Random Points in Isotropic Convex Sets

JEAN BOURGAIN

Abstract. Let $K$ be a symmetric convex body of volume 1 whose inertia tensor is isotropic, i.e., for some constant $L$ we have $\int_K \langle x, y \rangle^2 dx = L^2 |y|^2$ for all $y$. It is shown that if $m$ is about $n(\log n)^3$ then with high probability, this tensor can be approximately realised by an average over $m$ independent random points chosen in $K$,

$$\frac{1}{m} \sum_{i=1}^{m} \langle x_i, y \rangle^2.$$

Our aim is to prove the following fact:

Proposition. Let $K \subset \mathbb{R}^n$ be a convex centrally symmetric body of volume 1, in isotropic position, i.e.,

$$\int_K \langle x, e_i \rangle \langle x, e_j \rangle dx = L^2 \delta_{ij} \quad \text{where} \quad L = L_K(\gtrsim 1). \quad (1)$$

Fix $\delta > 0$ and choose $m$ random points $x_1, \ldots, x_m \in K$, where

$$m > C(\delta)n(\log n)^3. \quad (2)$$

Then, with probability $> 1 - \delta$,

$$(1 - \delta)L^2 < \frac{1}{m} \sum_{i=1}^{m} |\langle x_i, y \rangle|^2 < (1 + \delta)L^2 \quad (3)$$

for all $y \in S^{n-1} = \{|y| = 1\}$.

We first use the following probabilistic estimate:

Lemma 1. Let $f_1, \ldots, f_m$ be independent copies of a random variable $f$ satisfying

$$\int f^2 = 1, \quad (4)$$

$$\|f\|_{\psi_1} < C \quad \text{(where} \psi_1(t) = e^t), \quad (5)$$

$$\|f\|_{\infty} < B. \quad (6)$$
Let $\varepsilon > 0$ and assume $B > 1/\varepsilon$ say. Then
\[
\operatorname{mes}\left[(1 - \varepsilon)m < \sum_{i=1}^{m} f_i^2 < (1 + \varepsilon)m \right] > 1 - e^{-c B^m}.
\] (7)

**Proof (standard).** For real $\lambda$ (to be specified),
\[
\int e^\lambda (\sum_{i=1}^{m} f_i^2 - 1) = \left( \int e^\lambda (f^2 - 1) \right)^m.
\] (8)

By (4)
\[
\int e^\lambda (f^2 - 1) = 1 + \sum_{j \geq 2} \frac{1}{j!} \lambda^j \int (f^2 - 1)^j.
\] (9)

From (5) and (6),
\[
\int (1 + |f|)^{2j} < \min((Cj)^{2j}, (1 + B)^j (Cj)^j).
\] (10)
for each $j$. Hence, substituting (10) in (9),
\[
\int e^\lambda (f^2 - 1) < 1 + \sum_{j \geq 2} (C\lambda)^j (j \wedge B)^j < 1 + C\lambda^2
\] (11)
provided
\[
\lambda < \frac{c}{B}.
\] (12)

for an appropriate $c$. Thus (8) $< (1 + C\lambda^2)^m < e^{C\lambda^2 m}$ and from this fact and Tchebychev’s inequality
\[
\operatorname{mes}\left[\frac{1}{m} \sum_{i=1}^{m} (f_i^2 - 1) > \varepsilon \right] < e^{-\lambda m \varepsilon} e^{C\lambda^2 m} < e^{\frac{c}{B} m}
\] (13)
for appropriate $\lambda$ satisfying (12) (and since $1/\varepsilon < B$). □

Recall the important fact (following from the Brunn–Minkowski inequality) that, for $K$ convex with $\operatorname{Vol} K = 1$, there is equivalence
\[
\|\langle y, x \rangle\|_{L^\psi(K, dx)} \sim \|\langle y, x \rangle\|_{L^2(K, dx)}
\] (14)
(with an absolute constant). Hence, in our situation
\[
\|\langle y, x \rangle\|_{L^\psi(K, dx)} < C L \quad \text{if } |y| = \|y\|_2 \leq 1.
\] (15)

It follows that
\[
\operatorname{mes}\left[\{x \in K \mid |x| > \lambda L \sqrt{n}\} \right] < e^{-C\lambda} \quad \text{for } \lambda > 1.
\] (16)

The next estimate may be refined significantly in terms of an estimate on the $\ell^2$-operator norm (see remark at the end) but for our purposes the following cruder form is sufficient.
Lemma 2. Let $K$ be as above and $x_1, \ldots, x_m$ random points in $K$. Then, with probability $> 1 - \delta$,

$$\left| \sum_{i \in E} x_i \right| < C(\delta) L \log n \left( |E|^{1/2} n^{1/2} + |E| \right)$$

(17)

holds for all subsets $E \subset \{1, \ldots, m\}$.

Proof. Write

$$\left| \sum_{i \in E} x_i \right|^2 = \sum_{i \in E} |x_i|^2 + 2 \sum_{i \neq j, i,j \in E} \langle x_i, x_j \rangle.$$  

(18)

From (15), we may clearly assume

$$|x_i| < C L \log n^{1/2}$$

for all $i = 1, \ldots, m$. Hence the first term of (18) may be assumed bounded by

$$C L^2 \log n n^{1/2} |E|.$$  

(19)

To estimate the second term of (18), we use a standard decoupling trick. We can find subsets $E_1, E_2$ of $E$ satisfying $E_1 \cap E_2 = \emptyset$, $|E_1| \geq |E_2|$, and

$$\sum_{i \neq j, i,j \in E} \langle x_i, x_j \rangle \leq 4 \sum_{i \in E_1} \left| \sum_{j \in E_2} x_j \right| \langle x_i, y_{E_2}(x) \rangle.$$  

(20)

Hence we are reduced to bounding expressions of the form (19).

Rewrite

$$\sum_{i \in E_1} \left| \sum_{j \in E_2} x_j \right| = \sum_{j \in E_2} \left| \sum_{i \in E_1} \langle x_i, y_{E_2}(x) \rangle \right|$$

(21)

where

$$y_{E_2}(x) = \sum_{j \in E_2} x_j; \quad \text{thus} \quad |y_{E_2}| = 1.$$  

Observe that the system $(x_i)_{i \in E_1}$ is independent of $y_{E_2}$, since $E_1 \cap E_2 = \emptyset$. Fix size scales $|E_1| \sim m_1$, $|E_2| \sim m_2$, $m \geq m_1 \geq m_2 \geq 1$.

Thus for fixed $m_1 > m_2$, $(E_1, E_2)$ run over at most $m^{Cm_1}$ pairs of subsets of \{1, \ldots, m\}. For given $y, |y| = 1$, (15) easily implies that

$$\int e^{\mu \sum_{i \in E_1} \langle x_i, y \rangle} \prod_{i \in E_1} dx_i < 2^{|E_1|};$$  

(22)

hence, for $\mu > C$,

$$\text{mes} \left[ (x_i)_{1 \leq i \leq m} \in K^m \left| \sum_{i \in E_1} \langle x_i, y_{E_2}(x) \rangle > \mu L |E_1| \right. \right] < e^{-\mu |E_1|}.$$  

(23)
Consequently, from (20) and the preceding, we may write

\[ \sum_{i \in E_1} \left| \left\langle x_i, \sum_{j \in E_2} x_j \right\rangle \right| < \sum_{j \in E_2} \mu L \frac{|E_1|}{|E_2|} \]  

for all \( |E_1| \sim m_1, |E_2| \sim m_2, E_1 \cap E_2 = \emptyset \) provided

\[ m^{Cm_1} e^{-\mu m_1} < 2^{-m_1}; \quad \text{thus } \mu \sim \log m \sim \log n. \]  

(25)

Thus, letting \( \mu \sim \log n \), (24) may be assumed valid for all \( E_1, E_2 \subset \{1, \ldots, m\} \) with \( E_1 \cap E_2 = \emptyset \).

Substituting (19), (24) in (18) thus yields, for all \( E \subset \{1, \ldots, m\} \),

\[ \left| \sum_{j \in E_2} x_j \right|^2 \leq C L^2 (\log n)^2 |E| + C L (\log n) |E| \max_{E_2 \subset E} \left| \sum_{j \in E_2} x_j \right| \]  

(26)

and (17) immediately follows. \( \square \)

**Proposition.** Fix \( \delta > 0 \) and choose random points \( x_1, \ldots, x_m \in K \), with \( m > C(\delta) n (\log n)^3 \).

Then with probability \( > 1 - \frac{1}{m} \frac{1}{m} \sum_{i=1}^m f^y(x_i) \leq (1 + \delta) L^2 \) for all \( y \in S^{n-1} \).

(27)

**Proof.** Restrict \( y \) to a \( \delta \)-dense set \( F_\delta \) in the unit sphere \( S^{n-1} \), \( \#F_\delta < \left( \frac{C}{\delta} \right)^n \).

Fix \( y \in F \) and define

\[ f = f^y(x) = \begin{cases} \frac{1}{L} |\langle x, y \rangle| & \text{if } |\langle x, y \rangle| < C_1 (\log n) L, \\ 0 & \text{otherwise} \end{cases} \]  

(28)

(with \( C_1 \) to be specified).

Thus

\[ 1 - \int f^2 = \frac{1}{L^2} \int_{K \cap |\langle x, y \rangle| > C_1 (\log n) L} |\langle x, y \rangle|^2 dx < e^{-c_1 \log n}. \]  

(29)

Applying Lemma 1 with \( B = C_1 \log n, \varepsilon = \frac{\delta}{m} \), it follows that for a random choice \( x_1, \ldots, x_m \) of points in \( K \), with probability \( > 1 - e^{-c(\varepsilon/\log n)m} \),

\[ \int f^2 - \varepsilon < \frac{1}{m} \sum_{i=1}^m f^y(x_i)^2 < \left( \int f^2 + \varepsilon \right); \]  

(30)

hence, by (28) and (29),

\[ \left| 1 - \frac{1}{L^2 m} \sum \langle x_i, y \rangle^2 \left| \left| \langle x_i, y \rangle \right| < C_1 L \log n \right| \right| < \varepsilon + \left( 1 - \int f^2 \right) < 2\varepsilon. \]  

(31)

Letting

\[ \left( \frac{C}{\delta} \right)^n e^{-c(\varepsilon/\log n)m} \ll 1, \quad \text{i.e., } m \gtrsim \frac{1}{\varepsilon} \log \frac{1}{\delta} (\log n) n, \]  

(32)

we may then assume (31) for all \( y \in \mathcal{F}_\delta \).
On the other hand, from Lemma 2, a random choice \( \{x_i \mid i = 1, \ldots, m\} \) of \( m \) points in \( K \) will also with probability \( 1 - \delta \) satisfy (17) for all \( E \subset \{1, \ldots, m\} \).

This permits to estimate \( \#E_\beta \), where for given \( y \) satisfying \( |y| = 1 \),

\[
E_\beta = E_\beta(y) = \{i = 1, \ldots, m \mid |\langle x_i, y \rangle| > \beta\}, \quad \beta > C_1 (\log n) L. \tag{33}
\]

Indeed, it follows from (17) that

\[
\frac{1}{2} |E_\beta| < C L \log n (|E_\beta|^{1/2} n^{1/2} + |E_\beta|) \tag{34}
\]

hence

\[
|E_\beta| < C \frac{L^2 (\log n)^2 n}{\beta^2} \tag{35}
\]

from the choice of \( \beta \). Consequently

\[
\frac{1}{L^2 m} \sum_{i=1}^m \left( |\langle x_i, y \rangle|^2 \right) \geq C_1 L \log n < \frac{1}{L^2 m} \sum_{n > \beta > C_1 \text{dyadic}} \beta^2 |E_\beta|
\]

\[
< C(\delta) (\log n)^3 \frac{n}{m} < \frac{\delta}{10} \tag{36}
\]

by the choice of \( m \).

Finally, combining (36) and (31), it follows that for all \( y \in F_\delta \)

\[
\left| 1 - \frac{1}{L^2 m} \sum_{i=1}^m |\langle x_i, y \rangle|^2 \right| < 2\varepsilon + \frac{\delta}{10} < \frac{\delta}{3} \tag{37}
\]

and therefore also (27). \( \square \)

**Remark.** By refining a bit the method of proof of Lemma 2, one may obtain the following result: Let \( x_1, \ldots, x_n \) be a choice of \( n \) independent vectors in \( \mathbb{R}^n \) according to a probability measure \( \mu \) on \( \mathbb{R}^n \) satisfying

\[
\|\langle x, y \rangle\|_{L^1(\mu(dx))} < \frac{1}{\sqrt{n}} \quad \text{for all } y \in S^{n-1}. \tag{38}
\]

Then, with probability \( 1 - \delta \), one gets for the matrix \( (x_1, \ldots, x_n) \) the bound

\[
\|(x_1, \ldots, x_n)\|_{B(\mathbb{R}^2)} < C(\delta) \left( \int \left( \max_{1 \leq i \leq n} |x_i| \right) d\mu + 1 \right). \tag{39}
\]

This is the same estimate as one would get assuming an \( L^{\psi_2} \)-bound

\[
\|\langle x, y \rangle\|_{L^{\psi_2}(\mu(dx))} < \frac{1}{\sqrt{n}} \quad \text{for } y \in S^{n-1} \tag{40}
\]

instead of (38).