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A Remark about the Scalar-Plus-Compact Problem

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Abstract. In [GM] a Banach space X was constructed such that every operator from a subspace $Y \subset X$ into the space is of the form $\lambda I_{Y\to X} + S$, where $I_{Y\to X}$ is the inclusion map and S is strictly singular. In this paper we show that there is an operator T from a subspace $Y \subset X$ into X which is not of the form $\lambda I_{Y\to X} + K$ with K compact.

1. Introduction

It is an open problem whether there exists an infinite-dimensional Banach space X such that every bounded linear operator from X to itself is of the form $\lambda I + K$, where λ is a scalar, I is the identity on X and K is a compact operator. The strongest property of a similar nature that has been obtained is that a space may be *hereditarily indecomposable* (see [GM] for this definition and several others throughout the paper), which implies [GM] that every operator on it is of the form $\lambda I + S$, where S is strictly singular, and even [F1] that every operator from a subspace into the space is a strictly singular perturbation of a multiple of the inclusion map. (These results assume complex scalars but several examples are known where the conclusion holds with real scalars.) In this note, we show that the first hereditarily indecomposable space to be discovered [GM], which we shall call X, has a subspace Y such that there is a non-compact strictly singular operator from Y into X. Therefore this operator is not a compact perturbation of a multiple of the inclusion map. Since all we are doing is showing that one particular space does not give an example of a stronger property than that required by the problem, the existence of this note needs some justification, which we shall now provide.

First, if one is trying to solve the problem with an example, then a natural line of attack is to try to construct a hereditarily indecomposable space such that every strictly singular operator is compact. To ensure the second property, a natural sufficient condition is the following: if $u_1 < u_2 < \cdots$ and $v_1 < v_2 < \cdots$

are any pair of normalized block bases such that $(u_n)_1^\infty$ dominates $(v_n)_1^\infty$, then they are actually equivalent. However, if such an example existed, then it would also give an example of the stronger property about maps from subspaces, so the stronger property is worth considering.

Second, the known hereditarily indecomposable spaces (for example [AD, F2, G1, G2, GM, H]) are obvious places to start in any search for a counterexample. Since not much was known about any of them in this respect, this note performs a modest, but necessary function.

Third, the method of proof does not rely very much on the detailed properties of the space X, so it is highly likely that it can be generalized, perhaps even to some very wide class of spaces such as reflexive ones. Indeed, it is the author's belief (but this is just a guess) that every reflexive space has a subspace such that there is a map from the subspace into the space which is not a compact perturbation of the inclusion map.

Nevertheless, since the result of this note is rather specific, we shall assume familiarity with the paper [GM], including its notation (although it is not necessary to have followed everything), and in some places we shall sketch easy arguments rather than proving them in full.

2. Construction of the Subspace and Operator into X

The main properties we shall use of the space X are the following two. Let $f(n) = \log_2(n+1)$. Then, for every $x \in X$,

$$\|x\| \le \|x\|_\infty \vee \sup\Big\{ f(k)^{-1/2} \sum_{i=1}^{k} \|E_i x\| : k \ge 2,\, E_1 < \cdots < E_k \Big\},$$

which implies that the norm on X is dominated by the norm on Schlumprecht's space defined with the function $\sqrt{f}$.

The second property is that for every $\varepsilon > 0$ and every $m \in \mathbb{N}$ there is a normalized block basis $u_1 < u_2 < \cdots$ of X such that if $a_1, \dots, a_m$ are scalars and $i_1 < \cdots < i_m$, then, setting $a = \sum_{j=1}^m a_j u_{i_j}$, there exist k and intervals $E_1 < \cdots < E_k$ such that

$$\|a\| \ge f(k)^{-1} \sum_{r=1}^{k} \|E_r a\| \ge (1+\varepsilon)^{-1} \sum_{j=1}^{m} |a_j|.$$

(To sketch the proof: let $v_1 < v_2 < \cdots$ be an infinite sequence in X such that every subsequence of length $M >> m$ is a rapidly increasing sequence with constant $1 + \varepsilon/2$. Let each u_i be a block consisting of M/m of the v_js added together.)

Now let $N_1 < N_2 < N_3 < \cdots$ be a sufficiently fast-growing sequence of integers. (It will be clear later that suitable choices of N_i exist.) For each integer s, let $u_1^{(s)} < u_2^{(s)} < u_3^{(s)} < \cdots$ be a block basis satisfying the condition above, with $m = N_s$ and $\varepsilon = 1$. Now let us choose vectors $y_1, y_2, y_3, \dots$ satisfying the following conditions.

(i) There is some function $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that $y_n = \sum_{s=1}^{\infty} 2^{-s} u^{(s)}_{\phi(s,n)}$ for every n, s.
(ii) Any pair of distinct $u^{(s)}_{\phi(s,n)}$ have disjoint ranges.
(iii) If $m < n$ then $u^{(s)}_{\phi(s,m)} < u^{(s)}_{\phi(s,n)}$.

It is not hard to show that all these properties can be satisfied simultaneously.

Consider a sum of the form $\sum_{i=1}^{N} a_i y_{n_i}$. If $N \le N_s$, then

$$\left\| \sum_{i=1}^{N} a_i y_{n_i} \right\| \ge 2^{-s} \ge 2^{-(s+1)} \sum_{i=1}^{N} |a_i|,$$

because we can estimate the norm on the left-hand side by isolating the contribution from the block basis $(u^{(s)}_j)$ and using the intervals $E_1 < \dots < E_k$ guaranteed by the condition on this block basis. (Note that we are not simply projecting onto the span of the $u^{(s)}_j$, which would not be allowed as X does not have an unconditional basis.)

Therefore, given any monotone function $\omega : \mathbb{N} \to [4, \infty]$ such that $\omega(n)$ tends to infinity with n, one can choose the sequence $N_1 < N_2 < \cdots$ in such a way that

$$\left\| \sum_{i=1}^{\infty} a_i y_i \right\| \ge \tfrac{1}{4} \sup_{A \subset \mathbb{N}} \omega(|A|)^{-1} \sum_{i \in A} |a_i|$$

whenever the left-hand side makes sense.

Our next aim is to show that if ω is sufficiently slow-growing, then the norm in X of any vector $\sum_{i=1}^{\infty} a_i e_i$ is at most $C \sup_{A \subset \mathbb{N}} \omega(|A|)^{-1} \sum_{i \in A} |a_i|$ for some absolute constant C. This will imply that there is a bounded linear map from Y to X taking y_n to e_n. This map is certainly not compact. Moreover, it follows easily from the above estimate that it is infinitely singular, and hence, since X is hereditarily indecomposable, strictly singular also. Thus, once we have the estimate, the proof is finished.

Let $L(\omega)$ be the space of all scalar sequences $a = (a_1, a_2, \dots)$ with

$$\|a\| = \sup_{A \subset \mathbb{N}} \omega(|A|)^{-1} \sum_{i \in A} |a_i|.$$

(This space is the dual of a Lorentz sequence space.) Let S be Schlumprecht's space, defined using the function $g(n) = (\log_2(n+1))^{1/2}$. Let S' be the symmetrization of S. That is, $\|a\|_{S'}$ is the supremum of $\|b\|_S$ over all rearrangements b of a. Then we have $\|a\|_X \le \|a\|_S \le \|a\|_{S'}$. Therefore, it is enough to show that the formal identity from $L(\omega)$ to S' is continuous.

To prove this, it is enough to consider extreme points in the unit ball of $L(\omega)$. Such a point has a decreasing rearrangement $a = (a_1, a_2, \dots)$, say, and it is easy to see that whenever $\sum_{i=1}^n a_i < \omega(n)$, we must have $a_n = a_{n+1}$. Therefore, a must be of the form

$$\sum_{i=1}^{n_1} \frac{\omega(n_1)}{n_1} e_i + \sum_{i=n_1+1}^{n_2} \frac{\omega(n_2)-\omega(n_1)}{n_2-n_1} e_i + \sum_{i=n_2+1}^{n_3} \frac{\omega(n_3)-\omega(n_2)}{n_3-n_2} e_i + \cdots .$$

Furthermore,

$$\sum_{i=n_{r-1}+1}^{n_r} \frac{\omega(n_r)-\omega(n_{r-1})}{n_r-n_{r-1}} e_i = \mathbb{E} \sum_{i=n_{r-1}+1}^{n_r} (\omega(i)-\omega(i-1)) e_{\pi(i)},$$

where the average is over all permutations π of the set $\{n_{r-1}+1, \ldots, n_r\}$, so in fact every extreme point of the ball of $L(\omega)$ has as its decreasing rearrangement the sequence $(\omega(1), \omega(2)-\omega(1), \omega(3)-\omega(2), \ldots)$.

Since the unit vector basis is 1-symmetric in both $L(\omega)$ and S', all that remains is to choose a decreasing sequence $b_m \to 0$ such that $\sum_{m=1}^\infty b_m = \infty$ and $\|(b_m)\|_{S'} < \infty$, so that we can set $\omega(n) = \sum_{m=1}^n b_m$. The existence of such a sequence is an easy exercise, given that the norm of $\sum_{i=1}^r e_i$ in S' is $r/f(r)$.

3. Further Questions

It would be nice of course to solve the whole problem, but if this cannot immediately be done, then to get more of a feel for the technicalities involved, it would be good to obtain results similar to those of this paper for other known hereditarily indecomposable spaces. For some of them the argument carries through with only minor modifications (this is certainly true of [G2] and probably of [F] and [H] as well). However, at least three known spaces present difficulties, each of a different kind. One is the dual of the space X considered here, another is the asymptotic ℓ_1-space constructed by Argyros and Delyanni and a third is the non-reflexive space constructed in [G1]. Some of these difficulties appear to be merely technical, but the problems should still be investigated.

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