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On the Order-Seven Transformation of Elliptic Functions

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This is a translation of Klein's *Ueber die Transformation siebenter Ordnung der elliptischen Funktionen*, first published in *Mathematische Annalen* **14** (1879), 428–471, and dated early November 1878. It follows the text printed in his *Gesammelte Mathematische Abhandlugen*, except where typos not present in the original had crept into formulas. I redrew all the figures (they had already been redrawn for the *Abhandlugen*: see caption on page 320), except for Figure 9 and the bottom figures on pages 315 and 316.

I have not attempted to modernize the terminology, except on a few occasions when the use of current language allowed me to replace a long-winded phrase by something crisper and clearer. Nor have I tried to approximate the English mathematical style of the time. The goal has been to produce a readable translation, as close to the original ideas as possible. Bibliographic citations have been converted to the house format, the editors of the *Abhandlugen* having taken similar liberties.

Brackets, if not delimiting bibliographic tags, indicate interpolated text, written either for the *Abhandlugen* (unsigned, or K. = Klein, B.-H. = Bessel-Hagen) or for this edition (L. = Levy).

I'm grateful to Jeremy J. Gray for many excellent suggestions.

In the study of the fifth-order transformation of elliptic functions we encounter, along with the modular equation of sixth degree and its well-known resolvent of fifth degree, the Galois resolvent of degree 60, called the *icosahedral equation*, which governs both. Starting from the icosahedral equation one sees with great ease the rule of formation and the properties of those lower-degree equations.

In this work I would like to further the theory of the transformation of the *seventh* order up to the same point. I have already shown in [Klein 1879a] how one can construct the modular equation of degree eight in its simplest form in terms of this theory. The corresponding resolvent of seventh degree was considered in [Klein 1879b]. The question now is *to construct the corresponding Galois resolvent of degree* 168 *in a suitable way, and to derive from it those lower-degree equations.*

As is well-known, the root η of this Galois resolvent, regarded as a function of the period ratio ω, has the characteristic property of remaining invariant under exactly those linear substitutions

$$\frac{\alpha\omega + \beta}{\gamma\omega + \delta}$$

that are congruent to the identity modulo 7. This will be for me in the sequel the definition of the irrationality η.

Therefore I begin (Section 1) with a short investigation of linear substitutions modulo 7. This investigation is thoroughly elementary, but should be included for the sake of completeness.[1] From this follows (Section 2) the way in which η is branched as a function of the absolute invariant J, and above all the fact that the equation linking η and J, which has genus $p = 3$, is sent to itself under 168 one-to-one transformations having an *a priori* specifiable arrangement.

This leads to a remarkable curve of order four, which is sent to itself under 168 collineations of the plane (Section 3) and which, as a consequence, enjoys a number of particularly simple properties (Sections 4 and 5). From the knowledge of the existence of those 168 collineations one can construct with little effort the whole system of covariants belonging to the curve (Section 6), and one obtains the equation of degree 168 in question in a particularly clear way, by intersecting the ground curve with a covariant pencil of curves of order 42 (still Section 6).

If one wants to descend from the equation so obtained to the modular equation of degree eight or to the resolvent of degree seven, certain results valid for the general curve of order four and dealing with contact curves of order three and with certain arrangements of bitangents (Sections 7–10) are particularly relevant. The roots of the equations under consideration thus turn out to be rational functions of the coordinates of *one* point on the curve, and to me the essential advance lies in this explicit representation achieved for the transformation of order seven.

The next several sections (Sections 11–15) attempt to sketch as intuitive as possible a picture of the branching of the Riemann surface defined by η as a function of J, and which is discussed more abstractly in Section 2. The figures that I have obtained in this way play the same important role in the understanding of the questions expounded here as the *shape* of the icosahedron plays in the related problem of degree five.

The most important results discussed here have already been announced in a note submitted on May 20, 1878 to the Erlangen Society.[2] There I had already shown how one can explicitly reduce those equations of degree seven having the same group as the modular equation to the modular equation itself.[3] In this article I do not yet go into this and other connected questions; I intend to return to them in more detail before long. [See [Klein 1879c].]

[1] Compare the more general investigations in [Serret 1866].

[2] [Klein 1878b]

[3] [In an earlier communication [Klein 1878a], I had only established the possibility of this reduction by abstract arguments. –K.]

1. Classification of the Substitutions $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$ Modulo 7

By a substitution $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$ pure and simple I will always mean a substitution

$$\omega' = \frac{\alpha\omega+\beta}{\gamma\omega+\delta}$$

where the coefficients are integers and the determinant is one. Moreover, for brevity, I will use the following expression. Two substitutions S_1 and S_2 are called *equivalent* if there is a third substitution S such that

$$S_1 = S^{-1}\cdot S_2 \cdot S.$$

In [Klein 1879a, § 8] I distinguished three kinds of such substitutions: elliptic, parabolic, and hyperbolic. The following propositions are straightforward:

Equivalent substitutions have the same sum $\alpha+\delta$.

All elliptic substitutions of period 2 (*and so satisfying* $\alpha+\delta=0$) *are equivalent*.

If elliptic substitutions of period 3 (*and so satisfying* $\alpha+\delta=\pm1$) *are taken in pairs*, *so that one is the second iterate of the other*, *all such pairs are equivalent*.

Parabolic substitutions ($\alpha+\delta=\pm2$) *fall into infinitely many classes*, *each contaning one representative among*

$$\omega'=\omega, \quad \omega'=\omega\pm1, \quad \omega'=\omega\pm2, \quad \ldots.$$

From now on we will consider substitutions $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$ only modulo 7, so we will regard two substitutions

$$\frac{\alpha\omega+\beta}{\gamma\omega+\delta} \quad \text{and} \quad \frac{\alpha'\omega+\beta'}{\gamma'\omega+\delta'}$$

as identical if $\alpha\equiv\alpha'$, $\beta\equiv\beta'$, $\gamma\equiv\gamma'$, $\delta\equiv\delta'$. Accordingly, we will not require that $\alpha\delta-\beta\gamma$ be equal to 1, but only that it be congruent to 1 modulo 7. In any case:

Substitutions that were formerly equivalent remain equivalent when considered modulo 7.

Now there are only finitely many substitutions, which can be easily counted:

The number of substitutions is 168.

Clearly, exactly one of these has period one, the identity $\omega'=\omega$. We will denote it by S_1.

To obtain the substitutions of period two, we introduce their characteristic condition, $\alpha+\delta=0$. There are 21 period-2 substitutions that are distinct modulo 7; since their period cannot change by considering them modulo 7, we have:

There are 21 *equivalent substitutions of period two*, which we denote by S_2. An example is $-1/\omega$.

In a similar way, applying the condition $\alpha+\delta=\pm1$, which characterizes elliptic substitutions of period three, we obtain:

There are 28 *equivalent pairs of substitutions* S_3 *of period three*. An example of a pair is $-\frac{2}{3}\omega$, $-\frac{3}{2}\omega$.

For parabolic substitutions we had $\alpha+\delta=\pm 2$, which leads to 49 substitutions that are distinct modulo 7. One is the identity. Each of the others is equivalent to one of $\omega \pm 1$, $\omega \pm 2$, and $\omega \pm 3$, and so has period 7. Thus:

There are 48 *substitutions* S_7 *of period seven*, *divided into eight equivalent sextuples*. One sextuple would be $\omega+1$, $\omega+2$, ..., $\omega+6$.

There remain $168-1-21-56-48=42$ substitutions, for which $\alpha+\delta=\pm 3$. The second iterate of a substitution of this kind satisfies $\alpha'+\delta'=0$, and so has period two; thus the 42 substitutions have period 4. I will pair each with its inverse. Thus:

There are 21 *equivalent pairs of substitutions* S_4 *of period four*, *each pair being associated with one* S_2.

For example, $\dfrac{2\omega+2}{-2\omega+2}$ and $\dfrac{2\omega-2}{2\omega+2}$ are associated with $-\dfrac{1}{\omega}$.

Connected with this classification of substitutions by their period is the construction of the *groups* they form. First we have the groups generated by a single element:

1. *One* G_1, *consisting of the identity only*: $\omega'=\omega$.
2. *Twenty-one* G_2*s with two substitutions each*; *for example* ω *and* $-1/\omega$.
3. *Twenty-eight* G_3*s with three substitutions each*; *for example* ω, $-\frac{2}{3}\omega$, $-\frac{3}{2}\omega$.
4. *Twenty-one* G_4*s with four substitutions each*; *for example*

$$\omega, \quad \frac{2\omega+2}{-2\omega+2}, \quad -\frac{1}{\omega}, \quad \frac{2\omega-2}{2\omega+2}.$$

5. *Eight* G_7*s with seven substitutions each*; *for example* ω, $\omega+1$, ..., $\omega+6$.

Among these groups, any two that have the same number of elements are equivalent. For this reason, to prove each the following results, it is enough to exhibit one example satisfying the given description.

1. *Every* S_2 *commutes with exactly four other* S_2*s*. *These four fall into two pairs such that the elements of each pair commute with each other*.

Example: The substitution $-1/\omega$ commutes with

$$\frac{2\omega+3}{3\omega-2}, \quad \frac{3\omega-2}{-2\omega-3}, \quad \frac{2\omega-3}{-3\omega-2}, \quad \frac{3\omega+2}{2\omega-3}.$$

The first two of these commute, as do the last two.

2. *Thus there are* 14 *groups* G_4' *of four elements such that every element different from the identity has period two*.[5] Examples:

$$\omega, \quad -\frac{1}{\omega}, \quad \frac{2\omega+3}{3\omega-2}, \quad \frac{3\omega-2}{-2\omega-3},$$

[5] [Such a group is called a *four-group* in my later terminology, which was picked up by other authors. –K.]

or

$$\omega,\quad -\frac{1}{\omega},\quad \frac{2\omega-3}{-3\omega-2},\quad \frac{3\omega+2}{2\omega-3}.$$

These 14 groups are *not* all equivalent; seven are equivalent to one of the examples just given, and seven to the other. Every G_2 is contained in a G'_4 from each class.

3. *Every group G_3 commutes with exactly three S_2s. Thus there are 28 groups G'_6 of six elements, all of them equivalent. Each S_2 is contained in four G'_6s.* Example:

$$\omega,\quad -\frac{3\omega}{2},\quad -\frac{2\omega}{3},\quad -\frac{1}{\omega},\quad \frac{2}{3\omega},\quad \frac{3}{2\omega}.$$

4. *The four substitutions S_2s that, by item 1 above, commute with a given S_2 also commute with the G_4 that contains the given S_2. This gives 21 equivalent groups G'_8 containing eight elements.*

Example: the four substitutions listed under item 4 on page 290, together with the four listed under item 1 on page 290.

5. *Each group G_7 commutes with 14 S_3s. This gives eight equivalent groups G'_{21} with 21 elements. Each S_3 lies in two of them.*

Example: $\omega+k$, $-\frac{2}{3}(\omega+k)$, $-\frac{3}{2}(\omega+k)$, for $k=0,1,\ldots,6$; or again the set of all substitutions of the form

$$\frac{\alpha\omega}{\gamma\omega+\delta}.$$

6. *The $2\cdot 7$ groups G'_4 (see item 2 above) give rise to $2\cdot 7$ groups G''_{24} with 24 elements*, as follows. We take one G'_4 and add to it:

(a) the six S_4s whose second iterates are in the chosen G'_4;
(b) the six S_2s that commute with some S_2 from the G'_4, but are not themselves in the G'_4;
(c) the compositions of the six S_2s just mentioned, which together make four pairs of S_3s.

Adding up, we have $4+6+6+4\cdot 2=24$.

For example, take G'_4 to consist of $\omega, -\dfrac{1}{\omega}, \dfrac{2\omega+3}{3\omega-2}, \dfrac{3\omega-2}{-2\omega-3}$. Then:

S_4s that belong to $-\dfrac{1}{\omega}$: $\dfrac{2\omega+2}{-2\omega+2}, \dfrac{2\omega-2}{2\omega+2}$.

S_4s that belong to $\dfrac{2\omega+3}{3\omega-2}$: $\dfrac{\omega+1}{\omega+2}, \dfrac{-2\omega+1}{\omega-1}$.

S_4s that belong to $\dfrac{3\omega-2}{-2\omega-3}$: $\dfrac{3\omega-3}{-3\omega+1}, \dfrac{\omega+3}{3\omega+3}$.

New S_2s that commute with $-\dfrac{1}{\omega}$: $\dfrac{2\omega-3}{-3\omega-2}, \dfrac{3\omega+2}{2\omega-3}$.

New S_2s that commute with $\dfrac{2\omega+3}{3\omega-2}$: $\dfrac{-\omega+1}{-2\omega+1}, \dfrac{\omega+2}{-\omega-1}$.

New S_2s that commute with $\dfrac{3\omega-2}{-2\omega-3}$: $\dfrac{3\omega-1}{3\omega-3}$, $\dfrac{-3\omega-3}{\omega+3}$.

Pairs of S_3s that arise by composition:

$$\frac{-3\omega-1}{2},\ \frac{-2\omega-1}{3};\quad \frac{2\omega}{\omega-3},\ \frac{3\omega}{\omega-2};\quad \frac{2}{3\omega+1},\ \frac{-\omega+2}{3\omega};\quad \frac{-\omega+3}{2\omega},\ \frac{-3}{2\omega+1}.$$

We see that the 24 substitutions making up a G''_{24} are related in the same way as the 24 permutations of four elements, or as the 24 rotations that take a regular octahedron to itself. I will make use later of both of these comparisons. These G''_{24} are obviously none other than the groups that I used in [Klein 1879b]. I wrote about these groups at that time in reference to Betti's work, in a slightly different form: namely, I did not stipulate that $\alpha\delta-\beta\gamma\equiv 1 \pmod 7$, but only that $\alpha\delta-\beta\gamma$ be congruent to a quadratic residue modulo 7—a distiction that has no significance in the context of fractional substitutions.

7. Finally, one can show by well-known methods that *the subgroups discussed above are the only ones to be found in the group of* 168 *substitutions in question*.[6]

2. The Function $\eta(\omega)$ and its Branching with Respect to J

Now let η be an algebraic function of J that is branched in such a way that, considered as a function of ω, it has the following properties:

1. It is single-valued everywhere in the positive half-plane ω.

2. It is sent to itself by exactly those substitutions $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$ that are congruent to the identity modulo 7.

I will denote by $\eta(\omega)$ one of the values corresponding to a given J. To obtain all other such values, it is enough to substitute for ω each of the 167 expressions $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$ that differ from ω modulo 7, because all the values of ω corresponding to the given J are of this form. It follows that:

η and J are related by an equation of degree 168 *in* η. We can denote the 168 roots (in some arbitrary order) by

$$\eta_1, \eta_2, \dots, \eta_{168}.$$

Now the result of making J go around a closed path in the complex plane is to replace any of the associated ωs by $\dfrac{\alpha'\omega+\beta'}{\gamma'\omega+\delta'}$. Accordingly, the ηs undergo a certain permutation, as a result of the substitution of this fractional expression for ω in $\eta\left(\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}\right)$. If, after this permutation, *one* of the η_i coincides with its initial value (assuming J to be generic), the substitution $\begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$ must be congruent to the identity modulo 7; therefore in this case *all* the η_i coincide with

[6] [An unsigned footnote in the *Abhandlungen* corrects this by mentioning the alternating group found within each symmetric group G''_{24}, and refers the reader to [Gierster 1881] –L.]

their initial values. For, if S denotes any substitution $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$ and S_0 any such substitution congruent to the identity modulo 7, the substitution S_0' such that

$$SS_0 = S_0'S$$

is likewise congruent to the identity modulo 7. Another way to express this is:

All roots η_i are equally branched with respect to J.[7]

The branch points themselves can only be at $J = 0, 1, \infty$, according to [Klein 1879a, § 2]. When J goes aroud 0, ω undergoes an elliptic substitution of period 2; when J goes around 1, ω undergoes an elliptic substitution of period 3; and when J goes around ∞, an appropriately chosen ω undergoes the parabolic transformation $\omega' = \omega + 1$. It follows that:

At $J = 0$ the 168 leaves of the Riemann surface that represents η are grouped into 56 cycles of three; at $J = 1$ they are grouped into 84 cycles of two; and at $J = \infty$, into 24 cycles of seven.

The *genus* of the equation that relates η and J is therefore found to be three:

$$p = \tfrac{1}{2}(2 - 2\cdot168 + 56\cdot2 + 84\cdot1 + 24\cdot6) = 3.$$

Two algebraic functions of J that have the same branching behavior are related by a rational expression in J. Thus:

Any root η_i of our equation is a rational function of any other root η_k and J.

Or, in other words:

One can construct 168 rational functions $R(\eta, J)$ with numerical coefficients, such that, if η is any of the roots, the others are given by

$$\eta_1 = R_1(\eta, J), \quad \eta_2 = R_2(\eta, J), \quad \ldots, \quad \eta_{168} = R_{168}(\eta, J).$$

Thus, corresponding to the 168 substitutions studied in Section 1, *there are 168 one-to-one transformations of our Riemann surface into itself*. The consequences that we are about to derive rest on the fact that we know the grouping of these substitutions from Section 1, and that these groups must have counterparts in terms of the one-to-one transformations of the Riemann surface that we will now consider.

We start with the following observation. The transformations take a point in the Riemann surface to another point directly above or below it [that is, one lying over the same J –L.]. If we ask, then, whether there are points that are left fixed by some transformations (or, equivalently, that are sent to less than 168 distinct images), the answer is simply the branch points: for they are the

[7] [The expression “equally branched” [gleichverzweigt] here means not merely that the branch locus of each root has the same arrangement and number of sheets as that of any other, but also that the *relationship* among the sheets for any two roots is the same. More precisely: If a point P describes on the (say) N-sheeted Riemann surface any simple closed curve, and if any other of the $N-1$ points exactly above or below P is made to move on the surface so as to always shadow P, it too will come back to itself. –B.-H.]

only points that belong to more than one sheet at once. Given what we said a moment ago about the branch locus, we conclude that:

Among the orbits of points arising from the group of transformations there is one with 24 *points of multiplicity seven*, *corresponding to* $J = \infty$; *one with* 56 *triple points*, *corresponding to* $J = 0$; *and one with* 84 *double points*, *corresponding* $J = 1$. *There are no other multiple points.*

I will give special names to these points, because of their importance: they will be called a-points, b-points, and c-points, respectively. Each a-point is fixed by a transformation of period 7, and thus by all the transformations of a G_7. Similarly, every b-point is fixed by a G_3, and every c-point by a G_2. But we know that there are only eight groups G_7, twenty-eight groups G_3 and twenty-one groups G_2, apart from twenty-one groups G_4. We conclude:

Each G_7 *leaves three* a*-points individually fixed*; *each* G_3 *leave two* b*-points fixed*; *and each* G_2 *leaves four* c*-points fixed.*

A G_4 *leaves no points fixed.*

Every G_7 was a normal subgroup of a G'_{21}, which apart from that contained only substitutions of period 3. The three a-points that are left fixed by a G_7 cannot be fixed by the other transformations in this G'_{21}; otherwise there would be only eight a-points in total, not 24. Therefore the three a-points are permuted by these other transformations, and since their period is three, the permutation is *cyclic*. Thus:

Every G'_{21} *has an associated triple of* a*-points that it leaves invariant as a set.*

In the same way one concludes:

Every G'_6 *has an associated invariant pair of* b*-points.*

Each G'_6 contains transformations of period 3, which fix the b-points individually, and transformations of period 2, which permute the points of the pair.

Several other results along the same lines can be deduced. I will only mention one more.

Every S_2 commutes with exactly four other S_2s, and with exactly four G_3s. This implies that:

Under a transformation of period two, *in addition to the four individually fixed* c*-points*, *there are also four quadruples of* c*-points and four pairs of* b*-points that are invariant as sets.*

Finally, recall that a G''_{24} contains four G_3s. Accordingly, we get four pairs of related b-points, and by what has been said about the G''_{24}, it is clear that *these four pairs of points are permuted in every possible way by the transformations of the* G''_{24}.

In all the statements above it is implicit that there are no more of each type of invariant set or fully permuted set than the ones stated.

3. The Normal Curve of Order Four

As the variable η in our equation of degree 168 we can choose any algebraic function that is single-valued on the Riemann surface just described and takes 168 distinct values on a generic orbit of the group of transformations. We will in any case want to select the simplest function when it comes to actually constructing the equation, and therefore I will first deal with the problem of finding the *normal curve of lowest order* from which the equation between η and J can be derived. This problem is settled, as we will soon see, by means of a series of simple deductions made possible by the fact that *a lot is known about algebraic functions of genus* $p = 3$.[8]

Regarding the normal curve two types of algebraic functions of genus 3 should be distinguished: *hyperelliptic* and *general*. In the hyperelliptic case the normal curve is a [plane] curve C_5 of the fifth order with a triple point, and in the general case it is a [plane] curve of the fourth order [with no multiple points].[9]

I claim, first of all, that *our normal curve cannot be hyperelliptic*. For our curve must, like the equation between J and η from which it is derived, be mapped to itself by 168 one-to-one transformations forming a group whose structure we already know. But a hyperelliptic curve has a one-parameter family of pairs of points that is invariant under one-to-one transformations (for the curve C_5, which has a triple point, this family is given by the intersection of the curve with rays that go through the triple point). Therefore the pencil of rays emanating from the triple point would be mapped to itself in 168 ways.[10] But a pencil of rays is a rational one-dimensional variety; therefore (by a reasoning that I have often used before) there must exist a group of 168 linear transformations behaving in exactly the same way as the group of transformations of the surface. In particular, there should be no transformation of period greater than seven. But it is well-known that such a group cannot exist.

Thus our normal curve has order four.

Now the theory of algebraic functions[11] says that in general, under a one-to-one transformation of a curve to itself, what Riemann called the φ functions transform linearly. For a curve of fourth order, the φ functions take a given value at (generically) four points, and the quadruples thus determined can be regarded as the intersections of the curve with lines going through a certain point of the plane. Thus every linear transformation of φ gives rise to a map of the plane

[8] See [Weber 1876].

[9] [See the exposition in [Clebsch and Gordan 1866, p. 65] and in [Clebsch 1876, vol. 1, pp. 687, 712]. –K.]

[10] Conceivably some S_2 might interchange the two intersection points on each ray, so there would be only 84 transformations of the pencil of rays, not 168. The reasoning given in the text would still work, but in any case this possibility does not arise because [such an S_2 would be central and –L.] our group is simple.

[11] See [Brill and Nöther 1874].

that associates to each straight line a straight line and to each point a point—in other words, a *collineation* in the usual sense. Therefore:

Our curve of order four is sent to itself by a group of 168 *collineations*, *having the structure we already know*. In particular, *there exists a group of* 168 *collineations of the plane none of which has period greater than seven*.[12]

On our curve of order four, most points are grouped in orbits of 168 points each, under the action of these collineations. But there are three smaller orbits: one with only 24 points (which we have labeled a), one with 56 points (labeled b), and one with 84 points (labeled c).

On the other hand, one knows that a curve of order four has distinguished sets with 24, 56 and 84 points: there are 24 *inflection points*, 56 *contact points of bitangents*, and 84 *sextatic points*. [A point is sextatic if some conic makes contact of order six with the curve there. For instance, if a smooth curve has an axis of symmetry — say the y-axis — its intersections with this axis are sextatic points: writing y as a function of x the first, third and fifth derivatives vanish, and among the conics tangent to the surface at the point and symmetric with respect to the same axis we still have two parameters with which to control the second and fourth derivatives. –L.] Each of these sets is characterized by a property that does not change under collineations, and therefore each is invariant, as a set, under our group of 168 collineations. Consequently:

The points a are the inflection points, *the points b are the contact points of the bitangents*, *and the points c are the sextatic points*.

One might object that conceivably the inflection points could be a subset of the contact points of bitangents or of the sextatic points, or the last two could be a subset of one another. But this cannot happen because, from what we know about the Riemann surface, only orbits of 24, 56, or 84 points can occur, and 56 is not divisible by 24, nor is 84 a sum of multiples of 24 and 56.

We can also give a simple geometric interpretation to the *triples* of points a, to the *pairs* of points b, and to the *quadruples* of points c.

Regarding the triples, note that every inflection tangent of our curve C_4 intersects the curve in exactly one other point. We thus obtain 24 points, one corresponding to each inflection, and they form an orbit. Since the only 24-point orbit consists of the inflection points themselves, *the intersection points of the inflection tangents coincide with the inflection points*, *in some permutation*. This permutation cannot fix any inflection point, otherwise the order of contact there would be four, and the inflection points would not all be distinct from one another, nor from the contact points of the bitangents. Thus:

An inflection tangent to the curve C_4 *intersects the curve in another inflection point*.

[12] This group is missing from the list of all finite groups of linear substitutions in three variables [that is, subgroups of $\mathrm{SL}(3,\mathbb{C})$ –L.] given in [Jordan 1878]. (As Jordan has pointed out to me, the error appears on page 167 of his article, line 8 from below, where Ω need not be divisible by 9φ, only by 3φ. (Added in proof December 1878.))

Now there exist collineations of C_4 into itself that fix a given inflection point a. Such a collineation must also fix the inflection point associated to that a by the process just described, and the inflection point associated to that, and so on. But the *triples* of points a are characterized precisely by the property that a transformation that fixes one point of the triple fixes all three. [See page 294 –L.] We conclude that:

The 24 *inflection points of* C_4 *fall into eight triples*, *corresponding to the triples of a-points*. *Each triple of inflection points forms the vertices of a triangle whose edges are the inflection tangents*.

Even simpler is the meaning of the pairs of b-points. If a collineation fixes one of the contact points of a bitangent, it must also fix the other. Therefore:

The 28 *pairs of b-points correspond to the* 28 *pairs of contact points of the bitangents*.

Finally, to interpret the quadruples of c-points, we use the easily proved fact that any plane collineation of period two is a perspective transformation. Thus, corresponding to an S_2, we have a perspective *center* and *axis*. There are 21 such centers and axes. Each perspective axis intersects C_4 in four points, and these are the points fixed by the corresponding perspective. Thus:

The 84 *sextatic points are the intersections of* C_4 *with* 21 *straight lines*. *The four points on each of these lines correspond to a quadruple of c-points*.

Finally, we revisit the statements given at the end of Section 2. They have the following counterparts:

Each perspective center lies on four perspective axes; *conversely*, *each axis contains four centers*.

Each bitangent contains three centers, *while each center lies on four bitangents*.

The 24 *collineations of a* G''_{24} *permute in all possible ways a certain set of four bitangents*.

4. Equations for the Curve of Order Four

The results already stated are more than enough to allow us to construct, for the curve C_4, several equations for which the different types of collineations stand out.

First, we might choose as our coordinate triangle a *triangle of inflection tangents*. Let its sides be $\lambda = 0$, $\mu = 0$ and $\nu = 0$, with the side $\lambda = 0$ osculating the curve at the intersection with $\mu = 0$, and so on cyclically. Then the equation of the curve must have the form

$$A\lambda^3\mu + B\mu^3\nu + C\nu^3\lambda + \lambda\mu\nu(D\lambda + E\mu + F\nu) = 0.$$

Now the curve is invariant under a cyclic permutation of λ, μ, ν. Replacing λ, μ and ν by appropriate multiples, we can arrange to have $A = B = C$ and $D = E = F$. Next, the curve is sent to itself by six collineations of period seven that leave each side of the triangle invariant. These collineations can be expressed analytically in such a way that the ratios $\lambda : \mu : \nu$ are multiplied by appropriate seventh roots of unity. Such a substitution cannot take the term $\lambda\mu\nu(\lambda+\mu+\nu)$ to a multiple of itself, so this term cannot in fact appear. *Therefore the equation reads simply*

$$0 = f = \lambda^3\mu + \mu^3\nu + \nu^3\lambda. \tag{1}$$

I will always express the collineations that take f to itself in such a way that the determinant is one. We first have the collineation of period three given by

$$\lambda' = \mu, \quad \mu' = \nu, \quad \nu' = \lambda, \tag{2}$$

and then the period-seven collineation

$$\lambda' = \gamma\lambda, \quad \mu' = \gamma^4\mu, \quad \nu' = \gamma^2\nu, \tag{3}$$

where $\gamma = e^{2\pi i/7}$. If we combine these two collineations and their iterates in all possible ways, *we obtain the G'_{21} that leaves invariant the chosen inflection triangle.*

To highlight the six elements of a G'_6 I will choose a new coordinate triangle, whose sides are defined by the property of being each fixed by the permutations (2). Thus I start by setting

$$x_1 = \frac{\lambda+\mu+\nu}{\alpha-\alpha^2}, \quad x_2 = \frac{\lambda+\alpha\mu+\alpha^2\nu}{\alpha-\alpha^2}, \quad x_3 = \frac{\lambda+\alpha^2\mu+\alpha\nu}{\alpha-\alpha^2}, \tag{4}$$

where $\alpha = e^{2\pi i/3}$.

The equation of the curve becomes

$$0 = f = \tfrac{1}{3}\big(x_1^4 + 3x_1^2x_2x_3 - 3x_2^2x_3^2 + x_1((1+3\alpha^2)x_2^3 + (1+3\alpha)x_3^3)\big). \tag{5}$$

To get rid of the cube roots of unity, we further set

$$x_1 = \frac{y_1}{\sqrt[3]{7}}, \quad x_2 = y_2\sqrt[3]{3\alpha+1}, \quad x_3 = y_3\sqrt[3]{3\alpha^2+1} \tag{6}$$

and get

$$0 = f = \frac{1}{21\sqrt[3]{7}}\big(y_1^4 + 21y_1^2y_2y_3 - 147y_2^2y_3^2 + 49y_1(y_2^3 + y_3^3)\big). \tag{6a}$$

We immediately see that $y_1 = 0$ is a bitangent, with contacts at $y_2 = 0$ and $y_3 = 0$, and that *the six substitutions of the corresponding G'_6 are generated by*

$$y_1' = y_1, \qquad y_2' = \alpha y_2, \quad y_3' = \alpha^2 y_3, \tag{7}$$

$$y_1' = -y_1, \quad y_2' = -y_3, \quad y_3' = -y_2 \tag{8}$$

(the first of these coincides with (2)).

The three perspective centers lying on $y_1 = 0$ are given by

$$y_2 + y_3 = 0, \quad y_2 + \alpha y_3 = 0, \quad y_2 + \alpha^2 y_3 = 0,$$

while the corresponding perspective axes have the equations

$$y_2 - y_3 = 0, \quad y_2 - \alpha y_3 = 0, \quad y_2 - \alpha^2 y_3 = 0.$$

In order to locate a G''_{24}, I will first of all find the bitangents that go through the perspective centers just listed. Each center lies on four bitangents, but one of them is just the line $y_1 = 0$, so there remain nine bitangents to be found. First we consider those that go through the center $y_1 = y_2 + y_3 = 0$, and which therefore have an equation of the form

$$\sigma y_1 + (y_2 + y_3) = 0.$$

To determine σ, we substitute the value of y_1 from the preceding equation into the equation of the curve, then sort by powers of $y_2 y_3/(y_2 + y_3)^2$ to obtain a quadratic equation in this quantity, and finally set its determinant to zero. We get

$$28\sigma^3 - 21\sigma^2 - 6\sigma - 1 = 0,$$

whose roots are $\sigma = 1$ and $\sigma = \frac{1}{8}(-1 \pm 3\sqrt{-1/7})$. We conclude that *the bitangents that go through the point* $y_1 = y_2 + y_3 = 0$ *have equations*

$$y_1 + y_2 + y_3 = 0 \quad \textit{and} \quad (-7 \pm 3\sqrt{-7})y_1 + 56y_2 + 56y_3 = 0.$$

The remaining six bitangents (going through the other two centers) are obtained from these three by two applications of (7).

Now I claim that $y_1 = 0$, *together with any three of the bitangents just discussed that are sent to one another by* (7), *form a quadruple of bitangents whose eight contact points lie on a conic*. More generally, the six points in any orbit of the G'_6, together with the contact points of $y_1 = 0$, lie on a conic, because the substitutions (7) and (8) preserve the quadratic expression $y_1^2 + k y_2 y_3$, for each k. The preceding claim is a particular case of this fact, because the six contact points form an orbit of the G'_6 (each given bitangent goes through the perspective center of an S_2, and so is invariant under it, its two contact points being interchagned).

In view of this, we can write the equation of our C_4 in three different ways in the form $pqrs - w^2 = 0$, where p, q, r, s are bitangents and w is the conic that goes through the contact points. The first such expression is

$$0 = \frac{1}{21\sqrt[3]{7}}\bigl(49y_1(y_1+y_2+y_3)(y_1+\alpha y_2+\alpha^2 y_3)(y_1+\alpha^2 y_2+\alpha y_3) - 3(4y_1^2-7y_2y_3)^2\bigr), \tag{9}$$

and the other two are

$$\begin{aligned}0 = \frac{1}{21\sqrt[3]{7}}\Biggl(\frac{y_1}{7\cdot 8^3}&\bigl((-7 \pm 3\sqrt{-7})y_1 + 56y_2 + 56y_3\bigr)\\ &\times\bigl((-7 \pm 3\sqrt{-7})y_1 + 56\alpha y_2 + 56\alpha^2 y_3\bigr)\\ &\times\bigl((-7 \pm 3\sqrt{-7})y_1 + 56\alpha^2 y_2 + 56\alpha y_3\bigr)\\ &\qquad -3\Bigl(\frac{1 \pm 3\sqrt{-7}}{16}y_1^2 - 7y_2y_3\Bigr)^2\Biggr).\end{aligned} \tag{10}$$

Equation (9) will be important in Section 15; the other one *yields, as I will now show, the substitutions in a* G''_{24}. Set

$$\begin{cases} \mathfrak{z}_1 = (21 \mp 9\sqrt{-7})\, y_1, \\ \mathfrak{z}_2 = (-7 \pm 3\sqrt{-7})\, y_1 + 56 y_2 + 56 y_3, \\ \mathfrak{z}_3 = (-7 \pm 3\sqrt{-7})\, y_1 + 56\alpha y_2 + 56\alpha^2 y_3, \\ \mathfrak{z}_4 = (-7 \pm 3\sqrt{-7})\, y_1 + 56\alpha^2 y_2 + 56\alpha y_3, \end{cases} \tag{11}$$

so that $\sum \mathfrak{z}_i = 0$. Then (10) becomes, apart from a scalar factor,

$$\left(\sum \mathfrak{z}_i^2\right)^2 - (14 \pm 6\sqrt{-7})\mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_3\mathfrak{z}_4 = 0, \tag{12}$$

and this equation is invariant under the 24 *collineations determined by the permutations of the* $\mathfrak{z}_i$. *These, therefore, are the collineations of the* G''_{24} *in question.*

We see that the collineations of a G''_{24} always leave invariant a certain conic

$$\sum \mathfrak{z}_i^2 = 0,$$

which goes through the contact points of the corresponding bitangents. Since there are $2\cdot 7$ groups G''_{24} and all bitangents have equal title, there are $2\cdot 7$ such conics, and by taking any seven together and intersecting with the curve C_4 we get all the contact points of bitangents. These conics will be very important in the sequel.

5. The 168 Collineations in Relation to the Inflection Triangle. Other Formulas

From (4) and (6) we obtain the following equations connecting the variables λ, μ, ν with y_1, y_2, y_3:

$$\begin{cases} -\sqrt{-3}\sqrt[3]{7}\,\lambda = y_1 + \sqrt[3]{7(3\alpha+1)}\, y_2 + \sqrt[3]{7(3\alpha^2+1)}\, y_3, \\ -\sqrt{-3}\sqrt[3]{7}\,\mu = y_1 + \alpha^2 \sqrt[3]{7(3\alpha+1)}\, y_2 + \alpha \sqrt[3]{7(3\alpha^2+1)}\, y_3, \\ -\sqrt{-3}\sqrt[3]{7}\,\nu = y_1 + \alpha \sqrt[3]{7(3\alpha+1)}\, y_2 + \alpha^2 \sqrt[3]{7(3\alpha^2+1)}\, y_3. \end{cases} \tag{12a}$$

If we now apply the substitution (8), replacing y_1, y_2, y_3 by $-y_1, -y_2, -y_3$, we get

$$\begin{aligned} -\sqrt{-3}\sqrt[3]{7}\,\lambda' &= y_1 - \sqrt[3]{7(3\alpha^2+1)}\, y_2 - \sqrt[3]{7(3\alpha+1)}\, y_3, \\ -\sqrt{-3}\sqrt[3]{7}\,\mu' &= y_1 - \alpha \sqrt[3]{7(3\alpha^2+1)}\, y_2 - \alpha^2 \sqrt[3]{7(3\alpha+1)}\, y_3, \\ -\sqrt{-3}\sqrt[3]{7}\,\nu' &= y_1 - \alpha^2 \sqrt[3]{7(3\alpha^2+1)}\, y_2 - \alpha \sqrt[3]{7(3\alpha+1)}\, y_3. \end{aligned}$$

Eliminating y_1, y_2, y_3 by combining the two systems, *we obviously find the change from one triangle of inflection tangents*, $\lambda\mu\nu = 0$, *to another*, $\lambda'\mu'\nu' = 0$. The calculation yields a very simple result if we use the well-known expressions for the cube roots on the right-hand sides in terms of third and seventh roots of unity.[13] Setting

$$A = \frac{\gamma^5 - \gamma^2}{\sqrt{-7}}, \quad B = \frac{\gamma^3 - \gamma^4}{\sqrt{-7}}, \quad C = \frac{\gamma^6 - \gamma}{\sqrt{-7}}, \tag{13}$$

$$\sqrt{-7} = \gamma + \gamma^4 + \gamma^2 - \gamma^6 - \gamma^3 - \gamma^5,$$

one easily gets

$$\begin{cases} \lambda' = A\lambda + B\mu + C\nu, \\ \mu' = B\lambda + C\mu + A\nu, \\ \nu' = C\lambda + A\mu + B\nu. \end{cases} \tag{14}$$

If we now combine this substitution (*which has period two*) *in all possible ways with arbitrary iterates of the substitutions* (2) *and* (3),

$$\lambda' = \mu, \quad \mu' = \nu, \quad \nu' = \lambda,$$
$$\lambda' = \gamma\lambda, \quad \mu' = \gamma^4\mu, \quad \nu' = \gamma^2\nu,$$

we get in explicit form all the 168 *collineations that preserve our curve of order four*, *or rather*, *the ternary quartic form*

$$f = \lambda^3\mu + \mu^3\nu + \nu^3\lambda.$$

It follows from this result that the coordinates of all the singular elements of our curve can be deduced without further ado: one need only determine the coordinates of one element of the desired kind and apply to them these 168 collineations. In this way it is straightforward to compute the coordinates of the inflection points and corresponding inflection tangents. As for the bitangents, let me remark that the bitangent $y_1 = 0$ of the preceding section, has the equation $\lambda + \mu + \nu = 0$ in terms of our inflection triangle, and that the contact points have coordinates $1 : \alpha : \alpha^2$ and $1 : \alpha^2 : \alpha$. Finally, in order to determine the 21 perspective axes and corresponding centers, it is enough to compute these elements for the substitution (14). We find for the perspective axis

$$\lambda' + \lambda = \mu' + \mu = \nu' + \nu = 0, \tag{15}$$

and for the corresponding perspective center

$$-B - C : B : C \quad \text{or} \quad B : -B - A : A \quad \text{or} \quad C : A : -C - A,$$

all of which indicate the same point.

In the sequel I will mainly use the expressions in λ, μ, ν that, when set to zero, represent the *eight inflection triangles* and the *two times seven conics*, respectively, discussed at the end of the preceding section. I will set down these

[13] $\sqrt[3]{7(3\alpha+1)} = (\gamma+\gamma^6) + \alpha(\gamma^2+\gamma^5) + \alpha^2(\gamma^4+\gamma^3)$ and likewise with α and α^2 interchanged.

equations here as they arise from one another by means of the 168 substitutions of determinant 1.

Denote by δ_∞ the inflection triangle to be used as a coordinate triangle, and write

$$\delta_\infty = -7\lambda\mu\nu, \tag{16}$$

introducing on the right a factor that will later prove convenient. The following formulas then arise for the remaining inflection triangles, where $x = 0, 1, \ldots, 6$:

$$\begin{aligned} \delta_x &= -7(A\gamma^x\lambda + B\gamma^{4x}\mu + C\gamma^{2x}\nu)(B\gamma^x\lambda + C\gamma^{4x}\mu + A\gamma^{2x}\nu)(C\gamma^x\lambda + A\gamma^{4x}\mu + B\gamma^{2x}\nu) \\ &= +\lambda\mu\nu - (\gamma^{3x}\lambda^3 + \gamma^{5x}\mu^3 + \gamma^{6x}\nu^3) + (\gamma^{6x}\lambda^2\mu + \gamma^{3x}\mu^2\nu + \gamma^{5x}\nu^2\lambda) \\ &\qquad + 2(\gamma^{4x}\lambda^2\nu + \gamma^x\nu^2\mu + \gamma^{2x}\mu^2\lambda). \end{aligned} \tag{17}$$

Next we obtain equations for two of the 14 conics, by taking the equation $\sum \mathfrak{z}^2 = 0$ of the preceding section and expressing it first in terms of the y_i and from there in terms of λ, μ, ν:

$$(\lambda^2 + \mu^2 + \nu^2) + \frac{-1 \pm \sqrt{-7}}{2}(\mu\nu + \nu\lambda + \lambda\mu) = 0.$$

Correspondingly, if we denote the left-hand side of the conics by c_x, for $x = 0, 1, 2, \ldots, 6$, we get

$$c_x = (\gamma^{2x}\lambda^2 + \gamma^x\mu^2 + \gamma^{4x}\nu^2) + \frac{-1 \pm \sqrt{-7}}{2}(\gamma^{6x}\mu\nu + \gamma^{3x}\nu\lambda + \gamma^{5x}\lambda\mu). \tag{18}$$

It is these two expressions that will later lead to the simplest resolvents of eighth and seventh degree.

6. Construction of the Equation of Degree 168[14]

As already mentioned, for the role of the variable η in the equation of degree 168 we can choose any single-valued function on our C_4 — and so any rational function of $\lambda : \mu : \nu$ — that takes in general distinct values at the 168 points of an orbit of the group of collineations. It seems simplest to choose λ/μ or λ/ν. But the result gains greatly in clarity if we introduce not *one* such ratio but *both* at once, the two being connected by the equation

$$f = \lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0.$$

For then J can be expressed as a rational function of order 42 of $\lambda : \mu : \nu$,

$$J = R(\lambda, \mu, \nu), \tag{19}$$

where R has a *very simple* form, and the *order*-42 *equation* (19), *together with the order-four equation* $f = 0$, *replaces the* one *degree*-168 *equation that we have*

[14] [Sections 6 through 10 may be compared with [Klein 1879c], which appeared a half year after the present article (March 1879) and is closely connected with it, but unfortunately had to be separated from it in these collected works. –K.]

been talking about so far. A similar procedure appears always to be appropriate when one is dealing with the construction of an equation of genus p greater than zero.

The function $R(\lambda, \mu, \nu)$ must first of all have the property of invariance under the 168 collineations. Thus, to find R, I will first discuss the construction of *all* functions of λ, μ, ν that have this property. Here we assume, of course, that the 168 collineations have been chosen to have determinant *one*. We know *one* such function,

$$f = \lambda^3\mu + \mu^3\nu + \nu^3\lambda;$$

we also know that the *covariants* of f always have the same invariance property. A short argument then shows that the covariants of f can be covered by the desired functions, and allows us at the same time to construct the whole system of functions with the relations that hold between the forms of the system. The rational function R proves to be the simplest combination of dimension zero that can be formed from the covariants. This is the same method that Gordan and I have repeatedly used in our recent works.

The first covariant of f is the Hessian ∇ of order six:

$$\nabla = \frac{1}{54}\begin{vmatrix} \dfrac{\partial^2 f}{\partial\lambda^2} & \dfrac{\partial^2 f}{\partial\lambda\,\partial\mu} & \dfrac{\partial^2 f}{\partial\lambda\,\partial\nu} \\ \dfrac{\partial^2 f}{\partial\mu\,\partial\lambda} & \dfrac{\partial^2 f}{\partial\mu^2} & \dfrac{\partial^2 f}{\partial\mu\,\partial\nu} \\ \dfrac{\partial^2 f}{\partial\nu\,\partial\lambda} & \dfrac{\partial^2 f}{\partial\nu\,\partial\mu} & \dfrac{\partial^2 f}{\partial\nu^2} \end{vmatrix} = 5\lambda^2\mu^2\nu^2 - (\lambda^5\nu + \nu^5\mu + \mu^5\lambda). \tag{20}$$

When set equal to zero, this expression determines the 24 inflection points on the surface $f = 0$, which indeed form an orbit. Now, there was no other orbit having 24 points on $f = 0$, and none having fewer. We conclude that there can be no invariant polynomial function of order less than six, and that any invariant polynomial function of order six must be a multiple of ∇. For if there were another function of order six, it would be expressible in the form

$$k\nabla + l\varphi f,$$

where k and l are constants — since when set to zero it must determine on $f = 0$ the same 24 inflection points. Here φ would be an invariant function of degree two, and such a function, as already remarked, does not exist. In exactly the same way we conclude that *the next higher invariant polynomial function has degree* 14 *and, when set to zero, it determines on* $f = 0$ *the* 56 *contact points of the bitangents*.

There are different ways in which a covariant of order 14 can be constructed. As is well known, Hesse has constructed for a general curve of order four a curve of order 14 that goes through the contact points of the bitangents. In our case this property holds for *any* covariant of order 14 that is not a multiple of $f^2\nabla$, so we can choose any of them. I choose

$$C = \frac{1}{9}\begin{vmatrix} \dfrac{\partial^2 f}{\partial\lambda^2} & \dfrac{\partial^2 f}{\partial\lambda\,\partial\mu} & \dfrac{\partial^2 f}{\partial\lambda\,\partial\nu} & \dfrac{\partial\nabla}{\partial\lambda} \\ \dfrac{\partial^2 f}{\partial\mu\,\partial\lambda} & \dfrac{\partial^2 f}{\partial\mu^2} & \dfrac{\partial^2 f}{\partial\mu\,\partial\nu} & \dfrac{\partial\nabla}{\partial\mu} \\ \dfrac{\partial^2 f}{\partial\nu\,\partial\lambda} & \dfrac{\partial^2 f}{\partial\nu\,\partial\mu} & \dfrac{\partial^2 f}{\partial\nu^2} & \dfrac{\partial\nabla}{\partial\nu} \\ \dfrac{\partial\nabla}{\partial\lambda} & \dfrac{\partial\nabla}{\partial\mu} & \dfrac{\partial\nabla}{\partial\nu} & 0 \end{vmatrix} = (\lambda^{14} + \mu^{14} + \nu^{14}) + \cdots. \tag{21}$$

I also form *a function of degree* 21, *the functional determinant of* f, ∇, *and* C:

$$K = \frac{1}{14}\begin{vmatrix} \dfrac{\partial f}{\partial\lambda} & \dfrac{\partial\nabla}{\partial\lambda} & \dfrac{\partial C}{\partial\lambda} \\ \dfrac{\partial f}{\partial\mu} & \dfrac{\partial\nabla}{\partial\mu} & \dfrac{\partial C}{\partial\mu} \\ \dfrac{\partial f}{\partial\nu} & \dfrac{\partial\nabla}{\partial\nu} & \dfrac{\partial C}{\partial\nu} \end{vmatrix} = -(\lambda^{21} + \mu^{21} + \nu^{21}) + \cdots. \tag{22}$$

When set equal to zero, K determines the 84 sextatic points on $f = 0$. Again, one can infer that apart from K there is no invariant function of order 21; for if there were it would be expressible in the form

$$k\,K - l\varphi f^{\nu},$$

where k and l are constants. Here φ would be an invariant function of degree $21 - 4\nu$, and so when set to zero it would intersect $f = 0$ in a number of points divisible by 4 but not by 8. But the only eligible orbits have 24 or 56 elements; this yields a contradiction.

Now recall that earlier we found the 84 sextatic points as the intersections of $f = 0$ with 21 straight lines, *the* 21 *perspective axes*; see (15). Therefore:

The equation $K = 0$ *represents the union of the* 21 *axes*.

If one wants to determine on $f = 0$ a *general* orbit of 168 points, it is clearly sufficient to consider the pencil of curves

$$\nabla^7 = k\,C^3,$$

for varying k. From this it follows first of all that *under the condition* $f = 0$ *we have*, *for appropriate values of* k *and* l, *a relation of the form*

$$\nabla^7 = k\,C^3 + lK^2; \tag{23}$$

and then it follows further that f, ∇, C, *and* K, *which are connected by this one equation*, *generate the whole system of forms under consideration*, *and a fortiori the whole system of covariants of* f.

To determine the constants k and l that appear in (23), I start by setting $\lambda = 1$, $\mu = 0$, $\nu = 0$. Formulas (20), (21), (22) yield

$$\nabla = 0, \quad C = 1, \quad K = -1, \tag{23a}$$

and moreover $f = 0$. Thus

$$k = -l.$$

Next I take f in the form (6a),

$$f = \frac{1}{21\sqrt[3]{7}}\big(y_1^4 + 21y_1^2y_2y_3 - 147y_2^2y_3^2 + 49y_1(y_2^3 + y_3^3)\big),$$

and compute some terms of ∇, C, and K, obtaining

$$\nabla = \tfrac{1}{27}(7^2y_3^6 - 3\cdot5\cdot7y_1y_2y_3^4\ldots), \quad C = \frac{2^3\cdot7^5\cdot\sqrt[3]{7}}{3^6}y_2y_3^{13}\ldots, \quad K = \frac{-2^3\cdot7^7}{3^9}y_3^{21}\ldots.$$

Now put $y_1 = 0$, $y_2 = 0$, $y_3 = 1$ in these equations, to obtain, besides $f = 0$,

$$\nabla = \frac{7^2}{3^3}, \quad C = 0, \quad K = -\frac{2^3\cdot7^7}{3^9}, \tag{23b}$$

so that

$$l = \frac{1}{2^6\cdot3^3}, \quad k = \frac{-1}{2^6\cdot3^3}.$$

Thus the relation among ∇, C *and* K *is*

$$(-\nabla)^7 = \left(\frac{C}{12}\right)^3 - 27\left(\frac{K}{216}\right)^2. \tag{24}$$

Based on this relation the function $R(\lambda, \mu, \nu) = J$ can now be determined immediately. J must be equal to 0 at the contact points of the bitangents, equal to 1 at the sextatic points, and equal to ∞ at the inflection points; it should take any other value on some 168-point orbit, and only there. *Thus we have the equation*

$$J : J-1 : 1 = \left(\frac{C}{12}\right)^3 : 27\left(\frac{K}{216}\right)^2 : -\nabla^7, \tag{25}$$

and this equation, together with $f = 0$, represents the problem of degree 168 that we had set out to formulate.

If we use, instead of J, the invariants g_2, g_3, Δ of elliptic integrals, we can write

$$g_2 = \frac{C}{12}, \quad g_3 = \frac{K}{216}, \quad \sqrt[7]{\Delta} = -\nabla. \tag{26}$$

7. Lower-Degree Resolvents

The group of 168 collineations contains subgroups G'_{21} and G'_{24} of 21 and 24 elements. Accordingly, our problem of degree 168 has resolvents of degree eight and of degree seven. There can be no question about what is the simplest form of these resolvents; they must be exactly the equations of degree eight and seven given respectively in [Klein 1879a, Equation (15)] and [Klein 1879b, Equations (5) to (7)], and which I constructed directly starting from the ω-substitutions. So all that is left to find out is how to pass from our current description to the equations given in those earlier articles. As always in such cases, this can be done in two ways.

The first approach is to seek the simplest *rational* function $r(\lambda, \mu, \nu)$ that takes the same value at all the points of any orbit of the subgroup G'_{21} or G'_{24} under consideration, and then ask how this function is related to J.

The second is to find the lowest-degree *polynomial* function of λ, μ, ν that remains invariant under the substitutions in the desired subgroup, and then determine its relationship with ∇, C, K or with Δ, g_2, g_3.

Each method has its advantages, and in the sequel we use the second to complement the first.

8. The Resolvent of Degree Eight

Consider the G'_{21} generated by the two substitutions

$$\lambda' = \mu, \qquad \mu' = \nu, \qquad \nu' = \lambda,$$
$$\lambda' = \gamma\lambda, \qquad \mu' = \gamma^4\mu, \qquad \nu' = \gamma^2\nu.$$

It leaves invariant the inflection triangle $\delta_\infty = -7\lambda\mu\nu$ of (16), and of course ∇, so also the rational function $\sigma = \delta_\infty^2/\nabla$. The latter has the property that it takes a prescribed value at only 21 points of the curve $f = 0$, because the pencil of order-six curves $\delta_\infty^2 - \sigma\nabla$ has three fixed points (the vertices of the coordinate triangle) in common with $f = 0$, each with multiplicity one. Thus, if we use σ as a variable, *J becomes a rational function of σ, of degree eight*:

$$J = \frac{\varphi(\sigma)}{\psi(\sigma)}. \tag{27}$$

Now we determine the multiplicity of the individual factors in φ, ψ, and $\varphi - \psi$, as I have done several times in similar problems.

J becomes infinite with multiplicity seven at the 24 inflection points. At three of these points — the vertices of the coordinate triangle — σ vanishes with multiplicity seven, since δ_∞ has a fourfold zero and ∇ a simple zero. At the remaining 21 inflection points σ becomes infinite with multiplicity one because of the denominator ∇. Therefore $\psi(\sigma)$ consists of a simple factor and a sevenfold one, the first vanishing at $\sigma = 0$ and the second at $\sigma = \infty$. *Thus, apart from a constant factor,* $\psi(\sigma)$ *equals* σ.

J vanishes with multiplicity three at the 56 contact points of the bitangents. With respect to the group G'_{21} the bitangents fall into two classes, one with 7 and one with 21 elements,[15] so the contact points are divided into two orbits with 7 points each and 2 with 21 each. At points of the first kind σ takes a certain value with multiplicity three, and at points of the second with multiplicity one. This means that φ *contains two simple and two threefold linear factors*.

Finally, J takes the value 1 with multiplicity two at the 84 sextatic points. With respect to G'_{21} these points fall into four orbits of 21, and at each point σ takes its value with multiplicity one. Therefore $\varphi - \psi$ *is the square of an expression of degree four of nonzero discriminant*.

Now these are the same conditions on φ, ψ, $\varphi - \psi$ that led me in [Klein 1879a, Abschnitt II, § 14] to the construction of the modular equation of degree eight:

$$J : J-1 : 1 = (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3 : (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau. \tag{28}$$

We arrive at this same equation in the present case, if we denote an appropriate multiple of σ *by* τ.

To determine this multiple, I now return to the y-coordinate system of (12a). The value of $7^2\lambda^2\mu^2\nu^2$ is $(5-3\alpha)\cdot 7^2/3^3$ when $y_1 = 0$, $y_2 = 0$, $y_3 = 1$, and $(5-3\alpha^2)\cdot 7^2/3^3$ when $y_1 = 0$, $y_2 = 1$, $y_3 = 0$. In both cases $\nabla = 7^2/3^3$ by (23b), so σ has the values $(5-3\alpha)$ and $(5-3\alpha^2)$, respectively. But the points $(y_1, y_2, y_3) = (0,0,1)$ and $(0,1,0)$ are the contact points of the bitangent $\lambda + \mu + \nu = 0$, which is one of the seven distinguished bitangents with respect to the chosen G'_{21}. Accordingly, J vanishes at these points and in particular the simple factor $\tau^2 + 13\tau + 49$ in (28) also vanishes. Its roots equal $3\alpha - 5$ and $3\alpha^2 - 5$. *Therefore we have simply*

$$\tau = -\sigma,$$

or, put another way:

One root τ *of Equation* (28) *has the value*

$$\tau_\infty = -\frac{\delta_\infty^2}{\nabla} = -\frac{7^2\lambda^2\mu^2\nu^2}{\nabla}. \tag{29}$$

Then (17) implies that *the remaining roots* τ_x *have the values*

$$\tau_x = -\frac{\delta_x^2}{\nabla} = -\frac{\left(\begin{array}{r}\lambda\mu\nu - (\gamma^{3x}\lambda^3 + \gamma^{5x}\mu^3 + \gamma^{6x}\nu^3) + (\gamma^{6x}\lambda^2\mu + \gamma^{3x}\mu^2\nu + \gamma^{5x}\nu^2\lambda)\\ +2(\gamma^{4x}\lambda^2\nu + \gamma^{x}\nu^2\mu + \gamma^{2x}\mu^2\lambda)\end{array}\right)^2}{\nabla}. \tag{30}$$

[15] This and similar statements can be verified easily using the formulas given earlier.

and so, as promised in the introduction, we have expressed the roots of the modular equation of degree eight as a rational function of one *point on the curve* $f = 0$.

As mentioned in [Klein 1879a, Abschnitt II, § 18], Equation (28) can be transformed as follows: write z^2 instead of τ, $27g_3^2/\Delta$ instead of $J-1$, and take the square root of both sides, to obtain

$$z^8 + 14z^6 + 63z^4 + 70z^2 - \frac{216g_3}{\sqrt{\Delta}}z - 7 = 0. \tag{31}$$

We can further replace $216g_3/\sqrt{\Delta}$ with $K/\sqrt{-\nabla^7}$, by (26); and replacing also z by its value $\delta/\sqrt{-\nabla}$, given by (29), (30), the result is

$$\delta^8 - 14\delta^6\nabla + 63\delta^4\nabla^2 - 70\delta^2\nabla^3 - \delta K - 7\nabla^4 = 0. \tag{32}$$

To see that the penultimate term should appear with a negative sign one can, for example, set $(\lambda, \mu, \nu) = (1, 0, 0)$ and replace δ by any of the values δ_x.

We would have arrived at the same equation (32) *if we had taken the polynomial approach.* For the simplest *polynomial* function of λ, μ, ν that is left invariant by G'_{21} is $\delta_\infty = -7\lambda\mu\nu$. Under the 168 collineations δ takes eight distinct values, whose symmetric function must be a polynomial function of ∇, C, K (since f is taken to equal 0). Therefore δ satisfies an equation of the eighth degree, which, in view of the degrees of ∇, C, K, must have the form

$$\delta^8 + a\nabla\delta^6 + b\nabla^2\delta^4 + c\nabla^3\delta^2 + dK\delta + e\nabla^4 = 0,$$

and if the coefficients a, b, c, d, e are determined by substituting for δ, ∇, K their values in terms of λ, μ, ν and taking into account that $f = 0$, we recover (32). This derivation has the advantage that it shows a priori why only certain powers of δ appear in (32).

9. Contact Curves of the Third Order. Solution of the Equation of Degree 168.

The eight roots of (32) can be expressed as follows, by virtue of (16) and (17):

$$\begin{cases} \delta_\infty = -7\lambda\mu\nu, \\ \delta_x = \lambda\mu\nu - \gamma^{-x}(\nu^3 - \lambda^2\mu) - \gamma^{-4x}(\lambda^3 - \mu^2\nu) - \gamma^{-2x}(\mu^3 - \nu^2\lambda) \\ \qquad + 2\gamma^x\nu^2\mu \qquad + 2\gamma^{4x}\lambda^2\nu \qquad + 2\gamma^{2x}\mu^2\lambda. \end{cases} \tag{33}$$

Now, I have already stated in [Klein 1879a, Abschnitt II, end of § 18] that Equation (31), and therefore also (32), is a *Jacobian equation of degree eight*, that is, the square roots of its roots can be written in terms of four quantities

A_0, A_1, A_2, A_3 as follows:[16]

$$\begin{cases} \sqrt{\delta_\infty} = \sqrt{-7}\, A_0, \\ \sqrt{\delta_x} = \qquad A_0 + \gamma^{\rho x} A_1 + \gamma^{4\rho x} A_2 + \gamma^{2\rho x} A_3 \end{cases} \tag{34}$$

(where ρ is any integer not divisible by 7). One may ask how this assertion, which I had deduced from the transcendent solution of (31) (loc. cit.) can be verified algebraically. This is done by considering certain *contact curves or order three*[17] of our curve $f = 0$, or, in other words, by considering *certain root functions of order three that exist on the curve* $f = 0$.

It is known that a curve of order four possesses 64 triply infinite families of contact curves of order three, of which 36 have *even* and 28 *odd* characteristic.[18] *In our case one family of even characteristic is singled out by the fact that it contains the eight inflection triangles as contact curves.*

We can certainly regard an inflection triangle as a contact curve of third order, in that its 12 intersections with our curve of order four actually coalesce into only *three* points, four at a time. Now consider, say, the triangle δ_∞. Through its intersections with C_4 we place the triply infinite family of curves of third order that have contact with C_4 at those points; their equation is

$$k\,\lambda\mu\nu + a\lambda^2\mu + b\mu^2\nu + c\nu^2\lambda = 0. \tag{35}$$

Each cubic meets the C_4 in another six points, and it is well-known that these are the contact points of another contact cubic belonging to the same family as δ_∞; and in this way one obtains *all* the cubics in the family. Now we have the identity

$$\begin{aligned}(k\,\lambda\mu\nu + a\lambda^2\mu + b\mu^2\nu + c\nu^2\lambda)^2 - (a^2\lambda\mu + b^2\mu\nu + c^2\nu\lambda)f \\ = \lambda\mu\nu\big(k^2\lambda\mu\nu - (a^2\mu^3 + b^2\nu^3 + c^2\lambda^3) + 2(bc\mu\nu^2 + ca\nu\lambda^2 + ab\lambda\mu^2) \\ + ((2ak - b^2)\lambda^2\mu + (2bk - c^2)\mu^2\nu + (2ck - a^2)\nu^2\lambda)\big).\end{aligned} \tag{36}$$

Therefore the totality of the contact cubics in this family is represented by the equation

$$\begin{aligned}0 = k^2\lambda\mu\nu - (a^2\mu^3 + b^2\nu^3 + c^2\lambda^3) + 2(bc\mu\nu^2 + ca\nu\lambda^2 + ab\lambda\mu^2) \\ + \big((2ak - b^2)\lambda^2\mu + (2bk - c^2)\mu^2\nu + (2ck - a^2)\nu^2\lambda\big).\end{aligned} \tag{37}$$

---

[16]Regarding the degree-eight Jacobian equation, see [Brioschi 1868] and the commentary in [Jung and Armenante 1869], as well as a remark at the end of [Klein 1878a] not yet used in the text [and also [Brioschi 1879]]. I hope to return to subject in detail soon. [See [Klein 1879c].] [See also the recent [Brioschi 1878/79].]

[17]That is, curves of order three that have six first-order contacts with $f = 0$. [The developments in the text follow the investigations in [Hesse 1855].]

[18][See [Riemann 1861/62] and Section 15 in this article.]

If we set

$$k = 1, \quad a = \gamma^{-x}, \quad b = \gamma^{-4x}, \quad c = \gamma^{-2x},$$

the right-hand side becomes the expression of δ_x, showing that all eight inflection triangles belong to the same family of contact cubics, as claimed.[19]

Now the formulas in (34) easily follow from the statement that *the set of root functions for a family of even characteristic is linearly generated by four independent elements*. Indeed, choose the four root functions correponding to the curves (37) for which, in turn,

$$\begin{aligned} &k = 1, \quad a = 0, \quad b = 0, \quad c = 0, \\ &k = 0, \quad a = 1, \quad b = 0, \quad c = 0, \quad \text{etc.}, \end{aligned}$$

and accordingly set

$$\begin{cases} A_0 = \sqrt{\lambda\mu\nu}, \\ A_1 = \sqrt{-\mu^3 - \nu^2\lambda}, \quad A_2 = \sqrt{-\nu^3 - \lambda^2\mu}, \quad A_3 = \sqrt{-\lambda^3 - \mu^2\nu}. \end{cases} \tag{38}$$

Then, by choosing the signs appropriately and using the condition $f = 0$, one obtains

$$\begin{cases} A_0A_1 = \lambda^2\mu, \quad A_0A_2 = \mu^2\nu, \quad A_0A_3 = \nu^2\lambda, \\ A_1A_2 = \lambda\mu^2, \quad A_2A_3 = \mu\nu^2, \quad A_2A_1 = \nu\lambda^2,\text{[20]} \end{cases} \tag{39}$$

and Equation (37) can be written in the following irrational form:

$$kA_0 + aA_1 + bA_2 + cA_3 = 0. \tag{40}$$

In particular, taking (33) into account,

$$\begin{cases} \sqrt{\delta_\infty} = \sqrt{-7}\, A_0, \\ \sqrt{\delta_x} = \qquad A_0 + \gamma^{-x}A_1 + \gamma^{-4x}A_2 + \gamma^{-2x}A_3. \end{cases} \tag{41}$$

These are exactly the same formulas as (34), *except that the formerly unspecified integer* ρ *now has been set to* -1.

[19] That the family has *even* characteristic follows from the irrational form of its equation, which we are about to state.

[20] Consequently A_0, A_1, A_2, A_3 satisfy a series of identities, all of which can be obtained by setting to zero the determinants of the 3×3 minors of

$$\begin{pmatrix} A_1 & A_0 & -A_2 & 0 \\ A_2 & 0 & A_0 & -A_3 \\ A_3 & -A_1 & 0 & A_0 \end{pmatrix}.$$

One can use these formulas to solve our equation of degree 168 explicitly in terms of elliptic functions.[21] The roots δ of (32) are proportional to the roots z of (31), and for the latter an expression in $q = e^{i\pi\omega}$ was given in [Klein 1879a, Abschnitt II, §§ 17, 18]. Using that expression we obtain here

$$\delta_\infty : \delta_x = -7\sqrt[6]{q^7}\prod(1-q^{14n})^2 : \sqrt[6]{\gamma^x q^{1/7}}\prod(1-\gamma^{2nx}q^{2n/7})^2. \tag{42}$$

The products on the right can be rewritten using the series development

$$q^{1/12}\prod(1-q^{2n}) = \sum_{-\infty}^{\infty}(-1)^n q^{(6n+1)^2/12},$$

so we can write the ratios among A_0, A_1, A_2, A_3 in terms of these series; using the equations

$$\frac{\lambda}{\mu} = \frac{A_0}{A_2}, \quad \frac{\mu}{\nu} = \frac{A_0}{A_3}, \quad \frac{\nu}{\lambda} = \frac{A_0}{A_1}, \tag{43}$$

arising from (39), *we obtain the following solutions for the equation of degree* 168:

$$\begin{cases} \dfrac{\lambda}{\mu} = q^{4/7}\dfrac{\sum_{-\infty}^{\infty}(-1)^{h+1}q^{21h^2+7h}}{\sum_{-\infty}^{\infty}(-1)^h q^{21h^2+h} + \sum_{-\infty}^{\infty}(-1)^h q^{21h^2+13h+2}}, \\ \dfrac{\mu}{\nu} = q^{2/7}\dfrac{\sum_{-\infty}^{\infty}(-1)^{h+1}q^{21h^2+7h}}{\sum_{-\infty}^{\infty}(-1)^{h+1} q^{21h^2+19h+4} + \sum_{-\infty}^{\infty}(-1)^h q^{21h^2+37h+16}}, \\ \dfrac{\nu}{\lambda} = q^{1/7}\dfrac{\sum_{-\infty}^{\infty}(-1)^{h+1}q^{21h^2+7h}}{\sum_{-\infty}^{\infty}(-1)^h q^{21h^2+25h+7} + \sum_{-\infty}^{\infty}(-1)^{h+1} q^{21h^2+31h+11}}. \end{cases} \tag{44}$$

It suffices to compute this one solution, *since the other* 167 *can be obtained from this one by applying the collineations of Section* 5.

Here I have only computed the ratios $\lambda : \mu : \nu$; if one wishes to start from the formulation represented by Equation (26), one of course gets formulas for the *actual values* of λ, μ, ν.

[21] The equation should also be solvable by means of a linear differential equation of third order; how is the latter to be constructed? [In the *Abhandlungen* this is complemented by a reference to a long footnote to [Klein 1879c], which reads in part as follows: The corresponding differential equation for $f = 0$ has been constructed by Halphen in a letter that reached me on 11 June 1884 [Halphen 1884] and later by Hurwitz [1886]. Let J be as in the text and set $\eta_i = y_i\nabla^8/(C^2K)$, for $i = 1, 2, 3$; then, according to Hurwitz, the η_i are certain solutions of

$$J^2(J-1)^2\frac{d^3\eta}{dJ^3} + (7J-4)J(J-1)\frac{d^2\eta}{dJ^2} + \left(\tfrac{72}{7}(J^2-J) - \tfrac{20}{9}(J-1) + \tfrac{3}{4}J\right)\frac{d\eta}{dJ} + \left(\tfrac{792}{7^3}(J-1) + \tfrac{5}{8} + \tfrac{2}{63}\right) = 0.$$

–L.]

10. The Resolvent of Degree Seven

The substitutions of a G''_{24} always leave invariant a conic c_x that goes through the contact points of four bitangents. By (18) we can write

$$c_x = (\gamma^{2x}\lambda^2 + \gamma^x\mu^2 + \gamma^{4x}\nu^2) + \frac{-1 \mp \sqrt{-7}}{2}(\gamma^{6x}\mu\nu + \gamma^{3x}\nu\lambda + \gamma^{5x}\lambda\mu). \qquad (45)$$

Now form the rational function

$$\xi = \frac{c_x^3}{\nabla}. \qquad (46)$$

Since the numerator and denominator are invariant under the substitutions in the G''_{24}, and since the pencil of sixth-order curves $\nabla - \xi\, c_x^3 = 0$ has no fixed intersection with the curve C_4, we conclude that ξ takes a given value at exactly the points of an orbit of the G''_{24}. Therefore:

J is a rational function of degree seven of ξ:

$$J = \frac{\varphi(\xi)}{\psi(\xi)}. \qquad (47)$$

We now consider again the values $J = \infty, 0, 1$.

The 24 inflection points, where J becomes infinite with multiplicity seven, form a single orbit of the G''_{24}, each point appearing once. Thus $\psi(\xi)$ is the seventh power of a linear factor. But ξ is itself infinite at the inflection points, because of (46). *Therefore $\psi(\xi)$ is a constant.*

Of the 56 contact points of the 28 bitangents eight lie on $c_x = 0$, so ξ vanishes with order three at those points. The other 48 split into 2 orbits of as 24 (each corresponding to 12 tangents). Thus φ *contains the simple factor ξ and the cube of a quadratic factor of nonzero discriminant*.

Finally, the 84 sextatic points fall into three orbits of 12 points each and two of 24 points each. Thus $\varphi-\psi$ *contains a simple cubic factor and the square of a quadratic factor*.

Again, these are the requirements on φ and ψ that led in [Klein 1879b, § 7] to the construction of the simplest equation of degree seven, which has the following form:

$$\begin{aligned} J : J-1 : 1 = {}& \mathfrak{z}\big(\mathfrak{z}^2 - 2^2{\cdot}7^2(7 \mp \sqrt{-7})\mathfrak{z} + 2^5{\cdot}7^4(5 \mp \sqrt{-7})\big)^3 \\ &: \big(\mathfrak{z}^3 - 2^2{\cdot}7{\cdot}13(7 \mp \sqrt{-7})\mathfrak{z}^2 + 2^6{\cdot}7^3(88 \mp 23\sqrt{-7})\mathfrak{z} - 2^8{\cdot}3^3{\cdot}7^4(35 \mp 9\sqrt{-7})\big) \\ &\quad \times \big(\mathfrak{z}^2 - 2^4{\cdot}7(7 \mp \sqrt{-7})\mathfrak{z} + 2^5{\cdot}7^3(5 \mp \sqrt{-7})\big)^2 \\ &: \mp 2^{27}{\cdot}3^3{\cdot}7^{10}\sqrt{-7}. \end{aligned} \qquad (48)$$

We conclude that the variable $\mathfrak{z}$ coincides with ξ apart from a multiplicative factor, though it is still in question whether the upper sign in front of the $\sqrt{-7}$ in (45) *corresponds to the upper or the lower sign in* (48).

To eliminate this ambiguity, I first transform (48) by setting $\mathfrak{z} = z^3$ and $J = g_2^3/\Delta$, and then I take the cube root of both sides:

$$z^7 - 2^2{\cdot}7^2(7 \mp \sqrt{-7})z^4 + 2^5{\cdot}7^4(5 \mp \sqrt{-7}) \mp 2^9{\cdot}3{\cdot}7^3\sqrt{-7}\,\frac{g_2}{\sqrt[3]{\Delta}} = 0. \tag{49}$$

Now, following (26), we substitute $C/(12\sqrt[3]{-\nabla^7})$ for $g_2/\sqrt[3]{\Delta}$ and $kc/\sqrt[3]{\nabla}$ for z, where k is a constant to be determined; the result is

$$k^7c^7 - 2^2{\cdot}7^2(7 \mp \sqrt{-7})k^4\nabla c^4 + 2^5{\cdot}7^4(5 \mp \sqrt{-7})k\nabla^2 c \pm 2^7{\cdot}7^3\sqrt{-7}\,C = 0. \tag{50}$$

This gives

$$\begin{aligned} k^3 \textstyle\sum c_x^3 &= 3{\cdot}2^2{\cdot}7^2(7 \mp \sqrt{-7})\big(5\lambda^2\mu^2\nu^2 - (\lambda^5\nu + \nu^5\mu + \mu^5\lambda)\big), \\ k^7 \textstyle\prod c_x &= \mp 2^7{\cdot}7^3\sqrt{-7}\,(\lambda^{14} + \mu^{14} + \nu^{14} + \cdots) \end{aligned}$$

(naturally under the assumption that $f = 0$). *So the two equations are reconciled when we choose*

$$k = \pm 2\sqrt{-7}$$

and make the sign of k *correspond to the upper sign in* (49) *and to the lower sign in* (45).

In other words: *The roots* z *of* (49) *and* $\mathfrak{z}$ *of* (48) *have the following values in terms of* λ, μ, ν:

$$z = \mathfrak{z}^{1/3} = \frac{\pm 2\sqrt{-7}\Big((\gamma^{2x}\lambda^2 + \gamma^x\mu^2 + \gamma^{4x}\nu^2) + \dfrac{-1\mp\sqrt{-7}}{2}(\gamma^{6x}\mu\nu + \gamma^{3x}\nu\lambda + \gamma^{5x}\lambda\mu)\Big)}{\sqrt[3]{\nabla}}, \tag{51}$$

and so, as promised in the introduction, we have explicitly written the $\mathfrak{z}$*'s as rational functions of one point on our* C_4.

Equation (50) becomes

$$c^7 + \tfrac{7}{2}(-1 \mp \sqrt{-7})\nabla c^4 - 7\Big(\frac{5 \mp \sqrt{-7}}{2}\Big)\nabla^2 c - C = 0.^{22} \tag{52}$$

Naturally, the polynomial approach would have led to the same equation. Indeed, the lowest polynomial function of λ, μ, ν that remains invariant under a G''_{24} is exactly the corresponding c_x, and this c_x must satisfy an equation of degree seven, whose coefficients are polynomials in ∇, C, K, and which therefore has the form

$$c^7 + \alpha\nabla c^4 + \beta\nabla^2 c + \gamma C = 0,$$

where α, β, γ are to be determined by the substitution of values for λ, μ, ν. Again, this approach has the advantage of showing a priory a great number of terms must be absent from (52) and (49).

[22] [The corresponding equation for $f \neq 0$ is given in [Klein 1879c, (12)].]

11. Replacement of the Riemann Surface of Section 2 by a Regularly Tiled Cover

Now I would like to explain the relationship between the irrationality $\lambda : \mu : \nu$ and the absolute invariant J, as well as with the roots τ and $\mathfrak{z}$ of the eighth- and seventh-degree equations, in as visual and intuitive a way as possible, using topology.

First recall the figures appearing in [Klein 1879a] for the eight-degree equation and in [Klein 1879b] for the seventh-degree equation. [They are reproduced on the next two pages. –L.] I will start with a general explanation concerning *Riemann surfaces that are related to their Galois resolvent by a rational parameter* [Klein 1879a, Abschnitt III]. Let $F(\eta, z) = 0$ be such a surface of degree N; by definition, it has the property that each root η_i is ramified with respect to the parameter z exactly like any other root η_k, so the surface is mapped to itself by N one-to-one transformations (compare Section 2).[23]

We regard the complex values of z as laid out on the plane, and denote by z_1, z_2, ..., z_n the branch points. The branching is the same for all sheets; assume that the sheets come together ν_1 at a time at z_1, ν_2 at a time at z_2, and so on. Now draw on the z-plane any simple closed curve that goes once through each of z_1, z_2, ..., z_n — in other words, a branch cut. It divides the z-plane and each of the N sheets of the η Riemann surface stretched over it into two regions. We think of the first region as being shaded, the second unshaded. Then transform the surface, which lay in sheets above the z-plane, so it now sits in space and is smoothly curved; but maintain the shading and the connectivity of the regions. The resulting surface is therefore divided into $2N$ alternately shaded and unshaded n-gons, which meet at the various vertices in groups of $2\nu_1$, $2\nu_2$, ..., $2\nu_n$ wedges, and which are, in the topological sense, alternately identical with and the mirror image of a given polygon; the edges of the polygons are the images of the branch cut we drew on the z-plane. The N one-to-one transformations of the equation $F(\eta, z) = 0$ into itself are reflected in that the surface thus obtained can be mapped one-to-one onto itself in N ways. Indeed, fix a (say) shaded polygon of the surface and map it to any chosen shaded polygon [preserving the numbering of the vertices –L.]; then declare that adjacent polygons should map to adjacent polygons. This assigns to each polygon a unique image in a consistent way, and the resulting correspondence of polygons is determined by the initial choice of an image for the base polygon. I will call covers that are divided in this sense into alternating regions *regularly* [*symmetric*] *tiled covers;* they comprise as particular cases, when the genus p is zero, the tilings of the sphere into 24, 48, and 120 triangles, associated with the tetrahedron, octahedron, and icosahedron.

We can state the following general theorem:

Any Galois resolvent $F(\eta, z) = 0$ *admits a regularly tiled cover*.

[23] [See footnote 7 on page 293. –B.-H.]

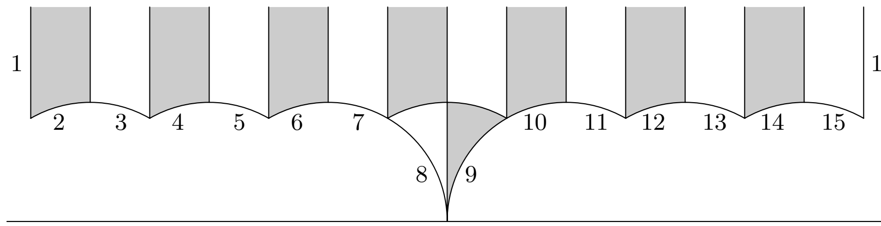


Figure 11 of [Klein 1879a].

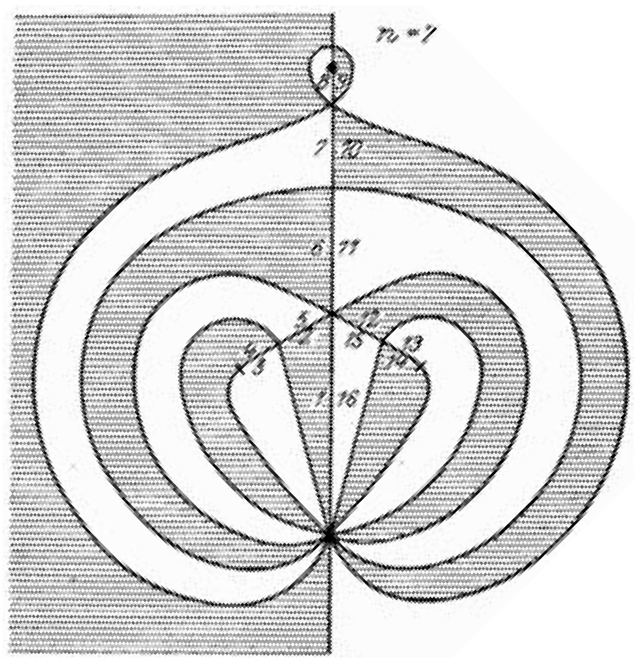


Figure 12 of [Klein 1879a].

[The top figure shows the fundamental polygon in the ω-plane for the modular equation of degree eight. The identification of the sides is given by Klein as follows: $\omega' = \omega+7$ maps 1 to 16, and $\omega' = \omega/(\omega+1)$ maps 6, 7, 8 to 11, 10, 9; these are parabolic transformations. Then $\omega' = (2\omega-7)/(\omega-3)$ maps 15, 14 to 12, 13, and $\omega' = -(2\omega+7)/(\omega+3)$ maps 2, 3 to 5, 4; these are elliptic transformations of period 3. The quotient of the upper half-plane by the group Γ generated by these transformations is a (punctured) sphere, parametrized by the variable τ; the bottom figure shows how the edges of the fundamental polygon become identified in the τ-plane (the figure is combinatorially but not conformally accurate). Thus on the τ-plane there are two order-three branch points of the quotient map (at the lower end of the edges $3 = 4$ and $13 = 14$) and one cusp (at the upper end of the edge $8 = 9$). The Klein surface — the quotient of the ω-plane by the group $\Gamma(7)$ of substitutions congruent to the identity modulo 7 — sits between the ω-plane and the τ-plane: it covers the τ-plane with multiplicity 21, since $\Gamma(7)$ has index 21 in Γ (the quotient $\Gamma/\Gamma(7)$ is the G'_{21} of Sections 2 and 8). –L.]

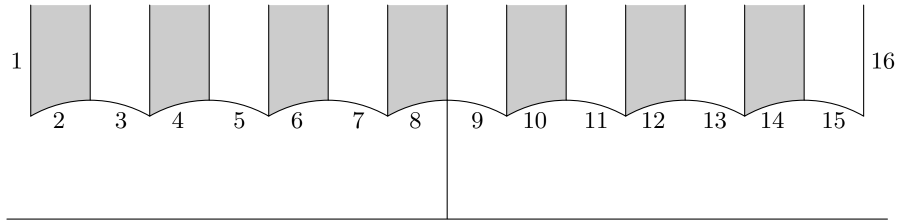


Figure 4 of [Klein 1879b].

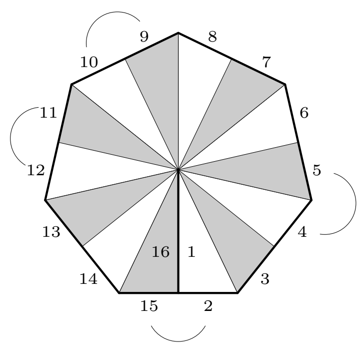


Figure 5 of [Klein 1879b].

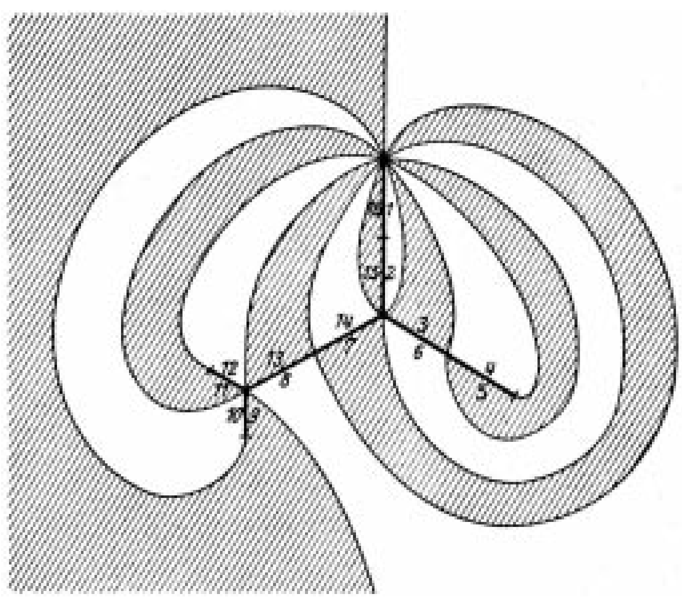


Figure 6 of [Klein 1879b].

[Top: fundamental polygon in the ω-plane for the modular equation of degree seven. Middle: identification of the sides (plus 7, 8 go to 14, 13). Bottom: the $\mathfrak{z}$-plane. –L.]

And conversely: *Every regularly tiled cover defines a particular Galois resolvent with one parameter*. This is because it defines a branching of η with respect to z having the property that each root η_i can be expressed rationally in terms of any other root η_k and of the parameter z.[24]

In the particular case we are considering, there are 168 sheets and three branch points, $J = 0, 1, \infty$. (I will continue to write J instead of z.) At $J = 0$ the sheets are grouped in threes, at $J = 1$ in twos, and at $J = \infty$ in sevens. Thus our surface is covered by $2 \cdot 168$ triangles, which come together in groups at 14 at 24 vertices, in groups of 6 at 56 vertices, and in groups of 4 at 84 vertices. The vertices of these triangles are none other than the a-points, b-points, and c-points of Section 2, and I will maintain this notation here.

From now on we assume that the branch cut in the J-plane is taken to coincide with the real axis. Then the two types of triangles that cover our surface correspond to the two J half-planes, *and the edges of the triangles correspond to real values of J*. I will (as always in the past) shade those triangles that correspond to the upper half-plane ($\operatorname{Im} J > 0$). Thus, for shaded triangles, we have this sequence of vertices:

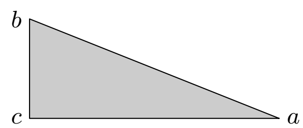


Figure 1.

If we compare this two-triangle surface with the decomposition of the ω-plane into infinitely many triangles from [Klein 1879a] [reproduced at the top of the next page; the labels correspond to the values of J –L.], it is clear that our irrationality J moves over *one* shaded or unshaded triangle when ω traverses a shaded or unshaded triangle, respectively.

Now, the figure at the top of page 315 explained the relation between ω and the root τ of the modular equation of degree eight, and the top and middle figure of page 316 did the same for the root $\mathfrak{z}$ of the modular equation of degree seven. If we move these figures onto our regularly tiled surface and observe that τ and $\mathfrak{z}$ are *rational* functions of $\lambda : \mu : \nu$, so that to any point of our surface there corresponds only one value of τ and one of $\mathfrak{z}$, we obtain the following results:

[24] I think it would be a very useful enterprise to list all the regularly tiled covers of low genus p and find out the corresponding equations $F(\eta, z) = 0$. [This problem was solved by W. Dyck in his Inaugural Dissertation [Dyck 1879]; see also the related [Dyck 1880a]. However, there is an error common to those works and the present article: the distinction between *regular* and *regular symmetric* tilings of surfaces had not been yet clearly grasped, and for this reason only the latter type was considered. This error was corrected in [Dyck 1882]; see particularly page 30 and the note therein.

In this connection I would like to stress that Dyck had already devoted a monograph to the study of Riemann surfaces that correspond to *Galois resolvents of modular equations* and achieved a general way to describe them clearly; see [Dyck 1881]. See also [Klein 1923, pp. 166 ff.]. –K.]

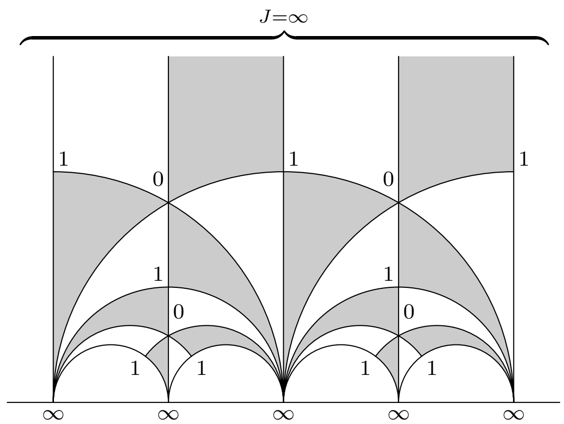


Figure 7 of [Klein 1879a].

Our regularly tiled surface can be divided into 21 *domains such as the one in Figure* 2. *It can also be divided into* 24 *heptagons such as the one in Figure* 3.[25]

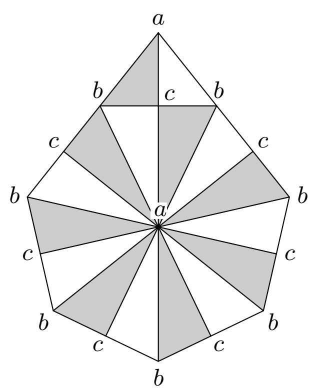


Figure 2.

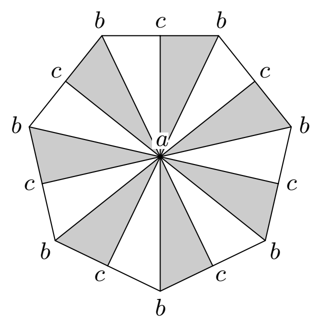


Figure 3.

Figure 2 is divided into two symmetric halves by its middle line. One of the halves is shown on the right. We can therefore say that *our surface is covered with* 42 *alternately congruent and symmetric regions of the type defined by this figure*.[26] I will use this decomposition to develop a completely visual and clear picture of the surface.

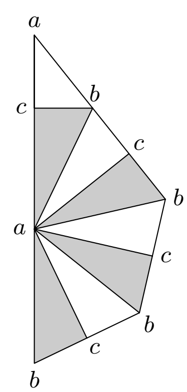


Figure 4.

[25]The grouping into 24 heptagons is analogous to taking the 120 triangles that tile the icosahedron and considering the groups of 10 that surround each of the 12 vertices of the icosahedron.

[26]Thus every region of this type corresponds to the right τ half-plane; see the bottom figure on page 316.

12. Explanation of the Main Figure

The regions just mentioned arrange themselves on our regularly tiled cover in groups of three as follows:

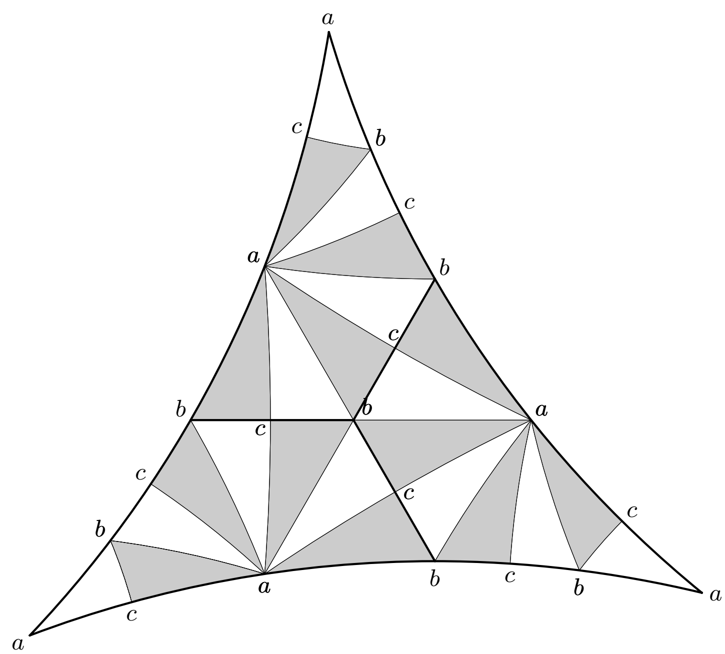


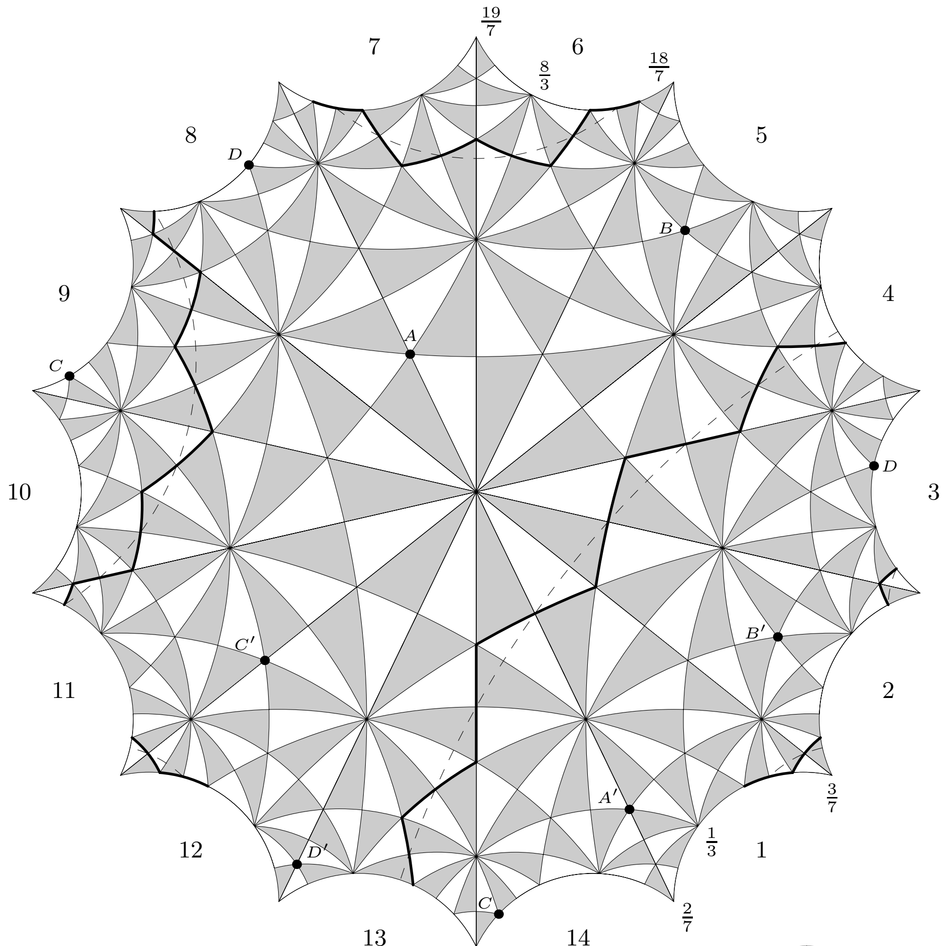
Figure 5.

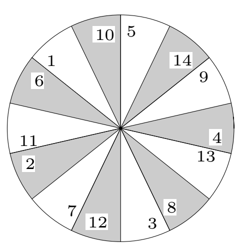
The Main Figure of this article, shown on the next page, is constructed by placing fourteen of these large triangles, with alternate symmetry, around the center point. For the sake of clarity I have made each small triangle out of arcs of circle, having angles of $\pi/7$, $\pi/3$ and $\pi/2$. I now claim that *this figure is a depiction of our regularly tiled surface*, *provided we think of the* 14 *boundary arcs as being identified with each other in the manner stated*.

In fact, our figure contains $2\cdot168$ small curved triangles, which exhibit the prescribed behavior at the points where they come together. Starting from this observation, one can look for a suitable correspondence between the boundary arcs, and then carry out the proof that there is no other possible grouping of the $2\cdot168$ triangles.

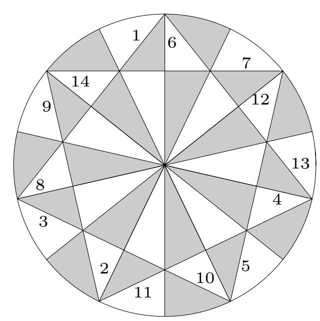
But in order not to make these considerations too abstract, I will resort again to the ω-plane and show on it the same collection of elementary triangles that makes up Figure 5. This is done in Figure 6.

If we now arrange 14 copies of this figure, with alternating symmetry, side by side on the ω-plane, we get the same configuration of triangles shown in the Main Figure. So we must check — and this can be done at once — that this arrangement of 336 triangles in the ω-plane can serve as a *fundamental polygon*





Edge identifications:
1 with 6
3 with 8
5 with 10
7 with 12
9 with 14
11 with 2
13 with 4



Vertices of one type

Vertices of the other type

Main Figure. [Note the three "eightfold ways", discussed in Section 14. Klein's original drawing can be found on page 115. The *Abhandlungen* version is as shown here, differing from the original by a $\pi/7$ rotation. (All the figures were redrawn for the *Abhandlungen*, for the most part with less care; but Figure 5 was improved in that originally the triangles had straight sides and widely different angles, so although combinatorially correct it was harder to grasp than the later version. See also footnote 31, p. 326.) –L.]

for our irrationality; that is, all 168 shaded triangles can be obtained from one of them by means of substitutions

$$\frac{\alpha\omega+\beta}{\gamma\omega+\delta},$$

all of which are distinct modulo 7; and likewise for all 168 unshaded triangles. We must also determine how the edges labeled 1, 2, ..., 14 in the Main Figure match up. Each such edge corresponds on the ω-plane to a *pair* of semicircles meeting the real axis perpendicularly; for example, edge 1 corresponds to a semicircle with endpoints $\omega = \frac{2}{7}, \frac{1}{3}$ and one with endpoints $\omega = \frac{1}{3}, \frac{3}{7}$. Thus, when I claim that edges 1 and 6 match, I must show that the corresponding pairs of semicircles

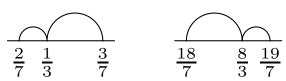


Figure 7.

in the ω-plane are mapped to each other by a substitution that is congruent to the identity modulo 7. This is indeed the case: the substitution

$$\omega' = \frac{113\omega - 35}{42\omega - 13}$$

maps $\frac{2}{7}$ to $\frac{19}{7}$ and $\frac{1}{3}$ to $\frac{8}{3}$, and so maps the semicircle that meets the real axis at $\frac{2}{7}$ and $\frac{1}{3}$ to the semicircle that meets the real axis at $\frac{19}{7}$ and $\frac{8}{3}$.

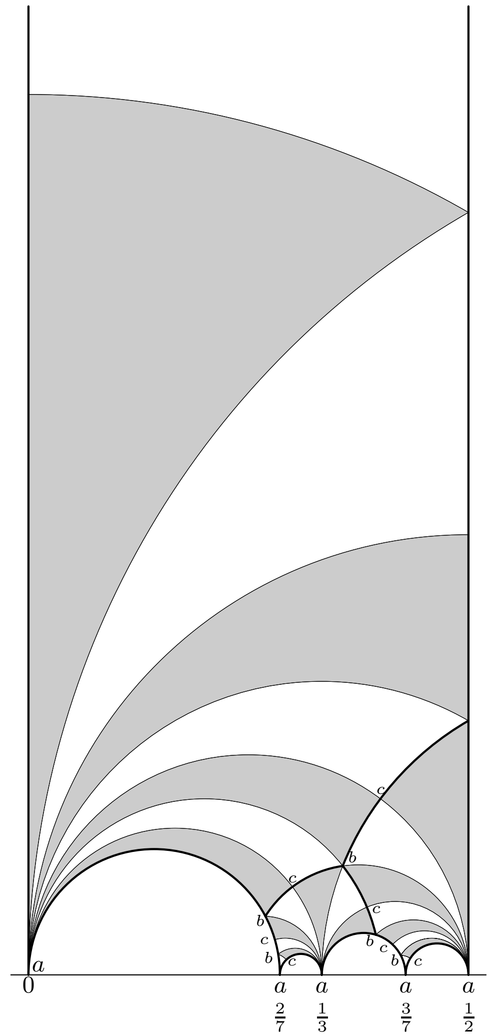


Figure 6.

Similarly, the substitution

$$\omega' = \frac{55\omega - 21}{21\omega - 8}$$

maps $\frac{1}{3}$ to $\frac{8}{3}$ and $\frac{3}{7}$ to $\frac{18}{7}$, which shows that the second halves of the pairs of semicircles match. I have marked the points $\frac{2}{7}$, $\frac{1}{3}$, $\frac{3}{7}$ and $\frac{18}{7}$, $\frac{8}{3}$, $\frac{19}{7}$ at the corresponding places in the Main Figure. *This argument shows that edges* 1 *and* 6 *are to be identified in such a way that* $\frac{2}{7}$ *coincides with* $\frac{19}{7}$ *and* $\frac{3}{7}$ *with* $\frac{18}{7}$.

In a similar way[27] one finds that *the following pairs of edges are to be identified*:

1 *with* 6, 3 *with* 8, 5 *with* 10, 7 *with* 12, 9 *with* 14, 11 *with* 2, 13 *with* 4,

and in each case vertices of the same type come together. What I mean by "vertices of the same type" is made clear by the figure; the small circles below the Main Figure illustrate how the 14 vertices come together according to their type, forming two a points.

If one were to actually bend the polygon of the Main Figure and glue the edges together, the result would be a very confusing figure. It is better to remain on the plane and complement the Main Figure with the edge identifications and the two small figures showing the incidence at the vertices. In this way one reaches the results compiled in the next section.

13. The 28 Symmetry Lines

By a *symmetry line* of our covering surface I will mean a line made up of triangle edges and not having kinks anywhere — going straight, so to speak, through a, b, and c-points. The surface is indeed symmetric with respect to such lines: as an example of a symmetry line we can take the vertical center line of the Main Figure, so long as we make it into a closed curve by adding edge 5, or, equivalently, edge 10; these two edges are symmetrically placed with respect to the center line, and moreover the gluing scheme for the remaining edges is symmetric with respect to this line.

This example also shows that such a symmetry line must contain six points of each type a, b, c, in the sequence indicated in Figure 8.

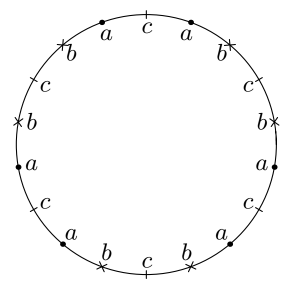


Figure 8.

Next we have, most importantly:

There are 28 *symmetry lines*. Together they comprise all the triangle edges, and so they exhaust the points on the surface that correspond to *real* values of J.

These symmetry lines are, for many purposes, the easiest means of orientation on our surface; I will use them here to characterize the groupings of a, b,

[27] [Each point $k+\frac{1}{3}$, for $k=0,1,\ldots,6$, lies on the edge labeled $2k+1$ on the Main Figure, and each point $k'-\frac{1}{3}$, for $k'=1,2,\ldots,7$, lies on the edge $2k$. When is a point $k+\frac{1}{3}$ equivalent to a point $k'-\frac{1}{3}$ under our group? The condition

$$k+\tfrac{1}{3}=\frac{(7a+1)(k'-\frac{1}{3})+7b}{7c(k'-\frac{1}{3})+(7d+1)}$$

yields, when considered modulo 7,

$$k'-k\equiv\tfrac{2}{3}\equiv 3 \pmod 7, \quad \text{or} \quad 2k'-(2k+1)\equiv 5 \pmod 7;$$

that is, edges $2k+1$ and $2k+6$ are to be identified. –B.-H.]

and c points. Then it is easy to form an idea of the corresponding one-to-one transformations of our surface into itself.

The 7 *symmetry lines that meet at an a-point also meet at the other two points in the same triple* (coming from a G'_{21}: see page 294). An example of such a triple is given by the center of our figure together with the two a-points coming from the two kinds of vertices along the boundary (seven of each kind). The process for passing from the closed regularly tiled surface to the Main Figure can be described as follows: Choose on the surface two out of a triple of a-points and cut along the seven pieces of symmetry lines that go from one of these a-points to the other. Since the surface has genus $p = 3$, the result is simply connected and has one boundary curve, and when stretched out on the plane, it becomes our Main Figure of page 320.

Clearly, any *two* triples of a-points determine exactly *one* symmetry line, on which the points of the two triples alternate.

The 3 *symmetry lines that meet at a b-point also meet at the other b-point with which it forms a pair*. Examples of pairs of b-points are given in the figure by A, A'; B, B'; C, C'; D, D'; we will return to them later. To each such triple of symmetry lines, and so to each pair of b-points, is associated a symmetry line, characterized by the fact that it intersects the lines of the triple in two c-points. This gives a one-to-one correspondence between the 28 pairs of b-points and the 28 symmetry lines.

The 2 *symmetry lines that meet at a c-point meet again at another c-point. There are two more symmetry lines that do not intersect the first two and that meet each other at another pair of c-points. In this way one obtains the quadruples of c-points.*

14. Definitive Shape of Our Surface

The more regular a figure is, the more it tends to be intuitive and easy to grasp. Thus I would like to put our regularly tiled surface into a shape that allows as many as possible of the 168 one-to-one transformations to be realized as *rotations*. Now, we know all the finite groups that can be realized by rotations: they correspond to the regular polyhedra. There is no group of 168 rotations [in three-dimensional space] in the sense we are talking about. On the other hand, we have already remarked in Section 1 that the 24 substitutions in a G''_{24} stand in the same relation to one another as the rotations that take an octahedron to itself. This suggests that it may be possible *to give our surface such a shape that it is sent to itself by the rotations of an octahedron.*

For this purpose we must first find four b-points that are permuted by the substitutions of a G''_{24}. This can be accomplished easily if we group the 14 triangles that meet at each a-point, making 24 heptagons that together cover the whole surface, as discussed earlier. *Then there are* $2 \cdot 7$ *ways to choose four pairs of b-points so that all* 24 *heptagons have one of the chosen b-points as a* *vertex*.[28] The four point pairs A, A', B, B', C, C', D, D', already mentioned, form such a quadruple. Six more are obtained by rotating the figure around the center in multiples of $2\pi/7$, and the remaining seven by reflecting the first seven in any symmetry line, say the vertical center line.

Now cut the surface (after having glued the three zigzag paths shown in the Main Figure as thick lines weaving around the dashed curves. The result is a sextuply connected surface with six boundary curves [a sphere minus six disks –L.], *and this surface can be stretched symmetrically onto a sphere in such a way that the eight points* A, A', *etc. coincide with the vertices of an inscribed cube*, *the vertices of the dual octahedron remain uncovered*, *and the twelve midpoints of the spherical octahedron's edges coincide with the c-points of the surface*. For greatest clarity I have sketched a drawing showing only one of the octants of the sphere (see Figure 9).

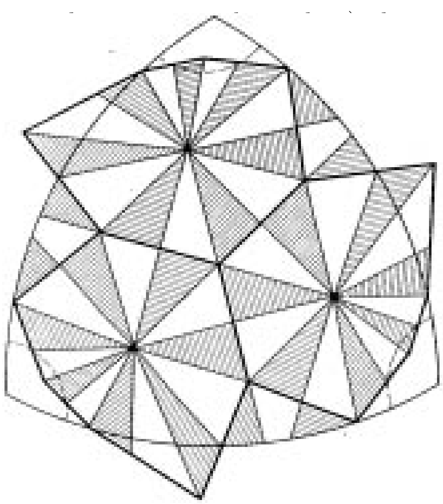


Figure 9.

The three heptagons that meet at the center of the octant fall partly outside the octant. But since this is true also about the heptagons that cover the neighboring octants, the only part of the octant that is not covered by the surface is the corners.

To obtain an image of the surface as a whole we must know how the boundary curves that surround the corners of the octahedron are to be joined together. The answer can be read off by comparing with the earlier figure, and it is very simple: *each point must be identified with the diametrically opposed point*.

These identifications can be carried out without breaking the desired octahedral symmetry: *one just has to bring together the boundary curves through infinity in such a way that the intersection with the plane at infinity consists of the curves shown in the Main Figure and in Figure* 9 *as dashed lines*. Therefore the heptagons that spread out from the center of the octant reach out in part beyond infinity, so that a total of twelve c-points lie on the plane at infinity. So the surface itself goes out to infinity in much the same way as the union of three congruent hyperboloids of rotation whose axes meet at right angles.[29] [See Figure 8 of [Gray 1982], page 127 in this volume. –L.]

[28] Also, the existence of the resolvent of degree eight can easily be proved using these heptagons: *There are eight ways to choose three heptagons so that the remaining 21 heptagons are adjacent to one of the three.*

[29] [Dyck prepared at the time a nice model of the surface in this form, for the Mathematics Institute of the Technische Hochschule München. –K.]

If one wishes to check that the 24 transformations expressible by rotations of the octahedron fix the number of points asserted earlier, one should keep in mind that *a rotation of period two fixes not only points on the rotation axis but also points on the line at infinity that lies perperdicular to the rotation axis*. Our surface is not intersected by the diagonals of the octahedron, but it is intersected four times by each line at infinity perpendicular to a diagonal of the octahedron. The diameters going through the midpoints of the edges of the octahedron intersect the surface twice, as do the lines at infinity perpendicular to these diameters. Finally, the diagonals of the cube have exactly two intersections with the surface. Therefore a rotation of period four fixes *no points*, one of period two fixes *four points*, and one of period three fixes *two points*. This is all as it should be.

15. The Real Points of the Curve of Order Four

I would like to conclude by showing how these relative positions stand out when we consider *the real points of the order-four curve*

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0.$$

The coordinate triangle may be taken to be equilateral, the coordinates being proportional to the distance to the sides. Then the bitangent $\lambda+\mu+\nu = 0$ is the line at infinity; its contact points $1:\alpha:\alpha^2$ and $1:\alpha^2:\alpha$ are the two cyclic points. The line at infinity is therefore an *isolated* bitangent. The six collineations of the corresponding G_6' are the only real ones among the 168; they consist of the three rotations through 120 degrees about the center of the coordinate triangle and of the reflections in three lines going through this same center. These lines are the only three real perspective axes; the related perspective centers lie at infinity, orthogonally to the lines. From the inflection triangle $\lambda\mu\nu = 0$, the three reflections give rise to a second real inflection triangle $\lambda'\mu'\nu' = 0$.

We now consider form (9) of the curve's equation:

$$49y_1(y_1 + y_2 + y_3)(y_1 + \alpha y_2 + \alpha^2 y_3)(y_1 + \alpha^2 y_2 + \alpha y_3) - 3(4y_1^2 - 7y_2y_3)^2 = 0.$$

We replace y_1 by 1 (since $y_1 = 0$ is the line at infinity), and replace y_2, y_3 by $x + iy$, $x - iy$, since the corresponding axes go through the cyclic points. We obtain

$$49(2x+1)(-x+\sqrt{3}\,y+1)(-x-\sqrt{3}\,y+1) - 3(4-7(x^2+y^2))^2 = 0.$$

The bitangents

$$2x+1 = 0, \quad -x+\sqrt{3}\,y+1, \quad -x-\sqrt{3}\,y+1 = 0$$

again form an equilateral triangle, of altitude $\frac{3}{2}$ and side length $\sqrt{3}$. Its intersection with the circle of radius $2/\sqrt{7}$ around the center consists of contact points of bitangents, *so we have three nonisolated bitangents*. One can check that all other

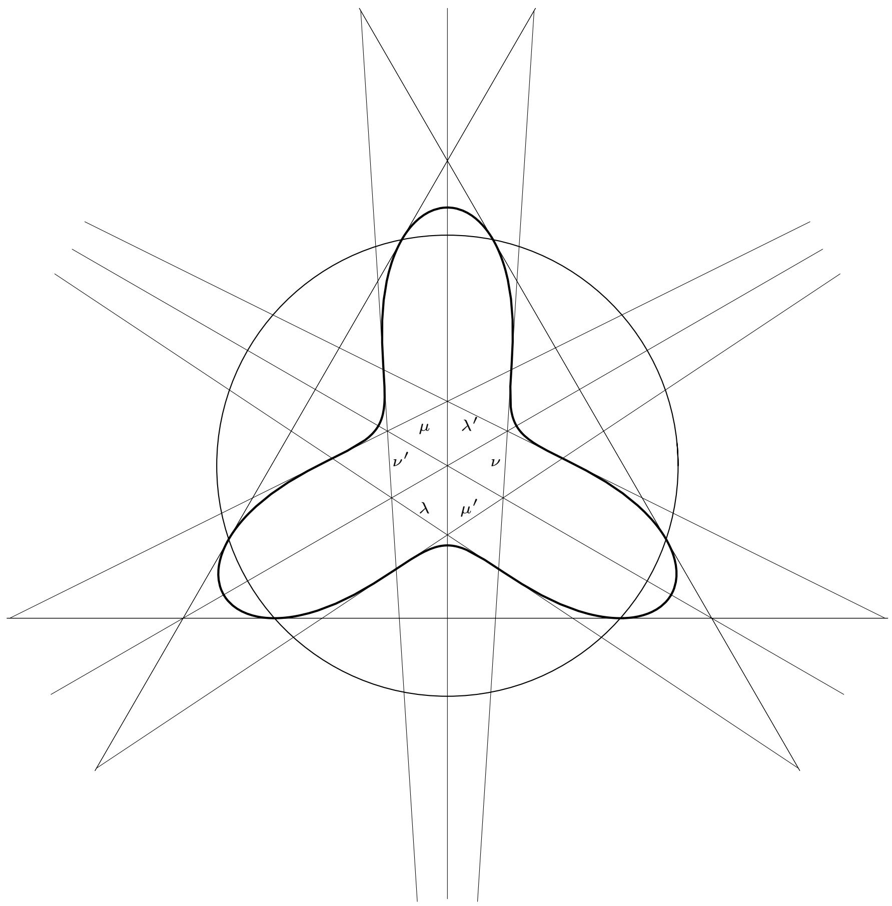


Figure 10.

bitangents are imaginary. *Thus our curve has one branch*[30] *and is inscribed in the triangle of nonisolated bitangents*. The accompanying diagram[31] (Figure 10) shows, in addition to the bitangents in question, the circle through the contact points, the three real perspective axes, and the two real inflection triangles.

The real curve so obtained has a very simple interpretation in terms of the Riemann surface: *it represents one of the* 28 *symmetry lines*. Indeed, real values of λ, μ, ν yield real values of J, and the symmetry lines are characterized by J real.

[30] See [Zeuthen 1874].

[31] [The schematic illustration in the original was replaced in the reprint by a figure precisely computed by Haskell, which appears in his Göttingen dissertation [Haskell 1891]. In this work, done at my instigation, Haskell applies the ideas developed in [Klein 1874; 1876] to the curve $\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$ and clarifies the results with illustrations. –K.]

This symmetry line is clearly associated with the isolated bitangent at infinity and so, like the bitangent, it is sent to itself by the substitutions of a G_6'. It contains six each of the a, b, and c-points, and it can be seen in Figure 10 that these points follow one another in the sequence expected for a symmetry line (Figure 8).

Munich, early November 1878.

[Additional Remarks Concerning Some of the Literature]

[The mathematical literature concerning the fourth-order curve $\lambda^3\mu + \mu^3\nu + \nu^3\lambda$ and the thus defined algebraic structure having 168 one-to-one transformations has multiplied since the publication of this article, particularly in its geometric aspects. It is not possible to cover it in detail here, but I will at least mention some highlights.

In what concerns the algebraic side of the question, we refer to Gordan's extensive investigations, discussed in [Klein 1922, pp. 426 ff.]. Here I will add a discussion of the role played by n-th roots of unit, which come up in the articles reprinted in the first half of [Klein 1923] and also in [Klein 1922]. When one considers Galois problems, these are "natural" irrationalities: for example, the fifth root of unity can be represented, by virtue of the icosahedral substitutions, as a quotient of appropriately chosen roots of the icosahedral equation. The same is true of the partition equations of elliptic functions, as a consequence of the so-called "Abel relations". See [Klein 1885, footnote 37]. For the modular equations of the functions $J(\omega)$, however, the n-th roots of unity are no longer "natural", but the Gaussian sum $\sqrt{(-1)^{(n-2)/2}n}$ formed from them is. See, for example, [Fricke 1922, p. 462]. This is also true of the special resolvents of fifth, seventh, and eleventh degree, treated in [Klein 1879b; 1879d]. (Cf. for instance [Fricke 1922, p. 482].) These results are important in order to determine in individual cases not only the monodromy group,[32] to which the exposition in the text has limited itself, but also the Galois group, taking as a basis the domain of rationality of the rational numbers.

Another line of research concerns the three globally finite integrals of our fourth-order curve. It seems particularly remarkable that their periods can be explicitly given. Poincaré [1883] and Hurwitz [1886, p. 123] find, for an appropriate choice of crosscuts, the period matrix

$$\begin{matrix} 1 & 0 & 0 & \tau & \tau-1 & -\tau \\ 0 & 1 & 0 & \tau-1 & -\tau & \tau \\ 1 & 0 & 0 & -\tau & \tau & \tau-1 \end{matrix}$$

[32] [The concept of monodromy group was introduced by Hermite in [Hermite 1851]. The name appears for the first time, so far as I know, in [Jordan 1870, p. 278]. –K.]

where τ denotes the quadratic irrational number $\frac{1}{4}(1+i\sqrt{7})$. This implies, in particular, that our Riemann surface has a multiple cover by an elliptic surface of singular modulus $\omega = \frac{1}{2}(-1+i\sqrt{7}) = (\tau-1)/\tau$, and therefore having the rational invariant $J(\omega) = -5^3/2^6$. Moreover Hurwitz [1885] has studied the integral of first type as a function of ω and in the coefficients of its power series development in $q^2 = e^{2\pi i\omega}$ he found those number-theoretic functions that Gierster ran into in the construction of class number relations of rank seven. See [Klein 1923, p. 5]. More details on the subject can be found in the "Modular functions".

Perhaps our curve achieves the greatest prominence in that the Main Figure on page 320, when placed inside a disk whose boundary is orthogonal to its arcs, provides the first concrete example of uniformization of an algebraic curve of higher genus. For this reason it became for me the best prop in building the general uniformization results in [Klein 1882a; 1882b; 1883].

The considerations in the text find an immediate continuation in a note by Dyck [1880b] about the normal curve $\lambda^4+\mu^4+\nu^4$ pertaining to the main congruence group of rank eight and admitting 96 one-to-one transformations onto itself, and particularly in Fricke's investigations about the ternary Valentiner–Wiman group and the transformation theory of triangle functions for a triangle with angles $\pi/5$, $\pi/2$, $\pi/4$. (Published as an appendix in [Fricke and Klein 1912]. See also [Klein 1922, pp. 501–502].) –K.]

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