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Stability Theory and its Variants

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Abstract. Dimension theory plays a crucial technical role in stability theory and its relatives. The abstract dependence relations defined, although combinatorial in nature, often have surprising geometric meaning in particular cases. This article discusses several aspects of dimension theory, such as categoricity, strongly minimal sets, modularity and the Zil'ber principle, forking, simple theories, orthogonality and regular types and in the third, stability, definability of types, stable groups and 1-based groups.

One of the achievements of the branch of model theory known as stability theory is the use of numerical invariants, dimensions, in a broad setting. In recent years, this dimension theory has been expanded to include the so-called simple theories. In this paper, I wish to give just a brief overview of the elements of this theory. In the first section, the special case of strongly minimal sets is considered. In the second section, the combinatorial definition of dividing is given and how it leads to a general independence relation is outlined. Only in the third section do stable theories appear and the theory surrounding them is developed there with an eye to other papers in this volume.

1. Strongly Minimal Sets

Categorical Theories. One of the simplest questions one can ask about a first order theory is how many models it has of a given cardinality. If T is a countable theory with an infinite model then, by the Lowenheim–Skolem Theorem, it will have at least one model of every infinite power. The situation we will look at first is when a theory has exactly one model of some fixed power.

Definition 1.1. A theory T is *λ-categorical* if T has exactly one model up to isomorphism of cardinality λ. T is said to be *totally categorical* if T is λ-categorical for every infinite cardinal λ. We will say that T is *uncountably categorical* if T is λ-categorical for all uncountable λ.

Example 1.2. 1. The theory of a set in a language which has only equality is totally categorical.

1. Suppose that D is a countable division ring. The theory of a vector space over D is uncountably categorical and is totally categorical if D is finite (that is, D is a finite field). Here the language contains a binary function symbol for addition and a unary function symbol for each scalar in D.

Two variants of this example are the following; fix an infinite dimensional vector space V over a finite field F:

2. Infinite dimensional projective space over a finite field. $P(V)$ is the set of 1-dimensional subspaces of V. Define an $n+1$-ary relation R_n on $P(V)$ as follows: $R_n(X_1, X_2, \ldots, X_n, Y)$ holds for $X_1, \ldots, X_n, Y \in P(V)$ if Y is contained in the subspace generated by $X_1, \ldots, X_n$. The theory of $P(V)$ together with all the R_n's is totally categorical.

A more general version of this example involves the Galois group of F; we describe it via its automorphism group. By a projective geometry, we will mean a structure whose underlying set is $P(V)$ and whose automorphism group is $\mathrm{PGL}(V) \rtimes \mathrm{Gal}(F/L)$ where L is some subfield of F. All of these examples are totally categorical.

3. Infinite dimensional affine space over a finite field. Let $\tau(x, y, z) = x-y+z$ and for every $f \in F$, let $\lambda_f(x, y) = \lambda x + (1-\lambda)y$. The theory of $(V, \tau, \{\lambda_f : f \in F\})$ is also totally categorical.

As with the projective space examples above, there are more general affine space examples. By an affine geometry we will mean a structure whose underlying set is V and whose automorphism group is $\mathrm{AGL}(V) \rtimes \mathrm{Gal}(F/L)$ where L is a subfield of F. Again, all these examples are totally categorical.

Here are a few other examples that will be commented on later:

5. The theory of $(Z/4Z)^\omega$ as an abelian group; this theory is totally categorical.
6. The theory of an algebraically closed field of a fixed characteristic; this theory is uncountably categorical.

It is possible for a theory to be ω-categorical without being totally categorical; here are two examples:

7. The theory of the rationals as a linear order and the theory of an equivalence relation with two classes, each infinite.

Of course there are many theories which are not categorical in any power, for example:

8. the theory of the real field, Peano arithmetic and the theory of the integers as an abelian group.

Łos conjectured that there were four possibilities for the categoricity spectrum: a theory would be either totally categorical, uncountably categorical but not ω-categorical, ω-categorical but not uncountably categorical or not categorical in any power. This was proved by Morley and is known as the Morley Categoricity Theorem:

THEOREM 1.3. *If T is a countable theory which is λ-categorical for some uncountable λ then T is λ-categorical for all uncountable λ.*

One intuition behind this conjecture is that for an uncountably categorical theory, the uncountable models are completely controlled by a single dimension and that by specifying that dimension, one specifies the isomorphism type of the model. I want to discuss the key ingredients of the proof of this theorem but first I want to indicate another theorem related to categoricity.

THEOREM 1.4 (ŁOS–VAUGHT TEST). *If T is categorical and has no finite models then T is complete.*

It follows then that examples 5, 6 and 7 above are complete. The first four examples above have finite models and are not complete. One can check that these four examples are all finitely axiomatized. The theory of the infinite models of these theories is then axiomatized by these finitely many axioms together with infinitely many sentences expressing the fact that the models are infinite. Various conjectures arose of which I will only state two:

1. Is there a finitely axiomatized, totally categorical theory?
2. Is every totally categorical theory finitely axiomatized modulo "axioms of infinity"?

We will say more about these conjectures later but to answer them, one needs a very detailed understanding of the models of totally categorical theories. Historically, this arose through an understanding of the proof of the Morley Categoricity Theorem.

We will work throughout this paper with a countable, complete theory T with infinite models. For convenience, fix a structure $\mathcal{M}$ which is a λ-saturated model of T for some big cardinal λ. λ will be larger than any set of parameters or any submodel we will choose throughout our discussions. The following is a critical definition.

DEFINITION 1.5. An infinite definable set $X \subseteq M^n$ for some n is called *strongly minimal* if every definable subset of X is either finite or cofinite. Here the definable sets may be definable with parameters.

1. In the examples we saw early, the entire model (the set defined by $x = x$) is strongly minimal in the following cases: an infinite pure set, an infinite vector space over a countable division ring, an infinite projective space over a finite field, an infinite affine space over a finite field and any algebraically closed field of a fixed characteristic. The case of either projective or affine space over a finite field follows from the corresponding case of an infinite vector space. For an infinite set, an infinite vector space or an algebraically closed field, it

is enough to check atomic formulas by quantifier elimination in each case and the solution set of an atomic formula is finite in all cases.

2. In the case of the theory of $(Z/4Z)^\omega$ as an abelian group, the entire model is not strongly minimal but the set defined by $x+x=0$ is. Again, this example has quantifier elimination which makes it easy to check this set is strongly minimal.
3. A theory does not have to be uncountably categorical to have a strongly minimal set: in the theory of an equivalence relation with two classes each infinite, either equivalence class is strongly minimal.

I wish to discuss dimension theory for strongly minimal sets so let me remind you of the definition of a pregeometry.

Definition 1.6. A pregeometry (A, cl) is a set A together with a closure operator $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$ such that:

1. for any $B \subseteq A$, $B \subseteq \mathrm{cl}(B) = \mathrm{cl}(\mathrm{cl}(B))$,
2. if $B \subseteq C \subseteq A$ then $\mathrm{cl}(B) \subseteq \mathrm{cl}(C)$,
3. if $B \subseteq A$ and $b \in \mathrm{cl}(B)$ there is a finite $B_0 \subseteq B$ such that $b \in \mathrm{cl}(B_0)$, and
4. (Steinitz exchange) if $B \subseteq A$, $b \in \mathrm{cl}(B \cup \{c\}) \setminus \mathrm{cl}(B)$ then $c \in \mathrm{cl}(B \cup \{b\})$.

Definition 1.7. If (A, cl) is a pregeometry, we say that $B \subseteq A$ is *independent* if $b \notin \mathrm{cl}(B \setminus \{b\})$ for any $b \in B$. A basis for a set C is an independent set $B \subseteq C$ such that $C \subseteq \mathrm{cl}(B)$.

Proposition 1.8. *Suppose* (A, cl) *is a pregeometry. Then for any* $B \subseteq A$, *any maximal independent subset of* B *is a basis for* B. *Moreover, any two bases for* B *have the same cardinality.*

We now want to consider the closure operator acl_X on a definable set X. Suppose that X is defined over a set of parameters $\bar{d}$. Then $\mathrm{acl}_X(Y) = \mathrm{acl}(Y \cup \{\bar{d}\}) \cap X$ for $Y \subseteq X$. It is routine to check that acl_X is a closure operator on X which satisfies the first three properties from the definition of pregeometry.

Proposition 1.9. *If* X *is a strongly minimal set then* acl_X *defines a pregeometry on* X.

Proof. We need only to check the exchange property. Suppose that $a \in \mathrm{acl}_X(C \cup \{b\}) \setminus \mathrm{acl}_X(C)$. Suppose that $\varphi(x, y)$ is a formula with parameters from C which expresses that fact that a is algebraic over b. We can choose φ so that $\varphi(a, b)$ holds and there is an N so that for every b' there are at most N realizations of $\varphi(x, b')$. Now suppose that $\{a_i : i < N+1\}$ is a set of distinct realizations of $\mathrm{tp}(a/C)$ (remember that a is not algebraic over C).

For the sake of contradiction, suppose that $b \notin \mathrm{acl}_X(C \cup \{a\})$. This means, in particular, that $\varphi(a, M) \cap X$ is infinite. Since X is strongly minimal, this set is cofinite, so $\varphi(a_i, M) \cap X$ is cofinite for each $i < N+1$. Hence there is $b' \in X$ so that $\varphi(a_i, b')$ holds for all $i < N+1$ which contradicts the choice of N. So in fact, $b \in \mathrm{acl}_X(C \cup \{a\})$. □

The following is easy by compactness.

Proposition 1.10. *If X is strongly minimal and $\varphi(x, y)$ is any formula then there is a number N so that for any b, if $\left|\{a \in X : \varphi(a, b) \text{ holds}\}\right| > N$ then $\varphi(X, b)$ is infinite.*

How will strongly minimal sets be used in trying to understand the models? Suppose that X is a strongly minimal set defined over parameters $\bar{d}$ and further, that I and J are two sets of elements from X, independent with respect to acl_X. The claim is that any injection from I to J is a $\bar{d}$-elementary map. What this amounts to showing is that for any set $B \subseteq X$ there is a unique nonalgebraic type in X over $B \cup \{\bar{d}\}$. The latter is true because for any formula over $B \cup \{\bar{d}\}$, either it or its negation is an algebraic formula.

To summarize then, suppose X is a strongly minimal set defined over $\bar{d}$. Now if N is a model containing $\bar{d}$ then to understand the elementary type of $X \cap N$ over $\bar{d}$ one needs only to know the dimension of $X \cap N$ for then $X \cap N$ is algebraic over any basis.

But do uncountably categorical theories contain strongly minimal sets? Suppose that M itself is not strongly minimal. Then one can find an infinite, co-infinite definable set $X \subseteq M$. Let $X_0 = X$ and $X_1 = M \setminus X$. In general suppose that we have defined an infinite set X_η where η is a finite sequence of 0's and 1's. If X_η is not strongly minimal then it contains an infinite, co-infinite subset Y which we label $X_{\eta 0}$ and let $X_{\eta 1} = X_\eta \setminus Y$. In this way, we produce uncountably many consistent partial types over countably many parameters. Compare this with the following fact due to Ehrenfeucht.

Fact 1.11. *For any countable theory T with infinite models and any uncountable cardinal λ, there is a model of T, M of size λ with the property that for every countable set $A \subseteq M$, the number of types realized in M over A is countable.*

This naturally leads to the following definition.

Definition 1.12. A theory T is said to be ω-stable if for every countable $A \subseteq M$, $S(A)$ is countable.

So above we have proved the following proposition.

Proposition 1.13. *A countable theory T which is uncountably categorical is ω-stable.*

Corollary 1.14. *Any countable, uncountably categorical theory has strongly minimal sets.*

Now there are two questions I want to address in the rest of this section and in some sense they represent two different directions in what is known as stability theory:

1. In an uncountably categorical theory, how is the rest of the model related to the strongly minimal set?

2. What can the strongly minimal set look like/be?

Let's consider the first question. One of the means of measuring dimension in model theory is via ranks. The first and the one which most closely resembles dimension from algebraic geometry is Morley rank (written RM for Rang Morley). In the following definition, remember that we are working in a saturated model of the theory at hand and that definable means definable in that model.

DEFINITION 1.15. For a nonempty definable set X, we define $\mathrm{RM}(X)$ inductively:

1. $\mathrm{RM}(X) \geq 0$;
2. for a limit ordinal δ, $\mathrm{RM}(X) \geq \delta$ if and only if $\mathrm{RM}(X) \geq \alpha$ for all $\alpha < \delta$;
3. $\mathrm{RM}(X) \geq \alpha + 1$ if and only if there are definable subsets Y_i of X for $i \in \omega$ which are pairwise disjoint and such that $\mathrm{RM}(Y_i) \geq \alpha$.

$\mathrm{RM}(X) = \alpha$ if $\mathrm{RM}(X) \geq \alpha$ and $\mathrm{RM}(X) \not\geq \alpha$. $\mathrm{RM}(X) = \infty$ if $\mathrm{RM}(X) \geq \alpha$ for all α. If $\varphi(x)$ is a consistent formula then by $\mathrm{RM}(\varphi(x))$ we mean the Morley rank of the set this formula defines.

FACT 1.16. 1. *Morley rank is invariant; that is, if $\bar{a}$ and $\bar{b}$ have the same type over the empty set and $\varphi(x, \bar{a})$ is consistent then* $\mathrm{RM}(\varphi(x, \bar{a})) = \mathrm{RM}(\varphi(x, \bar{b}))$.
2. *If $X \subseteq Y$ then* $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$.
3. $\mathrm{RM}(\bigcup_{i=1}^{n} X_i) = \max\{\mathrm{RM}(X_i) : i = 1, \ldots, n\}$.
4. *There is an ordinal α that depends only on the theory T such that if* $\mathrm{RM}(X) \geq \alpha$ *then* $\mathrm{RM}(X) = \infty$.

REMARKS 1.17. In the case of algebraically closed fields, Morley rank and algebro-geometric dimension coincide, that is, $\mathrm{RM}(V) = \dim(V)$ for V an algebraic variety.

DEFINITION 1.18. If $\mathrm{RM}(X) = \alpha < \infty$ then the Morley degree of X, $\mathrm{dM}(X)$, is the largest number k so that there are pairwise disjoint definable subsets of X, Y_i, with $\mathrm{RM}(Y_i) = \alpha$ for $i = 1, \ldots, k$.

FACT 1.19. *If* $\mathrm{RM}(X) < \infty$ *then* $\mathrm{dM}(X)$ *is well-defined.*

PROOF. Suppose that $\mathrm{RM}(X) = \alpha$. For this proof say that a definable subset Y of X is α-irreducible if $\mathrm{RM}(Y) = \alpha$ and there do not exist $Z_1, Z_2 \subseteq Y$ which are disjoint and have Morley rank α. It is easy to show that the fact that $\mathrm{RM}(X) \not\geq \alpha$ implies both that there are α-irreducible subsets of X and that any maximal collection of such is finite. So suppose that $\{Y_1, \ldots, Y_m\}$ and $\{Z_1, \ldots, Z_n\}$ are two maximal collections of α-irreducible subsets of X. We may assume that $\bigcup_{i=1}^{m} Y_i = \bigcup_{j=1}^{n} Z_j = X$. So

$$Z_j = \left(\bigcup_{i=1}^{m} Y_i \right) \cap Z_j = \bigcup_{i=1}^{m} (Y_i \cap Z_j)$$

so there is at least one i so that $\mathrm{RM}(Y_i \cap Z_j) = \alpha$. Since Z_j is α-irreducible, there is at most one such i. So the map sending j to that i such that $\mathrm{RM}(Z_j \cap Y_i) = \alpha$ is well-defined and injective. Symmetry shows that it is a bijection and so $k = m$. □

Note that in the previous proof we don't show that the decomposition of X into Morley degree 1 pieces is unique but only that it is "α-unique": $\mathrm{RM}(Y_i \triangle Z_j) < \alpha$. It is also worth noting that a strongly minimal set has Morley rank and Morley degree 1.

PROPOSITION 1.20. *T is ω-stable if and only if* $\mathrm{RM}(X) < \infty$ *for every X.*

PROOF. Suppose there is a definable set X such that $\mathrm{RM}(X) = \infty$. By Fact 1.16 there is an α such that if $\mathrm{RM}(Y) \geq \alpha$ then $\mathrm{RM}(Y) = \infty$. So since $\mathrm{RM}(X) \geq \alpha + 1$, choose two disjoint subsets of X both of which have Morley rank greater than or equal to α. By the choice of α, both of these sets have Morley rank ∞. Repeating this argument one can build a binary tree of height ω of definable sets such that each branch is consistent and no two branches are mutually consistent. This contradicts ω-stability.

Now suppose that $\mathrm{RM}(X) < \infty$ for every X. Fix a countable model M of T. For every $p \in S(M)$, associate a formula $\varphi_p \in p$ so that φ_p has the least Morley rank of all formulas in p and among those has the least Morley degree. It is easy to see then that this formula uniquely determines p. But there are only countably many formulas over M so $S(M)$ is countable. □

Now we will address how a structure is built or constructed over a strongly minimal set.

DEFINITION 1.21. 1. A type $p \in S_n(A)$ is said to be isolated if there is a formula $\varphi \in p$ such that if $\varphi \in q \in S_n(A)$ then $p = q$ (p is isolated in the Stone space topology on $S_n(A)$).
2. If $A \subseteq N \prec M$ then N is said to be a prime model over A if whenever $A \subseteq N' \prec M$ then there is an elementary map $f : N \to N'$ fixing A.
3. N is said to be constructible over A if $N = \{a_i : i < \alpha\}$ and, for every i,

$$\mathrm{tp}(a_i / A \cup \{a_j : j < i\})$$

is isolated.

FACT 1.22. 1. *If N is constructible over A then N is prime over A.*
2. *If T is ω-stable then there are constructible models over all sets.*
3. *If N_1 and N_2 are constructible over A then N_1 and N_2 are isomorphic over A.*

PROOF. The first is straightforward. For the second, the main point is that if you fix a set A and any consistent formula $\varphi(x)$ over A then the type of least Morley rank and degree containing $\varphi(x)$ is isolated. The third fact was proved by J. P. Ressayre. □

The main structure theorem for uncountably categorical theories is:

Theorem 1.23 [Baldwin and Lachlan 1971]. *If T is uncountably categorical then there is an isolated type $p(\bar{y})$ over the empty set and a formula $\varphi(x,\bar{y})$ such that whenever N is a model of T and $\bar{a} \in N$ realizes p then $\varphi(x,\bar{a})$ defines a strongly minimal set and N is constructible and minimal over $\varphi(N,\bar{a})$.*

That is, the isomorphism type of N is determined by the dimension of the set $\varphi(N,\bar{a})$.

Example 1.24. An example of what the above theorem does not say is given by the theory of $(Z/4Z)^\omega$ as an abelian group. It would be nice if every model of an uncountably categorical theory was algebraic over a basis for a strongly minimal subset of the model (as in the case of algebraically closed fields) but $(Z/4Z)^\omega$is a counterexample to this. For in this group, every element a is isolated over the strongly minimal set $2x = 0$ by the formula, $2x = 2a$. But this formula has infinitely many solutions.

Returning now to the question of what a strongly minimal set can be, we make a definition which makes sense for any pregeometry.

Definition 1.25. A pregeometry (X, cl) is said to be modular if whenever $A, B \subseteq X$ are finite dimensional, closed subsets of X then

$$\dim(A) + \dim(B) = \dim(A \cap B) + \dim(A \cup B)$$

(X, cl) is said to be locally modular if the above equality holds whenever $A \cap B$ is nonempty.

A strongly minimal set is said to be modular or locally modular if the associated pregeometry is.

Example 1.26. 1. An infinite set, an infinite vector space over a division ring or a projective space over a finite field are all examples of modular strongly minimal sets.

If the pregeometry (X, cl) satisfies $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$ for all $A \subseteq X$ then the pregeometry is said to be trivial or degenerate. An infinite set is such a pregeometry.

2. An affine space over a finite field is locally modular but not modular; two distinct parallel lines are a counter-example to modularity.

3. An algebraically closed field is not locally modular (see Example 1.8 on page 77 of [Pillay 1996]).

One of the guiding principles of geometric model theory is the Zil'ber Principle (for a general discussion, see [Peterzil and Starchenko 1996]):

Zil'ber Principle. *Under suitable geometric or topological conditions, a non-locally modular strongly minimal set interprets an infinite field.*

A particular instance of this principle can be applied to simple, algebraic groups. Suppose that G is a simple, algebraic group over an algebraically closed field F.

Viewed as an abstract group G is not locally modular and in fact, G interprets a field F' which is isomorphic to F.

More information regarding the Zil'ber Principle appears in [Marker 2000]. In the remaining part of this section, I wish to point out a special case of the Zil'ber Principle in ω-categorical theories. The following theorem is due to Cherlin, Harrington and Lachlan [Cherlin et al. 1985] and Zil'ber [1984].

THEOREM 1.27. *An ω-categorical, strongly minimal set is locally modular.*

One can in fact say more but we need a definition first.

DEFINITION 1.28. A strongly minimal set X is said to be strictly minimal if $\mathrm{acl}_X(a) = \{a\}$ for all $a \in X$.

For instance, a projective or affine geometry over a finite field is strictly minimal but a vector space over a finite field is not. In general, to obtain a strictly minimal set from an ω-categorical strongly minimal set, one first removes the algebraic closure of the empty set (a finite set in the ω-categorical case) and then quotients by the definable equivalence relation of interalgebraicity. In this way, one can see that, except for finitely many points, an ω-categorical, strongly minimal set is a finite cover of a strictly minimal set.

The following theorem provides an enumeration of all ω-categorical, strictly minimal sets.

THEOREM 1.29. *An ω-categorical strictly minimal set is either a pure set, or a projective or affine geometry over a finite field.*

One can use this more precise information about ω-categorical, strictly minimal sets to understand the structure of the prime model discussed above in the case of a totally categorical theory. Fix a totally categorical theory T and any model, M, of T.

FACT 1.30. *For any finite, algebraically closed set $B \subseteq M$ and $a \notin B$, there is $c \in \mathrm{acl}(B \cup \{a\})$ such that $\mathrm{tp}(c/B)$ is strictly minimal.*

A more global version of this local fact is:

FACT 1.31. *For any $a \in M$, there is a number n and $a_0, \ldots, a_n \in \mathrm{acl}(a)$ such that $a = a_n$ and for every i, $\mathrm{tp}(a_i/a_0, \ldots, a_{i-1})$ is either algebraic or strictly minimal.*

EXAMPLE 1.32. To see how this works, let's return to the example of the abelian group, $(Z/4Z)^\omega$. Recall that it is not algebraic over the strongly minimal set $2x = 0$. However, if we fix any element a and let $2a = b$ then we see that the number n in the previous fact is 2: b is either 0 (and hence algebraic over the empty set) or $\mathrm{tp}(b/\varnothing)$ is strictly minimal. In either case, $\mathrm{tp}(a/b)$ is strictly minimal; in the latter case, the strictly minimal set is an affine geometry.

The picture then to have of this example is a base set represented by the strictly minimal set $2x = 0$ and then above each element of this set, a "fibre" which is itself a strictly minimal set. To completely understand the structure then one must know what the strictly minimal sets are (and this is given by the Theorem above) and how the fibres interact (often the hardest part).

Hrushovski used this type of analysis to prove the following:

THEOREM 1.33 [Hrushovski 1989]. *Any totally categorical theory is finitely axiomatized modulo "axioms of infinity"* (*which express the fact that the strongly minimal set is infinite*).

2. Dividing

Before we can introduce the general notion of dimension, we must introduce a basic model theoretic definition.

DEFINITION 2.1. Suppose that $(I, <)$ is an infinite linear order and A is a subset of a model M. A sequence $\{a_i : i \in I\}$ of tuples from M is said to be *indiscernible over* A if, whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ are sequences from I, $\mathrm{tp}(a_{i_1} \ldots a_{i_n}/A) = \mathrm{tp}(a_{j_1} \ldots a_{j_n}/A)$.

EXAMPLE 2.2. 1. A transcendence basis in an algebraically closed field, ordered in any way, is an example of an indiscernible sequence over the empty set.
2. $(Q, <)$ with the usual order is also an example of an indiscernible sequence over the empty set.

We now introduce the most important definition on our way towards a general dimension theory. This definition is due to Shelah [1978].

DEFINITION 2.3. 1. A formula $\varphi(x, a)$ (and the set it defines) is said to *divide* over A if there is an sequence I, indiscernible over A, with $a \in I$ such that $\{\varphi(x, b) : b \in I\}$ is inconsistent.
2. A definable set X (and the formula which defines it) is said to *fork* over A if X is contained in the union of finitely many definable sets, each of which divides over A.
3. A type p divides (forks) over A if there is $\varphi \in p$ which divides (forks) over A.

We introduce a ternary relation symbol $\mathop{\smile}\limits^{\mid}$ between subsets of M as follows:

$$A \mathop{\smile}\limits^{\mid}_C B \text{ if and only if } \mathrm{tp}(A/B \cup C) \text{ does not divide over } C.$$

We will say that A and B are independent over C if $A \mathop{\smile}\limits^{\mid}_C B$ although the justification for this terminology will only come later.

EXAMPLE 2.4. 1. Suppose that X is a strongly minimal set defined over A. Then any infinite, definable subset of X does not divide over A.

To see this, by compactness, it is enough to see that any finitely many A-conjugates of X have a nonempty intersection. But X, and each of its conjugates, is cofinite and so the intersection is nonempty.

2. In an ω-stable theory, a type $p \in S(B)$ does not divide over $A \subseteq B$ if $\mathrm{RM}(p) = \mathrm{RM}(p|A)$.

To see this, suppose that p divides over A. Then there is a sequence $\{p_i : i \in \omega\}$ of A-conjugates of p so that for some n, any n distinct p_i's are mutually inconsistent. We can assume that $p_i \in S(B_i)$ and that the sequence $\{B_i : i \in \omega\}$ is indiscernible over A. Choose $\varphi \in p|A$ so that $\mathrm{RM}(\varphi) = \mathrm{RM}(p|A)$. Now since the p_i's are n-inconsistent, we can find $\psi_i(x, b_i) \in p_i$ and $m \leq n$ with the following properties:

1. $\psi(x, b_i)$ strengthens φ for every i;
2. if we let $\theta_k = \bigwedge_{mk \leq i < m(k+1)} \psi_i(x, b_i)$ then $\mathrm{RM}(\theta_k) = \mathrm{RM}(\varphi)$ for all k;
3. $\mathrm{RM}(\theta_k \wedge \theta_l) < \mathrm{RM}(\varphi)$ for all $k \neq l$

From the properties of Morley rank, we see that the formulas

$$\left\{\theta_k \wedge \bigwedge_{l<k} \neg\theta_l : k \in \omega\right\}$$

are pairwise inconsistent, strengthen φ and have the same Morley rank as φ which is a contradiction.

3. Suppose our model is an infinite vector space over a finite field with a non-degenerate symplectic bilinear form. Then any nonalgebraic one-type does not divide over the empty set. For suppose that p is a nonalgebraic 1-type over a subspace A (in general, it is enough to check that all 1-types do not divide over small sets). Let $\{A_i : i \in \omega\}$ be any indiscernible sequence with $A_0 = A$ and let B be their common intersection. Then it follows that $\{A_i : i \in \omega\}$ is linearly disjoint over B. Now p is determined by the linear map f it defines on A (for all $a \in A$, $f(a) = \alpha$ if and only if $\langle x, a\rangle = \alpha \in p$). If we let p_i be the conjugate of p over A_i and f_i be the corresponding conjugate of f then the consistency of $\bigcup_{i\in\omega} p_i$ is equivalent to the ability to extend all the maps f_i to a linear map on the subspace generated by A_i's. The latter is clear by the linear disjointness of the A_i's over B.

Dividing and forking satisfy many properties in all theories.

1. Both dividing and forking are invariant under automorphisms of the large, saturated model that we are working in.
2. If $X \subseteq Y$ are definable sets and Y divides (forks) over A then so does X.
3. (Extension) If $A \subseteq B \subseteq C$ and $p \in S(B)$ does not fork over A then p has an extension in $S(C)$ which does not fork over A.
4. (Finite Character) $A \mathop{\smile\!\!\!\!\!\mid}_C B$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}_c b$ for every finite $a \in A$, $b \in B$ and $c \in C$.
5. If $A \subseteq B$, $a \in \mathrm{acl}(B)$ and $a \mathop{\smile\!\!\!\!\!\mid}_A B$ then $a \in \mathrm{acl}(A)$.
6. (Weak transitivity) If $B \subseteq C \subseteq D$ and $A \mathop{\smile\!\!\!\!\!\mid}_B D$ then $A \mathop{\smile\!\!\!\!\!\mid}_B C$ and $A \mathop{\smile\!\!\!\!\!\mid}_C D$.
7. (Left transitivity) If $C \subseteq B \subseteq A$, $A \mathop{\smile\!\!\!\!\!\mid}_B D$ and $B \mathop{\smile\!\!\!\!\!\mid}_C D$ then $A \mathop{\smile\!\!\!\!\!\mid}_C D$.

The first two properties listed above follow immediately from the definitions. The extension property is the entire reason for defining forking and is easily verified by compactness. Both finite character and weak transitivity are easily verified. Left transitivity is extremely useful (see [Hart et al. 1998], section 4); a proof can be found in [Shelah 1980] and [Kim 1998].

DEFINITION 2.5. 1. A sequence $\{a_i : i < \alpha\}$ is *independent over* A if, for every j, $a_j \perp_A \{a_i : i < j\}$.
2. A *Morley sequence* for a type $p \in S(A)$ is a sequence of realizations of p which is both independent and indiscernible over A.

Key general assumptions

1. (Forking equals dividing) Whenever a definable set forks over a set A, it divides over A.
2. (The Kim property, KP) A formula $\varphi(x, a)$ does not divide over A if and only if there is a Morley sequence I in $\mathrm{tp}(a/A)$ such that $\{\varphi(x, b) : b \in I\}$ is consistent.

In a theory which satisfies the key general assumptions the following properties hold, trivially, dividing satisfies the extension property. Far less trivially, we have

THEOREM 2.6. *In a theory which satisfies the key general assumptions, dividing is symmetric, that is,* $A \perp_C B$ *if and only if* $B \perp_C A$.

As a corollary of the left transitivity property,

COROLLARY 2.7. *In a theory which satisfies the key general assumptions, dividing is transitive; that is, if* $C \subseteq B \subseteq A$ *and* $D \perp_B A$ *and* $D \perp_C B$ *then* $D \perp_C A$.

In fact recently in [Kim 1999], Kim has shown the the key general assumptions are equivalent to dividing being symmetric.

The most important of the properties that dividing satisfies in certain theories is the following (proved in [Kim and Pillay 1997]):

THEOREM 2.8. *(Type amalgamation over a model; also known as the Independence Theorem) Fix a theory which satisfies the key general assumptions. Suppose that* A *and* B *both contain a model* M *and are independent over* M*. If* p *and* q *are types over* A *and* B *respectively and both are nonforking extensions of a common type over* M *then* p *and* q *have a common nonforking extension.*

Simple theories

DEFINITION 2.9. A theory T is simple if every type does not divide over a set of size at most $|T|$. We say that in such a theory, dividing (or forking) satisfies local character.

THEOREM 2.10. *Simple theories satisfy the key general assumptions; that is, forking equals dividing and the Kim property holds.*

Example 2.11. 1. Any ω-stable theory is simple. This follows from Example 2.4.2.

2. The theory of an infinite vector space over a finite field with a nondegenerate symplectic bilinear form is simple. This follows from Example 2.4.3.

3. The theory of the random graph is simple. One can show this in a manner similar to (but easier than) Example 2.4.3; see the remarks after the next theorem for an alternative approach.

4. The generic triangle-free graph is **not** an example of a simple theory. There are several ways to see this; the one I present will be an application of type amalgamation.

The first observation is that the only indiscernible sequence of singletons in any model of this theory has no edges between the points. Otherwise, any three points will form a triangle.

So suppose that the theory of the generic triangle-free graph is simple. Fix a model M (we do this only to match the form of the type amalgamation theorem). I claim that any two points a and b, not in M, with an edge between them are independent over M. This follows from the first observation since if we consider an M-indiscernible sequence $\{b_i : i \in \omega\}$ starting with b then there are no edges between the b_i's so there is nothing inconsistent about the type which contains the type of a over M and the statement that x is connected to each of the b_i's. But then if we let p_a be the type of an element connected to a but not to any element of M and p_b be the similar type, connected to b but nothing in M, then by what we just said, p_a and p_b are nondividing extensions of their common restriction to M. However, any point which would realize p_a and p_b would form a triangle. Since type amalgamation fails, this theory cannot be simple.

5. The theory of the real field is **not** simple; the theory of $(Q, <)$ is **not** simple.

Let's show that $(Q, <)$ is not simple (real closed fields can be done in a similar manner). We will show that dividing is not symmetric. Consider the type $p(x; y, z)$ determined by "$y < x < z$". For any a it is fairly clear that $p(a; y, z)$ does not divide over the empty set (for any sequence of singletons in any model of this theory it is consistent that there is something bigger than and something smaller than them all). However, if $a < b$ then $p(x; a, b)$ does divide over the empty set. For we can choose an indiscernible sequence $a = a_0 < b = b_0 < a_1 < b_1 < a_2 < b_2 < \cdots$ for which even $p(x; a, b) \cup p(x, a_1, b_1)$ is not consistent.

The following characterization theorem shows the connection between the existence of an independence relation satisfying many of the properties we have mentioned up until now and simpicity. Its proof is due to Kim and Pillay and appears in [Kim and Pillay 1997].

Theorem 2.12. *A theory is simple if and only if there is an invariant ternary relation on sets which has finite and local character*, *is symmetric and transitive* *and satisfies extension and type amalgamation over models. If such a ternary relation exists then it must be nondividing.*

The usefulness of this theorem cannot be overstated. In practice, when one encounters a theory "in nature", it often comes with a suggestion for an independence relation. A case in point is the theory of algebraically closed fields with a generic automorphism (ACFA, see [Chatzidakis 2000]). It is frequently easier to check that the example satisfies the properties listed above then it is to work through the definition of dividing. For instance, for the random graph, suppose A is a subgraph of B and C, and say that B and C are independent over A if and only if $B \cap C = A$. Then it is straightforward to check that all the properties listed in the characterization theorem are satisfied and so the random graph is simple (and this independence relation is the dividing relation).

Orthogonality, supersimplicity and regular types. Any sufficiently general notion of independence leads to a derived notion of orthogonality; see the almost axiomatic treatments in [Makkai 1984] and [Shelah 1978]. In the case of simple theories, the details are worked out in [Hart et al. 1998] where the following definitions appear.

Definition 2.13. 1. If $p, q \in S(A)$ then p and q are *almost orthogonal* if whenever a and b realize p and q respectively then a and b are independent over A.
2. p and q are said to be *orthogonal* if all their nonforking extensions to common domains are almost orthogonal.
3. p is regular if it is orthogonal to all its forking extensions.

It is shown in [Hart et al. 1998] that regular types are the dimension carrying objects in simple theories.

Proposition 2.14. *If $p \in S(A)$ is regular then independence over A is a pregeometry on the realizations of p.*

Example 2.15. 1. The nonalgebraic type over the empty set in a strongly minimal set is a regular type.
2. The unique rank ω 1-type over the empty set in the theory of differentially closed fields of characteristic zero is a regular type. (See [Marker 2000] in this volume.)

There is a rank which is more general than Morley rank called SU-rank; it is also defined inductively:

Definition 2.16. For a type p,

1. $\mathrm{SU}(p) \geq 0$.
2. $\mathrm{SU}(p) \geq \alpha + 1$ if and only if there is q, a forking extension of p, such that $\mathrm{SU}(q) \geq \alpha$.
3. For a limit ordinal δ, $\mathrm{SU}(p) \geq \delta$ if and only if $\mathrm{SU}(p) \geq \alpha$ for all $\alpha < \delta$.

4. $\mathrm{SU}(p) = \alpha$ if $\mathrm{SU}(p) \geq \alpha$ and $\mathrm{SU}(p) \ngeq \alpha + 1$; $\mathrm{SU}(p) = \infty$ if $\mathrm{SU}(p) \geq \alpha$ for all α.

A theory for which $\mathrm{SU}(p) < \infty$ for all p is called *supersimple*. The following appears in [Hart et al. 1998] and shows that it is supersimple theories that have “enough” regular types.

PROPOSITION 2.17. *If T is simple and $\mathrm{SU}(p) < \infty$ then p is nonorthogonal to a regular type.*

3. Stability

The rationale for this section stems from the need for many of concepts in [Chatzidakis 2000]. On the other hand, many discussions of dimension theory in a model theoretic context will revolve around the notions here; we start with one which predates simplicity.

DEFINITION 3.1. A partial type p is stable if there is a λ such that, for any set A of size λ, the number of extensions of p over A is of size at most λ.

EXAMPLE 3.2. 1. The partial type $x = x$ is stable in all of the following theories: algebraically closed fields of any fixed characteristic, differentially closed fields of characteristic zero, any strongly minimal set. In such a case, one says that the theory is stable.
2. Consider a theory whose universe is partitioned by two unary predicates U and V; U contains a an algebraically closed field and V contains a copy of a real closed field. It is easy to see that the partial type $U(x)$ is stable.
3. Consider a theory again with two unary predicates which partition the universe, call them V and V^*. The model (V, V^*) will be an infinite vector space over a finite field, V and its dual V^*. Neither of these predicates is stable even though the theory of a vector space over any field is stable in its own right.

Definability of types. Independence on realizations of stable types has a more intrinsic definition than the one found in the last section which depends on the following definition.

DEFINITION 3.3. A type $p \in S(A)$ is definable if, for every formula $\varphi(x, y)$, there is a formula $d_\varphi(y, a)$ with $a \in A$ such that, for every $b \in A$, $\varphi(x, b) \in p$ if and only if $d_\varphi(b, a)$ holds.

EXAMPLE 3.4. Suppose that M is a left R-module for some ring R. M is considered a structure in the language with $0, +, -$ and a unary function symbol for every $r \in R$. The following is a theorem of Baur ([Baur 1976]).

THEOREM 3.5. *If $T = \mathrm{Th}(M)$ then any definable set is a boolean combination of cosets of definable subgroups. In fact, each coset is an instance of a formula $\varphi(x, y)$ of the form $\exists z(Axz = y)$ where A is an R-matrix and the subgroup is defined by $\varphi(x, 0)$.*

Now if p is a type over M and $\varphi(x,y) = \exists z(Axz = y)$ then the φ-definition d_φ is

1. false, if p does not contain an instance of φ and
2. $\varphi(b,y)$, if $\varphi(x,c) \in p$ for some $c \in M$ and $\varphi(b,c)$ holds for some $b \in M$.

Here are two key equivalences to the notion of a stable type.

THEOREM 3.6. *The following are equivalent*:

1. *p is a stable type.*
2. *There is no formula $\varphi(x,y)$ and realizations a_i of p and elements b_i for $i \in \omega$ such that $\varphi(a_i,b_j)$ holds if and only if $i < j$.*
3. *Every extension of p is definable.*

In a stable theory or for a stable type, nonforking and the definability of types are very closely related.

FACT 3.7. *If $q \in S(B)$ is a stable type and $A \subseteq B$ then q does not divide over A if and only if q is defined almost over A.*

COROLLARY 3.8. *If q is a stable type then q does not divide over a set of size at most $|T|$.*

COROLLARY 3.9. *Stable theories are simple.*

Stability has added advantages over simplicity and this is no more evident than in the notion of multipicity.

DEFINITION 3.10. If $p \in S(A)$ then the multiplicity of p is the supremum, if it exists, over all B, $A \subseteq B$ of $|\{q \in S(B) : q$ is a nonforking extension of $p\}|$.

FACT 3.11. *A type p is stable if and only if every extension of p has bounded multiplicity and does not divide over a set of size $|T|$.*

A stable type with multiplicity one is called *stationary.* A stable type over a model is stationary.

For the rest of this paper, I will assume that we are working inside a stable type p. All of the forking technology discussed in the previous section goes through in this context. If the reader likes, there is no real loss in assuming that the ambient theory is simple.

The canonical base

DEFINITION 3.12. 1. If $\varphi(x,a)$ defines X then the canonical parameter of X, written $\ulcorner\varphi(x,a)\urcorner$ or just $\ulcorner\varphi\urcorner$, is the element of M^{eq}, a/E_φ, where $E_\varphi(x,y) := \forall z(\varphi(z,x) \longleftrightarrow \varphi(z,y))$.
2. For a stationary type p, the canonical base of p, $Cb(p) = \text{dcl}\{\ulcorner d_\varphi\urcorner : \varphi\}$.

The canonical base of a stationary type p has many properties.

FACT 3.13. *Suppose that p is a stationary type and $\mathbf{p}$ is the nonforking extension of p to a large saturated model M.*

1. *p does not fork over $Cb(p)$.*
2. *$Cb(p)$ is the unique subset C of M^{eq} such that for all automorphisms σ of M, σ fixes C if and only if σ fixes* $\mathbf{p}$.
3. *$Cb(p)$ is contained in the definable closure of any Morley sequence in p.*

It is immediate that if T has elimination of imaginaries and is stable then the canonical base of any stationary type lies in M not M^{eq}.

1-based theories and types. In this last subsection, we make an attempt to tie together many of the concepts from this paper. Additionally, the Theorem below is used critically in [Chatzidakis 2000] where the concepts are discussed in more detail.

DEFINITION 3.14. 1. If $p \in S(C)$ and A is a set of realizations of p we will write $\bar{A}$ for $\mathrm{acl}^{\mathrm{eq}}(A \cup C)$.
2. $p \in S(C)$ is *1-based* if for every set A of realizations of p and every model M containing A, $Cb(A/M) \subseteq \bar{A}$.

FACT 3.15. *$p \in S(C)$ is* 1*-based if and only if for every pair A and B of sets of realizations of p, $\bar{A} \perp_{\bar{A} \cap \bar{B}} \bar{B}$.*

REMARKS 3.16. As the previous fact points out, 1-based types have an independence relation which is as simple as possible. The terminology "1-based" stems from the fact that in general, for a stationary type, the canonical base lies in the closure of any Morley sequence in the type; for 1-based types, the canonical base of any extension lies in the closure of a single realization of the type.

EXAMPLE 3.17. 1. A modular, strongly minimal set is 1-based.
2. Any complete theory of modules is 1-based.

DEFINITION 3.18. If G is the set of realizations of a partial type p then we say G is a stable group if there is a definable, binary function $*$ so that $(G, *)$ is a group.

REMARKS 3.19. A stable group may have more structure than just a group structure; for example, by the previous definition, a field is a stable group.

The following Theorem, found in [Hrushovski and Pillay 1987], shows how the strong model theoretic assumptions we have mentioned in this section impact on the definable sets in a stable group.

THEOREM 3.20. *A type-definable group G is stable and* 1*-based if and only if every definable subset is equivalent to a boolean combination of cosets of definable subgroups of G^n for some n.*

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