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Isotropic Constants of Schatten Class Spaces

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Abstract. We study the value of the isotropic constant of the unit ball in the Schatten class spaces C_p^n. We prove that, for $2 \le p \le \infty$, this value is bounded by a fixed constant, whereas for $1 \le p < 2$ it is bounded by $c\,(\log n)^{1/p-1/2}$, where c is a fixed constant.

The isotropic constant of a convex symmetric body $K \subset \mathbb{R}^d$ is a highly important quantity. One of its equivalent definitions is

$$L_K = \min_{T\in SL(d)} \frac{1}{\sqrt{d}\,|K|^{1/d}} \sqrt{\frac{1}{|K|}\int_K \|Tu\|_2^2\,du},$$

where $|K|$ stands for the volume of K and $\|\cdot\|_2$ is the usual Euclidean norm. See [MP] for other formulations and a full discussion.

Estimation of L_K is one of the central problems on the border between local theory and convexity. Bourgain [Bo] has shown that $L_K \lesssim d^{1/4}\log d$ for any $K \subset \mathbb{R}^d$, where by $A \lesssim B$ we mean that $A \le cB$ for some universal constant c. He also raised the problem of universal boundedness of the isotropic constant independent of the dimension.

Till now there was no improvement of Bourgain's result in the general case, but boundedness of the isotropic constant was established for some families of bodies (see [MP,Ba,J1,J2], for example), covering the unit balls of most of the classical spaces.

The case of the Schatten class spaces C_p^n, however, was left out.

In [D] we showed that $L_{B_{C_1^n}} \lesssim \sqrt{\log n}$. Here we extend the result to all values of p using a simpler argument.

First we shall recall the definiton of the Schatten class spaces. If u is an $n \times n$ matrix, u^*u is a positive definite symmetric matrix, i.e., orthogonally diagonalizable with nonnegative eigenvalues $\eta_1 \ge \eta_2 \ge \dots \ge \eta_n \ge 0$. Set $\lambda_i(u) = \sqrt{\eta_i}$. The numbers $\lambda_1(u) \ge \lambda_2(u) \ge \cdots \ge \lambda_n(u) \ge 0$ are called the characteristic values of u.

The Schatten class space C_p^n is the n^2-dimensional space of all $n \times n$ real matrices equipped with the norm $\|u\|_{C_p^n} = (\sum_{i=1}^n \lambda_i(u)^p)^{1/p}$.

We'll denote by K_p the unit ball of C_p^n. It is well known (see [TJ], for example) that $|K_p|^{1/n^2} \sim 1/n^{(1/2)+(1/p)}$.

Proposition. *For $2 \le p \le \infty$, L_{K_p} is bounded by a universal constant.*

For $1 \le p < 2$, we have $L_{K_p} \lesssim (\log n)^{(1/p)-(1/2)}$.

Proof. For $2 \le p \le \infty$, we have

$$K_p \subset n^{(1/2)-(1/p)} K_2 \sim \sqrt{n^2}|K_p|^{1/n^2} B_{l_2^{n^2}}.$$

Hence

$$L_{K_p} \le \frac{1}{\sqrt{n^2}|K_p|^{1/n^2}} \sqrt{\frac{1}{|K_p|}\int_{K_p} \|u\|_2^2\,du} \lesssim \sqrt{\frac{1}{|K_p|}\int_{K_p} \|u\|_{K_p}^2\,du} \le 1.$$

Our treatment of the case $1 \le p < 2$ will rely on the following fact:

$$\text{For any } t \ge 1, \quad \left| K_p \setminus \left(\frac{2pt\log n}{n}\right)^{1/p-1/2} B_{l_2^{n^2}} \right| \lesssim \frac{1}{n^{2(t-2)n}} |K_p|. \qquad (*)$$

Before proving this, let's see that it implies the desired result. Indeed,

$$\begin{aligned}
\frac{1}{|K_p|}\int_{K_p} \|u\|_2^2\,du &= \int_0^\infty \frac{\left|K_p \setminus rB_{l_2^{n^2}}\right|}{|K_p|} 2r\,dr \\
&= \left(\frac{2p\log n}{n}\right)^{2(1/p-1/2)} 2\left(\frac{1}{p}-\frac{1}{2}\right) \\
&\qquad \times \int_0^\infty \frac{\left|K_p \setminus (\frac{2pt\log n}{n})^{1/p-1/2} B_{l_2^{n^2}}\right|}{|K_p|} t^{(2/p)-2}\,dt \\
&\le \left(\frac{2p\log n}{n}\right)^{2(1/p-1/2)} 2\left(\frac{1}{p}-\frac{1}{2}\right)\left(2 + \int_2^\infty \frac{t^{(2/p)-2}\,dt}{n^{(t-2)n}}\right) \\
&\sim \left(\frac{\log n}{n}\right)^{2(1/p-1/2)} \sim n^2 |K_p|^{2/n^2} (\log n)^{2(\frac{1}{p}-\frac{1}{2})}.
\end{aligned}$$

Therefore, $L_{K_p} \lesssim (\log n)^{1/p-1/2}$.

It remains to prove $(*)$:

Any $u \in K_p$ can be decomposed as $w+(u-w)$ in such a way that w is a rank-one operator with norm $\lambda_1(u)$ not greater than 1 and $u-w$ has characteristic values $\lambda_2(u), \dots, \lambda_n(u), 0$.

If $u \in K_p \setminus \left(\frac{2pt\log n}{n}\right)^{1/p-1/2} B_{l_2^{n^2}}$, then

$$\lambda_1(u) \ge \frac{\|u\|_2^{2/2-p}}{\|u\|_p^{p/2-p}} \ge \left(\frac{2pt\log n}{n}\right)^{1/p},$$

so we'll have

$$\|u-w\|_p = \left(\|u\|_p^p - \lambda_1(u)^p\right)^{1/p} \le \left(1 - \frac{2pt\log n}{n}\right)^{1/p} \le 1 - \frac{2t\log n}{n}$$

for this rank-one operator $w \in K_p$.

Now take a $\frac{\log n}{n}$-net $\mathcal{N}$ for $B_{l_2^n}$. This can be done with

$$|\mathcal{N}| \le \left(1 + \frac{2}{\frac{\log n}{n}}\right)^n = \left(1 + \frac{2n}{\log n}\right)^n.$$

Then $\mathcal{N} \otimes \mathcal{N}$ is a $\frac{2\log n}{n}$-net for the rank-one operators with norm at most 1, so we have

$$K_p \setminus \left(\frac{2pt\log n}{n}\right)^{1/p-1/2} B_{l_2^{n^2}} \subset \bigcup_{w\in\mathcal{N}\otimes\mathcal{N}} \left(w + \left(1 - \frac{2(t-1)\log n}{n}\right)K_p\right).$$

Hence

$$\begin{aligned}\left|K_p \setminus \left(\frac{2pt\log n}{n}\right)^{1/p-1/2} B_{l_2^{n^2}}\right| &\le |\mathcal{N}\otimes\mathcal{N}| \left(1 - \frac{2(t-1)\log n}{n}\right)^{n^2} |K_p| \\ &\le \left(1 + \frac{2n}{\log n}\right)^{2n} \frac{1}{n^{2(t-1)n}} |K_p| \\ &\lesssim \frac{1}{n^{2(t-2)n}} |K_p|,\end{aligned}$$

and we are done. □

Added in proof. Boundedness of the isotropic constant of the Schatten classes was very recently established using more complicated methods by H. Konig, M. Meyer and A. Pajor.

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