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# $1\times n$ Konane: A Summary of Results

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ABSTRACT. We look at $1\times n$ Konane positions consisting of three solid patterns each separated by a single space and present forumulas for the values of certain positions.

## Introduction

Combinatorial game theory [Berlekamp et al. 1982, Conway 1976] has discovered a fascinating array of mathematical structures that provide explicit winning strategies for many positions in a wide variety of games. The theory has been most successful on those games that tend to decompose into sums of smaller games. After assigning a game-theoretic mathematical value to each of the summands, these values are added to determine the value of the entire position. This approach has provided powerful new insights into a wide range of popular games, including Go [Berlekamp and Wolfe 1994], Dots-and-Boxes [Berlekamp 2000a], and even certain endgames in Chess [Elkies 1996].

Some positions on two-dimensional board games breakup into one-dimensional components, and the values of these components often have their own interesting structures. Examples include Blockbusting [Berlekamp 1988], Toads and Frogs [Erickson 1996] and Amazons [Berlekamp 2000b]. The same is true for Konane as suggested by previous analyses of one-dimensional positions [Ernst 1995 and Scott 1999].

Konane is an ancient Hawaiian game similar to checkers. It is played on an $18\times 18$ board with black and white stones placed in an alternating fashion so that no two stones of the same color are in adjacent squares. Two adjacent pieces adjacent to the center of the board are removed to begin the game. A player moves by taking one of his stones and jumping, in the horizontal or vertical direction, over an adjacent opposing stone into an empty square. The jumped stone is removed. A player can make multiple jumps on his turn but cannot change direction mid turn. The first player who cannot make a move loses.

Our goal was to find values for all possible $1\times n$ Konane positions that consist of three solid patterns each separated by a single space, which we call an "almost solid pattern with two spaces". We assumed that the positions were far away from the edges of the board so that there was no interference.

Let $\bullet$ represent a black stone, $\circ$ a white stone and $\cdot$ represent an empty square. One example of a solid pattern is $\circ\bullet\circ\bullet\circ$ and an example of an almost solid pattern with two spaces is $\circ\bullet\cdot\bullet\circ\bullet\cdot\bullet\circ\bullet$. We represent positions with $k$ stones in the first fragment, $m$ stones in the second and $n$ stones in the third with $\mathbf{S}(k,m,n)$. The convention is to start the left end of the position with a white stone.

Like almost solid patterns with one space, in which a segment's parity (odd or even in length) determines the value of a game, almost solid patterns with two spaces exhibit the same trend. Thus we categorize a game by its parity e.g. odd-odd-odd or even-odd-even, etc.

Odd-odd-odd and odd-even-odd games have been completely solved. For even-odd-even games and even-even-even games, we have found patterns and proved some of them. For the rest, we have discovered patterns and some partial proofs [Chan and Tsai 2000]. The value of $\mathbf{S}(k,m,n)$, for most values of the arguments, is a number or a number plus *, a common and well-known infinitesimal. However, for odd even even and even odd odd cases, we encounter more complicated infinitesimals.

Our proofs for the most part depend on the decomposition of games of the form $\mathbf{S}(k,m\cdot,n)$. These are games where there are two solid patterns separated by a single space followed by two spaces and another solid pattern, e.g. $\mathbf{S}(4,2\cdot,3)$ $=\circ\bullet\circ\bullet\cdot\bullet\circ\cdot\cdot\bullet\circ\bullet$. From the data we have collected on this topic, we propose a new decomposition theorem. Specifically, we conjecture that such games decompose into $\mathbf{S}(k,m)+\mathbf{S}(n)$ except for the cases $\mathbf{S}(2j,2j\cdot,2k)$ where $j$ is an odd integer and $k$ is any non-negative integer. We have been able to prove only restricted cases of this conjecture. We have verified it empirically in many other cases, but it is still conceivable that there are some additional exceptions.

Here are some forumlas for $1\times n$ Konane positions that play a part in our proofs.

**Theorem.** [Ernst 1995] *Let* $\mathbf{S}(n)$ *represent a solid pattern of stones of length* $n$*, beginning with a white stone. Then*

$$\mathbf{S}(2k+1)=k, \qquad \mathbf{S}(2k)=k\cdot *.$$

**Theorem.** [Scott 1999] *Let* $\mathbf{S}(n,m)$ *represent two solid patterns, of length* $n$ *and* $m$ *respectively, starting with a white stone and separated by a single space.*

$$\mathbf{S}(2j+1,2k+1)=j+k,$$

$$\mathbf{S}(2j+1,2k)=\begin{cases} k\cdot\uparrow+k\cdot * & \textit{if } j=0,\\ j-k & \textit{if } j>k,\\ 2^{j-k-1} & \textit{otherwise.}\end{cases}$$

*For* $j \le k$*:*

$$\mathbf{S}(2j, 2k) = \begin{cases} j+k\cdot * & \text{if } j<k, \\ k & \text{if } j=k \text{ and } k \text{ is even}, \\ (k-1) & \text{if } j=k \text{ and } k \text{ is odd}. \end{cases}$$

The tables of data included in the paper are formatted as follows: for a given game $\mathbf{S}(k, m, n)$, the numbers in the leftmost column are the $k$ values, the numbers along the top are the $n$ values and the number in the upper left had corner is the $m$ value. We present a table for each of pattern of parities (mod2) of the arguments.

## 1. Odd-Odd-Odd $\mathbf{S}(2i+1, 2j+1, 2k+1)$

| **3** | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 5 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 7 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 9 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 11 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

| **5** | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 7 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 9 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 11 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 13 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

**Table 1.** Sample values for odd-odd-odd games.

For all $i, j, k$, $\mathbf{S}(2i+1, 2j+1, 2k+1) = i+j+k$.

White (right) has no move. Black (left) can either move to $\mathbf{S}(2i-1, 2j+1, 2k+1)$ when $i > 0$ or to $\mathbf{S}(2i+1, 2j+1, 2k-1)$ when $k > 0$. If both $i, k = 0$, black moves to $\mathbf{S}(2j-1, 1)$ or $\mathbf{S}(1, 2j-1)$ both of which have value $j-1$.

## 2. Odd-Even-Even $\mathbf{S}(2i+1, 2j, 2k)$

| **4** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | $2*$ | 2 | $2*$ | 2 | $2*$ |
| 3 | $\frac{5}{4}$ | $\frac{9}{4}$ | $\frac{5}{2}$ | 3 | $3*$ | 3 | $3*$ |
| 5 | $\frac{3}{2}$ | $\frac{5}{2}$ | 3 | 4 | $4*$ | 4 | $4*$ |
| 7 | 2 | 3 | 4 | 5 | $5*$ | 5 | $5*$ |
| 9 | 3 | 4 | 5 | 6 | $6*$ | 6 | $6*$ |
| 11 | 4 | 5 | 6 | 7 | $7*$ | 7 | $7*$ |
| 13 | 5 | 6 | 7 | 8 | $8*$ | 8 | $8*$ |

| **6** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 1 | $1\uparrow 3*$ | $2\uparrow 3*$ | $\frac{5}{2}$ | 3 | $3*$ | 3 | $3*$ |
| 3 | $\frac{9}{8}$ | $\frac{17}{8}$ | 3 | 4 | $4*$ | 4 | $4*$ |
| 5 | $\frac{5}{4}$ | $\frac{9}{4}$ | $\frac{17}{4}$ | $\frac{9}{2}$ | 5 | 5 | $5*$ |
| 7 | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{7}{2}$ | $\frac{9}{2}$ | 5 | 6 | $6*$ |
| 9 | 2 | 3 | 4 | 5 | 6 | 7 | $7*$ |
| 11 | 3 | 4 | 5 | 6 | 7 | 8 | $8*$ |
| 13 | 4 | 5 | 6 | 7 | 8 | 9 | $9*$ |

**Table 2.** Sample values for odd-even-even games.

Conjecture:

$$\mathbf{S}(2i{+}1, 2j, 2k) = \begin{cases} (j{+}i){+}k\cdot\ast & \text{if } k \geq 2j, \\ k{+}i{-}j & \text{if } i > j \text{ and } k < 2j. \end{cases}$$

## 3. Odd-Odd-Even $\mathbf{S}(2i{+}1, 2j{+}1, 2k)$

| **3** | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{128}$ |
| 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ |
| 5 | 3 | 2 | 1 | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}\ast$ | $\frac{1}{2}$ | $\frac{1}{2}\ast$ |
| 7 | 4 | 3 | 2 | $\frac{3}{2}$ | 1 | $1\ast$ | 1 | $1\ast$ |
| 9 | 5 | 4 | 3 | $\frac{5}{2}$ | 2 | $2\ast$ | 2 | $2\ast$ |
| 11 | 6 | 5 | 4 | $\frac{7}{2}$ | 3 | $3\ast$ | 3 | $3\ast$ |
| 13 | 7 | 6 | 5 | $\frac{9}{2}$ | 4 | $4\ast$ | 4 | $4\ast$ |

| **5** | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ |
| 3 | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{3}{16}$ | $\frac{1}{8}$ | $\frac{1}{8}\ast$ |
| 5 | 4 | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{1}{4}$ | $\frac{1}{4}\ast$ |
| 7 | 5 | 4 | 3 | 2 | 1 | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}\ast$ |
| 9 | 6 | 5 | 4 | 3 | 2 | $\frac{3}{2}$ | 1 | $1\ast$ |
| 11 | 7 | 6 | 5 | 4 | 3 | $\frac{5}{2}$ | 2 | $2\ast$ |
| 13 | 8 | 7 | 6 | 5 | 4 | $\frac{7}{2}$ | 3 | $3\ast$ |

**Table 3.** Sample values for odd-odd-even games.

White has only one move to $\mathbf{S}(2i{+}1, 2j{+}1, 2k{-}2)$. When $k > 2j$ black moves to $\mathbf{S}$(2i-1, 2j+1, 2k). When $k \leq 2j$, black moves to $\mathbf{S}(2i{+}1, 2j{+}1\bullet\cdot, 2k{-}2)$ which decomposes into the sum

$$\mathbf{S}(2i{+}1, 2(j{+}1)){+}\mathbf{S}(2k{-}2) = \begin{cases} i{-}(j{+}1){+}(k{-}1)\cdot\ast & \text{if } i > j{+}1, \\ 2^{i-(j+1)-1}{+}(k{-}1)\cdot\ast & \text{if } i \leq j{+}1. \end{cases}$$

## 4. Even-Odd-Even $\mathbf{S}(2i, 2j{+}1, 2k)$

| **3** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | 0 | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | $\frac{15}{16}$ | $\frac{31}{32}$ | $\frac{63}{64}$ |
| 4 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | $\frac{7}{4}$ | $\frac{15}{8}$ | $\frac{31}{16}$ | $\frac{63}{32}$ |
| 6 | $\frac{3}{4}$ | $\frac{3}{2}$ | 2 | $2\ast$ | 2 | $2\ast$ | 2 |
| 8 | $\frac{7}{8}$ | $\frac{7}{4}$ | $2\ast$ | 2 | $2\ast$ | 2 | $2\ast$ |
| 10 | $\frac{15}{16}$ | $\frac{15}{8}$ | 2 | $2\ast$ | 2 | $2\ast$ | 2 |
| 12 | $\frac{31}{32}$ | $\frac{31}{16}$ | $2\ast$ | 2 | $2\ast$ | 2 | $2\ast$ |
| 14 | $\frac{63}{64}$ | $\frac{63}{32}$ | 2 | $2\ast$ | 2 | $2\ast$ | 2 |

| **5** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | 0 | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | $\frac{15}{16}$ | $\frac{31}{32}$ | $\frac{63}{64}$ |
| 4 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | $\frac{7}{4}$ | $\frac{15}{8}$ | $\frac{31}{16}$ | $\frac{63}{32}$ |
| 6 | $\frac{3}{4}$ | $\frac{3}{2}$ | 2 | $\frac{5}{2}$ | $\frac{5}{2}\ast$ | $\frac{5}{2}$ | $\frac{5}{2}\ast$ |
| 8 | $\frac{7}{8}$ | $\frac{7}{4}$ | $\frac{5}{2}$ | 3 | $3\ast$ | 3 | $3\ast$ |
| 10 | $\frac{15}{16}$ | $\frac{15}{8}$ | $\frac{5}{2}\ast$ | $3\ast$ | 3 | $3\ast$ | 3 |
| 12 | $\frac{31}{32}$ | $\frac{31}{16}$ | $\frac{5}{2}$ | 3 | $3\ast$ | 3 | $3\ast$ |
| 14 | $\frac{63}{64}$ | $\frac{63}{32}$ | $\frac{5}{2}\ast$ | $3\ast$ | 3 | $3\ast$ | 3 |

**Table 4.** Sample values for even-odd-even games.

| **7** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | $-1$ | $0$ | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | $\frac{15}{16}$ | $\frac{31}{32}$ |
| 4 | $0$ | $1$ | $\frac{3}{2}$ | $\frac{7}{4}$ | $\frac{15}{8}$ | $\frac{31}{16}$ | $\frac{63}{32}$ |
| 6 | $\frac{1}{2}$ | $\frac{3}{2}$ | $2$ | $\frac{5}{2}$ | $\frac{11}{4}$ | $\frac{23}{8}$ | $\frac{47}{16}$ |
| 8 | $\frac{3}{4}$ | $\frac{7}{4}$ | $\frac{5}{2}$ | $3$ | $\frac{7}{2}$ | $\frac{15}{4}$ | $\frac{31}{8}$ |
| 10 | $\frac{7}{8}$ | $\frac{15}{8}$ | $\frac{11}{4}$ | $\frac{7}{2}$ | $4$ | $4*$ | $4$ |
| 12 | $\frac{15}{16}$ | $\frac{31}{16}$ | $\frac{23}{8}$ | $\frac{15}{4}$ | $4*$ | $4$ | $4*$ |
| 14 | $\frac{31}{32}$ | $\frac{63}{32}$ | $\frac{47}{16}$ | $\frac{31}{8}$ | $4$ | $4*$ | $4$ |

**Table 4.** Sample values for even-odd-even games (continued).

If $i, k \geq j$, $\mathbf{S}(2i, 2j{+}1, 2k) = (j{+}1) + (k{+}i)\cdot *$

Since $\mathbf{S}(2i, 2j, 2k) = -\mathbf{S}(2k, 2j, 2i)$, we'll assume that $k \geq i$. Black's best move is to $\mathbf{S}(2i, 2j{+}1, 2k{-}2)$. White moves to $\mathbf{S}(2i \circ \cdot, 2j{-}1, 2k)$ if $i \leq (j{+}1)$ and to $\mathbf{S}(2(i{-}1), \cdot \circ 2j, 2k)$ otherwise. If $k \neq (j{-}1)$ or $j$ is not even, $\mathbf{S}(2i \circ \cdot, 2j{-}1, 2k)$ decomposes into

$$\mathbf{S}(2i{+}1) + \mathbf{S}(2j{-}1, 2k) = \begin{cases} i{+}k\cdot\downarrow{+}k\cdot * & \text{if } j = 2, \\ i{-}(j{-}1{-}k) & \text{if } (j{-}1) > k, \\ i{-}2^{(j-1)-k-1} & \text{if } (j{-}1) \leq k. \end{cases}$$

If $k \neq (j{+}1)$ or $k$ not odd, $\mathbf{S}(2(i\text{-}1), \cdot \circ 2j{+}1, 2k)$ decomposes into

$$\mathbf{S}(2(i{-}1)) + \mathbf{S}(2(j{+}1), 2k) = (i{-}1)\cdot * + (j{+}1) + k\cdot * = (j{+}1) + (k{+}i{-}1)\cdot *$$

In the case where $j$ is even (ex. $\mathbf{S}(2i, 5, 2k)$), the values are slightly different when $i = (j{+}1)$ since the right and left followers of the game do not decompose.

## 5. Odd-Even-Odd $\mathbf{S}(2i{+}1, 2j, 2k{+}1)$

| **2** | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|---|
| 1 | $*2$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ |
| 3 | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-3$ | $-4$ |
| 5 | $1$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-3$ |
| 7 | $2$ | $1$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-1$ | $-2$ |
| 9 | $3$ | $2$ | $1$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-1$ |
| 11 | $4$ | $3$ | $2$ | $1$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| 13 | $5$ | $4$ | $3$ | $2$ | $1$ | $\frac{1}{2}$ | $0$ |

**Table 5.** Sample values for odd-even-odd games.

| **4** | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-3$ | $-4$ |
| 3 | $\frac{1}{4}$ | 0 | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-3$ |
| 5 | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ | $-2$ |
| 7 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ |
| 9 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 | $-\frac{1}{4}$ | $-\frac{1}{2}$ |
| 11 | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 | $-\frac{1}{4}$ |
| 13 | 4 | 3 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 |

**Table 6.** Sample values for ~~odd~~even~~odd~~ games (continued).

Assume $i \geq k$ since $\mathbf{S}(2i{+}1, 2j, 2k{+}1) = -\mathbf{S}(2k{+}1, 2j, 2i{+}1)$.

$$\mathbf{S}(2i{+}1, 2j, 2k{+}1) = \begin{cases} 0 & \text{if } i = k \text{ (except in the case } \mathbf{S}(1,2,1) = *2\text{)}, \\ -2^{-k-1} & \text{for } 2k{+}1 = 2i{+}3 \text{ to } 2k{+}1 = 2i{+}1{+}j, \\ -k & \text{for } 2k{+}1 > 2i{+}1{+}j. \end{cases}$$

When $i = k = 0$: if $j$ is even then $\mathbf{S}(1, 2j, 1)$ has value $\{\ \mathbf{S}(1, 2j{-}2\cdot, 2) \mid \mathbf{S}(2, \cdot 2j{-}2, 1)\ \}$ otherwise the canonical value is $\{\ \mathbf{S}(1, 2j{-}2) \mid \mathbf{S}(2j{-}1, 1)\ \}$

When $i = 0$ and $k \neq 0$, $\mathbf{S}(1, 2j, 2k{+}1)$ has value

$$\{\ \mathbf{S}(2j{-}2, 2k{+}1) \mid \mathbf{S}(1, 2j, 2k{-}1)\ \}.$$

If $j = 0$, white has no options.

When $i \neq 0$ and $k = 0$, $\mathbf{S}(2i{+}1, 2j, 1)$ has value

$$\{\ \mathbf{S}(2i{-}1, 2j, 1) \mid \mathbf{S}(2i{+}1, 2j{-}2)\ \}.$$

If $j = 0$, black has no options.

When $i, k \neq 0$ $\mathbf{S}(2i{+}1, 2j, 2k{+}1)$ has value

$$\{\ \mathbf{S}(2i{-}1, 2j, 2k{+}1) \mid \mathbf{S}(2i{+}1, 2j, 2k{-}1)\ \}.$$

## 6. Even-Even-Even $\mathbf{S}(2i, 2j, 2k)$

For $j > 1$, $j \equiv 2 mod 4$ and $k \geq i$:

$$\mathbf{S}(2i, 2j, 2k) = \begin{cases} (i{-}k){+}j\cdot * & \text{if } i, k < j, \\ (j{-}k{-}1) & \text{if } k = j \text{ and } i < j, \\ (j{-}k){+}k\cdot * & \text{if either } k \text{ or } i \text{ (but not both) is greater than } j. \end{cases}$$

Conjecture: For $i, k > j$, $\mathbf{S}(2i, 2j, 2k) = \mathbf{S}(2(i{-}j), 2, 2(k{-}j))$.

For $j > 2$, $j \equiv 0 mod 4$ and $4k \geq i$:

$$\mathbf{S}(2i, 2j, 2k) = \begin{cases} i{-}k & \text{if } i, k < j, \\ (j{-}k){+}k\cdot * & \text{if } i < j \text{ and } k > j. \end{cases}$$

| **2** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | $*$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-\frac{1}{2}*$ | $-\frac{1}{2}$ | $-\frac{1}{2}*$ |
| 4 | $0$ | $*$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{4}*$ | $-\frac{1}{4}$ | $-\frac{1}{4}*$ |
| 6 | $\frac{1}{4}$ | $0$ | $*$ | $0$ | $-\frac{1}{16}$ | $-\frac{1}{8}$ | $-\frac{1}{8}*$ |
| 8 | $\frac{1}{2}$ | $\frac{1}{4}$ | $0$ | $*$ | $0$ | $-\frac{1}{16}$ | $-\frac{1}{16}*$ |
| 10 | $\frac{1}{2}*$ | $\frac{1}{4}*$ | $\frac{1}{16}$ | $0$ | $*$ | $0$ | $\frac{1}{64}$ |
| 12 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $0$ | $*$ | $0$ |
| 14 | $\frac{1}{2}*$ | $\frac{1}{4}*$ | $\frac{1}{8}*$ | $\frac{1}{16}*$ | $\frac{1}{16}$ | $0$ | $*$ |

| **6** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | $*$ | $-1*$ | $-1$ | $-2$ | $-2*$ | $-2$ | $-2*$ |
| 4 | $1*$ | $*$ | $0$ | $-1$ | $-1*$ | $-1$ | $-1*$ |
| 6 | $1$ | $0$ | $*$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-\frac{1}{2}*$ |
| 8 | $2$ | $1$ | $0$ | $*$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{4}*$ |
| 10 | $2*$ | $1*$ | $\frac{1}{4}$ | $0$ | $*$ | $0$ | $-\frac{1}{16}$ |
| 12 | $2$ | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $0$ | $*$ | $0$ |
| 14 | $2*$ | $1*$ | $\frac{1}{2}*$ | $\frac{1}{4}*$ | $\frac{1}{16}$ | $0$ | $*$ |

**Table 7.** Sample values for even-even-even games. $\mathbf{S}(2i,2j,2k)$ where $2j \equiv 2 \mod 4$.

Conjecture: If $i,k > j$, $\mathbf{S}(2i,2j,2k) = \mathbf{S}(2(i-j),4,2(k-j))$.

Assume that $k \geq i$ since $\mathbf{S}$(2i, 2j, 2k) = -$\mathbf{S}$(2k, 2j, 2i). In general, if $i < j$ and $k < j$, $\mathbf{S}(2i,2j,2k)$ has canonical value $\{\ \mathbf{S}(2i,2j-2,\cdot\bullet 2k) \mid \mathbf{S}(2i\circ\cdot,2(j-1),2k)\}$.

$\mathbf{S}(2i,2(j-1),\cdot\bullet 2k)$ decomposes into the sum

$$\mathbf{S}(2i,2(j-1))-\mathbf{S}(2k+1) = i+(j-1)\cdot*-k$$

If $j$ is not even or $k \neq (j-1)$, $\mathbf{S}$(2i$\circ\cdot$, 2(j-1), 2k) decomposes into the sum

$$\mathbf{S}(2i+1)+\mathbf{S}(2(j-1),2k) = i-k+(j-1)\cdot*$$

So the value of the game is

$$\{\ i-k+(j-1)\cdot* \mid (i-k)+(j-1)\cdot*\} = (i-k)+j\cdot*$$

If $i < j < k$, black moves to $\mathbf{S}(2(i-1),2j,2k)$ and white moves to $\mathbf{S}(2i,2j\bullet\cdot,$ $2(k-1))$.

| **4** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | $0$ | $-1$ | $-1*$ | $-1$ | $-1*$ | $-1$ | $-1*$ |
| 4 | $1$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-\frac{1}{2}*$ | $-\frac{1}{2}$ | $-\frac{1}{2}*$ |
| 6 | $1*$ | $\frac{1}{4}$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{4}*$ | $-\frac{1}{4}$ | $-\frac{1}{4}*$ |
| 8 | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $0$ | $-\frac{1}{16}$ | $-\frac{1}{8}$ | $-\frac{1}{8}*$ |
| 10 | $1*$ | $\frac{1}{2}*$ | $\frac{1}{4}*$ | $\frac{1}{16}$ | $0$ | $-\frac{1}{16}$ | $-\frac{1}{16}*$ |
| 12 | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $0$ | $-\frac{1}{16}$ |
| 14 | $1*$ | $\frac{1}{2}*$ | $\frac{1}{4}*$ | $\frac{1}{8}*$ | $\frac{1}{16}*$ | $\frac{1}{16}$ | $0$ |

| **8** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 2 | $0$ | $-1$ | $-2$ | $-3$ | $-3*$ | $-3$ | $-3*$ |
| 4 | $1$ | $0$ | $-1$ | $-2$ | $-2*$ | $-2$ | $-2*$ |
| 6 | $2$ | $1$ | $0$ | $-1$ | $-1*$ | $-1$ | $-1*$ |
| 8 | $3$ | $2$ | $1$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-\frac{1}{2}*$ |
| 10 | $3*$ | $2*$ | $1*$ | $\frac{1}{4}$ | $0$ | $-\frac{1}{4}$ | $-\frac{1}{4}*$ |
| 12 | $3$ | $2$ | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $0$ | $-\frac{1}{16}$ |
| 14 | $3*$ | $2*$ | $1*$ | $\frac{1}{2}*$ | $\frac{1}{4}*$ | $\frac{1}{16}$ | $0$ |

**Table 8.** Sample values for even-even-even games. $\mathbf{S}(2i, 2j, 2k)$ where 2k $\equiv$ 0 mod 4.

$\mathbf{S}(2i, 2j \bullet \cdot, 2(k-1))$ decomposes into the sum

$$\mathbf{S}(2i, 2j+1)+\mathbf{S}(2(k-1)) = \begin{cases} i-j+(k-1)\cdot * & \text{if } j > i, \\ -(2^{j-i-1})+(k-1)\cdot * & \text{otherwise.} \end{cases}$$

If $i, j > k$ the canonical value is $\{\ \mathbf{S}(2i-2, 2j, 2k) \mid \mathbf{S}(2i, 2j, 2k-2)\ \}$.

## Conclusion

An open question is to determine precisely which patterns of $\mathbf{S}(k, m\cdot, n)$ decompose. That would eliminate the gaps in some of our proofs and help to complete the analysis for all $1\times n$ games.

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