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Integrals of Smooth and Analytic Functions over Minkowski's Sums of Convex Sets

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1. Introduction and Statement of Main Results

Let $\bar{K} = (K_1, K_2, \dots, K_s)$ be an s-tuple of compact convex subsets of $\mathbb{R}^n$. For any continuous function $F : \mathbb{R}^n \longrightarrow \mathbb{C}$, consider the function

$$M_{\bar{K}}F : \mathbb{R}^s_+ \longrightarrow \mathbb{C}\,, \text{where } \mathbb{R}^s_+ = \{(\lambda_1, \dots, \lambda_s) \mid \lambda_j \geq 0\}\,,$$

defined by

$$(M_{\bar{K}}F)(\lambda_1, \dots, \lambda_s) = \int_{\sum_{i=1}^s \lambda_i K_i} F(x)\,dx. \qquad (*)$$

This defines an operator $M_{\bar{K}}$, which we will call a Minkowski operator. Denote by $\mathcal{A}(\mathbb{C}^n)$ the Frechet space of entire functions in n variables with the usual topology of the uniform convergence on compact sets in $\mathbb{C}^n$, and $C^r(\mathbb{R}^n)$ the Frechet space of r times differentiable functions on $\mathbb{R}^n$ with the topology of the uniform convergence on compact sets in $\mathbb{R}^n$ of all partial derivatives up to the order r $(1 \leq r \leq \infty)$.

The main results of this work are Theorems 1 and 3 below.

THEOREM 1.
$(\imath)$ *If* $F \in \mathcal{A}(\mathbb{C}^n)$, *then* $M_{\bar{K}}F$ *has a (unique) extension to an entire function on* $\mathbb{C}^s$ *and defines a continuous operator from* $\mathcal{A}(\mathbb{C}^n)$ *to* $\mathcal{A}(\mathbb{C}^s)$ (see Theorem 3 below).
$(\imath\imath)$ *If* $F \in C^r(\mathbb{R}^n)$, *then* $M_{\bar{K}}F \in C^r(\mathbb{R}^s_+)$ *(it is smooth up to the boundary) and again* $M_{\bar{K}}$ *defines a continuous operator from* $C^r(\mathbb{R}^n)$ *to* $C^r(\mathbb{R}^s_+)$.

COROLLARY 2. *If* F *is a polynomial of degree* d, *then* $M_{\bar{K}}F$ *is a polynomial of degree at most* $d + n$.

Indeed, we can assume F to be homogeneous of degree d. Then $M_{\bar{K}}$ is an entire function, which is homogeneous of degree $d + n$, hence it is a polynomial.

In fact, this corollary is well known and it is a particular case of the Pukhlikov–Khovanskii Theorem ([P-Kh]; see another proof below).

Theorem 3. *Assume that a sequence* $F^{(m)} \in \mathcal{A}(\mathbb{C}^n)$ *(or* $C^r(\mathbb{R}^n)$*, respectively),* $m \in \mathbb{N}$ *is such that*

$$F^{(m)} \longrightarrow F \text{ in } \mathcal{A}(\mathbb{C}^n) \text{ (or } C^r(\mathbb{R}^n)).$$

Let $K_i^{(m)}$, K_i, $i = 1, 2, \ldots, s$, $m \in \mathbb{N}$ *be convex compact sets in* $\mathbb{R}^n$, *and suppose* $K_i^{(m)} \longrightarrow K_i$ *in the Hausdorff metric for every* $i = 1, \ldots, s$. *Then*

$$M_{\bar{K}^{(m)}} F^{(m)} \longrightarrow M_{\bar{K}} F$$

in $\mathcal{A}(\mathbb{C}^s)$ *(or* $C^r(\mathbb{R}^s_+)$*).*

Remarks. 1. It follows from Theorem 1 that, if K is a compact convex set, D is the standard Euclidean ball and γ_n is the standard Gaussian measure in $\mathbb{R}^n$, then $\gamma_n(K + \varepsilon \cdot D)$ is an entire function of ε and the coefficients of the corresponding power expansion are rotation invariant continuous valuations on the family of compact convex sets (see the related definitions in Section 4).

2. There is a different simpler proof of Theorem 1 in the case when all the K_i are convex polytopes. However, the standard approximation argument cannot be applied automatically, since Theorem 3 on the continuity does not follow from that simpler construction even for polytopes.

In Section 4 we present another proof of the Pukhlikov–Khovanskii Theorem.

2. Preliminaries

Before proving these theorems, let us recall some facts, which are probably quite classical, but we will follow Gromov's work [G] (see also [R]).

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called convex if for all $x, y \in \mathbb{R}^n$ and $\mu \in [0, 1]$,

$$f(\mu x + (1-\mu)y) \leq \mu f(x) + (1-\mu) f(y);$$

f is called strictly convex if

$$f(\mu x + (1-\mu)y) < \mu f(x) + (1-\mu) f(y)$$

whenever $x \neq y$ and $\mu \in (0, 1)$. Define a Legendre transform of the convex function f (which is also called a conjugate function of f)

$$Lf(y) := \sup_{x \in \mathbb{R}^n} ((y, x) - f(x)).$$

Then Lf is a convex function and $-\infty < Lf \leq +\infty$. A set $K_f := \{y \in \mathbb{R}^n \mid Lf(y) < +\infty\}$ is called the effective domain of Lf. Obviously, K_f is a convex set. For any convex set K, we will denote the relative interior of K by $\operatorname{Int} K$.

Lemma 4.
($\imath$) *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a strictly convex* C^2*-function. Then* K_f *is a convex set and the gradient map* $\nabla f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a one-to-one map of* $\mathbb{R}^n$ *onto* $\operatorname{Int} K_f$.
($\imath\imath$) *If* f_1, f_2 *are as in* ($\imath$), *then for all* λ_1 , $\lambda_2 > 0$,

$$\operatorname{Im}(\nabla(\lambda_1 f_1 + \lambda_2 f_2)) = \lambda_1 \operatorname{Im}(\nabla f_1) + \lambda_2 \operatorname{Im}(\nabla f_2).$$

Proof. ($\imath$) The injectivity of ∇f immediately follows from the strict convexity of f.

For any $x_0, x \in \mathbb{R}^n$,

$$f(x) \geq f(x_0) + (\nabla f(x_0), x - x_0).$$

Hence $Lf(\nabla f(x_0)) = (\nabla f(x_0), x_0) - f(x_0) < \infty$ and $\operatorname{Im}(\nabla f) \subset K_f$. In order to check that $\operatorname{Im}(\nabla f) \subset \operatorname{Int} K_f$, let us choose any $a \in \partial K_f$ and assume that there exists $b \in \mathbb{R}^n$ such that $\nabla f(b) = a$. Without loss of generality, one may assume that $a = 0 = b$ and $f(0) = 0$. Then $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Since K_f is convex and $0 \in \partial K_f$, one can find a unit vector $u \in \mathbb{R}^n$ such that $\lambda u \notin K_f$ for all $\lambda > 0$. Consider a new convex function on $\mathbb{R}^1$

$$\phi(t) := \inf \{f(y + tu) \mid y \perp u\}.$$

Clearly, $\phi(t) \geq 0$ everywhere and $\phi(0) = 0$.

Case 1. Assume that there exists $t_0 > 0$ such that $\phi(t_0) > 0$. Then, by the convexity of ϕ, $\phi(t) \geq \frac{\phi(t_0)}{t_0} t$ for $t \geq t_0$ and for $t \leq 0$. Hence

$$L\phi\Big(\frac{\phi(t_0)}{t_0}\Big) \leq \sup \Big\{ \frac{\phi(t_0)}{t_0} t - \phi(t) \Bigm| t \in [0, t_0] \Big\} < \infty.$$

But for the Legendre transform of f, one has

$$\begin{aligned} Lf\Big(\frac{\phi(t_0)}{t_0} u\Big) &= \sup_{x \in \mathbb{R}^n} \Big(\Big(\frac{\phi(t_0)}{t_0} u,\, x\Big) - f(x) \Big) \\ &= \sup_{s \in \mathbb{R},\, y \perp u} \Big(\Big(\frac{\phi(t_0)}{t_0} u,\, su + y\Big) - f(su + y) \Big) \\ &= \sup_{s \in \mathbb{R}} \Big(\frac{\phi(t_0)}{t_0} s - \phi(s) \Big) = L\phi\Big(\frac{\phi(t_0)}{t_0}\Big) < \infty. \end{aligned}$$

Thus $\frac{\phi(t_0)}{t_0} u \in K_f$, and this contradicts the choice of u.

Case 2. Assume that $\phi(t) = 0$ for all $t \geq 0$. Let us show that this case is impossible (this will finish the proof of part ($\imath$) of Lemma 4). It would follow from the fact that $f(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

If the last statement is false, then there exists a sequence of vectors $x_k \longrightarrow \infty$ such that $|f(x_k)| \leq C$ (where C is some constant). Passing to a subsequence, we may assume that

$$\frac{x_k}{|x_k|} \longrightarrow u \in \mathbb{R}^n,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. Since f is strictly convex, $f(0) = 0$ and $\nabla f(0) = 0$ by assumption, then $f(u) > 0$. Also, for all $x \in \mathbb{R}^n$,

$$f(x) \geq f(u) + (\nabla f(u),\, x - u).$$

Substituting $x = 0$ or $x = x_k$, we obtain

$$(\nabla f(u),\, u) \geq f(u) > 0,$$

$$f(x_k) \geq f(u) + (\nabla f(u),\, x_k - u).$$

The last inequality can be rewritten

$$f(x_k) \geq f(u) - (\nabla f(u),\, u) + |x_k| \left(\nabla f(u),\, \frac{x_k}{|x_k|}\right).$$

But $(\nabla f(u), \frac{x_k}{|x_k|}) \longrightarrow (\nabla f(u),\, u) > 0$, hence $f(x_k) \longrightarrow \infty$, which contradicts our assumptions.

($\imath\imath$) Under conditions of the lemma $\lambda_1 f_1 + \lambda_2 f_2$ is also a strictly convex function. By the part ($\imath$),

$$\mathrm{Im}(\nabla f_i) = \mathrm{Int}\, K_{f_i} \text{ for } i = 1, 2.$$

Then easily

$$\mathrm{Im}(\lambda_1 \nabla f_1 + \lambda_2 \nabla f_2) \subset \lambda_1 \,\mathrm{Int}\, K_{f_1} + \lambda_2 \,\mathrm{Int}\, K_{f_2} \subset$$

$$\mathrm{Int}(\lambda_1 K_{f_1} + \lambda_2 K_{f_2}) \subset \mathrm{Int}(K_{\lambda_1 f_1 + \lambda_2 f_2}) = \mathrm{Im}(\nabla(\lambda_1 f_1 + \lambda_2 f_2)).$$

Hence all the sets in the above sequence of inclusions coincide. □

Lemma 5. [G] *Let $K \subset \mathbb{R}^n$ be an open bounded convex set, let μ be the Lebesgue measure in $\mathbb{R}^n$. Define*

$$f(x) := \log \int_K \exp(x, y)\, d\mu(y). \tag{1}$$

Then f is a strictly convex C^∞-function and $\mathrm{Im}(\nabla f) = K$.

Now let K_i, $1 \leq i \leq s$ be compact convex subsets of $\mathbb{R}^n$. For every i, fix a point $a_i \in K_i$. Let μ_i denote $(\dim K_i)$-dimensional Lebesgue measure supported on $\mathrm{span}(K_i - a_i)$. Define

$$f_i(x) := (x,\, a_i) + \int_{K_i - a_i} \exp(x,\, y)\, d\mu_i(y).$$

For every i, $f_i(x)$ depends only on the orthogonal projection of x on $\mathrm{span}(K_i - a_i)$. Moreover, f_i is a convex function on $\mathbb{R}^n$ and strictly convex on $\mathrm{span}(K_i - a_i)$. Then it is easy to see that $K_{f_i} \subset a_i + \mathrm{span}(K_i - a_i)$. Thus, by Lemmas 5 and 4,

$$\mathrm{Im}\, \nabla f_i = \mathrm{Int}\, K_i = \mathrm{Int}\, K_{f_i}.$$

Corollary 6. *Let K_i, f_i, $1 \le i \le s$ be as above, $\lambda_i > 0$. Then*

$$\operatorname{Im}\Bigg(\nabla\Bigg(\sum_{i=1}^{s}\lambda_i f_i\Bigg)\Bigg) = \sum_{i=1}^{s}\lambda_i \operatorname{Int} K_i.$$

Proof. It is sufficient to consider all $\lambda_i = 1$. Set $L := \operatorname{span}\Big(\sum_i (K_i - a_i)\Big)$. Without loss of generality, we may assume that $L = \mathbb{R}^n$. Then obviously the function $f := \sum f_i$ is strictly convex on $\mathbb{R}^n$, and by Lemma 4, $\operatorname{Im}\nabla f = \operatorname{Int} K_f$ is an open and convex set. Clearly,

$$\operatorname{Im}\nabla f \subset \sum \operatorname{Im}\nabla f_i = \sum \operatorname{Int} K_i = \sum \operatorname{Int} K_{f_i} = \operatorname{Int}\Big(\sum K_{f_i}\Big)$$

(the last equality holds for general convex bounded subsets of $\mathbb{R}^n$). One can easily see that $\sum K_{f_i} \subset K_f$, hence

$$\operatorname{Im}\nabla f \subset \sum \operatorname{Int} K_i \subset \operatorname{Int} K_f = \operatorname{Im}\nabla f. \qquad \square$$

3. Proofs of Theorems 1 and 3

Proof of Theorem 1. For every K_i, choose $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ as above. Then $\nabla f_i = (\frac{\partial f_i}{\partial y_1}, \ldots, \frac{\partial f_i}{\partial y_n})$, and the Jacobian of the gradient map equals the Hessian of f_i,

$$H(f_i) = \left(\frac{\partial^2 f_i}{\partial y_p \partial y_q}\right)_{p,q=1}^{n},$$

which is a non-negative definite matrix, since f_i is convex.

We have for $\lambda_i > 0$,

$$\int_{\sum \lambda_i K_i} F(x)\,dx = \int_{\mathbb{R}^n} F\Big(\sum \lambda_i \nabla f_i(y)\Big) \det\Big(H\Big(\sum \lambda_i f_i(y)\Big)\Big)\,dy. \qquad (2)$$

Write for simplicity $H_i(y) = H(f_i(y))$, so that the last expression is

$$\int_{\mathbb{R}^n} F\Big(\sum \lambda_i \nabla f_i(y)\Big) \det\Big(\sum \lambda_i H_i(y)\Big)\,dy$$
$$= \sum_{j_1,\ldots,j_n} \lambda_{j_1}\ldots\lambda_{j_n} \int_{\mathbb{R}^n} F\Big(\sum \lambda_i \nabla f_i(y)\Big) D(H_{j_1}(y), \ldots, H_{j_n}(y))\,dy, \qquad (3)$$

where $D(H_{j_1}(y), \ldots, H_{j_n}(y))$ denotes the mixed discriminant of non-negative definite symmetric matrices $H_{j_1}(y), \ldots, H_{j_n}(y)$. But it is well known that the mixed discriminant of such matrices is nonnegative (see, e.g., [Al]).

Let us substitute $F \equiv 1$ into (2). We obtain

$$\operatorname{vol}\Big(\sum \lambda_i K_i\Big) = \sum_{j_1,\ldots,j_n} \lambda_{j_1}\ldots\lambda_{j_n} \int_{\mathbb{R}^n} D(H_{j_1}(y), \ldots, H_{j_n}(y))\,dy.$$

Hence $\int_{\mathbb{R}^n} D(H_{j_1}(y), \ldots, H_{j_n}(y))\,dy = V(K_{j_1}, \ldots, K_{j_n})$ (the right hand side denotes the mixed volume of $K_{j_1}, \ldots, K_{j_n}$; see, e.g., [B-Z], [Sch]).

Observe that the integrand in (3) makes sense also for $\lambda_i < 0$, if $F \in C^r(\mathbb{R}^n)$ and for all complex λ_i, if $F \in \mathcal{A}(\mathbb{C}^n)$. We only have to check the convergence of the integral for such λ_i and its convergence after taking partial derivatives with respect to λ_i. Then Theorem 1 ($\imath$) and($\imath\imath$) will be proved.

Let us show that for the integral in (3), and the same proof works for the partial derivatives with respect to the λ_i.

Since $\mathrm{Im}(\nabla f_i) \subset K_i$, there exists a constant C, such that $\big\|\sum \lambda_i \nabla f_i(y)\big\| \leq C \cdot \sum |\lambda_i|$ for all $y \in \mathbb{R}^n$, where $\|\cdot\|$ is some norm in $\mathbb{C}^n$ (or in $\mathbb{R}^n$). By the continuity of F, $F\big(\sum \lambda_i \nabla f_i\big)$ is bounded by some constant $K(R)$ if $\sum |\lambda_i| \leq R$ and $y \in \mathbb{R}^n$. Hence

$$\begin{aligned}\int \big|F\big(\textstyle\sum \lambda_i \nabla f_i(y)\big) D\big(H_{j_1}(y), \dots, H_{j_n}(y)\big)\big| \, dy \\ \leq K(R) \int_{\mathbb{R}^n} D(H_{j_1}(y), \dots, H_{j_n}(y)) \, dy \\ = K(R) \, V(K_{j_1}, \dots, K_{j_n}) < \infty. \qquad \square\end{aligned}$$

Remark. We have actually shown that, if $F \in C^r(\mathbb{R}^n)$, then the equality (2) gives us a smooth extension of $M_{\bar{K}} F(\lambda_1 \dots, \lambda_n)$ from $\mathbb{R}^s_+$ to $\mathbb{R}^s$. It turns out that this extension is natural in some sense, i.e. it does not depend on the choice of the functions f_i. Indeed, assume that we have two such extensions $M_{\bar{K}} F$ and $M'_{\bar{K}} F$ corresponding to f_i and f'_i. Choose a sequence of polynomials $\{P_m\}$ approximating F uniformly on compact sets in $\mathbb{R}^n$. Then for corresponding extensions, we have $M_{\bar{K}} P_m \longrightarrow M_{\bar{K}} F$ and $M'_{\bar{K}} P_m \longrightarrow M'_{\bar{K}} F$ uniformly on compact sets in $\mathbb{R}^s$. By Corollary 2, $M_{\bar{K}} P_m$ and $M'_{\bar{K}} P_m$ are polynomials and since they coincide on $\mathbb{R}^s_+$, they coincide everywhere on $\mathbb{R}^s$. Hence $M_{\bar{K}} P_m \equiv M'_{\bar{K}} P_m$ on $\mathbb{R}^s$ and $M_{\bar{K}} F \equiv M'_{\bar{K}} F$.

Proof of Theorem 3. *Step 1.* It is sufficient to prove the continuity of $M_{\bar{K}} F$ separately with respect to F and $\bar{K} = (K_1, \dots, K_s)$, because $M_{\bar{K}} F = M(F; K_1, \dots, K_s)$ can be considered as a map $M : L_1 \times \mathcal{K}^s \longrightarrow L_2$, where L_1 and L_2 are Frechet spaces, $L_1 = \mathcal{A}(\mathbb{C}^n)$ or $C^r(\mathbb{R}^n)$, $L_2 = \mathcal{A}(\mathbb{C}^s)$ or $C^r(\mathbb{R}^s)$, and $\mathcal{K}$ is the space of compact convex subsets of $\mathbb{R}^n$ with the Hausdorff metric. Since M is linear with respect to the first argument $F \in L_1$, and $\mathcal{K}^s$ is locally compact (by the Blaschke's selection theorem), then M is continuous as a function of two arguments (it is an easy and well known consequence of the Banach–Steinhaus theorem, which says that if L_1, L_2 are Frechet spaces, T is a locally compact topological space and $M : L_1 \times T \longrightarrow L_2$ is linear with respect to the first argument and continuous with respect to each argument separately, then M is continuous as a function of two variables).

Step 2. Let $K_1, \dots, K_s$ be fixed, $F^{(m)} \longrightarrow F$. Using formula (2) and simple estimates as in the proof of Theorem 1, one can easily see that $M_{\bar{K}} F^{(m)} \longrightarrow M_{\bar{K}} F$.

Step 3. Now suppose $F \in \mathcal{A}(\mathbb{C}^n)$ (respectively, $C^r(\mathbb{R}^n)$) is fixed, $K_i^{(m)} \longrightarrow K_i$ as $m \longrightarrow \infty$ for all $i = 1, \dots, s$. Let us choose $a_i^{(m)} \in K_i^{(m)}$, $a_i \in K_i$. Define

$$f_i(y) = (a_i,\, x) + \log \int_{K_i - a_i} \exp(x, y)\, d\mu_i(x),$$
$$f_i^{(m)}(y) = (a_i^{(m)},\, x) + \log \int_{K_i^{(m)} - a_i^{(m)}} \exp(x, y) d\mu_i^{(m)}(x),$$

where μ_i, $\mu_i^{(m)}$ are measures as in the discussion after Lemma 5 with K_i, $K_i^{(m)}$ instead of K. By (3), $M_{\bar{K}^{(m)}} F(\lambda_1, \dots, \lambda_s) =$

$$\sum_{j_1,\dots,j_n} \lambda_{j_1} \dots \lambda_{j_n} \int_{\mathbb{R}^n} F\Big(\sum \lambda_i \nabla f_i^{(m)}(y)\Big) D(H_{j_1}^{(m)}(y), \dots, H_{j_n}^{(m)}(y))\, dy$$

and $M_{\bar{K}} F(\lambda_1, \dots, \lambda_s) =$

$$\sum_{j_1,\dots,j_n} \lambda_{j_1} \dots \lambda_{j_n} \int_{\mathbb{R}^n} F\Big(\sum \lambda_i \nabla f_i(y)\Big) D(H_{j_1}(y), \dots, H_{j_n}(y))\, dy.$$

Since all K_i, $K_i^{(m)}$ are uniformly bounded, there exists a large Euclidean ball U containing all these sets. As in the proof of Theorem 1, if $\sum |\lambda_i| \le R$, then

$$\begin{aligned} |M_{\bar{K}^{(m)}} F(\lambda_1, \dots, \lambda_s)| &\le \sum_{j_1,\dots,j_n} R^n \max_{x \in R\cdot U} |F(x)| \cdot V(K_{j_1}^{(m)}, \dots, K_{j_n}^{(m)}) \\ &\le \big(\max_{x \in R\cdot U} |F(x)|\big) \cdot \Big(\sum_{j_1,\dots,j_n} R^n \operatorname{vol}(U)\Big) \\ &\le K(R) \cdot \max_{x \in R\cdot U} |F(x)|, \end{aligned}$$

where $K(R)$ is some constant depending on R.

The same estimate holds for $M_{\bar{K}} F$. Hence, for every $\varepsilon > 0$, one can choose a polynomial P_ε approximating F on the set $R\cdot U$, such that for all i, m, λ_i with $\sum |\lambda_i| \le R$ we have

$$|M_{\bar{K}^{(m)}} (F - P_\varepsilon)(\lambda_1, \dots, \lambda_s)| < \varepsilon, \tag{5}$$

$$|M_{\bar{K}} (F - P_\varepsilon)(\lambda_1, \dots, \lambda_s)| < \varepsilon. \tag{6}$$

But by Corollary 2, the degrees of $M_{\bar{K}^{(m)}} P_\varepsilon$ and $M_{\bar{K}} P_\varepsilon$ are independent of m. Obviously, by the definition $(*)$ in the Introduction, $M_{\bar{K}^{(m)}} P_\varepsilon$ converges to $M_{\bar{K}} P_\varepsilon$ uniformly on compact sets in the non-negative orthant $\mathbb{R}_+^s$. Hence because of the boundedness of their degrees, $M_{\bar{K}^{(m)}} P_\varepsilon \longrightarrow M_{\bar{K}} P_\varepsilon$ in $\mathbb{R}^s$ (respectively, $\mathbb{C}^s$). This and (5) and (6) imply that, for large m,

$$|M_{\bar{K}^{(m)}} F(\lambda_1, \dots, \lambda_s) - M_{\bar{K}} F(\lambda_1, \dots, \lambda_s)| < 3\varepsilon$$

whenever $\sum |\lambda_i| \le R$.

A similar argument can be applied to prove uniform convergence of partial derivatives of $M_{\bar{K}^{(m)}} F$ on compact sets. □

4. Polynomial Valuations

We are now going to present another proof of the Pukhlikov–Khovanskii Theorem. They introduced in [P-Kh] the notion of the polynomial valuation, generalizing the classical translation invariant and translation covariant valuations.

Let Λ be an additive subgroup of $\mathbb{R}^n$. Denote by $\mathcal{P}(\Lambda)$ the set of all polytopes with vertices in Λ. We will assume that $\operatorname{span}\Lambda = \mathbb{R}^n$.

Definition. (a) A function $\phi : \mathcal{P}(\Lambda) \longrightarrow \mathbb{R}$ is called a valuation, if for all $P_1, P_2 \in \mathcal{P}(\Lambda)$, such that $P_1 \cup P_2$ and $P_1 \cap P_2$ belong to $\mathcal{P}(\Lambda)$ we have

$$\phi(P_1 \cup P_2) + \phi(P_1 \cap P_2) = \phi(P_1) + \phi(P_2). \tag{7}$$

(b) The valuation ϕ is called fully additive if, for every finite family of polytopes $P_1, \dots, P_k$ in $\mathcal{P}(\Lambda)$ such that the intersection $\bigcap_{i\in\sigma} P_i$ over every nonempty subset $\sigma \subset \{1 \dots k\}$ and their union $\bigcup_{i=1}^k P_i$ lie in $\mathcal{P}(\Lambda)$, the following equation holds:

$$\phi\Big(\bigcup_{i=1}^k P_i\Big) = \sum_{\sigma\subset\{1,\dots,k\},\, \sigma\neq\varnothing} (-1)^{|\sigma|+1}\, \phi\Big(\bigcap_{i\in\sigma} P_i\Big), \tag{8}$$

where $|\sigma|$ is the cardinality of σ.

Obviously, for $k = 2$, (8) is equivalent to (7). We will consider only fully additive valuations; however it is true that, if $\Lambda = \mathbb{R}^n$, then every valuation on $\mathcal{P}(\Lambda)$ is fully additive (see [V], [P-S]). But it is not known to the author whether this holds in the general case. In the definitions (a) and (b) one can replace $\mathcal{P}(\Lambda)$ by the set of all convex compact sets $\mathcal{K}$. If ϕ is continuous with respect to the Hausdorff metric on $\mathcal{K}$, then (a) implies (b) [Gr].

(c) The valuation $\phi : \mathcal{P}(\Lambda) \longrightarrow \mathbb{R}$ is called polynomial of degree at most d, if for every fixed $K \in \mathcal{P}(\Lambda)$, $\phi(K + x)$ is a polynomial of degree at most d with respect to $x \in \Lambda$.

Examples. 1. Let μ be any signed locally finite measure on $\mathbb{R}^n$. Then $\phi(K) := \mu(K)$ is a fully additive valuation.

2. The mixed volume

$$\phi(K) = V(K[j],\, A_1, \dots, A_{n-j}),$$

where $K[j]$ means that K occurs j times, and A_l are fixed convex compact sets, is known to be a fully additive translation invariant continuous valuation.

3. Let $\Lambda = \mathbb{Z}^n \subset \mathbb{R}^n$ be an integer lattice, and let f be a polynomial of degree d. Then for $K \in \mathcal{P}(\Lambda)$,

$$\phi(K) := \sum_{x\in K\cap\mathbb{Z}^n} f(x)$$

is a fully additive polynomial valuation of degree d.

4. Let $\Lambda = \mathbb{Z}^n$, let Ω be a subset of $\mathbb{R}^n$, which is invariant with respect to translations to vectors in $\mathbb{Z}^n$, and let K and f be as in example 3. Then $\phi(K) := \int_{K\cap\Omega} f(x)\,dx$ is also a fully additive polynomial valuation of degree d.

For more information about valuations, especially those which are translation invariant and translation covariant, see the surveys [Mc-Sch] and [Mc2].

THEOREM 6. [P-Kh] *Let $\phi : \mathcal{P}(\Lambda) \longrightarrow \mathbb{R}$ be a fully additive polynomial valuation of degree d. Fix $K_1, \dots, K_s \in \mathcal{P}(\Lambda)$. Then $\phi\big(\sum_i \lambda_i K_i\big)$ is a polynomial in $\lambda_i \in \mathbb{Z}_+$ of degree at most $d+n$. Moreover, if $\mathbb{Q}\cdot\mathcal{P}(\Lambda) = \mathcal{P}(\Lambda)$, then it is a polynomial in $\lambda_i \in \mathbb{Q}_+$.*

REMARK. For translation invariant valuations this theorem was proved in [Mc1], and our proof uses some constructions of that work.

LEMMA 7. (Well known; see, e.g., [GKZ, p. 215].) *Let $P \subset \mathbb{R}^n$ be a polytope. Then there exists a family of k-simplices $\{S_\alpha\}_{\alpha\in I}$, $0 \le k \le n$, such that*
($\imath$) $P = \bigcup_{\alpha\in I} S_\alpha$;
($\imath\imath$) each vertex of each S_α is a vertex of P;
($\imath\imath\imath$) every two S_β and S_γ intersect in a common face;
($\imath v$) for all β and γ, $S_\beta \bigcap S_\gamma \in \{S_\alpha\}_{\alpha\in I}$.

LEMMA 8. *Let $K_1, \dots, K_s$ be polytopes in $\mathbb{R}^n$. Then for all $\lambda_i \ge 0$, $1 \le i \le s$, the set $K(\bar\lambda) := \sum_i \lambda_i K_i$ has a decomposition*

$$K(\bar\lambda) = \bigcup_{\alpha\in I} S_\alpha(\bar\lambda),$$

where $S_\alpha(\bar\lambda)$ are polytopes (not necessarily simplices) such that
($\imath$) they satisfy ($\imath$) – ($\imath v$) in Lemma 7;
($\imath\imath$) if for some $\bar\lambda^0$, $\lambda_i^0 > 0$, and $\alpha, \beta, \gamma \in I$, $S_\alpha(\bar\lambda^0) \cap S_\beta(\bar\lambda^0) = S_\gamma(\bar\lambda^0)$, then for all $\bar\lambda = (\lambda_i)$, $\lambda_i \ge 0$ we have $S_\alpha(\bar\lambda) \cap S_\beta(\bar\lambda) = S_\gamma(\bar\lambda)$;
($\imath\imath\imath$) each $S_\alpha(\bar\lambda)$ has the form

$$S_\alpha(\bar\lambda) = \sum_i \lambda_i S_{i,\alpha}$$

where $S_{i,\alpha}$ are simplices with vertices in K_i, independent of $\bar\lambda$ and $\dim S_\alpha(\bar\lambda) = \sum_i \dim(\lambda_i S_{i,\alpha})$ (note that $\dim(\lambda_i S_{i,\alpha}) = \dim S_{i,\alpha}$ for $\lambda_i > 0$).

PROOF. Because of the homogeneity it is sufficient to prove the lemma only for $\lambda_i \ge 0$, $\sum \lambda_i = 1$. Consider in $\mathbb{R}^s \oplus \mathbb{R}^n$ a convex polytope

$$P := \left\{ (\mu_1, \dots, \mu_s; x) \mid \mu_i \ge 0,\, \sum_{i=1}^s \mu_i = 1,\, x \in \sum_i \mu_i K_i \right\}$$

Now apply Lemma 7 to $P = \bigcup_\alpha S_\alpha$. Set

$$S_\alpha(\bar\lambda) := S_\alpha \cap \{(\mu_1, \dots, \mu_s; x) \mid \mu_i = \lambda_i \text{ for all } i\}\,.$$

One can easily check that $S_\alpha(\bar\lambda)$ satisfy all the properties in Lemma 8. $\square$

Lemma 9. *Let $P \in \mathcal{P}(\Lambda)$. Then $\phi(N\cdot P)$ is a polynomial in $N \in \mathbb{Z}_+$ of degree at most $d+n$.*

Proof. By Lemma 7, $P = \bigcup_{\alpha\in I} S_\alpha$, where the S_α are simplices. Hence

$$\phi(N\cdot P) = \sum_{\sigma\subset I,\, \sigma\neq\varnothing} (-1)^{|\sigma|-1}\phi\left(N\cdot\left(\bigcap_{\alpha\in\sigma} S_\alpha\right)\right)$$

But for fixed σ, there exists $\gamma \in I$ such that $\bigcap_{\alpha\in\sigma} S_\alpha = S_\gamma$. So we have to show that for every simplex $\Delta \in \mathcal{P}(\Lambda)$, $\phi(N\cdot\Delta)$ is a polynomial of degree at most $d+n$.

Fix Δ and write $k = \dim\Delta$. The proof will be by induction in k. If $k=0$, then $\Delta = \{v\}$ is a point and $\phi(N\cdot\{v\}) = \phi(\{0\}+Nv)$ is a polynomial of degree at most d by the definition of the polynomial valuation.

Let $k>0$. For simplicity of notation we will assume that $k=n$. In an appropriate coordinate system Δ has the form $\Delta = a+\tilde{\Delta}$, where $a\in\mathbb{R}^n$, $\tilde{\Delta} = \{(x_1,\dots,x_n)\mid 0\le x_1\le\dots\le x_n\le 1\}$. Thus

$$N\cdot\tilde{\Delta} = \{(x_1,\dots,x_n)\mid 0\le x_1\le\dots\le x_n\le N\}\,.$$

$N\cdot\tilde{\Delta}$ can be represented as a disjoint union

$$N\cdot\tilde{\Delta} = \bigcup_{z\in\mathbb{Z}^n\cap((N-1)\cdot\tilde{\Delta})}\left((z+\tilde{Q})\cap(N\cdot\tilde{\Delta})\right)\bigcup(N\cdot\Delta'), \tag{9}$$

where $\tilde{Q} := \{(x_1,\dots,x_n)\mid 0\le x_i<1 \text{ for all } i\}$ and

$$\Delta' = \{(x_1,\dots,x_n)\mid 0\le x_1\le\dots\le x_{n-1}\le x_n = 1\}\,.$$

Of course, $(z+\tilde{Q})\cap(N\cdot\tilde{\Delta})$ is not a compact polytope, so ϕ is not defined on it. But we can define ϕ on this set in the following natural way. First, for $\tau\subset\{1,\dots,n\}$, denote $F_\tau :=$

$$\{(x_1,\,\dots,\,x_n)\mid 0\le x_i\le 1 \text{ for all } i\in\{1,\dots,n\},\ \text{ and } x_j = 1 \text{ for all } j\in\tau\}$$

Clearly, F_τ is an $(n-|\tau|)$-dimensional face of the unit cube $[0\,,\,1]^n$. Now define

$$\phi\left((z+\tilde{Q})\cap(N\cdot\tilde{\Delta})\right) := \sum_{\tau\subset\{1,\dots,n\}}(-1)^{|\tau|}\phi\left((z+F_\tau)\cap(N\cdot\tilde{\Delta})\right).$$

Since in (9) we have a disjoint union,

$$\phi(N\cdot\Delta) = \phi(N\cdot a+N\cdot\Delta') + \sum_{z\in\mathbb{Z}^n\cap((N-1)\cdot\tilde{\Delta})}\phi\left(N\cdot a+(z+\tilde{Q})\cap N\cdot\tilde{\Delta}\right). \tag{10}$$

Every $z\in\mathbb{Z}^n\cap\left((N-1)\cdot\tilde{\Delta}\right)$ has the form $z=(z_i)_{i=1}^n$, where

$$z_1 = \dots = z_{j_1} < z_{j_1+1} = \dots = z_{j_2} < \dots < z_{j_{l-1}+1} = \dots = z_{j_l} \le N-1, \tag{11}$$

and $j_l = n$.

Set for $1 \le i \le j \le n$, $T_{i,j} :=$

$$\{(x_1, \dots, x_n) \mid 0 \le x_i \le x_{i+1} \le \dots \le x_j \le 1 \text{ and } x_l = 0 \text{ for } l < i \text{ or } l > j\}.$$

For a sequence $0 < j_1 < \dots < j_{l-1} < n$, denote (as in [Mc1]) $T_{j_1 \dots j_{l-1}} := T_{0 j_1} + \dots + T_{l-1,\, n}$. Now let $\tilde{T}_{j_1 \dots j_{l-1}} := T_{j_1 \dots j_{l-1}} \cap \tilde{Q}$. So if z belongs to $\mathbb{Z}^n \cap (N-1)\cdot\tilde{\Delta}$ and satisfies (11), then obviously $(z + \tilde{Q}) \cap N\cdot\tilde{\Delta} = z + \tilde{T}_{j_1 \dots j_{l-1}}$. Define $S_{j_1 \dots j_{l-1}}(N) = \{z \in \mathbb{Z}^n \mid z \text{ satisfies (11) }\}$. Then (10) can be rewritten:

$$\phi(N\cdot\Delta) = \phi(N\cdot a + N\cdot\Delta') + \sum_{0<j_1<\dots<j_{j-1}<n} \left(\sum_{z \in S_{j_1 \dots j_{l-1}}(N)} \phi(N\cdot a + z + \tilde{T}_{j_1 \dots j_{l-1}}) \right). \quad (12)$$

By the inductive hypothesis, $\phi(N\cdot a + N\cdot\Delta')$ is a polynomial in $N \in \mathbb{Z}_+$ of degree at most $d+n$. Now fix $0 < j_1 < \dots < j_{l-1} < n$. Then $\phi(x + \tilde{T}_{j_1 \dots j_{l-1}})$ is a polynomial in x of degree at most d, let us denote it $q(x)$. It is sufficient to show that $\sum_{z \in S_{j_1 \dots j_{l-1}}(N)} q(N\cdot a + z)$ is a polynomial in $N \in \mathbb{Z}_+$ of degree at most $d+n$.

We can write $q(N\cdot a + z) = \sum_{t=0}^{d} N^t q_t(z)$, where $q_t(z)$ is a polynomial of degree at most $d-t$. Recall that for any $z \in S_{j_1 \dots j_{l-1}}(N)$ and $m = 1, \dots l-1$, $z_{j_{m-1}+1} = \dots = z_{j_m}$. So set $w_m := z_{j_m}$. We have $0 \le w_1 < w_2 < \dots < w_l \le N-1$. Actually, $q_t(z)$ is a polynomial in the vector $w = (w_1, \dots, w_l) \in \mathbb{R}^l$. We will show that

$$f(N) := \sum_{0 \le w_1 < w_2 < \dots < w_l \le N-1} q_t(w)$$

is a polynomial in $N \in \mathbb{Z}_+$ of degree at most $deg\, q_t + l$ (note that, if $N \le l-1$ the sum is extended over an empty set and for such an N, we just define $f(N) := 0$). This and (12) will imply that $\phi(N\cdot\Delta)$ is a polynomial of degree at most $d+n$.

In order to prove that $f(N)$ is a polynomial of degree g, it is sufficient to show that $f(N+1) - f(N)$ is a polynomial of degree $g-1$.

Let us apply induction in l. If $l = 1$,

$$f(N+1) - f(N) = q_t(N) \text{ for } N \ge 0, \quad (13)$$

and the lemma follows.

Assume that $l > 0$. We have

$$f(N+1) - f(N) = \sum_{0 \le w_1 < \dots < w_{l-1} < w_l = N} q_t(w).$$

We may assume q_t to be a monomial $q_t(w) = w_1^{\alpha_1} \dots w_l^{\alpha_l}$, $\alpha_j \ge 0$. Hence

$$f(N+1) - f(N) = N^{\alpha_l} \cdot \sum_{0 \le w_1 < \dots < w_{l-1} \le N-1} w_1^{\alpha_1} \dots w_{l-1}^{\alpha_{l-1}}.$$

By the inductive hypothesis, the last sum is a polynomial of degree at most $l - 1 + \sum_1^{n-1} \alpha_j$. Hence $f(N)$ is a polynomial of degree at most $l + \sum_1^n \alpha_j$. $\square$

Proof of Theorem 6. Using the same notation as previously, we have to show that $\phi(K(\bar\lambda))$ is a polynomial in $\lambda_i \in \mathbb{Z}_+$ of degree at most $d+n$. By Lemma 8 and the full additivity of ϕ,

$$\phi(K(\bar\lambda)) = \phi\bigg(\bigcup_{\alpha\in I} S_\alpha(\bar\lambda)\bigg) = \sum_{\sigma\subset I,\,\sigma\neq\varnothing} (-1)^{|\sigma|+1}\phi\bigg(\bigcap_{\beta\in\sigma} S_\beta(\bar\lambda)\bigg).$$

Fix some $\sigma \subset I, \sigma \neq \varnothing$. By Lemma 8 (ii) there exists $\gamma \in I$, such that $\bigcap_{\beta\in\sigma} S_\beta(\bar\lambda) = S_\gamma(\bar\lambda)$ for every vector $\bar\lambda$ with nonnegative coordinates.

So it is sufficient to show that for any γ, $\phi(S_\gamma(\bar\lambda))$ is a polynomial. But $S_\gamma(\bar\lambda) = \sum_{i=1}^s \lambda_i \cdot S_{\gamma,i}$ as in Lemma 8 (iii).

Suppose that for $1 \leq i \leq p$, $\dim S_{\gamma,i} > 0$ and for $i > p$, $\dim S_{\gamma,i} = 0$, i.e. $S_{\gamma,i} = \{v_i\}$ is a point for $i > p$.

Define $\Delta_i := S_{\gamma,i} - v_{\gamma,i}$, where $v_{\gamma,i}$ is some vertex of $S_{\gamma,i}$. So $S_\gamma(\bar\lambda) = \sum_{i=1}^p \lambda_i\Delta_i + \sum_{i=1}^p \lambda_i v_{\gamma,i} + \sum_{i>p} \lambda_i u_i$. By Lemma 8 (iii),

$$\dim S_\gamma(\bar\lambda) = \sum_i \dim(\lambda_i\Delta_i).$$

This implies that $\sum \lambda_i\Delta_i$ is, in fact, a direct sum of the $\lambda_i\Delta_i$. So we have to check that

$$\phi\bigg(\bigoplus_{i=1}^s(\lambda_i\Delta_i) + \sum_{j=1}^l \mu_j u_j\bigg)$$

is a polynomial in λ_i, $\mu_j \in \mathbb{Z}_+$ of degree at most $d+n$, where the u_j are fixed integer vectors.

Let $L_1 = \bigoplus_{j=1}^{s-1} \operatorname{span}\Delta_j$, $L_2 = \operatorname{span}\Delta_s$. For any polytopes K_1, K_2, $K_i \subset L_i$, $i = 1, 2$, consider the polynomial $\phi(K_1 \oplus K_2 + x)$, which we will denote by $W_{K_1\oplus K_2}(x)$. Obviously, all the previous definitions of the valuation and the polynomial valuation can be formulated not only for the real valued functions on $\mathcal{P}(\Lambda)$, but also for the vector valued functions with values in a linear space (and even in an abelian semigroup). The proofs of all the previous lemmas of Section 4 will work without any change.

Then obviously $W_{K_1\oplus K_2}(x)$ is a fully additive polynomial valuation with respect to each argument K_1 and K_2, with values in the linear space of polynomials in x (here we use the fact that the sum of K_1 and K_2 is direct). Hence by Lemma 9 (applied in the vector valued case), $W_{K_1\oplus N\cdot K_2}(x)$ is a polynomial in N (at the moment we are not interested in its degree). In particular, this implies that $\phi(K_1 \oplus N\cdot K_2 + x)$ is a polynomial in N and x, where $N \in \mathbb{Z}_+$, $x \in \Lambda$. Then obviously if we decompose $W_{K_1\oplus N\cdot K_2}$ with respect to the powers in N, then its coefficients will be polynomial valued fully additive polynomial valuations with respect to K_1 (now K_2 is fixed). Applying an inductive argument in s, we see that

$$\phi\bigg(\bigoplus_{i=1}^s(\lambda_i\Delta_i) + \sum_{j=1}^l \mu_j u_j\bigg) \tag{14}$$

is a polynomial in λ_i, $\mu_j \in \mathbb{Z}_+$.

Let us estimate its degree. If λ_i and μ_j are fixed,

$$\phi\Bigg(\bigoplus_{i=1}^{s} (t\cdot\lambda_i\Delta_i) + \sum_{j=1}^{l} t\cdot\mu_j u_j \Bigg)$$

is a polynomial in $t \in \mathbb{Z}_+$ of degree at most $d+n$ by Lemma 9. Hence the degree of (14) cannot be bigger than $d+n$.

Now consider the case $\mathbb{Q}\cdot\mathcal{P}(\Lambda) = \mathcal{P}(\Lambda)$. Let $K_1,\dots,K_s \in \mathcal{P}(\Lambda)$. For any natural number m, $\phi\big(\sum_{i=1}^{s}\lambda_i(\frac{1}{m}K_i)\big)$ is a polynomial in $\lambda_i \in \mathbb{Z}_+$, hence $\phi\big(\sum_{i=1}^{s}\lambda_i K_i\big)$ is a polynomial in $\lambda_i \in \frac{1}{m}\cdot\mathbb{Z}_+$ for any $m \in \mathbb{N}$. Consequently, it is a polynomial in $\lambda_i \in \mathbb{Q}_+$. $\square$

Remarks. 1. The valuation ϕ can be defined not only on polytopes, but on the family of all convex compact sets. If ϕ is continuous with respect to the Hausdorff metric, then it is called a continuous valuation (this implies its full additivity, see [Gr]). If a continuous valuation is polynomial of degree at most d, then for all convex compact sets $K_1,\dots,K_s$, the function $\phi\big(\sum_i \lambda_i K_i\big)$ is a polynomial in $\lambda_i \in \mathbb{R}_+$ of degree at most $d+n$. This can be deduced immediately from Theorem 6 using approximation by polytopes.

2. We would like to recall here some results in the same spirit due to Khovanskii [Kh1, Kh2].

Let A and B be finite subsets of an abelian semigroup G. Denote by $N * A$ the sum of N copies of the set A. Let $\chi : G \longrightarrow \mathbb{C}$ be a multiplicative character, i.e. $\chi(x+y) = \chi(x)\cdot\chi(y)$. Let $f(N)$ denote the sum of values of the character χ over all elements of the set $B + N * A$.

Theorem 10. [Kh2] *For sufficiently large N, the function $f(N)$ is a quasi-polynomial in N, i.e. for large N, the function $f(N) = \sum q_i^N P_i(N)$, where q_i are values of the character χ on the set A, and P_i are polynomials of degree strictly less than the number of points in A, in which the value of χ is equal to q_i.*

Now let A and B be finite subsets of an abelian group G. Denote by $G(A)$ the subgroup of the group G consisting of the elements of the form $\sum n_i a_i$, where $a_i \in G, n_i \in \mathbb{Z}$ and $\sum n_i = 0$. Now take $\chi \equiv 1$, then $f(N)$ is equal to the cardinality of the set $B + N * A$.

Theorem 11. [Kh1] *Let G be the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and assume that $G(A) = \mathbb{Z}^n$. Then for large N, the function $f(N)$ is a polynomial of degree at most n and the coefficient of N^n is equal to the volume of the convex hull of A.*

The methods of [Kh1] and [Kh2] in fact imply the following more general versions of these theorems:

Theorem 10′. *Let G and χ be as in Theorem 10, and let $B, A_1, \dots, A_s$ be finite subsets of G. Let $f(N_1, \dots, N_s)$ be the sum of values of the character χ over all the elements of the set $B + N_1 * A_1 + \dots + N_s * A_s$. Then if all the N_i, $1 \leq i \leq s$, are sufficiently large, $f(N_1, \dots, N_s)$ is a quasi-polynomial.*

Theorem 11′. *Let B, A_i, $1 \leq i \leq s$ be finite subsets of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and $G\big(\bigcup A_i\big) = \mathbb{Z}^n$. Then, if all the N_i, $1 \leq i \leq s$ are sufficiently large, the cardinality of $\sum N_i * A_i$ is a polynomial of degree at most n, whose homogeneous component of degree n is equal to the polynomial* $\operatorname{vol}(N_1 \cdot \operatorname{conv} A_1 + \dots + N_s \cdot \operatorname{conv} A_s)$.

As we were informed by Prof. Khovanskii, these facts were known to him (unpublished).

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