Hankel Determinants as Fredholm Determinants

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Abstract. Hankel determinants which occur in problems associated with orthogonal polynomials, integrable systems and random matrices are computed asymptotically for weights that are supported in an semi-infinite or infinite interval. The main idea is to turn the determinant computation into a random matrix “linear statistics” type problem where the Coulomb fluid approach can be applied.

1. Introduction

Let \( w \) be a weight function supported on \( L \) (a subset of \( \mathbb{R} \)) that has finite moments of all orders

\[
\mu_n = \int_L x^n w(x) \, dx.
\]

With \( w(x) \) we associate the Hankel matrix \( (\mu_{i+j}) \), where \( i, j = 0, \ldots, n-1 \). The purpose of this paper is the determination of

\[
D_n[w] := \det(\mu_{i+j})_{i,j=0}^{n-1}
\]

for large \( n \) with suitable conditions on \( w \). If \( L \) is a single interval, say \([-1,1]\), then the asymptotic form of such determinants was computed by Szegö [1918] and later by Hirschmann [1966] for quite general \( w \).

Our main result is as follows. Suppose we replace \( w(x) \) by a function given in the form \( u_0(x)U(x) \) where \( u_0(x) \) is the weight \( e^{-x^2} \). Then for appropriate functions \( w \), the determinants are given asymptotically as \( n \to \infty \) by

\[
D_n[w] = \exp \left( c_1 n^2 \log n + c_2 n^2 + c_3 n \log n + c_4 n + c_5 n^{1/2} + c_6 \log n + c_7 + o(1) \right) \quad (1)
\]

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where
\[ c_1 = 1, \quad c_2 = -3/2, \quad c_3 = \nu, \quad c_4 = -\nu + \log 2\pi, \]
\[ c_5 = \frac{2}{\pi} \int_0^\infty \log(U(x^2)) \, dx, \quad c_6 = \nu^2/2 - 1/6, \]
\[ c_7 = \frac{4}{3} \log G(1/2) + (1/3 + \nu/2) \log \pi + (\nu/2 - 1/18) \log 2 \]
\[ - \log G(1 + \nu) - (\nu/2) \log U(0) + \frac{1}{2\pi^2} \int_0^\infty xS(x)^2 \, dx, \]
\[ S(x) = \int_0^\infty \cos(xy) \log U(y^2) \, dy, \]
and \( G \) is the Barnes function (see Section 2).

In Section 2 we establish an identity relating \( D_n[w] \), \( D_n[w_0] \), and a certain Fredholm determinant and a description of the Fredholm determinant from a “linear statistics” point of view. A computation of \( D_n[w_0] \) is also included. Then in Section 3 the Coulomb fluid approach is used to compute the asymptotics of the Fredholm determinant. This, along with the computation of \( D_n[w_0] \) allows us to give a heuristic, Coulomb fluid derivation of the formula. A rigorous proof based on operator theory techniques developed in [Basor 1997; Tracy and Widom 1994] will appear in a forthcoming paper. In Section 3 the Hermite case is also included.

2. Preliminaries

Let \( p_i(x) \) be polynomials orthonormal with respect to \( w_0(x) \) (the reference weight function) over \( L \)
\[ \int_L p_i(x)p_j(x)w_0(x) \, dx = \delta_{i,j}, \]
with strictly positive leading coefficients. For later convenience we also write \( \phi_j = \sqrt{w_0}p_j \) as the orthonormal functions. Consider the determinant,
\[ \det \left( \int_L p_i(x)p_j(x)w(x) \, dx \right)_{i,j=0}^{n-1}. \]

If \( p_i(x) = \sum_{j=0}^i c_{ij}x^j \), then
\[ \det \left( \int_L p_ip_jw \, dx \right)_{i,j=0}^{n-1} = \det \left( \int_L \sum_{k=0}^i \sum_{l=0}^j c_{ik}c_{jl}x^{k+l}w \, dx \right)_{i,j=0}^{n-1} \]
\[ = \left( \prod_{i=0}^{n-1} c_{ii} \right)^2 \det \left( \int_L x^{j+k}w \, dx \right)_{j,k=0}^{n-1} \]
\[ = \left( \prod_{i=0}^{n-1} c_{ii} \right)^2 \det (\mu_{j+k})_{j,k=0}^{n-1}. \]
\[ (2) \]
If \( w = w_0 \) then the left side of (2) is 1. So,

\[
\det \left( \int_L p_i p_j w \, dx \right)_{i,j=0}^{n-1} = \det \left( \mu_{i+j} \right)_{i,j=0}^{n-1} = \frac{D_{n}[w]}{D_{n}[w_0]},
\]

(3)

where

\[
\mu_i := \int_L x^i w_0 \, dx,
\]

are the moments of the reference weight \( w_0 \), and

\[
D_{n}[w_0] = \left( \prod_{i=0}^{n-1} c_{ii} \right)^{1/2}.
\]

We now express the left side of (3) as a Fredholm determinant.

\[
\det \left( \int_L p_i p_j w \, dx \right) = \det \left( \int_L \phi_i \phi_j \left( 1 - \frac{w}{w_0} \right) \, dx \right) = \det(\delta_{i,j} - M_{i,j}),
\]

where

\[
M_{i,j} := \int_L \phi_i \phi_j \left( 1 - \frac{w}{w_0} \right) \, dx =: \int_L \phi_i \phi_j F \, dx = \int_L \phi_i \phi_j \left( 1 - U \right) \, dx.
\]

We have the standard expansion

\[
- \log \det(\delta_{i,j} - M_{i,j}) = \sum_{p=1}^{\infty} \frac{\text{tr} \, M^p}{p},
\]

where the matrix \( M \) has elements \( M_{i,j} \). To compute \( \text{tr} \, M^p \), we first look at the simpler case of \( p = 3 \). We see that \( \text{tr} \, M^3 \) equals

\[
\sum_{i,j,k=0}^{n-1} M_{i,j} M_{j,k} M_{k,i} = \int dX \sum_{i,j,k=0}^{n-1} \phi_i(x_1) \phi_j(x_2) F(x_1) \phi_k(x_3) F(x_2) \phi_k(x_3) F(x_3)
\]

\[
= \int dX F(x_1) F(x_2) F(x_3) \sum_{i=0}^{n-1} \phi_i(x_3) \phi_i(x_1) \sum_{j=0}^{n-1} \phi_j(x_1) \phi_j(x_2) \sum_{k=0}^{n-1} \phi_k(x_2) \phi_k(x_3)
\]

\[
= \int dX K_n(x_1, x_2) K_n(x_2, x_3) K_n(x_3, x_1) F(x_1) F(x_2) F(x_3),
\]

where \( \int dX \) stands for \( \int_L \cdots \int_L dx_1 \, dx_2 \, dx_3 \) and where

\[
K_n(x, y) := \sum_{i=0}^{n-1} \phi_i(x) \phi_i(y) = a_n \frac{\phi_n(x) \phi_{n-1}(y) - \phi_n(y) \phi_{n-1}(x)}{x - y}.
\]

(4)
The last equality of (4) is the Christoffel–Darboux formula [Szego 1975] and $a_n$ are the off diagonal recurrence coefficients of $p_i$. The generalization to integer $p$ is obvious and we find,

$$\text{tr} M^p = \text{tr}(K_n F)^p,$$

where the operator $K_n F$ has kernel $K_n(x,y) F(y)$. So

$$\det(\delta_{i,j} - M_{i,j}) = \det(I - K_n F), \quad (5)$$

and $I$ in (5) has kernel $\delta(x-y)$. We now come to the linear statistics.

If $x_i$, for $i = 1, \ldots, n$, are random variables with the joint probability density function

$$p(x_1, \ldots, x_n) \propto \prod_{i=1}^n w_0(x_i) \prod_{1 \leq j,k \leq n} |x_j - x_k|^\alpha, \quad x_i \in L,$$

then $Q = \sum_{i=1}^n f(x_i)$ (the linear statistics) is also a random variable. Consider the generating function of $Q$, $\langle \exp(-Q) \rangle$, where

$$\langle \ldots \rangle = \frac{\int_L \ldots \int_L \ldots \int_L p(x_1, \ldots, x_n) \; dx_1 \ldots dx_n}{\int_L \ldots \int_L p(x_1, \ldots, x_n) \; dx_1 \ldots dx_n}. $$

Recall the Heine formula [Szego 1975] for Hankel determinants:

$$\det(\mu_{i+j}^{n-1})_{i,j=0} = \frac{1}{n!} \int_L \ldots \int_L \prod_{i=1}^n w(x_i) \prod_{1 \leq j,k \leq n} |x_j - x_k|^\alpha, \quad (7)$$

and the analogous one with $\mu_{i+j}$ replaced by $\mu_{i+j}^0$ and $w$ replaced by $w_0$. If we write $w = \exp(-v)$, $w_0 = \exp(-v_0)$ and $v = v_0 + f$ then

$$D_n[w] = \det(\mu_{i+j}^{n-1})_{i,j=0} \langle \exp(-Q) \rangle = D_n[w_0] \langle \exp(-Q) \rangle. \quad (8)$$

In this notation $f(x) = -\log U(x)$. So our strategy is to choose a suitable $w_0$ for which there is exact result for $D_n[w_0]$, and compute $\langle \exp(-Q) \rangle$ as a Fredholm determinant for $n$ large. In the next section we will use a heuristic method to give an indication how the results for $\langle \exp(-Q) \rangle$ for large $n$ can be found. If we take $w_0(x) = x^\nu \exp(-x)$, $\nu > -1$ and $L = [0, \infty)$ then $p_i(x)$ are the orthonormal Laguerre polynomials. It is well known [Szego 1975] that

$$c_{ii}^2 = \frac{1}{\Gamma(1 + i + \nu)\Gamma(1 + i)}.$$ 

So

$$D_n[w_0] = \prod_{i=0}^{n-1} \Gamma(1 + i + \nu)\Gamma(1 + i) = \frac{G(n + \nu + 1) \; G(n + 1)}{G(\nu + 1) \; G(1)}, \quad (9)$$

where the Barnes function $G$ [Barnes 1900; Whittaker and Watson 1962] satisfies the functional equation $G(z+1) = \Gamma(z)G(z)$, with the initial condition $G(1) = 1$. The asymptotics of the Barnes function are computed in [Whittaker and Watson 1962] and since $G(1 + a + n)$ is asymptotic to

$$n^{(n+a)^2/2-1/12} e^{-3/4n^2+am} (2\pi)^{(n+a)/2} G^{2/3}(1/2) \pi^{1/6} 2^{-1/36}$$
we can directly apply this formula with $a = 0$ and $a = \nu$ to obtain asymptotically

$$D_n[w_0] = \exp \left\{ d_1 n^2 \log n + d_2 n^2 + d_3 n \log n + d_4 n + d_5 \log n + d_6 + o(1) \right\}$$

(10)

where

$$d_1 = 1, \quad d_2 = -3/2, \quad d_3 = \nu, \quad d_4 = -\nu + \log 2\pi, \quad d_5 = \nu^2/2 - 1/6, \quad d_6 = (4/3) \log G(1/2) + (1/3 + \nu/2) \log \pi + (\nu/2 - 1/18) \log 2 - \log G(1 + \nu).$$

3. The Coulomb Fluid Method

For suitably chosen $f$, $\langle \exp(-Q) \rangle$ for large $n$ was computed in [Chen and Lawrence 1998] starting from the Heine formula. An alternative and shorter derivation is given here. Now if $Q$ is in some sense “small” then by expanding up to $Q^2$, we have, $\langle \exp(-Q) \rangle \approx 1 - \langle Q \rangle + 1/2 \langle Q^2 \rangle$. This can be reproduced by expanding

$$\exp \left( -\langle Q \rangle - \frac{\langle Q^2 \rangle}{2} \right),$$

up to $\langle Q^2 \rangle$ and $\langle Q \rangle^2$. With the introduction of the microscopic density $\rho(x) := \sum_{i=1}^{n} \delta(x - x_i)$, one finds

$$\langle Q \rangle = \int_{L} f(x) \langle \rho(x) \rangle \, dx,$$

$$\langle Q \rangle^2 - \langle Q^2 \rangle = \int_{L} \int_{L} f(x) (\langle \rho(x) \rangle \langle \rho(y) \rangle - \langle \rho(x) \rho(y) \rangle) f(y) \, dx \, dy.$$

In the Coulomb fluid approach, expected to be valid for large $n$, we replace $\langle \rho(x) \rangle$ by the equilibrium density $\sigma(x)$, which is supposed to be supported in a single interval $(a, b)$. It is then a simple exercise to show that the correlation function

$$\langle \rho(x) \rangle \langle \rho(y) \rangle - \langle \rho(x) \rho(y) \rangle$$

is replaced by

$$\frac{1}{2\pi^2 \sqrt{(b - x)(x - a)}} \partial_y \left( \frac{\sqrt{(b - y)(y - a)}}{x - y} \right).$$

Therefore,

$$\langle \exp(-Q) \rangle \sim \exp(-S_1 - S_2),$$

(11)

where

$$S_1 = \frac{1}{4\pi^2} \int_{a}^{b} \int_{a}^{b} \frac{f(x)}{\sqrt{(b - x)(x - a)}} \partial_y \left( \frac{\sqrt{(b - y)(y - a)}}{x - y} \right) f(y) \, dx \, dy,$$

$$S_2 = \int_{a}^{b} \sigma(x) f(x) \, dx.$$

The end points of the interval, $a$ and $b$, are determined by the normalization condition $\int_{a}^{b} \sigma(x) \, dx = n$ and a supplementary condition [Chen and Lawrence 1998].
So the constants $c_i$, $i = 1, \ldots, 6$ and part of $c_7$ are obtained from the asymptotic expansion of $D_n[w_0]$ while $c_5$ and the last two terms of $c_7$ are obtained from the large $n$ behaviour of $S_2$ and $S_1$ respectively. For $w_0(x) = x^\nu \exp(-x)$, $x \geq 0$ it is known that $a = 0$, $b = 4n + 2\nu$ and

$$\sigma(x) = -\nu \delta_+(x) + \frac{1}{2\pi} \sqrt{\frac{b-x}{x}}, \quad 0 \leq x < b.$$  

So,

$$S_2 = -\frac{\nu}{2} f(0) + \frac{1}{2\pi} \int_0^b f(x) \sqrt{\frac{b-x}{x}} \, dx$$

$$\quad \to -\frac{\nu}{2} \log U(0) - \frac{2n^{1/2}}{\pi} \int_0^\infty \log U(x^2) \, dx, \quad \text{as } n \to \infty. \quad (12)$$

As $n \to \infty$, $S_1$ tends to

$$\frac{1}{4\pi^2} \int_0^\infty \int_0^\infty f(x) \frac{\partial}{\partial y} \frac{\sqrt{y}}{x-y} \, dx \, dy.$$

Changing the integration variables $x = s^2$, $y = t^2$ and noting

$$-\frac{1}{2} \int_{-\infty}^{\infty} |x| \exp(-ixt) \, dx = \frac{1}{i^2},$$

we find

$$S_1 \to -\frac{1}{2\pi^2} \int_0^\infty x \left( \int_0^\infty \log U(s^2) \cos(xs) \, ds \right)^2 \, dx$$

$$\quad = -\frac{1}{2\pi^2} \int_0^\infty x S(x)^2 \, dx. \quad (13)$$

Therefore (1) follows from (10), (12) and (13).

We can also make use of the above information to determine the recurrence coefficients of the monic polynomials $P_j(x)$, orthogonal with respect to $x^\nu \exp(-tx) U(x)$. The parameter $t$ is introduced for later convenience. The recurrence relations reads

$$x P_n(x) = P_{n+1}(x) + \alpha_n(t) P_n(x) + \beta_n(t) P_{n-1}(x).$$

From the basic properties of the Hankel determinant generated by the weight $w(x, t) = x^\nu \exp(-tx) U(x)$, one finds

$$\alpha_n(t) = -\frac{d}{dt} \ln \frac{D_{n+1}(t)}{D_n(t)},$$

$$\beta_n(t) = \frac{D_{n+1}(t) D_{n-1}(t)}{(D_n(t))^2}.$$
where $D_n(t) = D_n[w(\cdot, t)]$. With the asymptotics, we find, as $n \to \infty$, that
\begin{align*}
\alpha_n(1) &= 2n + \nu + 1 - \left( \frac{1}{\pi} \int_0^\infty \log U(x^2) \, dx \right) n^{-1/2} + o(1) \\
\beta_n(1) &= n^2 + \nu n - \left( \frac{1}{2\pi} \int_0^\infty \log U(x^2) \, dx \right) n^{1/2} + o(1),
\end{align*}
are the recurrence coefficients of those monic polynomials orthogonal with respect to $w(x) = x^\nu \exp(-x) U(x)$. As a further application of the asymptotic formula, we study the short noise generating function of an $n$-channel disordered conductor [Mutalib and Chen 1996] where the $f$ of the linear statistics is
\[ f(x) := M \ln \frac{x + z}{x + 1}, \quad |z| = 1. \]
As $n \to \infty$, $S_1$ tends to
\[ -M^2 \log \frac{\sqrt{z} + 1}{2^{1/4}}, \]
while $S_2$ tends to
\[ \frac{\nu M}{2} \log z + 2\sqrt{n}M(1 - \sqrt{z}). \]
This generalises the results of [Mutalib and Chen 1996] to $\nu \neq 0$.

Now suppose $w$ is supported in $(-\infty, \infty)$. We adopt the same strategy to determine the large $n$ behaviour of the associated Hankel determinant:
\[ D_n[w] = D_n[w_0] \frac{D_n[w]}{D_n[w_0]}, \]
where the “reference” Hankel determinant is generated by the Hermite weight $w_0(x) = \exp(-x^2)$, where $x \in (-\infty, \infty)$. Now $a = -b = -\sqrt{2n}$ and $\sigma(x) = \frac{1}{\pi} \sqrt{x^2 - b^2}$. Thus, as $n \to \infty$,
\[ \frac{D_n[u]}{D_n[w_0]} \sim \exp(-S_1 - S_2), \]
where
\begin{align*}
S_1 &= -\frac{1}{8\pi^2} \int_{-\infty}^\infty |k| \hat{f}(-k) \hat{f}(k) \, dk \\
S_2 &= \frac{\sqrt{2n}}{\pi} \int_{-\infty}^\infty f(x) \, dx.
\end{align*}
(14)
Here $\hat{f}(k) = \int_{-\infty}^\infty \exp(ikx)f(x) \, dx$. Equations (14) are essentially those found by Kac and by Akhiezer [Akhiezer 1964]. Therefore the large $n$ behaviour of $D_n[w]$ follows from
\[ D_n[w_0] = (2\pi)^{n/2} 2^{-n/2} G(n + 1), \]
the asymptotics of the Barnes function, and Equation (14).
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