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ on M_1 such that $M_2 \cong M_1 \rtimes G$ and $M = M_1^G$.

8.3. Action on a subfactor. Equation (8–9) suggests the following definition of the action of B on M_1.

Proposition 8.3.1. *The map* $\triangleright : B \otimes_{\mathbb{C}} M_1 \to M_1$:

$$b \triangleright x = \lambda^{-1} E_{M_1}(bxe_2) \tag{8–10}$$

defines a left action of B *on* M_1 (*cf.* [S], *Proposition* 17).

Proof. The above map defines a left B-module structure on M_1, since $1 \triangleright x = x$ and

$$b \triangleright (c \triangleright x) = \lambda^{-2} E_{M_1}(bE_{M_1}(cxe_2)e_2) = \lambda^{-1} E_{M_1}(bcxe_2) = (bc) \triangleright x.$$

Next, using equation (8–9) we get

$$\begin{aligned} b \triangleright xy &= \lambda^{-1} E_{M_1}(bxye_2) = \lambda^{-2} E_{M_1}(b_{(1)}xe_2)E_{M_1}(b_{(2)}ye_2) \\ &= (b_{(1)} \triangleright x)(b_{(2)} \triangleright y). \end{aligned}$$

By [NV2], 4.4 and properties of S_B we also get

$$\begin{aligned} S(b)^\dagger \triangleright x^* &= \lambda^{-1} E_{M_1}(S(b)^\dagger x^* e_2) = \lambda^{-1} E_{M_1}(S_B(w)^{-1} S_B(wbw^{-1})^* S_B(w) x^* e_2) \\ &= \lambda^{-1} E_{M_1}(S_B(b^*) x^* e_2) = \lambda^{-1} E_{M_1}(e_2 x^* b^*) \\ &= \lambda^{-1} E_{M_1}(bxe_2)^* = (b \triangleright x)^*. \end{aligned}$$

Finally,

$$b \triangleright 1 = \lambda^{-1} E_{M_1}(be_2) = \lambda^{-1} E_{M_1}(\lambda^{-1} E_{M_1}(be_2)e_2) = \varepsilon_t(b) \triangleright 1,$$

and $b \triangleright 1 = 0$ if and only if $\varepsilon_t(b) = \lambda^{-1} E_{M_1}(be_2) = 0$. $\square$

PROPOSITION 8.3.2. *$M_1^B = M$, i.e. M is the fixed point subalgebra of M_1.*

PROOF. If $x \in M_1$ is such that $b \triangleright x = \varepsilon_t(b) \triangleright x$ for all $b \in B$, then $E_{M_1}(bxe_2) = E_{M_1}(\varepsilon_t(b)xe_2) = E_{M_1}(be_2)x$. Taking $b = e_2$, we get $E_M(x) = x$ which means that $x \in M$. Thus, $M_1^B \subset M$.

Conversely, if $x \in M$, then x commutes with e_2 and

$$b \triangleright x = \lambda^{-1} E_{M_1}(be_2x) = \lambda^{-1} E_{M_1}(\lambda^{-1} E_{M_1}(be_2)e_2x) = \varepsilon_t(b) \triangleright x,$$

therefore $M_1^B = M$. □

PROPOSITION 8.3.3. *The map $\theta : [x \otimes b] \mapsto xS(w)^{1/2}bS(w)^{-1/2}$ defines a von Neumann algebra isomorphism between $M_1 \rtimes B$ and M_2.*

PROOF. By definition of the action $\triangleright$ we have:

$$\begin{aligned}\theta([x(z \triangleright 1) \otimes b]) &= xS(w)^{1/2}\lambda^{-1}E_{M_1}(ze_2)bS(w)^{-1/2} \\ &= xS(w)^{1/2}zbS(w)^{-1/2} = \theta([x \otimes zb]),\end{aligned}$$

for all $x \in M_1$, $b \in B$, $z \in B_t$, so θ is a well defined linear map from $M_1 \rtimes B = M_1 \otimes_{B_t} B$ to M_2. It is surjective since an orthonormal basis of $B = M' \cap M_2$ over $B_t = M' \cap M_1$ is also a basis of M_2 over M_1 ([Po], 2.1.3). Finally, one can check that θ is a von Neumann algebra isomorphism ([NV2], 6.3). □

REMARK 8.3.4. (i) The action of B constructed in Proposition 8.3.1 is minimal, since we have $M_1' \cap M_1 \rtimes B = M_1' \cap M_2 = B_s$ by Proposition 8.3.3.

(ii) If $N' \cap M = \mathbb{C}$, then B is a usual Kac algebra (i.e., a Hopf C^*-algebra) and we recover the well-known result proved in [S], [L], and [D].

(iii) It is known ([Po], Section 5) that there are exactly 3 non-isomorphic subfactors of depth 2 and index 4 of the hyperfinite II_1 factor R such that $N' \cap M \neq \mathbb{C}$. One of them is $R \subset R \otimes_{\mathbb{C}} M_2(\mathbb{C})$, two others can be viewed as diagonal subfactors of the form

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix} \middle| x \in R \right\} \subset R \otimes M_2(\mathbb{C}),$$

where α is an outer automorphism of R such that $\alpha^2 = \mathrm{id}$, resp. α^2 is inner and $\alpha^2 \neq \mathrm{id}$.

An explicit description of the corresponding quantum groupoids can be found in ([NV1], 3.1, 3.2), ([NV2], 7). Also, in ([NV2], 7) we describe the C^*-quantum groupoid $B = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ corresponding to the subfactor with the principal graph A_3 ([GHJ], 4.6.5). This subfactor has index $4\cos^2\frac{\pi}{5}$ so that the index of B is equal to $16\cos^4\frac{\pi}{5}$.

9. C^*-Quantum Groupoids and Subfactors: The Finite Depth Case

Here we show that quantum groupoids give a description of arbitrary finite index and finite depth II_1 subfactors via a Galois correspondence and explain how to express subfactor invariants such as bimodule categories and principal graphs in quantum groupoid terms. Recall that the *depth* ([GHJ], 4.6.4) of a subfactor $N \subset M$ with finite index $\lambda^{-1} = [M : N]$ is

$$n = \inf\{k \in \mathbb{Z}^+ \mid \dim Z(N' \cap M_{k-2}) = \dim Z(N' \cap M_k)\} \in \mathbb{Z}_+ \cup \{\infty\},$$

where $N \subset M \subset M_1 \subset M_2 \subset \cdots$ is the corresponding Jones tower.

9.1. A Galois correspondence. The following simple observation gives a natural passage from arbitrary finite depth subfactors to depth 2 subfactors.

Proposition 9.1.1. *Let $N \subset M$ be a subfactor of depth n then for all $k \geq 0$ the inclusion $N \subset M_k$ has depth $d+1$, where d is the smallest positive integer $\geq \frac{n-1}{k+1}$. In particular, $N \subset M_i$ has depth 2 for all $i \geq n-2$, i.e., any finite depth subfactor is an intermediate subfactor of some depth 2 inclusion.*

Proof. Note that $\dim Z(N' \cap M_i) = \dim Z(N' \cap M_{i+2})$ for all $i \geq n-2$. By [PP2], the tower of basic construction for $N \subset M_k$ is

$$N \subset M_k \subset M_{2k+1} \subset M_{3k+2} \subset \cdots,$$

therefore, the depth of this inclusion is equal to $d+1$, where d is the smallest positive integer such that $d(k+1) - 1 \geq n-2$. $\square$

This result means that $N \subset M$ can be realized as an intermediate subfactor of a crossed product inclusion $N \subset N \rtimes B$ for some quantum groupoid B:

$$N \subset M \subset N \rtimes B.$$

Recall that in the case of a Kac algebra action there is a Galois correspondence between intermediate von Neumann subalgebras of $N \subset N \rtimes B$ and left coideal $*$-subalgebras of B [ILP],[E1]. Thus, it is natural to ask about a quantum groupoid analogue of this correspondence.

A unital $*$-subalgebra $I \subset B$ such that $\Delta(I) \subset B \otimes I$ (resp. $\Delta(I) \subset I \otimes B$) is said to be a left (resp. right) *coideal $*$-subalgebra*. The set $\ell(B)$ of left coideal $*$-subalgebras is a lattice under the usual operations:

$$I_1 \wedge I_2 = I_1 \cap I_2, \qquad I_1 \vee I_2 = (I_1 \cup I_2)''$$

for all $I_1, I_2 \in \ell(B)$. The smallest element of $\ell(B)$ is B_t and the greatest element is B. $I \in \ell(B)$ is said to be *connected* if $Z(I) \cap B_s = \mathbb{C}$.

To justify this definition, note that if $I = B$, then this is precisely the definition of a connected quantum groupoid, and if $I = B_t$, then this definition is equivalent to $\widehat{B}$ being connected ([N1], 3.10, 3.11, [BNSz], 2.4).

On the other hand, the set $\ell(M_1 \subset M_2)$ of intermediate von Neumann subalgebras of $M_1 \subset M_2$ also forms a lattice under the operations

$$K_1 \wedge K_2 = K_1 \cap K_2, \qquad K_1 \vee K_2 = (K_1 \cup K_2)''$$

for all $K_1, K_2 \in \ell(M_1 \subset M_2)$. The smallest element of this lattice is M_1 and the greatest element is M_2.

Given a left (resp. right) action of B on a von Neumann algebra N, we will denote (by an abuse of notation)

$$N \rtimes I = \operatorname{span}\{[x \otimes b] \mid x \in N,\, b \in I\} \subset N \rtimes B.$$

The next theorem establishes a *Galois correspondence*, i.e., a lattice isomorphism between $\ell(M_1 \subset M_2)$ and $\ell(B)$.

THEOREM 9.1.2. *Let $N \subset M \subset M_1 \subset M_2 \subset \cdots$ be the tower constructed from a depth 2 subfactor $N \subset M$, $B = M' \cap M_2$ be the corresponding quantum groupoid, and θ be the isomorphism between $M_1 \rtimes B$ and M_2 (Proposition 8.3.3). Then*

$$\phi : \ell(M_1 \subset M_2) \to \ell(B) : K \mapsto \theta^{-1}(M' \cap K) \subset B,$$
$$\psi : \ell(B) \to \ell(M_1 \subset M_2) : I \mapsto \theta(M_1 \rtimes I) \subset M_2$$

define isomorphisms between $\ell(M_1 \subset M_2)$ and $\ell(B)$ inverse to each other.

PROOF. First, let us check that ϕ and ψ are indeed maps between the specified lattices. It follows from the definition of the crossed product that $M_1 \rtimes I$ is a von Neumann subalgebra of $M_1 \rtimes B$, therefore $\theta(M_1 \rtimes I)$ is a von Neumann subalgebra of $M_2 = \theta(M_1 \rtimes B)$, so ψ is a map to $\ell(M_1 \subset M_2)$. To show that ϕ maps to $\ell(B)$, let us show that the annihilator $(M' \cap K)^0 \subset \widehat{B}$ is a left ideal in $\widehat{B}$.

For all $x \in A$, $y \in (M' \cap K)^0$, and $b \in M' \cap K$ we have

$$\begin{aligned}\langle\, xy,\, b\,\rangle &= \lambda^{-2}\tau(xye_2e_1wb) = \lambda^{-2}\tau(ye_2e_1wbx)\\ &= \lambda^{-3}\tau(ye_2e_1E_{M'}(e_1wbx)) = \langle\, y,\, \lambda^{-1}E_{M'}(e_1wbx)\,\rangle,\end{aligned}$$

and it remains to show that $E_{M'}(e_1wbx) \in M' \cap K$. By ([GHJ], 4.2.7), the square

$$\begin{array}{ccc} K & \subset & M_2 \\ \cup & & \cup \\ M' \cap K & \subset & M' \cap M_2 \end{array}$$

is commuting, so $E_{M'}(K) \subset M' \cap K$. Since $e_1wbx \in K$, then $xy \in (M' \cap K)^0$, i.e., $(M' \cap K)^0$ is a left ideal and $\phi(K) = \theta^{-1}(M' \cap K)$ is a left coideal $*$-subalgebra.

Clearly, ϕ and ψ preserve $\wedge$ and $\vee$, moreover $\phi(M_1) = B_t$, $\phi(M_2) = B$ and $\psi(B_t) = M_1$, $\psi(B) = M_2$, therefore they are morphisms of lattices.

To see that they are inverses for each other, we first observe that the condition $\psi \circ \phi = \mathrm{id}$ is equivalent to $M_1(M' \cap K) = K$, and the latter follows from applying

the conditional expectation E_K to $M_1(M' \cap M_2) = M_1 B = M_2$. The condition $\phi \circ \psi = \mathrm{id}$ translates into $\theta(I) = M' \cap \theta(M_1 \rtimes I)$. If $b \in I$, $x \in M = M_1^B$, then

$$\begin{aligned}\theta(b)x &= \theta([1 \otimes b][x \otimes 1]) = \theta([(b_{(1)} \triangleright x) \otimes b_{(2)}]) \\ &= \theta([x(1_{(2)} \triangleright 1) \otimes \varepsilon(1_{(1)} b_{(1)}) b_{(2)}]) = \theta([x \otimes b]) = x\theta(b),\end{aligned}$$

i.e., $\theta(I)$ commutes with M. Conversely, if $x \in M' \cap \theta(M_1 \rtimes I) \subset B$, then $x = \theta(y)$ for some $y \in (M_1 \rtimes I) \cap B = I$, therefore $x \in \theta(I)$. □

Corollary 9.1.3. *(*[NV3], 4.5)

(i) *$K = M \rtimes I$ is a factor if and only if $Z(I) \cap B_s = \mathbb{C}$.*
(ii) *The inclusion $M_1 \subset K = M_1 \rtimes I$ is irreducible if and only if $B_s \cap I = \mathbb{C}$.*

9.2. Bimodule categories. Here we establish an equivalence between the tensor category $\mathrm{Bimod}_{N-N}(N \subset M)$ of $N-N$ bimodules of a subfactor $N \subset M$ (which is, by definition, the tensor category generated by simple subobjects of ${}_N L_2(M_n)_N$, $n \geq -1$, where $M_0 = M$ and $M_{-1} = N$) and the co-representation category of the C^*-quantum groupoid B canonically associated with it as in Theorem 9.1.2.

First introduce several useful categories associated to B. Recall that a left (resp., right) B-comodule V (with the structure map denoted by $v \mapsto v^{(1)} \otimes v^{(2)}$, $v \in V$) is said to be *unitary*, if

$$\begin{gathered}(v_2^{(1)})^*(v_1, v_2^{(2)}) = S(v_1^{(1)}) g(v_1^{(2)}, v_2) \\ (\text{resp., } (v_1^{(2)})(v_1^{(1)}, v_2) = g^{-1} S((v_1^{(1)})^*)(v_1, v_2^{(1)})),\end{gathered}$$

where $v_1, v_2 \in V$, and g is the canonical group-like element of B. This definition for Hopf $*$-algebra case can be found, e.g., in ([KS], 1.3.2).

Given left coideal $*$-subalgebras H and K of B, we consider a category $\mathcal{C}_{H-K}$ of left relative $(B, H-K)$ Hopf bimodules (cf. [Ta]), whose objects are Hilbert spaces which are both $H-K$-bimodules and left unitary B-comodules such that the bimodule action commutes with the coaction of B, i.e., for any object V of $\mathcal{C}_{H-K}$ and $v \in V$ one has

$$(h \triangleright v \triangleleft k)^{(1)} \otimes (h \triangleright v \triangleleft k)^{(2)} = h_{(1)} v^{(1)} k_{(1)} \otimes (h_{(2)} \triangleright v^{(2)} \triangleleft k_{(2)}),$$

where $v \mapsto v^{(1)} \otimes v^{(2)}$ denotes the coaction of B on v, $h \in H$, $k \in K$, and morphisms are intertwining maps.

Similarly one can define a category of right relative $(B, H-K)$ Hopf bimodules.

Remark 9.2.1. Any left B-comodule V is automatically a $B_t - B_t$-bimodule via $z_1 \cdot v \cdot z_2 = \varepsilon(z_1 v^{(1)} z_2) v^{(2)}$, $v \in V, z_1, z_2 \in B_t$. For any object of $\mathcal{C}_{H-K}$, this $B_t - B_t$-bimodule structure is a restriction of the given $H-K$-bimodule structure; it is easily seen by applying $(\varepsilon \otimes \mathrm{id})$ to both sides of the above relation of commutation and taking $h, k \in B_t$. Thus, if $H = B_t$ (resp. $K = B_t$), we

can speak about right (resp. left) relative Hopf modules, a special case of weak Doi–Hopf modules [Bo].

PROPOSITION 9.2.2. *If B is a group Hopf C^*-algebra and H, K are subgroups, then there is a bijection between simple objects of $\mathcal{C}_{H-K}$ and double cosets of $H\backslash B/K$.*

PROOF. If V is an object of $\mathcal{C}_{H-K}$, then every simple subcomodule of V is 1-dimensional. Let $U = \mathbb{C}u$ ($u \mapsto g \otimes u$, $g \in B$) be one of these comodules, then all other simple subcomodules of V are of the form $h \triangleright U \triangleleft k$, where $h \in H$, $k \in K$, and

$$V = \oplus_{h,k} (h \triangleright U \triangleleft k) = \mathrm{span}\{HgK\}.$$

Vice versa, $\mathrm{span}\{HgK\}$ with natural $H-K$ bimodule and B-comodule structures is a simple object of $\mathcal{C}_{H-K}$. □

EXAMPLE 9.2.3. If H, V, K are left coideal $*$-subalgebras of B, $H \subset V, K \subset V$, then V is an object of $\mathcal{C}_{H-K}$ with the structure maps given by $h \triangleright v \triangleleft k = hvk$ and Δ, where $v^{(1)} \otimes v^{(2)} = \Delta(v)$, $v \in V$, $h \in H$, $k \in K$. The scalar product is defined by the restriction on V of the *Markov trace* of the connected inclusion $B_t \subset B$ ([JS], 3.2). Similarly, right coideal $*$-subalgebras of B give examples of right relative $(B, H-K)$ Hopf bimodules.

Given an object V of $\mathcal{C}_{H-K}$, it is straightforward to show that the conjugate Hilbert space $\overline{V}$ is an object of $\mathcal{C}_{K-H}$ with the bimodule action

$$k \triangleright \overline{v} \triangleleft h = \overline{h^* \triangleright v \triangleleft k^*} \quad (\forall h \in H,\, k \in K)$$

(here $\overline{v}$ denotes the vector $v \in V$ considered as an element of $\overline{V}$) and the coaction $\overline{v} \mapsto \overline{v}^{(1)} \otimes \overline{v}^{(2)} = (v^{(1)})^* \otimes \overline{v}^{(2)}$. Define V^*, the dual object of V, to be $\overline{V}$ with the above structures. One can directly check that $V^{**} \cong V$ for any object V. In Example 9.2.3 the dual object can be obtained by putting $\overline{v} = v^*$ for all $v \in V$.

DEFINITION 9.2.4. Let L be another coideal $*$-subalgebra of B. For any objects $V \in \mathcal{C}_{H-L}$ and $W \in \mathcal{C}_{L-K}$, we define an object $V \otimes_L W$ from $\mathcal{C}_{H-K}$ as a tensor product of bimodules V and W [JS], 4.1 equipped with a comodule structure

$$(v \otimes_L w)^{(1)} \otimes (v \otimes_L w)^{(2)} = v^{(1)} w^{(1)} \otimes (v^{(2)} \otimes_L w^{(2)}).$$

One can verify that we have indeed an object from $\mathcal{C}_{H-K}$ and that the operation of tensor product is (i) associative, i.e., $V \otimes_L (W \otimes_P U) \cong (V \otimes_L W) \otimes_P U$; (ii) compatible with duality, i.e., $(V \otimes_L W)^* \cong W^* \otimes_L V^*$; (iii) distributive, i.e., $(V \oplus V') \otimes_L W = (V \otimes_L W) \oplus (V' \otimes_L W)$ ([NV3], 5.5). The tensor product of morphisms $T \in \mathrm{Hom}(V, V')$ and $S \in \mathrm{Hom}(W, W')$ is defined as usual:

$$(T \otimes_L S)(v \otimes_L w) = T(v) \otimes_L S(w).$$

From now on let us suppose that B is biconnected and acts outerly on the left on a II_1 factor N. Given an object V of $\mathcal{C}_{H-K}$, we construct an $N{\rtimes}H - N{\rtimes}K$-bimodule $\widehat{V}$ as follows. We put

$$\widehat{V} = \mathrm{span}\{\Delta(1) \triangleright (\xi \otimes v) \mid \xi \otimes v \in L^2(N) \otimes_{\mathbb{C}} V\} = L^2(N) \otimes_{B_t} V$$

and denote $[\xi \otimes v] = \Delta(1) \triangleright (\xi \otimes v)$. Let us equip $\widehat{V}$ with the scalar product

$$([\xi \otimes v], [\eta \otimes w])_{\widehat{V}} = (\xi, \eta)_{\widehat{L^2(N)}} (v, w)_V$$

and define the actions of N, H, K on $\widehat{V}$ by

$$a[\xi \otimes v] = [a\xi \otimes v], \qquad [\xi \otimes v]a = [\xi(v^{(1)} \triangleright a) \otimes v^{(2)}],$$

$$h[\xi \otimes v] = [(h_{(1)} \triangleright \xi) \otimes (h_{(2)} \triangleright v)], \qquad [\xi \otimes v]k = [\xi \otimes (v \triangleleft k)],$$

for all $a \in N$, $h \in H$, $k \in K$. One can check that these actions define the structure of an $(N{\rtimes}H) - (N{\rtimes}K)$ bimodule on $\widehat{V}$ in the algebraic sense and that this bimodule is unitary.

For any morphism $T \in \mathrm{Hom}(V, W)$, define $\widehat{T} \in \mathrm{Hom}(\widehat{V}, \widehat{W})$ by

$$\widehat{T}([\xi \otimes v]) = [\xi \otimes T(v)].$$

Example 9.2.5. ([NV3], 5.6) For $V \in \mathcal{C}_{H-K}$ from Example 9.2.3, we have $\widehat{V} = N{\rtimes}V$ as $N{\rtimes}H - N{\rtimes}K$-bimodules.

The proof of the following theorem is purely technical (see [NV3], 5.7):

Theorem 9.2.6. *The above assignments $V \mapsto \widehat{V}$ and $T \mapsto \widehat{T}$ define a functor from $\mathcal{C}_{H-K}$ to the category of $N{\rtimes}H - N{\rtimes}K$ bimodules. This functor preserves direct sums and is compatible with operations of taking tensor products and adjoints in the sense that if W is an object of $\mathcal{C}_{K-L}$, then*

$$\widehat{V \otimes_K W} \cong \widehat{V} \otimes_{N{\rtimes}K} \widehat{W}, \qquad \text{and} \qquad \widehat{V^*} \cong (\widehat{V})^*.$$

According to Remark 9.2.1, $\mathcal{C}_{B_t - B_t}$ is nothing but the category of B-comodules, $\mathrm{Corep}(B)$. Let us show that this category is equivalent to $\mathrm{Bimod}_{N-N}(N \subset M)$.

Theorem 9.2.7. *Let $N \subset M$ be a finite depth subfactor with finite index, k be a number such that $N \subset M_k$ has depth ≤ 2, and let B be a canonical quantum groupoid such that $(N \subset M_k) \cong (N \subset N{\rtimes}B)$. Then $\mathrm{Bimod}_{N-N}(N \subset M)$ and $\mathrm{Rep}(B^*)$ are equivalent as tensor categories.*

Proof. First, we observe that

$$\mathrm{Bimod}_{N-N}(N \subset M) = \mathrm{Bimod}_{N-N}(N \subset M_l)$$

for any $l \geq 0$. Indeed, since both categories are semisimple, it is enough to check that they have the same set of simple objects. All objects of

$$\mathrm{Bimod}_{N-N}(N \subset M_l)$$

are also objects of $\mathrm{Bimod}_{N-N}(N \subset M)$. Conversely, since irreducible $N-N$ subbimodules of ${}_N L^2(M_i)_N$ are contained in the decomposition of ${}_N L^2(M_{i+1})_N$ for all $i \geq 0$, we see that objects of $\mathrm{Bimod}_{N-N}(N \subset M)$ belong to $\mathrm{Bimod}_{N-N}(N \subset M_l)$.

Hence, by Proposition 9.1.1, it suffices to consider the problem in the case when $N \subset M$ has depth 2 ($M = N \rtimes B$), i.e., to prove that

$$\mathrm{Bimod}_{N-N}(N \subset N \rtimes B)$$

is equivalent to $\mathrm{Corep}(B)$. Theorem 9.2.6 gives a functor from $\mathrm{Corep}(B) = \mathrm{Rep}(B^*)$ to $\mathrm{Bimod}_{N-N}(N \subset N \rtimes B)$. To prove that this functor is an equivalence, let us check that it yields a bijection between classes of simple objects of these categories.

Observe that B itself is an object of $\mathrm{Corep}(B)$ via $\Delta : B \to B \otimes_{\mathbb{C}} B$ and $\widehat{B} = {}_N L^2(M)_N$. Since the inclusion $N \subset M$ has depth 2, the simple objects of $\mathrm{Bimod}_{N-N}(N \subset M)$ are precisely irreducible subbimodules of ${}_N L^2(M)_N$. We have $\widehat{B} = {}_N L^2(M)_N = \oplus_i {}_N(p_i L^2(M))_N$, where $\{p_i\}$ is a family of mutually orthogonal minimal projections in $N' \cap M_1$ so every bimodule $p_i L^2(M)$ is irreducible. On the other hand, B is cosemisimple, hence $B = \oplus_i V_i$, where each V_i is an irreducible subcomodule. Note that $N' \cap M_1 = \widehat{B} = \sum p_i \widehat{B}$ and every $p_i B$ is a simple submodule of B (= simple subcomodule of B^*). Thus, there is a bijection between the sets of simple objects of $\mathrm{Corep}(B)$ and $\mathrm{Bimod}_{N-N}(N \subset M)$, so the categories are equivalent. □

9.3. Principal graphs. The principal graph of a subfactor $N \subset M$ is defined as follows ([JS], 4.2, [GHJ], 4.1). Let $X = {}_N L^2(M)_M$ and consider the following sequence of $N - N$ and $N - M$ bimodules:

$${}_N L^2(N)_N,\ X,\ X \otimes_M X^*,\ X \otimes_M X^* \otimes_N X, \ldots$$

obtained by right tensoring with X^* and X. The vertex set of the principal graph is indexed by the classes of simple bimodules appearing as summands in the above sequence. Let us connect vertices corresponding to bimodules ${}_N Y_N$ and ${}_N Z_M$ by l edges if ${}_N Y_N$ is contained in the decomposition of ${}_N Z_N$, the restriction of ${}_N Z_M$, with multiplicity l.

We apply Theorem 9.2.6 to express the principal graph of a finite depth subfactor $N \subset M$ in terms of the quantum groupoid associated with it.

Let B and K be a quantum groupoid and its left coideal $*$-subalgebra such that B acts on N and $(N \subset M) \cong (N \subset N \rtimes K)$. Then $\widehat{K} = {}_N L^2(M)_M$, where we view K as a relative (B, K) Hopf module as in Example 9.2.3

By Theorem 9.2.6 we can identify irreducible $N - N$ (resp. $N - M$) bimodules with simple B-comodules (resp. relative right (B, K) Hopf modules). Consider a bipartite graph with vertex set given by the union of (classes of) simple B-comodules and simple relative right (B, K) Hopf modules and the number of edges between the vertices U and V representing B-comodule and relative right

(B, K) Hopf module respectively being equal to the multiplicity of U in the decomposition of V (when the latter is viewed as a B-comodule):

$$\begin{array}{c}\text{simple } B\text{-comodules}\\ \cdots|\cdots|\cdots\\ \text{simple relative right } (B,K) \text{ Hopf modules}\end{array}$$

The principal graph of $N \subset M$ is the connected part of the above graph containing the trivial B-comodule.

It was shown in ([NV3], 3.3, 4.10) that:

(a) the map $\delta : I \mapsto [g^{-1/2}S(I)g^{1/2}]' \cap \widehat{B} \subset \widehat{B} \rtimes B$ defines a lattice anti-isomorphism between $\ell(B)$ and $\ell(\widehat{B})$;
(b) the triple $\delta(I) \subset \widehat{B} \subset \widehat{B} \rtimes I$ is a basic construction.

Proposition 9.3.1. *If K is a coideal $*$-subalgebra of B then the principal graph of the subfactor $N \subset N \rtimes K$ is given by the connected component of the Bratteli diagram of the inclusion $\delta(K) \subset \widehat{B}$ containing the trivial representation of $\widehat{B}$.*

Proof. First, let us show that there is a bijective correspondence between right relative (B, K) Hopf modules and $(\widehat{B} \rtimes K)$-modules. Indeed, every right (B, K) Hopf module V carries a right action of K. If we define a right action of $\widehat{B}$ by

$$v \triangleleft x = \langle\, v^{(1)},\, x \,\rangle v^{(2)}, \quad v \in V, x \in \widehat{B},$$

then we have

$$\begin{aligned}(v \triangleleft k) \triangleleft x &= \langle\, v^{(1)} k_{(1)},\, x \,\rangle (v^{(2)} \triangleleft k_{(2)}) = \langle\, v^{(1)},\, (k_{(1)} \triangleright x) \,\rangle (v^{(2)} \triangleleft k_{(2)})\\ &= (v \triangleleft (k_{(1)} \triangleright x)) \triangleleft k_{(2)},\end{aligned}$$

for all $x \in \widehat{B}$ and $k \in K$ which shows that kx and $(k_{(1)} \triangleright x)k_{(2)}$ act on V exactly in the same way, therefore V is a right $(\widehat{B} \rtimes K)$-module.

Conversely, given an action of $(\widehat{B} \rtimes K)$ on V, we automatically have a B-comodule structure such that

$$\begin{aligned}\langle\, v^{(1)} k_{(1)}, x \,\rangle (v^{(2)} \triangleright k_{(2)}) &= \langle\, v^{(1)},\, x_{(1)} \,\rangle \langle\, k_{(1)},\, x_{(2)} \,\rangle (v^{(2)} \triangleleft k_{(2)})\\ &= (v \triangleleft (k_{(1)} \triangleright x)) \triangleleft k_{(2)} = (v \triangleleft k) \triangleleft x\\ &= \langle\, x,\, (v \triangleleft k)^{(1)} \,\rangle (v \triangleleft k)^{(2)},\end{aligned}$$

which shows that $v^{(1)} k_{(1)} \otimes (v^{(2)} \triangleleft k_{(2)}) = (v \triangleleft k)^{(1)} \otimes (v \triangleleft k)^{(2)}$, i.e., that V is a right relative (B, K)-module.

Thus, we see that the principal graph is given by the connected component of the Bratteli diagram of the inclusion $\widehat{B} \subset \widehat{B} \rtimes K$ containing the trivial representation of $\widehat{B}$. Since $\widehat{B} \rtimes K$ is the basic construction for the inclusion $\delta(K) \subset \widehat{B}$, therefore the Bratteli diagrams of the above two inclusions are the same. $\square$

COROLLARY 9.3.2. *If $N \subset N \rtimes B$ is a depth 2 inclusion corresponding to the quantum groupoid B, then its principal graph is given by the Bratteli diagram of the inclusion $\widehat{B}_t \subset \widehat{B}$.*

PROOF. In this case $K = B$ and inclusion $\widehat{B}_t \subset \widehat{B}$ is connected, so that $\delta(K) = \widehat{B}_t$ (note that $\widehat{B}$ is biconnected). $\square$

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