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Unsolved Problems in Combinatorial Games

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We have retained the numbering from the list of unsolved problems given on pp. 183–189 of AMS *Proc. Sympos. Appl. Math.* **43**(1991), called PSAM **43** below, and on pp. 475–491 of this volume's predecessor, *Games of No Chance*, hereafter referred to as GONC. This list also contains more detail about some of the games mentioned below. References in brackets, e.g., Ferguson [1974], are listed in Fraenkel's Bibliography later in this book; WW refers to

Elwyn Berlekamp, John Conway and Richard Guy, *Winning Ways for your Mathematical Plays*, Academic Press, 1982. A.K.Peters, 2000.

and references in parentheses, e.g., Kraitchik (1941), are at the end of this article.

1. Subtraction games are known to be periodic. Investigate the relationship between the subtraction set and the length and structure of the period. The same question can be asked about *partizan* subtraction games, in which each player is assigned an individual subtraction set. See Fraenkel and Kotzig [1987].

See also Subtraction Games in WW, 83–86, 487–498 and in the Impartial Games article in GONC. A move in the game $S(s_1, s_2, s_3, \ldots)$ is to take a number of beans from a heap, provided that number is a member of the **subtraction-set**, $\{s_1, s_2, s_3, \ldots\}$. Analysis of such a game and of many other heap games is conveniently recorded by a **nim-sequence**,

$$n_0 n_1 n_2 n_3 \ldots,$$

meaning that the nim-value of a heap of h beans is n_h, $h = 0, 1, 2, \ldots$, i.e., that the value of a heap of h beans in this particular game is the **nimber** $*n_h$. To avoid having to print stars, we say that the nim-value of a position is n, meaning that its value is the nimber $*n$.

For examples see Table 2 in § **4** on p. 67 of the Impartial Games paper in GONC.

In subtraction games the nim-values 0 and 1 are remarkably related by **Ferguson's Pairing Property** [Ferguson [1974]; WW, 86, 422]: if s_1 is the least member of the subtraction-set, then

$$\mathsf{G}(n) = 1 \qquad \text{just if} \qquad \mathsf{G}(n - s_1) = 0.$$

Here and later "$\mathsf{G}(n) = v$" means that the nim-value of a heap of n beans is v.

It would now seem feasible to give the complete analysis for games whose subtraction sets have just three members, but the detail has so far eluded those who have looked at the problem.

2. Are all finite **octal games** ultimately periodic? Resolve any number of outstanding particular cases, e.g., ·**6** (Officers), ·**06**, ·**14**, ·**36**, ·**64**, ·**74**, ·**76**, ·**004**, ·**005**, ·**006**, ·**007** (One-dimensional tic–tac–toe, Treblecross), ·**016**, ·**106**, ·**114**, ·**135**, ·**136**, ·**142**, ·**143**, ·**146**, ·**162**, ·**163**, ·**172**, ·**324**, ·**336**, ·**342**, ·**362**, ·**371**, ·**374**, ·**404**, ·**414**, ·**416**, ·**444**, ·**564**, ·**604**, ·**606**, ·**744**, ·**764**, ·**774**, ·**776** and **Grundy's Game** (split a heap into two unequal heaps), which has been analyzed, mainly by Dan Hoey, as far as heaps of 5×2^{32} beans.

A similar unsolved game is John Conway's **Couples-Are-Forever** where a move is to split any heap except a heap of two. The first 50 million nim-values haven't displayed any periodicity. See Caines et al. [1999].

Explain the structure of the periods of games known to be periodic.

[If the binary expansion of the kth code digit in the game with code

$$\mathbf{d}_0 \cdot \mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3 \ldots$$

is

$$\mathbf{d}_k = 2^{a_k} + 2^{b_k} + 2^{c_k} + \cdots,$$

where $0 \le a_k < b_k < c_k < \cdots$, then it is legal to remove k beans from a heap, provided that the rest of the heap is left in exactly a_k or b_k or c_k or ... non-empty heaps. See WW, 81–115. Some specimen games are exhibited in Table 3 of § **5** of the Impartial Games paper in GONC.]

In GONC, p. 476, we listed ·**644**, but its period, 442, had been found by Richard Austin in his thesis [1976].

Gangolli and Plambeck [1989] established the ultimate periodicity of four octal games which were previously unknown: ·**16** has period 149459 (a prime!), the last exceptional value being $\mathsf{G}(105350) = 16$. The game ·**56** has period 144 and last exceptional value $\mathsf{G}(326639) = 26$. The games ·**127** and ·**376** each have period 4 (with cycles of values 4, 7, 2, 1 and 17, 33, 16, 32 respectively) and last exceptional values $\mathsf{G}(46577) = 11$ and $\mathsf{G}(2268247) = 42$.

Achim Flammenkamp has recently settled ·**454**: it has the remarkable period and preperiod of 60620715 and 160949018, in spite of only $\mathsf{G}(124) = 17$ for the last sparse value and 41 for the largest nim-value, and even more recently has determined that ·**104** has period and preperiod 11770282 and 197769598 but no sparse space. For information on the current status of each of these games, see Flammenkamp's web page at http://www.uni-bielefeld.de/~achim/octal.html.

In Problem 38 in *Discrete Math.*, **44**(1983) 331–334 Fraenkel raises questions concerning the computational complexity of octal games. In Problem 39, he and Kotzig define **partizan octal games** in which distinct octals are assigned to the two players. In Problem 40, Fraenkel introduces **poset games**, played on a partially ordered set of heaps, each player in turn selecting a heap and removing a positive number of beans from this heap and all heaps which are above it in the poset ordering. Compare Problem **23** below.

3. Examine some **hexadecimal games**.

[**Hexadecimal games** are those with code digits $\mathbf{d}_k$ in the interval from **0** to **F** (= **15**), so that there are options splitting a heap into three heaps. See WW, 116–117.]

Such games may be arithmetically periodic. That is, the nim-values belong to a finite set of arithmetic progressions with the same common difference. The number of progressions is the period and their common difference is called the **saltus**. Sam Howse has calculated the first 1500 nim-values for each of the 1-, 2- and 3-digit games. Richard Austin's theorem 6.8 in his 1976 thesis suffices to confirm the (ultimate) arithmetic periodicity of several of these games.

For example ·**XY**, where **X** and **Y** are each **A, B, E** or **F** and ·**E8**, ·**E9**, ·**EC** and ·**ED** are each equivalent to Nim.

·**0A**, ·**0B**, ·**0E**, ·**0F**, ·**1A**, ·**1B**, ·**48**, ·**4A**, ·**4C**, ·**4E**, ·**82**, ·**8A**, ·**8E** and ·**CZ**, where **Z** is any even digit, are equivalent to Duplicate Nim, while ·**0C**, ·**80**, ·**84**, ·**88**, and ·**8C** are like Triplicate Nim.

Some games displayed ordinary periodicity; ·**A2**, ·**A3**, ·**A6**, ·**A7**, ·**B2**, ·**B3**, ·**B7** have period 4, and ·**81**, ·**85**, ·**A0**, ·**A1**, ·**A4**, ·**A5**, ·**B0**, ·**B1**, ·**B5**, ·**D0**, ·**F0**, ·**F1** are all essentially She-Loves-Me-She-Loves-Me-Not.

·**9E**, ·**9F**, ·**BC**, ·**C9**, ·**CB**, ·**CD** and ·**CF** have (apparent ultimate) period 3 and saltus 2; ·**89**, ·**8D**, ·**A8**, ·**A9**, ·**AC**, ·**AD** each have period 4 and saltus 2, while ·**8B**, ·**8F** and ·**9B** have period 7 and saltus 4.

More interesting specimens are ·**28** = ·**29**, which have period 53 and saltus 16, the only exceptional value being $\mathsf{G}(0) = 0$; ·**9C**, which has period 36, preperiod 28 and saltus 16; and ·**F6** with period 43 and saltus 32, but its apparent preperiod of 604 and failure to satisfy one of the conditions of the theorem prevent us from verifying the ultimate periodicity.

The above accounts for nearly half of the two-digit genuinely hexadecimal (i.e., containing at least one **8**) games. There remain almost a hundred for which a pattern has yet to be established.

Kenyon's Game, ·**3F**, had been the only example found whose saltus of 3 countered the conjecture of Guy and Smith that it should always be a power of two. But Nowakowski has now shown that ·**3F3** has period 10 and saltus 5; ·**209**, ·**228** have period 9 with saltus 3; and ·**608** has period 6 and saltus 3. Further examples whose saltus is not a power of two may be ·**338**, probably with period 17 and saltus 6 and several, probably isomorphic, with period 9 and saltus 3.

The game ·**9** has not so far yielded its complete analysis, but, as far as analyzed, i.e. to 12000, exhibits a remarkable fractal-like set of nim-values. See Austin, Howse and Nowakowski (2002).

4. Extend the analysis of **Domineering**.

[Left and Right take turns to place dominoes on a checker-board. Left orients her dominoes North-South and Right orients his East-West. Each domino exactly covers two squares of the board and no two dominoes overlap. A player unable to play loses.]

See Berlekamp [1988] and the second edition of WW, 138–142, where some new values are given. For example David Wolfe and Dan Calistrate have found the values (to within '-ish', i.e., infinitesimally shifted) of 4×8, 5×6 and 6×6 boards. Lachmann, Moore and Rapaport (this volume) discover who wins on rectangular, toroidal and cylindrical boards of widths 2, 3, 5 and 7, but do not find their values.

Berlekamp asks, as a hard problem, to characterize all hot Domineering positions to within "ish". As a possibly easier problem he asks for a Domineering position with a new temperature, i.e., one not occurring in Table 1 on GONC, p. 477.

5. Analyze positions in the game of **Go**.

Compare Berlekamp [1988], his book with Wolfe [1994], and continuing discoveries, discussed in GONC and the present volume, which also contains Spight's analysis of an enriched environment Go game and Takizawa's rogue Ko positions.

6. Go-Moku. Solved by Allis, Herik and Huntjens [1996].

7. Complete the analysis of impartial **Eatcakes** (WW, 269, 271, 276–277).

[Eatcakes is an example of a **join** or **selective compound** of games. Each player plays in **all** the component games. It is played with a number of rectangles, $m_i \times n_i$; a move is to remove a strip $m_i \times 1$ or $1 \times n_i$ from each rectangle, either splitting it into two rectangles, or reducing the length or breadth by one. Winner removes the last strip.]

For fixed breadth the remoteness becomes constant when the length is sufficiently large. But 'sufficiently large' seems to be an increasing function of the breadth and doesn't, in the hand calculations already made, settle into any clear pattern. Perhaps computer calculations will reveal something.

8. Complete the analysis of **Hotcakes** (WW, 279–282).

[Also played with integer-sided rectangles, but as a **union** or **selective compound** in which each player moves in **some** of the components. Left cuts as many rectangles vertically along an integer line as she wishes, and then rotates one from each pair of resulting rectangles through a right angle. Right cuts as many rectangles as he wishes, horizontally into pairs of integer-sided rectangles and rotates one rectangle from each pair through a right angle. The **tolls** for rectangles with one dimension small are understood, but much remains to be discovered.]

9. Develop a **misère theory** for unions of partizan games.

[In a union of two or more games, you move in as many component games as you wish. In misère play, the last player *loses*.]

10. Extend the analysis of **Squares Off** (WW, 299).

[Played with heaps of beans. A move is to take a perfect square (> 1) number of beans from any number of heaps. Heaps of 0, 1, 2 or 3 cannot be further reduced. A move leaving a heap of 0 is an overriding win for the player making it. A move leaving 1 is an overriding win for Right, and one leaving 2 is an overriding win for Left. A move leaving 3 doesn't end the game unless all other heaps are of size 3, in which case the last player wins.]

11. Extend the analysis of **Top Entails** (WW, 376–377).

[Played with stacks of coins. Either split a stack into two smaller ones, or remove the top coin from a stack. In the latter case your opponent's move must use the same stack. Last player wins. Don't leave a stack of 1 on the board, since your opponent must take it and win, since it's now your turn to move in an empty stack!]

We are unable to report any advance on Julian West's discovery of loony positions at 2403 coins, 2505 coins, and 33,243 coins. The authors of *Winning Ways* did not know of a loony stack of more than 3 coins. These results are typical of the apparently quite unpredictable nature of combinatorial games, even when they have quite simple rules.

12. Extend the analysis of **All Square** (WW, 385).

[This game involves **complimenting moves** after which the same player has an extra **bonus move**. Note that this happens in Dots-and-Boxes when a box is completed. All Square is played with heaps of beans. A move splits a heap into two smaller ones. If both heap sizes are perfect squares, the player must move again: if he can't he loses!]

13. Extend the misère analysis of various octal games, e.g., **Officers**, **Dawson's Chess**, ..., and of **Grundy's Game** (WW, 411–421).

William L. Sibert made a breakthrough by completing the analysis of misère Kayles; see the post-script to Sibert and Conway [1992]. Plambeck [1992] has used their method to analyze a few other games, but there's a wide open field here. Recently, Allemang (2001) has extended the content of his 1984 thesis to include the complete analysis of ·**26**, ·**53**, ·**54**, ·**72**, ·**75** and **4·7**. (See also http://spdcc.com:8431/summary.html.)

We can ask the same question for Hexadecimal games (see Problem **3**).

14. Moebius, when played on 18 coins has a remarkable pattern. Is there any trace of pattern for larger numbers of coins? Can any estimate be made for the rate of growth of the nim-values?

[See Coin-turning games in WW, 432–435; and Vera Pless's lecture and the Impartial Games lecture in PSAM **43**. Moebius is played with a row of coins. A move turns 1, 2, 3, 4 or 5 coins, of which the rightmost must go from heads to tails (to make sure the game satisfies the Ending Condition). The winner is the player who makes all coins tails.]

15. Mogul has an even more striking pattern when played on 24 coins, which has some echoes when played on 40, 56 or 64 coins. Thereafter, is there complete chaos?

[See references for Problem **14**. A move turns 1, 2, ..., 7 coins.]

16. Find an analysis of **Antonim** with four or more coins (WW, 459–462).

[Played with coins on a strip of squares. A move moves a coin from one square to a smaller-numbered square. Only one coin to a square, except that square zero can have any number of coins. It is known that (a, b, c) is a P-position in Antonim just if $(a+1, b+1, c+1)$ is a P-position in Nim, but for more than 3 coins much remains to be discovered.]

17. Extend the analysis of **Kotzig's Nim** (WW, 481–483). Is the game eventually periodic in terms of the length of the circle for every finite move set? Analyze the misère version of Kotzig's Nim.

[Players alternately place coins on a circular strip, at most one coin on a square. Each coin must be placed m squares clockwise from the previously placed coin, provided m is in the given **move set**, and provided the square is not already occupied. The complete analysis is known only for a few small move sets.]

See Fraenkel, Jaffray, Kotzig and Sabidussi [1995].

18. Obtain asymptotic estimates for the proportions of N-, O- and P-positions in Epstein's **Put-or-Take-a-Square** game (WW, 484–486).

[Played with one heap of beans. At each turn there are just two options, to take away or add the largest perfect square number of beans that there is in the heap. 5 is a P-position, because 5 ± 4 are both squares; 2 and 3 are O-positions, a win for neither player, since the best play is to go from one to the other, and not to 1 or 4 which are N-positions.]

19. Simon Norton's game of **Tribulations** is similar to Epstein's game, but squares are replaced by triangular numbers. Norton conjectures that there are no O-positions, and that the N-positions outnumber the P-positions in golden ratio. True up to 5000 beans.

Investigate other put-or-take games. If the largest number of form $2^k - 1$ is put or taken, we have yet another disguise for She-Loves-Me-She-Loves-Me-Not, with the remoteness given by the binary representation of the number of beans. For Fibulations and Tribulations, see WW 501–503. If the largest number used is of form T_n+1, where T_n is a triangular number, the P-positions are the multiples of 3.

20. Complete the analysis of **D.U.D.E.N.E.Y**

[Played with a single heap of beans. Either player may take any number of beans from 1 to Y, except that the immediately previous move mustn't be repeated. When you can't move you lose. Analysis easy for Y even, and known (WW, 487–489) for 53/64 of the odd values of Y.]

Marc Wallace and Alan Jaffray made a little progress here, but is the situation one in which there is always a small fraction of cases remaining, no matter how far the analysis is pursued?

21. Schuhstrings is the same as D.U.D.E.N.E.Y, except that a deduction of zero is also allowed, but cannot be immediately repeated (WW, 489–490).

22. Analyze **Nim** in which you are not allowed to repeat a move. There are at least five possible forms (assume that b beans have been taken from heap H):

medium local: b beans may not be taken from heap H until some other move is made in heap H.

short local: b beans may not be taken from heap H on the next move.

long local: b beans may never again be taken from heap H.

short global: b beans may not be taken from any heap on the next move.

long global: b beans may never again be taken from any heap.

23. Burning-the-Candle-at-Both-Ends. John Conway and Aviezri Fraenkel ask us to analyze Nim played with a row of heaps. A move may only be made in the leftmost or in the rightmost heap. When a heap becomes empty, then its neighbor becomes the end heap.

Albert and Nowakowski [2001] have solved the impartial and partizan versions. But there is also **Hub-and-Spoke Nim**, proposed by Fraenkel. One heap is the hub and the others are arranged in rows forming spokes radiating from the hub.

There are several versions:

(a) beans may be taken only from a heap at the end of a spoke;

(b) beans may also be taken from the hub;

(c) beans may be taken from the hub only when all the heaps in a spoke are exhausted;

(d) beans may be taken from the hub only when just one spoke remains;

(e) in versions (b), (c) and (d), when the hub is exhausted, beans may be taken from a heap at either end of any remaining spoke; i.e. the game becomes the sum of a number of games of Burning-the-Candle-at-Both-Ends.

Albert notes that Hub-and-Spoke Nim can be generalized to playing on a forest, i.e., a graph each of whose components is a tree. The most natural variant is that beans may only be taken from a leaf (valence 1) or isolated vertex (valence 0).

24. Continue the analysis of **The Princess and the Roses** (WW, 490–494).

[Played with heaps of beans. Take one bean, or two beans, one from each of two different heaps. The rules seem trivially simple, but the analysis takes on remarkable ramifications.]

25. Extend the analysis of the Conway-Paterson game of **Sprouts** in either the normal or misère form. (WW, 564–568).

[A move joins two spots, or a spot to itself by a curve which doesn't meet any other spot or previously drawn curve. When a curve is drawn, a new spot must be placed on it. The valence of any spot must not exceed three.]

Applegate, Jacobson and Sleator [1999] have pushed the normal analysis to 11 initial spots and the misère analysis to 9.

number of spots	1	2	3	4	5	6	7	8	9	10	11
normal play	P	P	N	N	N	P	P	P	N	N	N
misère play	N	P	P	P	N	N	P	P	P		

where P and N denote previous-player and next-player winners. There is a temptation to conjecture that the patterns continue.

26. Extend the analysis of **Sylver Coinage** (WW, 575–597).

[Players alternately name different positive integers, but may not name a number which is the sum of previously named ones, with repetitions allowed. Whoever names 1 loses. See Section 3 of Richard Nowakowski's chapter in PSAM **43**.]

27. Extend the analysis of **Chomp** (WW, 598–599).

[Players alternately name divisors of N, which may not be multiples of previously named numbers. Whoever names 1 loses. David Gale offers a prize of US$100.00 for the first complete analysis of 3D-Chomp, i.e., where N has three distinct prime divisors, raised to arbitrarily high powers.]

Doron Zeilberger (www.ics.uci.edu/~eppstein/cgt) has analyzed Chomp for $N = 2^2 3^n$ up to $n = 114$. For an excursion into infinite Chomp, see Huddleston and Shurman [2001] in this volume.

28. Extend Úlehla's or Berlekamp's analysis of **von Neumann's game** from directed forests to directed acyclic graphs (WW, 570–572; Úlehla [1980]).

[von Neumann's game, or Hackendot, is played on one or more rooted trees. The roots induce a direction, towards the root, on each edge. A move is to delete a node, together with all nodes on the path to the root, and all edges incident with those nodes. Any remaining subtrees are rooted by the nodes that were adjacent to deleted nodes.]

Since Chomp and the superset game (Gale and Neyman [1982]) can be described in terms of directed acyclic graphs but not by directed forests, a partial analysis of such an extension of von Neumann's game could throw some light on these two unsolved games. Fraenkel and Harary [1989] discuss a similar game, but with the directions determined by shortest distances. They find winning strategies for trees in normal play, circuits in normal and misère play; and for complete graphs with rays of equal length in normal play.

29. Prove that Black doesn't have a forced win in **Chess**.

Andrew Buchanan has recently emailed that he has examined some simpler (sub-)problems in which the moves 1. e4, e5 are made followed by either a Bishop move by each player, or a Queen move by each player. He claims that at most six of each of these sets of positions can be wins for Black.

30. A **King and Rook v. King** problem. Played on a quarter-infinite board, with initial position WKa1, WRb2 and BKc3. Can White win? If so, in how few moves? It may be better to ask, "what is the smallest board (if any) that White can win on if Black is given a win if he walks off the North or East edges of the board?" Is the answer 9×11? In an earlier edition of this paper I attributed this problem to Simon Norton, but it was proposed as a kriegsspiel problem, with unspecified position of the WK, and with W to win with probability 1, by Lloyd Shapley around 1960.

31. David Gale's version of **Lion and Man**. L and M are confined to the non-negative quadrant of the plane. They move alternately a distance of at most one unit. For which initial positions can L catch M?

David Gale's Lion-and-Man has been solved by Jiří Sgall [2001].

Variation. Replace quadrant by wedge-shaped region.

32. Gale's Vingt-et-un. Cards numbered 1 through 10 are laid on the table. L chooses a card. Then R chooses cards until his total of chosen cards exceeds the card chosen by L. Then L chooses until her cumulative total exceeds that of R, etc. The first player to get 21 wins. Who is it?

[As posed here it is not clear if the object is to get 21 exactly or 21-or-more. Jeffery Magnoli, a student of Julian West, thought that the latter rule was the more interesting and found a first-player win in 6-card Onze and in 8-card Dix-sept.]

33. Subset Take-away. Given a finite set, players alternately choose proper subsets subject to the rule that once a subset has been chosen, none of *its* subsets may be chosen subsequently by either player. Last player wins.

[David Gale conjectures that it's a second player win—this is true for sets of less than six elements.]

34. Eggleton and Fraenkel ask for a theory of **Cannibal Games** or an analysis of special families of positions. They are played on an arbitrary finite digraph. Place any numbers of "cannibals" on any vertices. A move is to select a cannibal and move it along a directed edge to a neighboring vertex. If this is occupied, the incoming cannibal eats the whole population (**Greedy Cannibals**) or just one cannibal (**Polite Cannibals**). A player unable to move loses. Draws are possible. A partizan version can be played with cannibals of two colors, each eating only the opposite color.

35. Welter's Game on an arbitrary digraph. Place a number of monochromatic tokens on distinct vertices of a directed acyclic graph. A token may be moved to any *unoccupied* immediate follower. Last player wins. Make a dictionary of P-positions and formulate a winning strategy for other positions. See Kahane and Fraenkel [1987] and Kahane and Ryba [2001].

36. Restricted Positrons and Electrons. Fraenkel places a number of Positrons (Pink tokens) and Electrons (Ebony tokens) on distinct vertices of a Welter strip. Any particle can be moved by either player leftward to any square u provided that u is either unoccupied or occupied by a particle of the opposite type. In the latter case, of course, both particles become annihilated (i.e., they are removed from the strip), as physicists tell us positrons and electrons do. Play ends when the excess particles of one type over the other are jammed in the lowest positions of the strip. Last player wins. Formulate a winning strategy for those positions where one exists. Note that if the particles are of one type only, this is Welter's Game. As a strategy is known for Misère Welter [WW, 480–481] it may not be unreasonable to ask for a misère analysis as well. See Problem 47, *Discrete Math.*, **46** (1983) 215–216.

37. General Positrons and Electrons. As Problem **36** but played on an arbitrary digraph. Last player wins.

38. Fulves's Merger. Start with heaps of 1, 2, 3, 4, 5, 6 and 7 beans. Two players alternately transfer any number of beans from one heap to another, except that beans may not be transferred from a larger to a smaller heap. The player who makes all the heaps *even* in size is the winner.

The total number of beans remains constant, and is even (28 in this case, though one is interested in even numbers in general: a similar game can be played in which the total number is odd and the object is to make all the heaps odd in size).

No progress has been reported for the general game.

39. Sowing or **Mancala Games**. Kraitchik (1941, p. 282) describes Ruma, which he attributes to Mr. Punga. Bell and Cornelius (1988, pp. 22–38) list Congklak, from Indonesia; Mankal'ah L'ib Al-Ghashim (the game of the unlearned); Leab El-Akil (the game of the wise or intelligent); Wari; Kalah, from Sumeria; Gabata, from Ethiopia; Kiarabu, from Africa; as well as some solitaire games based on a similar idea. Botermans et al. (1989, pp. 174–179) describe Mefuhva from Malawi and Kiuthi from Kenya. Many of these games go back for thousands of years, but several should be susceptible to present day methods of analysis. See Jeff Erickson's article in GONC.

Conway starts with a **line** of heaps of beans. A typical move is to take (some of) a heap of size N and do something with it that depends on the game and on N. He regards the nicest move as what he calls the **African move** in which **all** the beans in a heap are picked up and 'sowed' onto successive heaps, **and** subject to the condition that the last bean must land on a nonempty heap. Beans are sowed to the right if you are Left, to the left if you are Right, or either way if you're playing impartially.

In the partizan version, the position 1 (a single bean) has value 0, of course; the position $1.1 = \{0.2 \mid 2.0\}$ has value $\{0 \mid 0\} = *$; and so does

$$1.1.1 = \{1.0.2, 2.1 \mid 1.2, 2.0.1\},$$

since $2.1 = \{\ \mid 3.0\}$, $3.0 = \{\ \mid\ \} = 0$ and $1.0.2 = \{\ \mid 2.1\}$, so that '3' has value 0, 2.1 has value -1, 1.0.2 value -2, 1.1.1 value $\{-2, -1 \mid 1, 2\} = 0$, 1.1.1.1 value 0, and 1.1.1.1.1 value $\pm\frac{1}{2}$.

Recent papers on mancala-type games are Björner and Lovász [1992], Broline and Loeb [1995] and Yeh Yeong-Nam [1995].

40. Chess. Noam Elkies has found endgames with values 0, 1, $\frac{1}{2}$, $*$, $*k$ for many k, $\uparrow$, $\uparrow *$, $\Uparrow *$, etc.; see his papers in GONC and this volume. See also Problems **29**, **30** and **45**.

41. Sequential compounds of games have been studied by Stromquist and Ullman. They mention a more general compound. Let $(P, <)$ be a finite poset and for each $x \in P$ let G_x be a game. Consider a game $G(P)$ played as follows. Moves are allowed in any single component G_x provided that no legal moves remain in any component G_y with $y > x$. A player unable to move loses. The sequential compound is the special case when $(P, <)$ is a chain (or linear order). The **sum** or disjunctive compound is the case where $(P, <)$ is an antichain. They have no coherent theory of games $G(P)$ for arbitrary posets. They list some more specific problems which may be more tractable. Compare Problem **23** above.

42. Beanstalk and **Beans-Don't-Talk** are games invented by John Isbell and by John Conway. See Guy [1986]. Beanstalk is played between Jack and the Giant. The Giant chooses a positive integer, n_0. Then J. and G. play alternately n_1, n_2, n_3, ... according to the rule $n_{i+1} = n_i/2$ if n_i is even, $= 3n_i \pm 1$ if n_i is odd; i.e. if n_i is even, there's only one option, while if n_i is odd there are just two. The winner is the person moving to 1. If the Giant chooses an odd number > 1, can Jack always win? Not by using the Greedy Strategy (always descend when it's safe to do so) as this can lead to cycles (draws).

In Beans-Don't-Talk, the move is from n to $(3n \pm 1)/2^*$ where 2^* is the highest power of two dividing the numerator; the winner is still the person moving to 1. Are there any drawn positions? There are certainly drawn **plays**, e.g., 7 (5) 7 (5) ... , but 5 is an N-position because there is the immediate winning option $(5 \times 3 + 1)/2^4 = 1$, and 7 is a P-position since the other option $(7 \times 3 + 1)/2 = 11$ is met by $(11 \times 3 - 1)/2^5 = 1$. What we want to know is: *are there any* **O-positions** (positions of infinite remoteness)?

[For remoteness see Chapter 9 of WW. There are several unanswered questions about the remotenesses of positions in these two games. Remoteness may also be the best tool we have for Problems **18** and **19** above.]

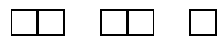
43. Inverting Hackenbush. John Conway turns Blue-Red Hackenbush, played on finite strings of edges, into a hot game by amending the move to 'remove an edge of your color and everything thus disconnected from the ground, and then turn the remaining string upside-down and replant it'. The analysis replaces the 'number tree' (WW, p. 25) by a similar tree, but with the smaller binary fractions replaced by increasingly hot games. The game can be generalized to play on trees: a move which prunes the tree at a vertex V includes replanting the tree with V as its root.

44. Konane. See the paper by Ernst and Berlekamp in GONC. There is much to be discovered about this fascinating and eminently playable game, which exhibits the values 0, $*$, $*2$, $\uparrow$, 2^{-n}, and many other infinitesimals and also hot values of arbitrarily high temperature. Chan and Tsai, in this volume, give some values for $1 \times n$ boards.

45. Elwyn Berlekamp asks for the **habitat** of $*2$. [$*2 = \{0, * | 0, *\}$.] It does not occur in Blockbusting, Hackenbush, Col or Go. It does occur in Konane and 6×6 Chess. What about Chilled Go, Domineering and 8×8 Chess? Elkies (see this volume) has modified and generalized the game Dawson's Chess to give games, on suitably large boards, of value $*k$ for many large k and conjectures that such values exist for all k.

46. There are various ways of playing **two-dimensional Nim**. One form is discussed on p. 313 of WW. Another is proposed by Berman, Fraenkel and Kahane in Problem 41, *Discrete Math.*, **45** (1983) 137–138. Start with a rectangular array of heaps of beans. At each move a row or column is selected and a positive number of beans is taken from some of the heaps in that row or column; see Fremlin [1973]. Ferguson's [2001] variant has the move as choosing a number and subtracting that number from all members of a row or column. He finds the outcomes for 2×2 matrices in both the impartial and partizan versions. Another variant is where beans may be taken only from contiguous heaps. Other variants are played on triangular or hexagonal boards; a special case of this last is Piet Hein's Nimbi, solved by Fraenkel and Herda [1980].

47. Many results are known concerning tiling rectangles with **polyominoes**. One can extend such problems to disconnected polyominoes. E.g., in GONC we asked if a rectangle can be tiled by

 or by 

If so, what are the rectangles of least size that can be so tiled?

Juha Saukkola showed that the former will not tile a rectangle, but that the latter can tile a 12×15 rectangle. Since then Joe Devincentis, Erich Friedman, Patrick Hamlyn, Mike Reid and others have examined numerous other cases. See http://www.stetson.edu/~efriedma/mathmagic/0299.html.

There is an obvious generalization of Domineering (see Problem **4** above) to a two-player game in which the players alternately place polyominoes of given shape and orientation on a rectangular or other board.

48. Find all words which can be reduced to 1 peg in 1-dimensional **Peg Solitaire**. E.g., 1, 011, 110, 1101, 110101, $1(10)^k1$. Here 1 represents a peg and 0 an empty space. A move is for a peg to jump over an adjacent peg into an empty adjacent space, and remove the jumped-over peg. E.g., 1101 $\rightarrow$ 0011 $\rightarrow$ 0100. Georg Gunther, Bert Hartnell and Richard Nowakowski found that for an $n \times 1$ board with one empty space, n must be even and the space next but one to the end. If the board is cyclic, the condition is simply n even. Christopher Moore and David Eppstein, indicate, in this volume, that this problem has been solved many times but does not seem to have been published. They coin the term Duotaire for one-dimensional peg solitaire played as a two-player game. They give some decomposition theorems and conjecture that arbitrarily high nim-values occur. J. P. Grossman notes that the position

$$\bigcirc\bigcirc\bullet(\bullet\bigcirc\bigcirc\bullet\bigcirc\bullet)^{20601}\bigcirc$$

i.e. a strip of 123610 squares, of which the first, second and last squares are occupied, together with the $(6n{+}5)$-th, $(6n{+}6)$-th and $(6n{+}8)$-th, for $0 \le n \le 20600$, has nim-value 197.

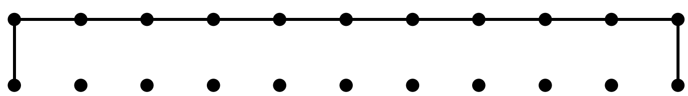
49. Elwyn Berlekamp asks if there is a game which has

1. simple, playable rules,
2. an intricate explicit solution, and
3. is provably NP or harder.

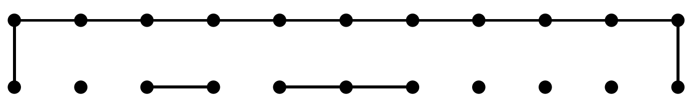
Is Phutball (WW, 688–691) such a game? See Demaine, Demaine and Eppstein and also Grossman and Nowakowski in this volume. Compare Problem **57** below.

50. John Selfridge asks: is **Four-File** a draw? Four-File is played on a chessboard with the chess pieces in their usual starting positions, but only on the a-, c-, e- and g-files. I.e. a rook, a bishop, a king, a knight and four pawns on each side. The moves are normal chess moves except that play takes place only on these four files. I.e., each move ends on one of the files a, c, e or g; pawns cannot capture and there is no castling. The aim is to checkmate your opponent's king.

51. Elwyn Berlekamp asks for a complete theory of closed $1 \times n$ **Dots-and-Boxes**. I.e., with starting position

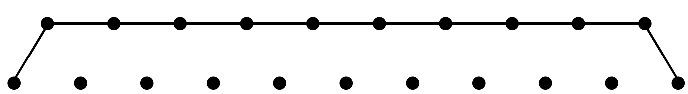


A sample position is



See WW, Chapter 16 and Berlekamp's book, *The Dots-and-Boxes Game* [2000]. Are there more nimber decomposition theorems? Compile a datebase of nim-values.

Nowakowski and Ottaway conjecture that closed $1 \times n$ Dots-and-Triangles (a row of n dots on top and n+1 on bottom with the top and sides already having lines) is a first player win except for $n = 2$.



52. How does one play **sums** of games with varied overheating operators? Berlekamp notes that overheating operators provide a very concise way of expressing closed-form solutions to many games, and David Moews observes that monotonicity and linearity depend on the parameters and the domain. Find a simple, elegant way of relating the operator parameters to the game. See WW, pp. 163–175, Berlekamp [1988], Berlekamp and Wolfe [1994] and Calistrate's paper in GONC.

53. N-heap Wythoff Game. Aviezri Fraenkel asks some questions and makes some conjectures. The set of all integers $\geq m$ is denoted by $Z_{\geq m}$ and $\oplus$ denotes Nim addition. For any subset $S \subset Z_{\geq 0}$, $S \neq Z_{\geq}0$, let $\operatorname{mex} S = \min(Z_{\geq}0 \setminus S) =$ least nonnegative integer not in S.

Define an N-heap Wythoff game as follows: Given $N \geq 2$ heaps of finitely many tokens, whose sizes are $p_1, \ldots, p_N$. The moves are to take any positive number of tokens from a *single* heap or to take $(a_1, \ldots, a_N) \in Z_{\geq 0}^N$ from *all* the heaps — a_i from the i-th heap — subject to the conditions: (i) $a_i > 0$ for some i, (ii) $a_i \leq p_i$ for all i, (iii) $a_1 \oplus \ldots \oplus a_N = 0$. The player making the last move wins and the opponent loses. Note that the classical Wythoff game is the case $N = 2$.

For $N = 3$, denote by (A_n, B_n, C_n) the P-positions of the game, with $A_n \leq B_n \leq C_n$. We conjecture that

> For every fixed $A_k = k \in Z_{\geq 1}$ there exists an integer $m = m(k) \in Z_{\geq 1}$ such that $B_n = \operatorname{mex}\bigl(\{B_i, C_i : i < n\} \cup T\bigr)$, $C_n = B_n + n$ for all $n \geq m$, where T is a (small) set of integers which depends only on k.

For example, for $k = 1$ we have $T = \{2\}$; and it seems that $m = 23$. A related conjecture is that

> For every fixed $A_k = k \in Z_{\geq 1}$ there exist integers $a = a(k)$, $j = j(k), m = m(k) \in Z_{\geq 1}$ with $j < a$, such that $B_n \in \{\lfloor n\phi \rfloor - (a + j), \lfloor n\phi \rfloor - (a + j - 1), \ldots, \lfloor n\phi \rfloor - (a - j + 1), \lfloor n\phi \rfloor - (a - j)\}$, $C_n = B_n + n$ for all $n \geq m$, where $\phi = (1 + \sqrt{5})/2$ (the golden section).

This appears to hold for $a = 4$, $j = 1$, $m = 64$ (perhaps a somewhat smaller value of m will do) when $k = 1$. Is perhaps $j = 1$ for all $k \geq 1$?

See also Coxeter [1953], Fraenkel and Ozery [1998] and Fraenkel and Zusman [2001].

54. Fox and Geese. Jonathan Welton notices that the conclusion of Chapter 20 of WW, namely that the value of Fox and Geese is $1 + 1/\mathbf{on}$, is incorrect. He believes that he can show that the geese can win with the fox having $1 + 1/32$ passes, and probably the actual value is still higher. What is the correct value?

[Fox and Geese is played on an ordinary checkerboard, the geese being four white checkers, moving diagonally forward, starting on squares a1, c1, e1, g1; while the fox is a black checker king moving in any diagonal direction, starting on d8. There is no capturing: the geese try to encircle the fox; the fox endeavors to break through.]

55. Amazons was invented by the Argentinian Walter Zamkauskas in 1988. It is played on a 10×10 board. Each player has four amazons. The white amazons are initially on a4, d1, g1, j4 and the black ones are on a7, d10, g10, j7. White moves first. Each move consists of two mandatory parts. First, an amazon moves just like a chess queen. After an amazon has moved she shoots a burning arrow, which also moves like a chess queen. The square where the arrow lands is burnt and is blocked for the rest of the game; neither an amazon nor an arrow can move to or over that square, nor to or over a square occupied by another amazon. There are no captures in Amazons. Nor are there draws: the aim is to control territory: the winner is the last player to complete a move.

Analyses of smaller boards with fewer amazons have been made. For example in Solving 5×5 Amazons (2001 preprint), Martin Müller shows that 5×5 Amazons (with amazons on a2, b1, d1, e2 and on a4, b5, d5, e4) is a first player win. Berlekamp [2000,2001] investigates sums of $2 \times n$ Amazon games. See also the papers of Müller and Tegos and of Snatzke in this volume.

56. Are there any draws in **Beggar-my-Neighbor**?

[Two players deal single cards in turn onto a common stack. If a court card (J, Q, K, A) is dealt, the next player must cover it with respectively 1, 2, 3, 4 cards. If one of these is a court card, the obligation to cover reverts to the previous player. If they are not court cards, the previous player acquires the stack, which he inverts and places beneath his own hand, and starts dealing again. A player loses if she is unable to play.]

This problem reappears periodically. It was one of Conway's 'anti-Hilbert problems' about 40 years ago, but must have suggested itself to players of the game over the several centuries of its existence.

Marc Paulhus [1999] exhibited some cycles with small decks, and used a computer to show that there were no cycles when the game is played with a half-deck, although the addition or subtraction of two non-court cards produced cycles. Michael Kleber found an arrangement of two 26-card hands which required the dealing of 5790 cards before a winner was declared.

57. Aviezri Fraenkel describes a game as **succinct** if its input size is logarithmic. Thus Nim is succinct, because its input size is the sum of the logarithms of its heap sizes. It has a polynomial time winning strategy, yet the loser can make length of play exponentially long. (A trivial example: two heaps of the same size, where Player I keeps removing a single token from one heap, which has to be matched by Player II taking a single token from the other heap.)

(a) Is there a nonsuccinct game with a polynomial winning strategy in which play can be made to last exponentially long?

(b) Node Kayles, on a general graph, was proved to be Pspace-complete by Schaefer [1978]. Its succinct form, the octal game $\cdot\mathbf{137}$, is polynomial. Is there a game which has a polynomial strategy on a general graph, but its succinct form is at least NP-hard?

[Node Kayles is played on a graph. A move is to place a counter on an unoccupied node that is not adjacent to any occupied node. Equivalently, to delete a node and all its neighbors. The game $\cdot\mathbf{137}$ is Dawson's Chess, i.e. Node

Kayles played on a path, and occurs in the analysis of several other games, notably Dots-and-Boxes. See WW, 92, 251, 466, 470, 532, 552.]

58. The one-dimensional version of **Clobber** is played on a $1 \times n$ strip of squares where there are blue and red pieces alternating, one to a square. Right moves the red pieces and Left the blue. A piece moves to an adjacent square but only if the square is occupied by an opposing piece. This piece is then removed from the board, i.e. it has been clobbered. Albert, Grossman and Nowakowski conjecture that $1 \times n$ Clobber is a first player win for $n \geq 13$. They also show that played on an arbitrary graph with one blue piece and the rest red, deciding the value of the game is NP-complete.

[Clobber is a special case of partizan Polite Cannibals (see Problem **34**) in which moves may only be made to occupied nodes.]

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