More Games of No Chance
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More Infinite Games

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ABSTRACT. Infinity in games introduces some weird and wonderful concepts. I examine a few here.

My PhD thesis was on transfinite numbers and ordered types and when I got my first job in Cambridge it was as a resident mathematical logician. This was very fortunate for me since I am also interested in wasting time, professionally, and I have invented some very powerful ways of wasting time. One of these is combinatorial game theory. Most combinatorial game theorists automatically have a finite mind set when they look at games — a game is a finite set of positions. However, as a logician, developing surreal numbers, this was irrelevant. I just took whatever was needed to make the theory work. For additive games the notion of sum worked very well. One does not need finiteness just, essentially, the idea that you cannot make an infinite sequence of legal moves.

One of the main results, the existence of strategies for games with no infinite chain of moves, has two proofs one of which works well for finite games the other for all games. The first involves drawing the game tree and, starting from the leaves, marking a position with a P, for *P*revious, if the next player to play from this position cannot win and otherwise mark it with an N, for a *N*ext player win. The other proof is a *reductio ad absurdum.* Suppose we have a game with no strategies. From this position look at the options: some will have strategies and some maybe not. If one option is marked with a P then we can move there and win so the original position is marked with an N. If all options are marked N then we can mark it as P. If there is no strategy then one of the options has no strategy and we iterate the argument and we get an infinite chain of moves which we supposing there wasn't. However, one of the interesting things about this argument is that a game can be perfectly well defined and computable but its winning strategy not be computable. The mathematical logician Michael Rabin studied this situation and found some interesting results.

There are games that are finite but the number of moves is not bounded. For example, choose an integer then reduce it by one on each turn afterward. Name

1000 and the game lasts for another 1000 moves. This is clearly unbounded but is boundedly unbounded — after one move you know how long the game will last. But there are games much worse than this. A boundedly unbounded game would have the property that at the beginning of the game, you could name a number n such that after n moves you know how long the game will last. An boundedly, unboundedly unbounded game would have a number n such that after n moves you could name a number m such that after m more moves you would know how long the game could last. It is easy to see that the order of unboundedness can be arbitrarily large, even infinite.

An example of this is Sylver Coinage, a game I invented when I wanted an example for a talk to a Cambridge Undergraduate Society. This game is played by two players who take turns in naming positive integers which cannot be made up from previously named numbers. The person who names 1 looses, otherwise the game is trivial. This game is finite, as can be shown by result of J. J. Sylvester. Naming 2 is a bad move because your opponent names 3 and now all even numbers cannot be named (since $4 = 2 + 2$, $6 = 2 + 2 + 2$, and so on) and neither can any odd number greater than 3 ($5 = 2+3$, $7 = 3+2+2$, and so on). Only naming 1 is left as a legal move, which loses.

Also, 4 and 6 are good replies to each. All the even numbers greater than 2 are eliminated and for $k \geq 1$, $4k + 1$, $4k + 3$ are good replies to one another. This seems like good evidence that the game is a loss for the first player.

However, a few weeks later, Hutchings came along and said "What was that game?" and we played. He started by naming 5 and he won. We played again, he named 5 and won. In fact, naming 5 can be shown to be a good move by a strategy stealing argument. So we know it is a good move but we don't know how. Indeed, this strategy stealing argument shows that all primes other than 2 and 3 are winning first moves for the first player. Since multiples of a prime are easy to answer, the only opening moves which are in doubt are those of the form 2^a3^b. We know that 2 and 3 are good answers to one another, so are 4 and 6 and so are 8 and 12. Is naming 16 a good move? Nobody knows.

It is easy to see that Sylver Coinage terminates; however it is unboundedly, unboundedly, ... , unboundedly unbounded! For example, naming 6^{1000} still leaves a game which is boundedly, unboundedly, unbounded since it could be followed by:

$$6^{999}, 6^{998}, \ldots, 6, \mathbf{3^n}, 3^{n-1}, \ldots, 3, \mathbf{2^m}, 2^{m-1}, \ldots, 2.$$

Is there an algorithm for Sylver Coinage that tells you, by looking over all (possibly infinite) options, what the status of the a position is and what if any are the winning moves? The answer is yes (see *Winning Ways*) but I do not know what the algorithm is.

What about other infinite games? Well, Sprague–Grundy theory (for impartial games) applies to heaps of infinite ordinals. Partizan theory has ordinary numbers and a mechanism for defining infinite numbers.

In Hackenbush, we can have strings of infinite length connected to the ground at one end. For example, a single blue edge would have value $\{0\,|\,\} = 1$, two blue edges would have value $\{0, 1\,|\,\} = 2$, etc., and an infinite beanstalk of blue edges would have value $\{0, 1, 2, \ldots\,|\,\} = \omega$, the first infinite ordinal. This is reminiscent of von Neumann's definition of the ordinals, that is, every ordinal is the set of all smaller ordinals. For us though, we have the vertical slash, Left membership and Right membership. For the ordinals we never have to use Right membership so the surreal numbers contain the reals. This is fantastic for a group. However, addition better not be Cantorian addition but I already knew about Hessenberg maximal addition which is the appropriate operation.

In ring theory, we get negatives, so we have $-\omega = \{\,|\,0, -1, -2, \ldots\}$. In game theory though, there are curious hybrids. We can take an infinite blue beanstalk and add a single red edge to get the game $\{0, 1, 2, \ldots\,|\,\omega\}$ and this game should be bigger than all the integers but smaller than ω. In fact, it is $\omega - 1$. So that ω is no longer the smallest infinite number but the smallest infinite ordinal. By adding an infinite red beanstalk on the top of the infinite blue beanstalk we get the game $\{0, 1, 2, \ldots\,|\,\omega, \omega - 1, \omega - 2, \ldots\} = \omega/2$.

Now every surreal number has a representation

$$\omega^{x_0} r_0 + \omega^{x_1} r_1 + \cdots + \omega^{x_\alpha} r_\alpha, \qquad \alpha < \beta$$

where x_i are surreal numbers, r_i are real numbers. This an analogue of Cantor's normal form. It doesn't really explain every number. Some are $\varepsilon = \omega^\varepsilon$ which is not explained by the normal form. No method of notation will explain every surreal number.

What kinds of infinities can occur? The first question is: how big can a game get? Well, a game can be as big as an ordinal but every game can be beaten by an ordinal For example, if $G = \{A, B, \ldots\,|\,D, E, \ldots.\}$ then $G < \{A, B, \ldots\,|\,\}$, why give your opponent a move! But now, by induction, A, B, etc., are smaller than some ordinal $\alpha \in \mathbf{On}$. Now every surreal number is less than some $\alpha \in \mathbf{On}$ therefore we also have $1/x > 1/\alpha$. So a related question is how small can positive games be?

There is this game $\uparrow\; = \{0|*\}$ where $* = \{0\,|\,0\}$. It is easy to show that $* \leq 1/4$ since $* = 0|0 \leq 0|1/2 \leq 1/4$. In fact, it is easy to show that $* \leq 1/\alpha$ for $\alpha \in \mathbf{On}$. Thus

$$\text{all negative numbers } < * < \text{ all positive numbers}$$

Likewise

$$0 < \uparrow < \text{positive numbers}$$

so $\uparrow$ is very small. Note that $\uparrow$ is not a number: it is the value of a game, which is a more subtle concept. Also note that $1/\uparrow$ is not defined since it would be bigger than all surreal numbers and there are no such numbers. (In fact, it does exist but is one of the *Oneiric* numbers.)

Indeed, $\uparrow = 0\|0|0$ and $+_x = 0\|0|-x$ is a positive game, in fact it is in the smallest subgroup of positive games.

There is a small set of notes in *On Numbers and Games* about infinite games which no one has taken up and I wish somebody would.

Let's take a copy of the Real line and somewhere at the end is ω then ω^2, ω^ω, etc., as Cantor did 120 years ago, all the way to the end of the line, which doesn't exist (but it does which we shall soon see). But what is the smallest infinite number? There is a lot of space between the first infinite ordinal, ω, and the real numbers. Here are some surreal numbers that fit in the gap: $\omega - 1$, $\sqrt{\omega}$ and $\log \omega$. Numbers like ω^{1/ω^α} are the smallest infinite numbers but not the smallest infinite games.

The smallest appear to be an analogues of $\uparrow$, and are of the form $Z\|Z|Z = \infty$ and $Z\|Z|0 = \infty_0$. Moreover,

$$Z\|Z|0 - Z\|Z|Z = Z|0 - Z\|Z|0 = Z|0 - Z|Z.$$

So $Z \,\|\, Z|Z = Z|Z + Z|0$ is an analogue of the upstart inequality $\uparrow + \uparrow + * = \{0|\uparrow\}$.

So we can ask the question what is in between the reals and ω^{1/ω^α}. Certainly, $\infty_\alpha = Z\|Z|{-\alpha}$. Some of the questions were answered in *On Numbers and Games* but I would love to have a theory to cover the situation with Z. I think there should be some analogue with the small games — some infinite thermography in units of ∞_0.

Let's take a look at loopy games. Recall that $1 = \{0\,|\,\}$, $2 = \{0, 1\,|\,\}$, $3 = \{0, 1, 2\,|\,\}$ etc. This is the way we generate all the ordinal numbers. Wouldn't it be nice to put yourself on the left. Well, we can: $G = \{G|\,\}$ or $G = \{\text{ pass }|\,\}$ and then by transfinite induction G is bigger than zero, all ordinals and, in fact, any game, The game is called $\text{On} = \{\,\text{On}\,|\,\}$. This is a loopy game, Left can move to On but Right cannot move. The existence of gaps in the line is solved by loopy games.

This game also gives $1/\text{On} = \{0|1/\text{On}\}$ and this game is the continuation of $1/2$, $1/4$, …, $1/\omega$, …. Therefore $1/\text{On}$ is absolutely the smallest positive game.

Each loopy game seems to have a left value and right value calling infinite number Left moves is a win for Left and an infinite number of moves for Right as a win for Right. We have some results but they are not true in full generality. The usual technique is to use sidling. Someone should follow this up.

Something else that intrigues me is multiplication of games. Addition of games is easy, $G + H = \{\,G^L + H, G + H^L\,|\,G^R + H, G + H^R\}$ Multiplication does not work with games in general but it does work for surreal numbers!

$$xy = \{x^Ly + xy^L - x^Ly^L, x^Ry_xy^R - x^Ry^R \mid x^L + xy^R - x^Lx^R, x^Ry + xy^L - x^Ry^L\}.$$

That is why we could not define $1/\uparrow$. It took me two years to come up with the definition of multiplication. Surprisingly this definition works for Nimbers.

	∗0	∗1	∗2	∗3	∗4	∗5	∗6	∗7	∗8
∗0	∗0	∗0	∗0	∗0	∗0	∗0	∗0	∗0	∗0
∗1	∗0	∗1	∗2	∗3	∗4	∗5	∗6	∗7	∗8
∗2	∗0	∗2	∗3	∗1	∗8	∗10	∗11	∗9	∗12
∗3	∗0	∗3	∗1	∗2	∗12	∗15	∗13	∗14	∗4
∗4	∗0	∗4	∗8	∗12	∗6	∗2	∗14	∗10	∗11

Let's drop the $*$. Note that $2 \times 2 = 3$ and $4 \times 4 = 6$. In fact, the nim-product of 2^{2^m} and 2^{2^m} is their normal product if $n \neq m$ and is $(3/2) \times 2^{2^n}$ otherwise. It all looks very finite. But to a logician it keeps on working into the infinite. In fact, I found that $\omega \times \omega \times \omega = 2$. So ω is the cube root of 2. This is a delightful part of the theory, that it still keeps surprising us with bizarre results, a wonderful mix of a coherent theory and hard calculation. I could only create artificial games with nim multiplication as part of their theory but Hendrik Lenstra has some quite natural games and knows more about these than anyone else. The ordinal that Cantor would call ω^3 let me write $[\omega^3]$ and our definition of cube inverts Cantor's multiplication, essentially $[\omega^3]^3 = \omega$. So our cube operation, moving from right to left, gives

$$2 \leftarrow \omega \leftarrow [\omega^3] \leftarrow [\omega^9] \leftarrow \cdots;$$

these tend to ω^ω and that turns out to be the fifth root of 4. As we now might expect, we have a chain of fifth powers

$$4 \leftarrow \omega^\omega \leftarrow [\omega^{\omega^5}] \leftarrow [\omega^{\omega^{2\times 5}}] \leftarrow \cdots.$$

These ordinals tend to ω^{ω^2} which is the seventh root of $\omega + 1$. Now let $2 = \alpha_3$, $4 = \alpha_5$ and $\omega + 1 = \alpha_7$. I proved the theorem that α_p is the p-th root of the first number which did not already have a p-th root. For example, 0 and 1 have cube roots but 2 does not so $2 = \alpha_3$. Since 2 and 3 have fifth roots but 4 does not so $4 = \alpha_5$. All integers (and ω) have seventh roots so $\alpha_7 = \omega + 1$. Lenstra has shown that $\alpha_p^{1/p}$ is computable and computed several of them.

This is a wonderful theory, there are many ways in which games can go infinite and what is most surprising is the incredible structure at all levels. Usually, in mathematics, when things go infinite, things smooth out. Except here, when suddenly it is the cube root of 2.

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