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Asymptotic Versions of Operators and Operator Ideals

VITALI MILMAN AND ROY WAGNER

Abstract. The goal of this note is to introduce new classes of operator ideals, and, moreover, a new way of constructing such classes through an application to operators of the asymptotic structure recently introduced by Maurey, Milman, and Tomczak-Jaegermann in *Op. Th. Adv. Appl.* **77** (1995), 149–175.

1. Preliminaries

1.1. Notation. We follow standard Banach-space theory notation, as outlined in [LTz].

Throughout this note X will be an infinite dimensional Banach-space with a shrinking basis $\{e_i\}_{i=1}^{\infty}$. The notation $[X]_n$ will stand for the head subspace $(\text{span}\{e_i\}_{i=1}^{n})$ and $[X]_{>n}$ for the tail subspace (the closure of $\text{span}\{e_i\}_{i=n+1}^{\infty}$). P_n and $P_{>n}$ are the coordinate-orthogonal projections on these subspaces respectively.

A basis $\{e_i\}$ is *equivalent* to a basis $\{f_i\}$ if

$$C_2 \left\| \sum a_i f_i \right\| \leq \left\| \sum a_i e_i \right\| \leq C_1 \left\| \sum a_i f_i \right\|$$

for any scalars $\{a_i\}$. The equivalence is quantified by the ratio C_1/C_2; the closer it is to 1, the better the equivalence.

A vector is called *a block* if it has *finite support*, that is, if it is a finite linear combination of elements of the basis. The blocks v and w are said to be *consecutive* ($v < w$) if the *support* of v (the set of elements of the basis that form v as a linear combination) ends before the support of w begins. $S(X)^n_<$ is the collection of all n-tuples of consecutive normalised blocks; thus $S(X)^1_<$ means normalised finite support vectors.

1.2. An intuitive introduction to asymptotic structure. The language of asymptotic structure has been introduced to study the essentially infinite dimensional structure of Banach-spaces, and to help bridge between finite dimensional and infinite dimensional theories. This approach does generalise spreading models, but takes an essentially different view. Formally introduced in [MMiT], it has already been studied, extended and applied in [KOS], [OTW], [T], [W1] and [W2]; it is closely related to the new surge of results in infinite dimensional Banach-space theory, and especially to [G] and [MiT]. For formal arguments, the most convenient terminology is the game terminology coming from [G], presented below. However, we choose to preface this by a less rigorous intuitive introduction.

The main idea behind this theory is a stabilisation at infinity of finite dimensional objects (subspaces, restrictions of operators), which repeatedly appear arbitrarily far and arbitrarily spread out along the basis. Piping these stabilised objects together gives rise to infinite dimensional notions: *asymptotic versions* of a Banach-space X or of an operator acting on X.

To define this structure we first have to choose a frame of reference in the form of a family of subspaces, $\mathcal{B}(X)$. It is most convenient to choose $\mathcal{B}(X)$ such that the intersection of any two subspaces from $\mathcal{B}(X)$ is in $\mathcal{B}(X)$. The family of tail subspaces is such a family; so is the family of finite codimensional subspaces, but here we will work with the former. The construction proceeds as follows.

Fix n and ε. Consider the tail subspace $[X]_{>N_1}$ for some "very large" N_1, and take a normalised vector in this tail subspace. Consider now a further tail subspace $[X]_{>N_2}$, with N_2 "very large", depending on the choice of x_1; choose again any normalised vector, x_2 in $[X]_{>N_2}$. After n steps we have a sequence of n vectors, belonging to a chain of tail subspaces, each subspace chosen 'far enough' with respect to the previous vectors.

The span of a sequence in X, $E = \operatorname{span}\{x_1, \dots, x_n\}$, is called *$\varepsilon$-permissible* if we can produce by the above process vectors $\{y_i\}_{i=1}^n$, which are $(1+\varepsilon)$-equivalent to the basis of E, regardless of the choice of tail subspaces.

Now we can explain how far is "far enough". The choice of the tail subspaces $[X]_{>N_i}$ is such that no matter what normalised blocks are chosen inside them, they will always form ε-permissible spaces. The existence of such "far enough" choices of tail subspaces is proved by a compactness argument.

We can now consider basic sequences which are $(1+\varepsilon)$-equivalent to ε-permissible sequences for every ε. These will be called n dimensional *asymptotic spaces*. Our ε-permissible sequences are $(1+\varepsilon)$-realizations in X of asymptotic spaces.

Finally, a Banach-space whose every head-subspace is an asymptotic space of X is called an *asymptotic version* of X.

The same construction can be made for an operator T as well. In this case, we would like to stabilise not only the domain (which is an asymptotic space), but also the image and action of the operator. More precisely, our permissible sequences will now be sequences $\{x_i\}_{i=1}^n$, such that we can find arbitrarily far and arbitrarily spread out sequences in our space, which are closely equivalent to $\{x_i\}_{i=1}^n$, and on top of that, are mapped by T to sequences closely equivalent to $\{T(x_i)\}_{i=1}^n$. The assumption that X has a shrinking basis promises that $\{T(x_i)\}_{i=1}^n$ are close to successive blocks. In turn this means that the asymptotic version of T, viewed as an operator from $[x_i]_{i=1}^n$ to $\big[T(x_i)/\|T(x_i)\|\big]_{i=1}^n$, is always a formally diagonal operator.

1.3. The Game. The following game (up to slight formalities) is used in [MMiT] to define asymptotic spaces. The game terminology comes from [G].

Definition 1. This is a game for two players. One is the subspace player, $\mathcal{S}$, and the other is the vector player, $\mathcal{V}$. The "board" of the game consists of a Banach-space with a basis, a natural number n, and two subsets of $S(X)^n_<$: Φ and Σ. Player $\mathcal{S}$ begins, and they play n turns. In the first turn player $\mathcal{S}$ chooses a tail subspace, $[X]_{>m_1}$. Player $\mathcal{V}$ then chooses a normalised block in this subspace, $x_1 \in [X]_{>m_1}$. In the k-th turn, player $\mathcal{S}$ chooses a tail subspace $[X]_{>m_k}$. Player $\mathcal{V}$ then chooses a normalised block, x_k, such that $x_k \in [X]_{>m_k}$ and $x_k > x_{k-1}$.

$\mathcal{V}$ wins if it produces a sequence of vectors in Φ.

$\mathcal{S}$ wins if it forces the choices of $\mathcal{V}$ to be in Σ.

Note that in this game it is not always true that one player wins and the other loses. Furthermore, in some cases, we are only interested in the winning prospects of one player, and therefore may ignore either Φ or Σ.

If $\mathcal{V}$ has a *winning strategy* in this game for Φ, that is a recipe for producing sequences in Φ considering any possible moves of $\mathcal{S}$, we call Φ *an asymptotic set of length n*. Formally this means:

$$\forall m_1\ \exists x_1 \in X_{>m_1}\ \forall m_2\ \exists x_2 \in X_{>m_2} \dots\ \forall m_n\ \exists x_n \in X_{>m_n} \quad \text{such that } (x_1,\dots,x_n)\in\Phi.$$

Note that this generalises an earlier notion of an asymptotic set for $n = 1$ (in this context of tail subspaces, rather than block subspaces; compare [GM]).

If $\mathcal{S}$ has a winning strategy in this game for the collection Σ, we call Σ *an admission set of length n*. Formally this means:

$$\exists m_1\ \forall x_1 \in X_{>m_1}\ \exists m_2\ \forall x_2 \in X_{>m_2} \dots\ \exists m_n\ \forall x_n \in X_{>m_n} \quad \text{such that } (x_1,\dots,x_n)\in\Sigma$$

This terminology comes from admissibility criteria in the study of Tsirelson's space and its variants. An admission set contains all vector sequences beginning "far enough" and spread out "far enough" — for some interpretation of the term "enough".

When the context is clear, we will omit the length, and simply write "an asymptotic (admission) set".

Remarks 2. (i) Note that a collection containing an admission set is an admission set, and that a collection containing an asymptotic set is an asymptotic set.

(ii) It is also useful to note that if $\mathcal{V}$ has a winning strategy for Φ, and $\mathcal{S}$ has a winning strategy for Σ, then playing these strategies against each other will necessarily produce sequences in $\Phi \cap \Sigma$.

In fact, in such a case $\mathcal{V}$ has a winning strategy for $\Phi \cap \Sigma$. Player $\mathcal{V}$'s strategy is as follows; player $\mathcal{V}$ plays his winning strategy for Φ, while pretending that the subspace dictated to him in each turn is the intersection of the subspace actually chosen by $\mathcal{S}$ with the subspace arising from the winning strategy for Σ.

The following is simply formal negation.

Lemma 3. *Let $\Sigma \subseteq S(X)^n_<$. Then either Σ is an admission set, or Σ^c is an asymptotic set. These options are mutually exclusive.*

Tail subspaces of a Banach-space form a filter. This allows us to demonstrate filter (cofilter) behaviour for admission (asymptotic) sets.

Lemma 4. (i) *Let $\Sigma_1, \dots, \Sigma_k \subseteq S(X)^n_<$ be admission sets. Then $\bigcap_{j=1}^k \Sigma_j$ is also an admission set.*

(ii) *Let $\Phi_1, \dots, \Phi_k \subseteq S(X)^n_<$, such that $\bigcup_{j=1}^k \Phi_j$ is asymptotic. Then, for some $1 \le j \le k$, Φ_j is asymptotic.*

Proof. (i) Suppose at any turn of the game player $\mathcal{S}$ has to choose $[X]_{>m_1}$ in order to win for Σ_1, $[X]_{>m_2}$ in order to win for $\Sigma_2, \dots$, and $[X]_{>m_k}$ in order to win for Σ_k. If player $\mathcal{S}$ chooses $[X]_{>m}$, where $m = \max\{m_1, m_2, \dots, m_k\}$, the vector sequence chosen by player $\mathcal{V}$ will have to be in $\bigcap_{j=1}^k \Sigma_j$.

(ii) Suppose for all $1 \le j \le k$, Φ_j is not asymptotic. Then Lemma 3 implies that Φ_j^c are all admission sets. By part 1 of the proof, $\bigcap_{j=1}^k \Phi_j^c$ is an admission set. Therefore, using Lemma 3 again, $\bigcup_{j=1}^k \Phi_j$ is not asymptotic, in contradiction. □

1.4. Asymptotic versions. The following notions come from [MMiT].

Definition 5. An n-dimensional Banach space F with a basis $\{f_i\}_{i=1}^n$ is called *an asymptotic space of a Banach-space* X, if for every $\varepsilon > 0$ the set of all sequences in $S(X)^n_<$, which are $(1+\varepsilon)$-equivalent to $\{f_i\}_i$, is an asymptotic set.

A space $\tilde{X}$ is an *asymptotic version of* X, if all the spaces $[\tilde{X}]_n$ are asymptotic spaces of X.

We use $\{X\}_\infty$ to denote the collection of asymptotic versions of a space X, and $\{X\}_n$ to denote the collection of its n-dimensional asymptotic spaces.

Remark 6. The existence of asymptotic versions and spaces for every Banach-space X is elementary, as observed in [MMiT]. Spreading models make obvious examples of asymptotic versions. Existence of some special asymptotic versions is dealt with in [MMiT]. In our existence theorem below we will prove that we can extract close realizations of an asymptotic version out of any sequence of increasingly long asymptotic sets.

1.5. König's Unendlichkeitslemma. The following combinatorial lemma comes from [Kö]. Its proof is an elementary exercise. A *rooted tree* is simply a connected tree with some vertex labelled as *root*. The root allows us to consider the *level* of vertices in the tree.

LEMMA 7. *A rooted tree has an infinite branch emanating from the root if*

(i) *there are vertices arbitrarily far from the root, and*
(ii) *the set of vertices with any fixed distance to the root is finite.*

This Lemma will be used to allow us to find asymptotic versions not only inside the whole space, but also inside asymptotic subsets. This in turn will be used to show, that if the collections of sequences with a certain property is asymptotic, an asymptotic version with the said property can be extracted.

2. Asymptotic Versions of Operators

DEFINITION 8. Let $T \in \mathcal{L}(X)$. Define $\tilde{T}$, *an asymptotic version of* T, to be a formally diagonal operator between asymptotic versions $\tilde{Y}, \tilde{Z} \in \{\tilde{X}\}_\infty$, such that for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, the following set is asymptotic:

All sequences in $S(X)^n_<$, which are $(1+\varepsilon)$-equivalent to the basis of $[\tilde{Y}]_n$, and whose images under T are $(1+\varepsilon)$-equivalent to the images of the basis of $[\tilde{Y}]_n$ under $\tilde{T}$.

A collection of such asymptotic sets, for arbitrarily small ε's and arbitrarily large n's will be referred to as *asymptotic sets realizing the asymptotic version* $\tilde{T}$.

The set of all asymptotic versions of an operator T will be denoted $\{T\}_\infty$.

Note that the asymptotic versions of the identity operator correspond simply to asymptotic versions of the space.

We are about to prove the basic existence theorem for asymptotic versions of operators. In order to formulate it, we need the following technical terminology.

DEFINITION 9. A *truncation* (*of length* k) of a given collection, $\Sigma \subseteq S(X)^n_<$, is the collection of sequences of the leading k blocks from all sequences in Σ.

A truncation of an asymptotic set is obviously asymptotic.

THEOREM 10 (THE EXISTENCE THEOREM). *Let* X *be a space with a shrinking basis. For every operator* $T \in \mathcal{L}(X)$ *and every sequence* $\{\Phi_n\}_{n=1}^\infty$ *of increasingly long asymptotic sets, there exists* $\tilde{T} \in \{T\}_\infty$ *realized by* **truncated** *subsets of the* Φ_n*'s.*

Before we prove the theorem, let us isolate the part of the proof which requires a shrinking basis.

LEMMA 11. *Let X be a space with a shrinking basis, and let $T \in \mathcal{L}(X)$. Let Σ be an admission set. The collection of all sequences, which T maps ε-close to sequences from Σ, is also an admission set.*

PROOF. As a first step we have to verify that, if a block is supported far enough, its image under T will be — up to an $\varepsilon/2$-perturbation — the first block of some sequence from Σ. This sounds reasonable, recalling that Σ is an admission set, and so any block which is supported sufficiently far can play the first vector of a sequence from Σ.

To verify, let $[X]_{>n_1}$ be the first step in the winning strategy of player $\mathcal{S}$ for Σ. We know that all blocks x_1 supported far enough have $\|P_{n_1}(T(x_1))\| < \varepsilon/2$. Indeed, if this weren't the case, we would have $\|P_{n_1}(T(x_1^k))\| \geq \varepsilon/2$ for a sequence $\{x_1^k\}_{k=1}^{\infty}$ of normalised vectors supported increasingly far (and hence weakly null). This would imply that one of the bounded functionals $P_{\{e_i\}}(T(x))$, $1 \leq i \leq n_1$, does not go to zero when applied to a sequence of weakly null vectors — a contradiction. Therefore, as long as x_1 is supported far enough, $T(x_1)$ fits (up to $\varepsilon/2$) as the first vector from a sequence in Σ. The first step is accomplished.

Let now $[X]_{>n_2}$ be the second step in the winning strategy of player $\mathcal{S}$ for Σ, given that player $\mathcal{V}$ has just chosen the appropriate perturbation of $T(x_1)$. The above reasoning shows that if x_2 is supported far enough, it's image under T will be supported (up to an $\varepsilon/4$ perturbation) after n_2, that is, far enough to fit as the second vector of a sequence from Σ, which begins with our slight perturbation of $T(x_1)$.

Repeating this argument we see, that if a sequence of vectors is sufficiently spread out (each vector is supported sufficiently far with respect to its predecessors), the image of that sequence will be a sequence of essentially consecutive vectors, ε-close to a sequence from Σ. We have thus proved that our collection is indeed an admission collection, and we're through. □

REMARK 12. In the proof we will apply the last Lemma to the collection of close realizations of asymptotic spaces in X. The paper [MMiT] explains that the collection $\Sigma_{n,\varepsilon}(X)$ of all block sequences of length n, which are $(1+\varepsilon)$-equivalent to asymptotic spaces, is indeed an admission set (we will point out a proof for this claim in Remark 13 below).

PROOF OF THE EXISTENCE THEOREM. The proof will split into three parts. First we have to make sure that we can restrict T to operate between asymptotic spaces. To do this we will extract from the Φ_n's asymptotic subsets of block sequences, whose normalised images under T are closely equivalent to asymptotic spaces of X (this is where we use the last Lemma and the shrinking property). Then we will use a compactness argument and Lemma 4 to extract asymptotic subsets of block sequences, where the norm of linear combinations, the normalised image under T and the action of T are almost fixed. Finally we will pipe such asymptotic sets of different lengths together by means of Lemma 7, to realize an asymptotic version of T.

First step: For every $\delta > 0$ and for every asymptotic set Ψ of length n, there is an asymptotic subset, Ψ', of sequences whose normalised images under T are $(1+\delta)$-equivalent to elements of $\{X\}_n$.

Proof of first step: We want to show that the following subset of Ψ is asymptotic:

> The intersection of Ψ with the collection of block sequences which T maps (up to $\delta/2$) into the admission collection $\Sigma_{n,\,\delta/2}(X)$ from Remark 12.

Ψ is already an asymptotic set. By the Lemma 11, the sequences mapped close to $\Sigma_{n,\,\delta/2}(X)$ form an admission set. Remark 2 says that their intersection must indeed be an asymptotic set.

Second step: Let M be the compactum of n-dimensional spaces with a normalised basis and a basic constant not worse than that of X (the metric on this space is given by equivalence of bases). Consider a finite covering $\{V_i\}_i$ of M. Consider a finite covering $\{I_k\}_k$ of the cube $[0, \|T\|]^n$. For every asymptotic set Φ of length n there is an asymptotic subset, Φ', of block-sequences with the following additional properties:

(i) The sequences in Φ' are contained in some fixed V_{i_0}.
(ii) The normalised images under T of sequences from Φ' are contained in some fixed W_{j_0}.
(iii) For all $\{x_i\}_{i=1}^n \in \Phi'$, the sequences $\{\|T(x_i)\|\}_{i=1}^n$ are contained in some fixed I_{k_0}.

Proof of second step: This is an easy application of Lemma 4 and the fact that the covering is finite. Our covering induces a covering of $M \times M \times [0, \|T\|]^n$. Each cell of this covering contains a subset of Φ of the form

$$\Phi_{i,j,k} = \left\{ \{x_i\}_{i=1}^n : \{x_i\}_{i=1}^n \in V_i,\ \left\{ \frac{T(x_i)}{\|T(x_i)\|} \right\}_{i=1}^n \in W_j,\ \{\|T(x_i)\|\}_{i=1}^n \in I_k \right\}.$$

By Lemma 4 one of those collections must be an asymptotic set.

Third step: For every operator $T \in \mathcal{L}(X)$ and every collection $\{\Phi_n\}_{n=1}^\infty$ of asymptotic sets of increasing lengths, there exists a formally diagonal operator $\tilde{T} \in \{T\}_\infty$, realized by truncated subsets of the Φ_n's.

Proof of third step: Fix a positive sequence converging to zero, $\{\varepsilon_n\}_n$. Next choose inductively open finite coverings as in step 2 above with cells of diameter less than ε_n. Our aim is that the cells of the coverings correspond to the vertices of a tree; we want a cell from the $(n-1)$-th covering to "split" into cells of the n-th covcering. To achieve that, we consider the projection from the n-order space to the $(n-1)$-order space that takes $(x_1, \dots, x_n; y_1, \dots, y_n; \lambda_1, \dots, \lambda_n)$ to $(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-1}; \lambda_1, \dots, \lambda_{n-1})$. We require that this projection maps the covering of the n-order space to a refinement of the covering of the $(n-1)$-order space.

If we choose ε_1 large enough (say $\varepsilon_1 = \|T\|$), The cells of such coverings do form an infinite rooted tree in an obvious way.

Take the collection $\{\Phi_n\}_n$ of asymptotic sets. Fix $k \leq n$. Applying step 2 to the k-truncation of Φ_n, we get that there's a vertex (i.e. a covering cell) which contains an asymptotic subset of the k-truncation of Φ_n. Note that if a certain vertex contains such an asymptotic subset, so must all the predecessors of that vertex (a truncation of an asymptotic set is an asymptotic set). This fact means that the vertices containing such asymptotic sets form a connected rooted subtree. Since k and n were arbitrary, our rooted subtree intersects every level of the original tree. We therefore have that our subtree satisfies both properties of Lemma 7, and must have an infinite branch.

Let $\{\Phi'_m\}_m$ be the truncated asymptotic subsets contained in the vertices of such an infinite branch. By step 1, we may assume that T sends sequences from Φ'_m to increasingly good realizations of asymptotic spaces of X. Thus, for every m, the truncated asymptotic subsets $\{\Phi'_m\}_m$ of $\{\Phi_n\}_n$ have the following properties:

(i) The sequences of blocks from $\{\Phi'_m\}_m$ are $(1+\varepsilon_m)$-equivalent to the basis of $[Y]_m$ for some fixed $Y \in \{X\}_\infty$.
(ii) The normalised images under T of all sequences in $\{\Phi'_m\}_m$ are $(1+\varepsilon_m)$-equivalent to the basis of $[Z]_m$ for some fixed $Z \in \{X\}_\infty$.
(iii) For any $(x_i)_i \in \{\Phi'_m\}_m$, the sequence $\|T(x_i)\|_i$ is fixed up to ε_m.

Hence, the result of this process is a sequence of asymptotic sets realizing a formally diagonal asymptotic version of T. □

Remark 13. Note that since we may start with any collection of asymptotic sets with increasing lengths, we may choose to extract our asymptotic version of T from asymptotic sets realizing a given asymptotic version of X. We thus have for any operator $T \in \mathcal{L}(X)$ and for any $\tilde{X} \in \{X\}_\infty$ an asymptotic version of T whose domain is $\tilde{X}$.

The proof of the existence theorem also implies that for every positive ε and natural n the collection $\Sigma_{n,\varepsilon}(X)$ (introduced in Remark 12 and used in the proof of step 1) is indeed an admission set.

If this were not the case, by Lemma 3 the complement of $\Sigma_{n,\varepsilon}(X)$ would be an asymptotic set, and by its own definition it could not contain sequences $(1+\varepsilon)$-equivalent to any n-dimensional asymptotic space. Then, using the proof of the existence theorem for the *identity* operator (this does not require the first step of the proof or the assumption that $\Sigma_{n,\varepsilon}(X)$ is an admission set), we extract asymptotic subsets realizing some fixed space. This space must (by definition) be asymptotic — a contradiction.

The same fact was proved in [MMiT, 1.5], by a different compactness argument.

3. Asymptotic Versions of Operator Ideals

3.1. Compact operators.

PROPOSITION 14. *If T is compact then $\{T\}_\infty = \{0\}$. If T is not compact then $\{T\}_\infty$ contains a non-compact operator.*

PROOF. If T is compact, take asymptotic sets, $\{\Phi_n\}_n$, realizing some asymptotic version of the operator. Let $\mathcal{V}$ play his winning strategy for Φ_n, and let $\mathcal{S}$ play tail subspaces $[X]_{>n}$ with n such that $\|T\|_{[X]_{>n}} < \varepsilon$.

The block-sequence resulting from this game will still be an approximate realization of the above asymptotic version. This shows that any asymptotic version of T can be approximated arbitrarily well by operators with norm smaller than any positive ε. Therefore the only asymptotic version of T is zero.

If T is not compact, there is an $\varepsilon > 0$ such that the set

$$\Phi_1 = \{x \in S(X)^1_< : \|T(x)\| \geq \varepsilon\}$$

is asymptotic (of length 1). For any n the collection Φ_n of sequences of n successive elements from Φ_1 is obviously asymptotic.

Using the existence theorem, extract from Φ_n asymptotic subsets realizing some asymptotic version of T. The norm of this asymptotic version will not be smaller than ε on any element of the basis of its domain, and will therefore be non-compact. □

Asymptotic versions induce a seminorm on operators, through the formula:

$$|||T||| = \sup \|\tilde{T}\|$$

where the supremum is taken over all asymptotic versions of T and the double-bar norm is the usual operator norm.

It is interesting to note that this gives a way of looking at the Calkin algebras $\mathcal{L}(X)/\mathcal{K}(X)$.

PROPOSITION 15. *Suppose X is a Banach space with a shrinking basis, where the norm of all tail projections is exactly* 1 (*the last property can be achieved by renorming, see* [LTz]). *Then the norm of the image of an operator T in the Calkin algebra is equal to $|||T|||$.*

PROOF. One direction is clear. If K is a compact operator on X, the norm of $T+K$ is at least the supremum of norms of asymptotic versions of $T+K$. The latter, by the proof of Proposition 14, are the same as asymptotic versions of T.

For the other direction we will show that for every T there exist compact operators K such that the norm of $T+K$ is almost achieved by asymptotic versions of $T+K$, which, again, are the same as asymptotic versions of T.

We will perturb T by a compact operator K, such that the set Φ of normalised blocks mapped by $T+K$ to vectors of norm greater than $\|T+K\|-\varepsilon$ will become an asymptotic set of length 1. We can then extract an asymptotic version of

$T+K$ from asymptotic sets containing only sequences of elements in Φ. Such asymptotic versions will almost achieve the norm of $T+K$, as required.

Take λ to be (up to ε) the largest such that

$$\{x \in S(X)^1_< : \|T(x)\| \geq \lambda\}$$

is asymptotic. By this we mean that the set

$$\{x \in S(X)^1_< : \|T(x)\| \geq \lambda + \varepsilon\}$$

does not have elements in some tail subspace $[X]_{>m}$. Consider $T' = T - T \circ P_m$, which is a compact perturbation of T. By our assumption on the basis we have $\|T'\| \leq \lambda + \varepsilon$, and the set

$$\{x \in S(X)^1_< : \|T'(x)\| \geq \lambda\}$$

is still asymptotic. The proof is now complete. □

3.2. Uniformly singular and asymptotically uniformly singular operators.

Definition 16. An operator T on a sequence space X is *asymptotically uniformly singular* if for every ε there exist $n(\varepsilon, T)$ and an admission set Σ_n of length n, such that any sequence in Σ_n contains a normalised vector, which T send ε-close to zero. Informally we will say, that T has an *almost kernel* in any sequence from Σ_n. T is called *uniformly singular* if it satisfies the above definition with $\Sigma_n = S(X)^n_<$.

In other words, an operator T is asymptotically uniformly singular, if , when restricted to the span of a sequence from Σ_n, T^{-1} is either not defined or has norm larger than $1/\varepsilon$.

An operator ideal close to the ideal of uniformly singular operators was defined in [Mi], and called σ_0. The difference is that the original definition referred to all n dimensional subspaces, rather than just block subspaces, as we read here.

Proposition 17. *Operators on a Banach-space X, which are asymptotically uniformly singular with respect to a given basis, form a Banach space with the usual operator norm, and a two sided ideal of $\mathcal{L}(X)$.*

Proof. Let T be asymptotically uniformly singular, and let S be bounded. ST is obviously asymptotically uniformly singular. Indeed, $n(\varepsilon, ST) \leq n(\varepsilon/\|S\|, T)$, and the admission sets for ST are the same as those for T.

To see that TS is asymptotically uniformly singular we use Lemma 11, and find admission sets Σ', whose normalised image under S are essentially contained in the admission sets Σ used in the definition of asymptotic uniform singularity for the operator T. TS will take some normalised block from the span of any sequence from Σ' to a vector with arbitrarily small norm. Indeed, S takes (essentially) Σ' to Σ, where T has an "almost kernel".

To show that the sum of two asymptotically uniformly singular operators is also asymptotically uniformly singular, we need to use the following claim:

If T is asymptotically uniformly singular, then for every ε and every k there exists an $N(k, \varepsilon, T)$, and an admission set Σ of length N, such that every sequence from Σ has a k-dimensional block subspace on which T has norm less than ε.

This claim is standardly proved by taking concatenations of the asymptotic sets from the definition of asymptotic uniform singularity for T. It is easier to think here of the uniformly singular case. Suppose T sends some normalised vector from any n_ε-dimensional block subspace into an ε-neighbourhood of zero. Then in any $\sum_{i=1}^k n_{\varepsilon_i}$-dimensional block subspace there is a k-dimensional block subspace, on which the norm of T is at most $C\sum_{i=1}^k \varepsilon_i$ (where C is the basic constant of X).

To complete the proof of the proposition, fix $\varepsilon > 0$ and consider asymptotically uniformly singular operators, T and S. From the definition of asymptotic uniform singularity produce an admission set Σ, of length $n(\varepsilon/2, S)$, such that S takes some normalised block in the span of any Σ-admissible sequence to a vector with norm less than $\varepsilon/2$. Take the admission set Ψ of length $N(n, \varepsilon/2, T)$ from the above claim. It is possible to extract an admission subset $\Psi' \subseteq \Psi$, such that any n consecutive blocks of a sequence in Ψ' are also in Σ (similarly to Remark 2).

We therefore have that in any block sequence in Ψ' there is an n-dimensional block subspace where T has norm less than $\varepsilon/2$. This subspace must be essentially the span of a sequence from Σ, so it contains a normalised vector, whose image under S has norm less than $\varepsilon/2$. This means that $T + S$ sends this vector ε-close to zero, and hence $T + S$ is asymptotically uniformly singular.

The fact that asymptotically uniformly singular operators form a closed subspace of $\mathcal{L}(X)$ is straightforward. □

REMARKS 18. (i) Every asymptotically uniformly singular operator T is actually uniformly singular on some block subspace.

Indeed, From Gowers' combinatorial lemma (in [G], see also [W1]) it follows that there is a block subspace Y, where $\Sigma_{n(\varepsilon,T)} = S(Y)_<^{n(\varepsilon,T)}$.

In a diagonal subspace we can find an n for every ε, such that any sequence of the form $e_n < x_1 < \ldots < x_n$ has a normalised block whose image under T has norm less than ε.

It is easy to see that on this block subspace T is uniformly singular; indeed, every sequence of n blocks contains a sequence of $[n/2]$ blocks supported after the $[n/2]$-th basic element.

(ii) It is not true, however, that the ideals of asymptotically uniformly singular and uniformly singular operators coincide.

Consider the following example: Let X be the ℓ_2 sum of increasingly long ℓ_2^n's, and let Y be their ℓ_3 sum. The formal identity from X to Y is not uniformly singular, but is asymptotically uniformly singular.

(iii) It is easy to extend Proposition 17 and show that asymptotically uniformly singular operators form a two sided operator ideal with the operator norm, when restricting our attention to the category of Banach spaces with shrinking bases. Uniformly singular operators will only form a left-sided ideal in the (shrinking) basis context. This is because a bounded operator multiplied to the right of a uniformly singular operator does not have to preserve blocks.

(iv) It is worth noting that in the Gowers-Maurey space (from [GM]), all operators are in fact uniformly singular perturbations of a scalar operator (this follows from Lemma 22 and Lemma 3 in [GM]). This point is even more interesting in light of Corollary 21 below, and the Remark which follows it.

The following proposition claims a strong dichotomy in the asymptotic structure of operators: either it contains an isomorphism, or it is composed only of uniformly singular operators.

PROPOSITION 19. *If T is an asymptotically uniformly singular operator then all operators in $\{T\}_\infty$ are uniformly singular. If T is not asymptotically uniformly singular, $\{T\}_\infty$ contains an isomorphism.*

PROOF. Let T be asymptotically uniformly singular. Take asymptotic sets Φ_n realizing an asymptotic version of T. Let player $\mathcal{V}$ play the winning strategy for Φ_n, while $\mathcal{S}$ plays the winning strategy for the admission sets Σ_n from the definition of asymptotic uniform singularity for T. The resulting vector sequences must approximately realize the above asymptotic version, but must also contain an "almost kernel" for T.

Keeping in mind that a restriction of an asymptotic version to a block subspace is still an asymptotic version, we conclude that T has an "almost kernel" on every block subspace of the appropriate dimension. Therefore any asymptotic version of T is uniformly singular.

Suppose T is not asymptotically uniformly singular. Then for some $\varepsilon > 0$ the sets Φ_n of all sequences in $S(X)^n_<$, on which T is an isomorphism with $\|T^{-1}\| \le 1/\varepsilon$, are asymptotic. Indeed, if they weren't, by Lemma 3, for some ε and for every n, Φ_n^c would be admission sets, and then T would be asymptotically uniformly singular, in contradiction. Using the existence theorem, extract from Φ_n asymptotic subsets Φ'_n, which realize an asymptotic version of T. This asymptotic version is an isomorphism. □

REMARK 20. Note that the proof shows that if T is asymptotically uniformly singular, all its asymptotic versions will be uniformly singular operators with the same $n(\varepsilon)$ as T.

We offer here an application of the theory developed above. Recall that an *asymptotic* ℓ_p space is a space, all whose asymptotic versions are isomorphic to ℓ_p.

COROLLARY 21. *Every asymptotically uniformly singular operator from an asymptotic ℓ_p space X to itself is compact.*

PROOF. Consider the set $\mathcal{AUS}(X) \subset \mathcal{L}(X)$ of asymptotically uniformly singular operators. The asymptotic versions of these operators, by Proposition 19 above, are all uniformly singular operators in $\mathcal{L}(\ell_p)$. In particular they are all asymptotically uniformly singular, and therefore belong to some proper closed two-sided ideal. It is well known [GoMaF] that the only proper closed two sided operator ideal in $\mathcal{L}(\ell_p)$ is the ideal of compact operators, but let us sketch a very simple proof for this particular context:

All asymptotic versions of T are diagonal uniformly singular operators in $\mathcal{L}(\ell_p)$. It is clear then, that given any $\varepsilon > 0$, only finitely many entries on the diagonal are larger than ε; otherwise, restricting to the span of the basic elements corresponding to the entries larger than ε we get an isomorphism. Therefore the entries on the diagonal go to zero, and the operator is compact.

We can now complete the proof of the corollary, using Proposition 14 once more.

$$\begin{aligned}\mathcal{AUS}(X) &= \{T \in \mathcal{L}(X) : \{T\}_\infty \subseteq \mathcal{US}(\ell_p)\}\\ &= \{T \in \mathcal{L}(X) : \{T\}_\infty \subseteq \mathcal{K}(\ell_p)\} = \mathcal{K}(X).\end{aligned}$$

where $\mathcal{K}(Z)$ is the set of all compact operators on Z, and $\mathcal{US}(Z)$ is the set of all uniformly singular operators on Z. □

REMARK 22. Note that, by this corollary, if an asymptotic-ℓ_p space with a shrinking basis has the property of the Gowers-Maurey space from the last point of Remark 18, then all bounded linear operators on this space will be compact perturbations of scalar operators. Whether such a space exists is an important open question.

3.3. A general theorem. We conclude with a theorem which explains that the above instances form part of a more general phenomenon. When referring to an operator ideal we invoke the categorical algebraic definition from [P]. An *injective* operator ideal J has the property that if $T : X \to Y$ is in J, then the same operator with a revised range, $T : X \to \overline{\mathrm{Im}}(T)$, is also in J. The following theorem states that the "asymptotic preimage" in $\mathcal{L}(X)$ of an injective ideal is itself an ideal in the algebra $\mathcal{L}(X)$. The preimage may be trivial, but previous examples show this needn't be the case.

THEOREM 23. *Let J be an injective operator ideal, and let X be a space with a shrinking basis. The set of operators $J' = \{T \in \mathcal{L}(X) : \{T\}_\infty \subseteq J\}$ is an operator ideal in $\mathcal{L}(X)$.*

Proof. If we multiply an operator $S \in J'$ with an operator $T \in \mathcal{L}(X)$, an asymptotic version of the product will always be a product of asymptotic versions.

Indeed, take the asymptotic sets realizing an asymptotic version, $\tilde{R}$, of $R = ST$ (or $R = TS$). Extract by the existence theorem asymptotic subsets, which realize an asymptotic version $\tilde{T}$ of T. Recall that the set of sequences which approximately realize asymptotic versions of S is an admission set (consult Remark 13). Thus Lemma 11 and the proof of the existence theorem allow to extract asymptotic subsets of sequences whose normalised images under T realize an asymptotic version $\tilde{S}$ of S. The product of these asymptotic versions, $\tilde{S}\tilde{T} \in J$, is realized by the same asymptotic subsets as well. But these asymptotic subsets must still realize $\tilde{R}$. Therefore $\tilde{R} \in J$, and R must be in J'.

Let T and S be in J'. Let $R = S + T$, and find asymptotic sets realizing $\tilde{R}$, an asymptotic version of R. Extract asymptotic subsets realizing asymptotic versions of S and T, $\tilde{S}$ and $\tilde{T}$ respectively. Note that we cannot say that $\tilde{R} = \tilde{S} + \tilde{T}$; in fact $\tilde{S}$ and $\tilde{T}$ may even have different ranges. However, we trivially have

$$\|\tilde{R}(x)\| \leq \|\tilde{S}(x)\| + \|\tilde{T}(x)\|. \tag{3.1}$$

This is enough in order to prove $\tilde{R} \in J$. Indeed, we can write

$$\tilde{R} = P \circ (i_1 \circ \tilde{S} + i_2 \circ \tilde{T}),$$

where

$$\tilde{R} : \tilde{X} \to \tilde{W}, \quad \tilde{S} : \tilde{X} \to \tilde{Y}, \quad \tilde{T} : \tilde{X} \to \tilde{Z},$$
$$i_1 : \tilde{Y} \to \tilde{Y} \oplus \tilde{Z}, \quad i_2 : \tilde{Z} \to \tilde{Y} \oplus \tilde{Z},$$
$$i_1(y) = (y, 0), \quad i_2(z) = (0, z),$$

and $P : \overline{\mathrm{Im}}(i_1 \circ \tilde{S} + i_2 \circ \tilde{T}) \to \tilde{W}$ is defined by the equation $P\big((i_1 \circ \tilde{S} + i_2 \circ \tilde{T})(x)\big) = \tilde{R}(x)$, and extended by continuity. Inequality (3.1) assures that P is well defined and continuous.

Now $\tilde{T}$ and $\tilde{S}$ are in J, so $i_1 \circ \tilde{S} + i_2 \circ \tilde{T}$ is also in J. By injectivity, we are allowed to modify the range as we compose with P, and still maintain that the result $\tilde{R}$ is in J. Thus $R \in J'$, and we conclude that J' is an ideal in $\mathcal{L}(X)$. □

References

[G] Gowers, W. T., *A new dichotomy for Banach spaces (research announcement)*, GAFA **6** (1996), 1083–1093.

[GM] Gowers, W. T. and Maurey, B., *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 851–874.

[GoMaF] Gohberg, I. C., Markus, A. S. and Feldmann, I. A., *Normally solvable operators and ideals associated with them* (Russian), Bul. Akad. Štiince RSS Moldove. **10** (76) (1960), 51–70.

[LTz] Lindenstaruss J. and Tzafriri, L., *Classical Banach spaces*, Vol. I, Ergeb. Math. Grenzgeb. **92**, Springer, Berlin.

[Mi] Milman V., *Spectrum of continuous bounded functions on the unit sphere of a Banach space*, Funct. Anal. Appl. **3** (1969), 67–79 (translated from Russian).

[MMiT] Maurey B., Milman V. and Tomczak-Jaegermann N., *Asymptotic infinite-dimensional theory of Banach spaces*, Operator Theory: Advances and Applications **77** (1995), 149–175.

[MiT] Milman V. and Tomczak-Jaegermann N., *Asymptotic ℓ_p spaces and bounded distortion*, Contemporary Math. **144** (1993), Bor-Luh Lin and W. B. Johnson eds., Amer. Math. Soc., 173–195.

[Kö] König, D., *Theorie der endlichen und unendlichen Graphen*, Akademische Verlagsgesellschaft, Leipzig, 1932; reprinted by Chelsea, New-York, 1950, page 81.

[KOS] Knaust, D., Odell, E. and Schlumprecht, Th., *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, preprint.

[OTW] Odell, E., Tomczak-Jaegermann N. and Wagner, R., *Proximity to ℓ_1 and distortion in asymptotic ℓ_1 spaces*, J. Functional Analysis **150** (1997), 101–145.

[P] Pietsch, A., *Operator Ideals*, North-Holland Mathematical Library **20**, North-Holland publishing company, 1980.

[T] Tomczak-Jaegermann, N., *Banach spaces of type p contain arbitrarily distortable subspaces*, GAFA **6** (1996), 1074–1082.

[W1] Wagner, R., *Gowers' dichotomy for asymptotic structure*, Proc. Amer. Math. Soc. **124** (1996), 3089–3095.

[W2] Wagner, R., *Finite high order games and an inductive approach towards Gowers' dichotomy*, preprint.

Vitali Milman
Department of Mathematics
Tel Aviv University
Raymond and Beverly Sackler Faculty of Exact Sciences
Ramat Aviv, Tel Aviv 69978
Israel
vitali@math.tau.ac.il

Roy Wagner
D.P.M.M.S.
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB
United Kingdom
r.wagner@dpmms.cam.ac.uk