Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

Threshold Intervals under Group Symmetries

JEAN BOURGAIN AND GIL KALAI

Abstract. This article contains a brief description of new results on threshold phenomena for monotone properties of random systems. These results sharpen recent estimates of Talagrand, Russo and Margulis. In particular, for isomorphism invariant properties of random graphs, we get a threshold whose length is only of order $1/(\log n)^{2-\varepsilon}$, instead of previous estimates of the order $1/\log n$. The new ingredients are delicate inequalities in the spirit of harmonic analysis on the Cantor group.

A subset A of $\{0,1\}^n$ is called monotone if the conditions $x \in A$, $x' \in \{0,1\}^n$ and $x_i \leq x'_i$ for $i = 1, \dots, n$ imply $x' \in A$. For $0 \leq p \leq 1$, define μ_p the product measure on $\{0,1\}^n$ with weights $1-p$ at 0 and p at 1. Thus

$$\mu_p(\{x\}) = (1-p)^{n-j} p^j \quad \text{where} \quad j = \#\{i = 1, \dots, n \mid x_i = 1\}. \tag{1}$$

If A is monotone, then $\mu_p(A)$ is clearly an increasing function of p. Considering A as a "property", one observes in many cases a threshold phenomenon, in the sense that $\mu_p(A)$ jumps from near 0 to near 1 in a short interval when $n \to \infty$. Well known examples of these phase transitions appear for instance in the theory of random graphs. A general understanding of such threshold effects has been pursued by various authors (see for instance Margulis [M] and Russo [R]). It turns out that this phenomenon occurs as soon as A depends little on each individual coordinate (Russo's zero-one law). A precise statement was given by Talagrand [T] in the form of the following inequality.

Define for $i = 1, \dots, n$

$$A_i = \{x \in \{0,1\}^n \mid x \in A,\ U_i x \notin A\} \tag{2}$$

where $U_i(x)$ is obtained by replacement of the i-th coordinate x_i by $1 - x_i$ and leaving the other coordinates unchanged. The number $\mu_p(A_i)$ is the *influence* of the i-th coordinate (with respect to μ_p). Let

$$\gamma = \sup_{i=1,\dots,n} \mu_p(A_i). \tag{3}$$

Then
$$\frac{d\mu_p(A)}{dp} \geq c\frac{\log(1/\gamma)}{p(1-p)\log\bigl(2/p(1-p)\bigr)}\,\mu_p(A)\ \bigl(1-\mu_p(A)\bigr), \tag{4}$$
where $c > 0$ is some constant.

A simple relation due to Margulis and Russo is
$$\frac{d\,\mu_p}{dp} = 2/p\sum_{i=1}^{n}\mu_p(A_i). \tag{5}$$

As the right side of (5) represents the sum of the influences it follows that a small threshold interval corresponds to a large sum of influences. In [T], (4) is deduced from an inequality of the form
$$\mu_p(A)\bigl(1-\mu_p(A)\bigr) \leq C(p)\sum_{i=1}^{n}\ \frac{\mu_p(A_i)}{\log\bigl(1/\mu_p(A_i)\bigr)}. \tag{6}$$
The proof of this last inequality relies on the paper by Kahn, Kalai and Linial [KKL], where it is shown that always
$$\sup_{1\leq i\leq n}\ \mu_{1/2}(A_i) \geq c\,\frac{\log n}{n}. \tag{7}$$
Friedgut and Kalai [FK] used an extension of (7) given in [BKKKL] to show that for properties which are invariant under the action of a transitive permutation group the threshold interval is $O(1/\log n)$ and proposed some conjectures on the dependence of the threshold interval on the group.

Our aim here is to obtain a refinement and strengthening of the preceding in the context of "G-invariant" properties. Let f be a $0,1$-valued function on $\{0,1\}^n$ and G a subgroup of the permutation group on n elements $\underline{n} = \{1,2,\ldots,n\}$. Say that f is G-invariant provided
$$f(x_1,\ldots,x_n) = f(x_{\pi(1)},\ldots,x_{\pi(n)})\ \text{ for all }\ x\in\{0,1\}^n, \pi\in G.$$
Given G, define for $1\leq t\leq n$
$$\phi(t) = \phi_G(t) = \min_{S\subset\underline{n},|S|=t}\log(\#\{\pi(S)\mid \pi\in G\})$$
and for all $\tau > 0$
$$a_\tau(G) = \sup\{\phi(t)\mid \phi(t) > t^{1+\tau}\}.$$
Observe that since $\phi(t)\leq \log\binom{n}{t}$, necessarily $a_\tau(G)\lesssim(\log n)^{1/\tau}$.

Theorem 1. *Assume G transitive and A a monotone G-invariant property. Then for all $\tau > 0$*
$$\frac{d\mu_p(A)}{dp} > c_\tau a_\tau(G)\,\mu_p(A)\bigl(1-\mu_p(A)\bigr),$$
provided $p(1-p)$ stays away from zero in a weak sense, say
$$\log\bigl(p(1-p)\bigr)^{-1}\lesssim\log\log n.$$

It follows that in particular the threshold interval is at most

$$C_\tau a_\tau(G)^{-1} \quad \textit{for all} \quad \tau > 0.$$

Previous results as mentioned above only yield estimates of the form $(\log n)^{-1}$ and the main point of this work is to provide a method going beyond this. For crossing the $(\log n)^{-1}$ bar we need a complicated harmonic analysis argument. This may be useful in related combinatorial problems.

Theorem 1 is deduced from (5) and the following fact, independent of monotonicity assumptions.

THEOREM 2. *Assume that A is G-invariant and* (12) *holds. Then for all $\tau > 0$*

$$\sum \mu_p(A_i) > c_\tau a_\tau(G)\mu_p(A)\big(1 - \mu_p(A)\big).$$

For primitive permutation groups Theorem 1 and the excellent knowledge of primitive permutation groups [C, KL] (based on the classification theorem for finite simple groups) imply a close to complete description of the possible threshold interval of a G-invariant property, depending on the structure of G. (Recall that a permutation group $G \subset S_n$ is primitive if it is impossible to partition $\underline{n}$ to blocks $B_1, \dots B_t$, $t > 1$ so that every element in G permute the blocks among themselves.) It turns out that there are some gaps in the possible behaviors of the largest threshold intervals. This interval is proportional to $n^{-1/2}$ for S_n and A_n but at least $\log^{-2} n$ for any other group. The worst threshold interval can be proportional to $\log^{-c} n$ for c belonging to arbitrary small intervals around the following values: 2, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, …, or for c which tends to zero as a function of n in an arbitrary way. This (and more) is summarized in the next theorem. First we need a few definitions. For a permutation group $G \subset S_n$ let

$$T_G(\varepsilon) = sup\{q - p : \mu_p(A) = \varepsilon, \mu_q(A) = 1 - \varepsilon\},$$

where the supremum is taken over all monotone subsets of $\{0,1\}^n$ which are invariant under G. A composition factor of group G is a quotient group H/H' where H is a normal subgroup of G and H' is a normal subgroup of H. A section of G is a quotient H/H' where H is an arbitrary subgroup of G and H' is a normal subgroup of H.

THEOREM 3. *Let $G \subset S_n$ be a primitive permutation group.*

1. *If $G = S_n$ or $G = A_n$ then $T_G(\varepsilon) = \log(1/\varepsilon)/n^{1/2}$.*
2. *If $G \neq S_n, A_n$, $T_G(\varepsilon) \geq c_1 \log(1/\varepsilon)/\log^2 n$.*
3. *For every integer $r > 0$ and real numbers $\delta > 0$ and $\varepsilon > 0$, if $T_G(\varepsilon) \leq c_2 \log(1/\varepsilon)/(\log n)^{(1+1/(r+1))}$ then already*

$$T_G(\varepsilon) \leq c_3(\delta)\log(1/\varepsilon)/(\log n)^{(1+1/r-\delta)}.$$

4. *If G does not involve as composition factors alternating groups of high order then $T_G(\varepsilon) \geq \log(1/\varepsilon)/\log n \log\log n$.*

5. *Let* $n = \binom{m}{r}$ *and* G *is* S_m *acting on* r*-subsets of* $[m]$*. Then for every* $\delta > 0$

$$(\log(1/\varepsilon)/\log^{(1+1/(r-1))} n) \le T_G(\varepsilon) \le c(\delta)(\log(1/\varepsilon)/\log^{(1+1/(r-1)-\delta)} n)$$

6. *For* $G = \mathrm{PSL}(m, q)$ *acting on the projective space over* F_q*, for fixed* q*,*

$$T_G(\varepsilon) = O(\log(1/\varepsilon)/\log n \log\log n)$$

7. *For every function* $w(n)$ *such that* $\log w(n)/\log\log n \to 0$ *there are primitive group* $G_n \subset S_n$ *such that* $T_{G_n}(\varepsilon)$ *behaves like* $\log(1/\varepsilon)/\log n \cdot w(n)$.
8. *For every* $w(n) > 1$ *such that* $w(n) = O(\log\log n)$ *there are primitive group* $G_n \subset S_n$ *which do not involve alternating groups of high order as composition factors such that* $T_{G_n}(\varepsilon)$ *behaves like* $\log(1/\varepsilon)/(\log n \cdot w(n))$.
9. *If* G *does not involve as sections alternating groups of high order then* $T_G(\varepsilon) \ge O(\log(1/\varepsilon)/\log n)$.

The preceding yields a particularly satisfying result on the size of the maximal threshold for monotone graph properties. In the particular case of monotone graph properties on N vertices, we get $n = \binom{N}{2}$ and G is induced by permuting the vertices. One gets essentially

$$\phi(t) \sim \log \binom{N}{\sqrt{t}}$$

in this situation and the conclusion of Theorem 1 is that any threshold interval is at most $C_\tau(\log N)^{-2+\tau}$, with $\tau > 0$. This is essentially the sharp result, since, fixing $M \sim \log N$, the property for a graph on N vertices to contain a clique of size M yields a threshold interval $\sim (\log N)^{-2}$.

More details and the proofs appear in [BK].

References

[B] B. Bollobás, *Random graphs*, Academic Press, London, 1985.

[BKKKL] J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, "The influence of variables in product spaces", *Israel J. Math.* **77** (1992), 55–64.

[BK] J. Bourgain and G. Kalai, "Influences of variables and threshold intervals under group symmetries," *Geometric and Functional Analysis* **7**:3 (1997), 438–461.

[C] P. Cameron, "Finite permutation groups and finite simple groups", Bull. London Math. Soc. **13** (1981), 1–22.

[FK] E. Friedgut and G. Kalai, "Every monotone graph property has a sharp threshold", *Proc. Amer. Math. Soc.* **124** (1996), 2993–3002.

[KKL] J. Kahn, G. Kalai, and N. Linial, "The influence of variables on Boolean functions", Proc. 29th IEEE FOCS **58-80**, IEEE, NY (1988).

[KL] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, Cambridge University Press, Cambridge and New York, 1990.

[M] G. Margulis, "Probabilistic characteristic of graphs with large connectivity", *Problemy Peredači Informacii* **10**:2 (1974) (in Russian); translation in *Problems Info. Transmission*, 1977.

[R] L. Russo, "An approximative zero-one law", *Z. Wahrsch. Verw. Gebiete* **61** (1982), 129–139.

[T] M. Talagrand, "On Russo's approximative zero-one law", *Annals of Prob.* **22**:3 (1994), 1576–1587.

Jean Bourgain
School of Mathematics
Olden Lane
Institute for Advanced Study
Princeton, NJ 08540
United States of America
bourgain@math.ias.edu

Gil Kalai
Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem
Israel
kalai@math.huji.ac.il