Holomorphic Spaces
MSRI Publications
Volume **33**, 1998

# Function Theory and Operator Theory on the Dirichlet Space

ZHIJIAN WU

Abstract. We discuss some recent achievements in function theory and operator theory on the Dirichlet space, paying particular attention to invariant subspaces, interpolation and Hankel operators.

## Introduction

In recent years the Dirichlet space has received a lot of attention from mathematicians in the areas of modern analysis, probability and statistical analysis. We intend to discuss some recent achievements in function theory and operator theory on the Dirichlet space. The key references are [Richter and Shields 1988; Richter and Sundberg 1992; Aleman 1992; Marshall and Sundberg 1993; Rochberg and Wu 1993; Wu 1993]. In this introductory section we state the basic results. Proofs will be discussed in the succeeding sections.

Denote by $\mathbb{D}$ the unit disk of the complex plane. For $\alpha \in \mathbb{R}$, the space $\mathcal{D}_\alpha$ consists of all analytic functions $f(z) = \sum_0^\infty a_n z^n$ defined on $\mathbb{D}$ with the norm

$$\|f\|_\alpha = \left( \sum_0^\infty (n+1)^\alpha |a_n|^2 \right)^{1/2}.$$

For $\alpha = -1$ one has $\mathcal{D}_{-1} = \mathcal{B}$, the Bergman space; for $\alpha = 0$, $\mathcal{D}_0 = \mathcal{H}^2$, the Hardy space; and for $\alpha = 1$, $\mathcal{D}_1 = \mathcal{D}$, the Dirichlet space. The space $\mathcal{D}_\alpha$ is referred to as a weighted Dirichlet space if $\alpha > 0$, and a weighted Bergman space if $\alpha < 0$. It is trivial that $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$ if $\alpha > \beta$. In particular, the Dirichlet space is contained in the Hardy space.

For any $w \in \mathbb{D}$, the point evaluation at $w$ is a bounded linear functional on $\mathcal{D}$. Therefore there is a corresponding reproducing kernel. It is given by

$$k_w(z) = k(z,w) = \frac{1}{\bar{w}z} \log \frac{1}{1-\bar{w}z}.$$

Research supported in part by the NSF Grant DMS-9622890.

Let

$$D(f) = \int_{\mathbb{D}} \left|f'(z)\right|^2 \, dA(z) = \sum_{n=1}^{\infty} n\,|a_n|^2,$$

where $dA(z) = (1/\pi)\,dx\,dy$ is normalized Lebesgue measure on $\mathbb{D}$. The square of the norm of a function $f$ in $\mathcal{D}$ can be also expressed as

$$\|f\|^2 = \|f\|_1^2 = \|f\|_0^2 + D(f).$$

Arazy, Fisher and Peetre proved in [Arazy et al. 1988] that the number $\sqrt{D(f)}$ is the Hilbert–Schmidt norm of the big Hankel operator on the Bergman space (introduced first in [Axler 1986]) with the analytic symbol $f$. Arazy and Fisher [1985] proved that the Dirichlet space $\mathcal{D}$ with the norm $\|\cdot\| = \sqrt{D(\cdot)}$ is the unique Möbius invariant Hilbert space on $\mathbb{D}$. A more general result in [Arazy et al. 1990] implies that for any Bergman type space $\mathcal{B}(\nu) = \mathcal{L}^2(\mathbb{D}, \nu) \cap \{\text{analytic functions on } \mathbb{D}\}$, which has a reproducing kernel $k_\nu(z, w)$, one has the formula

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |f(z) - f(w)|^2 |k_\nu(z, w)|^2 \, d\nu(z)\, d\nu(w) = D(f) \quad \text{for all } f \in \mathcal{D}.$$

The operator $M_z$ of multiplication by $z$ on the Dirichlet space, denoted sometimes by $(M_z, \mathcal{D})$, is a bounded linear operator. Moreover it is an analytic 2-isometry; that is,

$$\left\|M_z^2 f\right\|^2 - 2\left\|M_z f\right\|^2 + \|f\|^2 = 0 \quad \text{for all } f \in \mathcal{D}, \quad \text{and} \quad \bigcap_{n=0}^{\infty} M_z^n \mathcal{D} = \{0\}.$$

Richter [1991] proved that every cyclic analytic 2-isometry can be represented as multiplication by $z$ on a Dirichlet-type space $\mathcal{D}(\mu)$ with the norm $\|\cdot\|_\mu$ defined by

$$\|f\|_\mu^2 = \|f\|_0^2 + D_\mu(f).$$

Here $\mu$ is a nonnegative finite Borel measure on the unit circle $\partial\mathbb{D}$; the number $D_\mu(f)$ is defined by

$$D_\mu(f) = \int_{\mathbb{D}} |f'(z)|^2 h_\mu(z)\, dA(z),$$

where $h_\mu(z)$ is the harmonic extension of the measure $\mu$ to $\mathbb{D}$, defined as the integral of the Poisson kernel $P_z(e^{i\theta}) = (1 - |z|^2)/|e^{i\theta} - z|^2$ against $d\mu$:

$$h_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(e^{i\theta})\, d\mu(\theta).$$

It is not hard to see that $D_\mu(f) < \infty$ implies $f \in \mathcal{H}^2$. Therefore $\mathcal{D}(\mu) \subseteq \mathcal{H}^2$. Deep results involving $\mathcal{D}(\mu)$ can be found in [Richter and Sundberg 1991].

A closed subspace $\mathcal{N}$ of $\mathcal{D}$ is called invariant if $M_z$ maps $\mathcal{N}$ into itself. We shall discuss the following two theorems for invariant subspaces.

THEOREM 0.1. *Let* $\mathcal{N} \neq \{0\}$ *be an invariant subspace for* $(M_z, \mathcal{D})$. *Then*

$$\dim(\mathcal{N} \ominus z\mathcal{N}) = 1,$$

*that is*, $z\mathcal{N}$ *is a closed subspace of* $\mathcal{N}$ *of codimension one.*

For $f \in \mathcal{D}$, denote by $[f]$ the smallest invariant subspace of $\mathcal{D}$ containing $f$. An analytic function $\varphi$ defined on $\mathbb{D}$ is called a multiplier of $\mathcal{D}$ if $\varphi\mathcal{D} \subseteq \mathcal{D}$. The multiplier norm of $\varphi$ is defined by

$$\|\varphi\|_{\mathcal{M}} = \sup\{\|\varphi f\| : f \in \mathcal{D},\ \|f\| = 1\}.$$

THEOREM 0.2. *Every nonzero invariant subspace* $\mathcal{N}$ *of* $(M_z, \mathcal{D})$ *has the form*

$$\mathcal{N} = [\varphi] = \varphi\mathcal{D}(m_\varphi),$$

*where* $\varphi \in \mathcal{N} \ominus z\mathcal{N}$ *is a multiplier of* $\mathcal{D}$, *and* $dm_\varphi = |\varphi(e^{i\theta})|^2\, d\theta/2\pi$.

The codimension-one property for invariant subspaces of the Dirichlet space was first proved in [Richter and Shields 1988]. Another proof that works for the more general operators $(M_z, \mathcal{D}(\mu))$ and gives more information can be found in [Richter and Sundberg 1992]. Aleman [1992] generalized the argument in [Richter and Sundberg 1992] so that it works for the weighted Dirichlet spaces $\mathcal{D}_\alpha$ for $0 < \alpha \leq 1$ ($\alpha > 1$ is trivial). Recently, Aleman, Richter, and Ross provided another approach to the codimension-one property which is good for a large class of weighted Dirichlet spaces and certain Banach spaces. Part of Theorem 0.2 was proved in [Richter 1991]. That the generator $\varphi$ is a multiplier of $\mathcal{D}$ was proved in [Richter and Sundberg 1992]. (The result there is for the operator $(M_z, \mathcal{D}(\mu))$).

Carleson [1958] proved that, for a disjoint sequence $\{z_n\} \subset \mathbb{D}$, the interpolation problem

$$\varphi(z_n) = w_n, \quad \text{for } n = 1, 2, 3, \dots \tag{0–1}$$

has a solution $\varphi \in \mathcal{H}^\infty$ for every given $\{w_n\} \in \ell^\infty$ if and only if there are a $\delta > 0$ and a $C < \infty$ such that

$$\left|\frac{z_n - z_m}{1 - \bar{z}_n z_m}\right| \geq \delta \quad \text{for all} \quad n \neq m$$

and

$$\sum_{z_n \in S(I)} (1 - |z_n|^2) \leq C\,|I| \quad \text{for all arcs } I \subset \partial\mathbb{D}.$$

Here $|I|$ is the arc length of $I$ and $S(I)$ is the Carleson square based on $I$, defined as

$$S(I) = \bigl\{z \in \mathbb{D} : z/|z| \in I \ \text{ and } |z| > 1 - |I|/2\pi\bigr\}.$$

Let $\mathcal{M}_{\mathcal{D}}$ denote the space of multipliers of $\mathcal{D}$. It is clear that $\mathcal{M}_{\mathcal{D}}$ is an algebra and $\mathcal{M}_{\mathcal{D}} \subset \mathcal{H}^\infty$. We remark that $\mathcal{H}^\infty$ is in fact the space of all multipliers of $\mathcal{H}^2$.

A sequence $\{z_n\} \subset \mathbb{D}$ is called an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ if, for each bounded sequence of complex numbers $\{w_n\}$, the interpolation problem (0–1) has a solution $\varphi$ in $\mathcal{M}_{\mathcal{D}}$. By the closed graph theorem, we know that if $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ then there is a constant $C < \infty$ so that the interpolation can be done with a function $\varphi \in \mathcal{M}_{\mathcal{D}}$ satisfying $\|\varphi\|_{\mathcal{M}} \le C \,\|\{w_n\}\|_{\ell^\infty}$.

Axler [1992] proved that any sequence $\{z_n\} \subset \mathbb{D}$ with $|z_n| \to 1$ contains a subsequence that is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$. Marshall and Sundberg [1993] gave the following necessary and sufficient conditions for an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$.

THEOREM 0.3. *A sequence $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ if and only if there exist a $\gamma > 0$ and a $C_0 < \infty$ such that*

$$1 - \left|\frac{z_n - z_m}{1 - \bar{z}_n z_m}\right|^2 \le (1 - |z_n|^2)^\gamma \quad \text{for all } n \ne m \tag{0–2}$$

*and*

$$\sum_{z_n \in \cup S(I_j)} \left(\log \frac{1}{1 - |z_n|^2}\right)^{-1} \le C\left(\log \frac{1}{\operatorname{Cap}(\bigcup I_j)}\right)^{-1}. \tag{0–3}$$

*Here $\{I_j\}$ is any finite collection of disjoint arcs on $\partial\mathbb{D}$.*

Bishop also proved this theorem independently. Sundberg told me that a similar result for $\mathcal{M}_{\mathcal{D}_\alpha}$ with $0 < \alpha < 1$ is also true.

Condition (0–3) is a geometric condition for a Carleson measure for $\mathcal{D}$. Carleson measures for $\mathcal{D}_\alpha$ were first characterized by Stegenga [1980]. His result says that a nonnegative measure $\mu$ on $\mathbb{D}$ satisfies

$$\int_{\mathbb{D}} |g(z)|^2 \, d\mu(z) \le C \, \|g\|_\alpha^2 \quad \text{for all } g \in \mathcal{D}_\alpha$$

(in other words, is a Carleson measure for $\mathcal{D}_\alpha$) if and only if

$$\mu\Big(\bigcup S(I_j)\Big) \le C \operatorname{Cap}_\alpha\Big(\bigcup I_j\Big)$$

for any finite collection of disjoint arcs $\{I_j\}$ on $\partial\mathbb{D}$. Here $\operatorname{Cap}_\alpha(\cdot)$ denotes an appropriate Bessel capacity depending on $\alpha$. When $\alpha = 1$, the usual logarithmic capacity may be used for $\operatorname{Cap}_1(\cdot)$. Using Stegenga's theorem, condition (0–3) can be replaced by

$$\sum \left(\log \frac{1}{1 - |z_n|^2}\right)^{-1} \delta_{z_n} \quad \text{is a Carleson measure for } \mathcal{D}. \tag{0–4}$$

Hankel operators (small and big) on the Hardy and the Bergman spaces have been studied intensively in the past fifteen years. We refer the reader to [Luecking 1992; Peller 1982; Rochberg 1985; Zhu 1990] and references therein for more information. Denote by $P_{\mathcal{H}^2}$ the orthogonal projection from $\mathcal{L}^2(\partial\mathbb{D})$ onto $\mathcal{H}^2$. On the Hardy space, the Hankel operator with symbol $b \in \mathcal{L}^2(\partial\mathbb{D})$ can be written as

$$(I - P_{\mathcal{H}^2})(\bar{b}g) \quad \text{for } g \in \mathcal{H}^\infty.$$

Another existing definition is

$$\overline{P_{\mathcal{H}^2}(b\bar{g})} \quad \text{for } g \in \mathcal{H}^\infty.$$

In fact, if $\{\bar{b}_n\}$ is the sequence of Fourier coefficients of $b$, the two operators above correspond to the Hankel matrices $\{b_1, b_2, b_3, \dots\}$ and $\{b_0, b_1, b_2, \dots\}$, respectively. In more general spaces, these expressions define two different operators, called big Hankel and small Hankel, respectively. One basic question in the study of Hankel operators is that of understanding the "size" of the operators (for example boundedness or compactness) via the "smoothness" of their symbols.

We can view the Dirichlet space as a subspace of the Sobolev space $\mathcal{L}^{2,1}(\mathbb{D})$, defined as the completion of $C^1(\mathbb{D})$ under the norm

$$\|f\| = \left\{ \left| \int_{\mathbb{D}} f\, dA \right|^2 + \int_{\mathbb{D}} \left( |\partial_z f|^2 + |\partial_{\bar{z}} f|^2 \right) dA \right\}^{1/2}.$$

Note that the restriction of this norm to $\mathcal{D}$, which yields $\|f\|^2 = |f(0)|^2 + D(f)$ if $f \in \mathcal{D}$, is different from but equivalent to the norm of $\mathcal{D}$ introduced previously. Denote by $\mathcal{P}$ the set of all polynomials in $z$ on $\mathbb{D}$. Clearly, $\mathcal{P}$ is dense in $\mathcal{D}_\alpha$. Let $P_{\mathcal{D}}$ be the orthogonal projection from $\mathcal{L}^{2,1}(\mathbb{D})$ onto $\mathcal{D}$. On the Dirichlet space, the small Hankel operator with symbol $b$ is defined densely by

$$\mathcal{H}_b(g) = \overline{P_{\mathcal{D}}(b\bar{g})} \quad \text{for } g \in \mathcal{P}.$$

It turns out that the big Hankel operator on the Dirichlet space with an analytic symbol is easy to study; this is opposite to the situation on the Bergman space. Therefore we discuss here only the small Hankel operator. The following two theorems can be found in [Rochberg and Wu 1993; Wu 1993], where they are proved for weighted Dirichlet and Bergman spaces.

THEOREM 0.4. *Suppose $b$ is analytic on $\mathbb{D}$.*

(a) *The Hankel operator $\mathcal{H}_b$ is bounded on $\mathcal{D}$ if and only if $|b'(z)|^2\, dA(z)$ is a Carleson measure for $\mathcal{D}$.*
(b) *The Hankel operator $\mathcal{H}_b$ is compact on $\mathcal{D}$ if and only if $|b'(z)|^2\, dA(z)$ is a Carleson measure for $\mathcal{D}$ and satisfies*

$$\int_{\bigcup S(I_j)} |b'(z)|^2\, dA(z) = o\left( \left\{ \log \frac{1}{\operatorname{Cap}\left(\bigcup I_j\right)} \right\}^{-1} \right). \tag{0–5}$$

Denote by $\mathcal{W}_{\mathcal{D}}$ and $w_{\mathcal{D}}$, respectively, the sets of all analytic functions $b$ that satisfy conditions (a) and (b) in Theorem 0.4. Let $\mathcal{X}_{\mathcal{D}}$ be the set of all analytic functions $f$ on $\mathbb{D}$ that can be expressed as $f = \sum g_j h_j'$, where $g_j, h_j \in \mathcal{D}$ with $\sum \|g_j\|\, \|h_j\| < \infty$. Define the norm of $f$ in $\mathcal{X}_{\mathcal{D}}$ as

$$\|f\|_{\mathcal{X}_{\mathcal{D}}} = \inf \left\{ \sum \|g_j\|\, \|h_j\| : f = \sum g_j h_j' \quad \text{with } g_j, h_j \in \mathcal{D} \right\}.$$

THEOREM 0.5. *The dual of $\mathcal{X}_{\mathcal{D}}$ is $\mathcal{W}_{\mathcal{D}}$, realized by the pairing*

$$\langle f, b\rangle^* = \int_{\mathbb{D}} f(z)\overline{b'(z)}\, dA(z), \qquad \text{with } f \in \mathcal{X}_{\mathcal{D}} \text{ and } b \in \mathcal{W}_{\mathcal{D}};$$

*The dual of $w_{\mathcal{D}}$ is $\mathcal{X}_{\mathcal{D}}$, realized by the pairing*

$$^*\langle b, f\rangle = \int_{\mathbb{D}} b'(z)\overline{f(z)}\, dA(z), \qquad \text{with } b \in w_{\mathcal{D}} \text{ and } f \in \mathcal{X}_{\mathcal{D}}.$$

The form of the definition of $\mathcal{X}_{\mathcal{D}}$ is natural. For example, a result in [Coifman et al. 1976] suggests that the right way to look at a function $f$ in the Hardy space $\mathcal{H}^1$ (on the unit ball) is as $f = \sum f_j g_j$, where $f_j, g_j \in \mathcal{H}^2$, and

$$\|f\|_{\mathcal{H}^1} = \inf\left\{\sum \|f_j\|_{\mathcal{H}^2}\, \|g_j\|_{\mathcal{H}^2}\right\}.$$

Corresponding to the weighted Dirichlet spaces, one can define

$$\mathcal{W}_{\mathcal{D}_\alpha} = \{b \in \mathcal{D}_\alpha : g \mapsto b'g \text{ is bounded from } \mathcal{D}_\alpha \text{ to } \mathcal{D}_{\alpha-2}\}$$

and

$$w_{\mathcal{D}_\alpha} = \{b \in \mathcal{D}_\alpha : g \mapsto b'g \text{ is compact from } \mathcal{D}_\alpha \text{ to } \mathcal{D}_{\alpha-2}\}.$$

It is not hard to see that $\mathcal{W}_{\mathcal{H}^2} = \text{BMOA}$ and $w_{\mathcal{H}^2} = \text{VMOA}$. Therefore Theorem 0.5 is similar to the analytic versions of Fefferman's and Sarason's well-known theorems, saying that $(\mathcal{H}^1)^* = \text{BMOA}$ and $(\text{VMOA})^* = \mathcal{H}^1$.

NOTATION. In the rest of the paper the letter $C$ denotes a positive constant that many vary at each occurrence but is independent of the essential variables or quantities.

## 1. Invariant Subspaces

The codimension-one property for invariant subspaces of the Dirichlet space is related to the cellular indecomposibility of the operator $(M_z, \mathcal{D})$. This concept was first introduced and studied by Olin and Thomson [1984] for more general Hilbert spaces. Later Bourdon [1986] proved in that if the operator $M_z$ is cellular indecomposable on a Hilbert space $\mathcal{H}$ of analytic functions on $\mathbb{D}$ with certain properties, then every nonzero invariant subspace for $(M_z, \mathcal{H})$ has the codimension-one property. The required properties for $\mathcal{H}$ in Bourdon's paper are: the polynomials are dense in $\mathcal{H}$; the operator $M_z$ is a bounded linear operator on $\mathcal{H}$; if $zg \in \mathcal{H}$ and $g$ is analytic on $\mathbb{D}$ then $g \in \mathcal{H}$; and for each point $w \in \mathbb{D}$ the point evaluation at $w$ is a bounded linear functional on $\mathcal{H}$. Clearly our spaces satisfy these requirements. The operator $(M_z, \mathcal{H})$ is said to be cellular indecomposable if $\mathcal{N} \cap \mathcal{Q} \neq \{0\}$ for any two nonzero invariant subspaces $\mathcal{N}$ and $\mathcal{Q}$ of $(M_z, \mathcal{H})$. We note that $(M_z, \mathcal{H}^2)$ clearly is cellular indecomposable by Beurling's Theorem. However, from an example constructed by Horowitz [1974], we know that $(M_z, \mathcal{B})$ is not cellular indecomposable. We also know that $(M_z, \mathcal{B})$

does not have the codimension-one property; see, for example, [Bercovici et al. 1985] or [Seip 1993].

To prove Theorem 0.1, we show that the operator $(M_z, \mathcal{D})$ is cellular indecomposable. This is a consequence of the following result.

THEOREM 1.1. *If $f \in \mathcal{D}$, then $f = \varphi/\psi$, where $\varphi$ and $\psi$ are in $\mathcal{D} \cap \mathcal{H}^\infty$.*

This was proved in [Richter and Shields 1988] for the Dirichlet spaces on connected domains, in [Richter and Sundberg 1992] for $\mathcal{D}(\mu)$, and in [Aleman 1992] for weighted Dirichlet spaces. We discuss the proof later.

Let $\mathcal{N} \neq \{0\}$ be an invariant subspace for $(M_z, \mathcal{D})$ and $f \in \mathcal{N} \setminus \{0\}$. By Theorem 1.1, there are functions $\varphi, \psi \in \mathcal{D} \cap \mathcal{H}^\infty$ such that $f = \varphi/\psi$. We claim that $\varphi = \psi f$ is in $[f] \subseteq \mathcal{N}$. Let $\psi_r(z) = \psi(rz)$, for $0 < r < 1$. It is clear that we only need to show $\psi_r f \to \varphi$ in $\mathcal{D}$. Straightforward estimates show that

$$\begin{aligned}\|\psi_r f - \varphi\| &\le \|\psi_r(f - f_r)\| + \|\varphi_r - \varphi\| \\ &\le \|\psi_r'(f - f_r)\|_{\mathcal{B}} + \|\psi\|_{\mathcal{H}^\infty} \|f - f_r\| + \|\varphi_r - \varphi\|.\end{aligned}$$

It suffices to show the first term on the right goes to zero as $r \to 1_-$. Recall that $\mathcal{D}$ is contained in the little Bloch space, which consists of all the analytic functions $g$ on $\mathbb{D}$ such that $(1-|z|^2)|g'(z)| = o(1)$ as $|z| \to 1_-$. We have therefore

$$(1 - |rz|^2)|\psi_r'(z)| \le C \quad \text{for all } |rz| < 1.$$

Write $f(z) = \sum_0^\infty a_n z^n$. Then

$$\left\| \frac{f - f_r}{1 - |rz|^2} \right\|_{\mathcal{B}}^2 \le \sum_0^\infty |a_n|^2 \int_0^1 \frac{t^{2n} - (rt)^{2n}}{(1 - r^2 t^2)^2} t\,dt \le C \sum_0^\infty n\, |a_n|^2.$$

This is enough for the claim. The discussion proves in fact the following result.

LEMMA 1.2. *Suppose $\psi \in \mathcal{D} \cap \mathcal{H}^\infty$ and $f \in \mathcal{D}$. If $\psi f \in \mathcal{D}$, then $\psi f \in [f]$.*

From Theorem 1.1, we see that there is always a nonzero bounded function $\varphi \in \mathcal{N}$ if $\mathcal{N}$ is a nonzero invariant subspace for $(M_z, \mathcal{D})$. If $\mathcal{N}$ and $\mathcal{Q}$ are two nonzero invariant subspaces for $(M_z, \mathcal{D})$, there are nonzero bounded functions $\varphi \in \mathcal{N}$ and $\psi \in \mathcal{Q}$. Clearly $\varphi\psi$ is also nonzero, bounded and in $\mathcal{D}$, therefore in both $\mathcal{N}$ and $\mathcal{Q}$ by Lemma 1.2. Thus $\mathcal{N} \cap \mathcal{Q} \neq \{0\}$; that is, $(M_z, \mathcal{D})$ is cellular indecomposable.

To prove Theorem 1.1, we need several lemmas.

LEMMA 1.3. *Suppose $a_n, b_n \ge 0$, for $n = 0, 1, 2, \ldots$. If*

$$\sum b_n(1 - r^n) \le \sum a_n(1 - r^n)$$

*for every $r$ in the interval $(1-\delta, 1)$ with some $\delta \in (0, 1)$, then*

$$\sum n b_n \le \sum n a_n.$$

PROOF. Without loss of generality, we assume $\sum na_n < \infty$. Clearly we have

$$\sum_{n=1}^{N} nb_n = \lim_{r\to 1_-} \frac{1}{1-r}\sum_{n=1}^{N} b_n(1-r^n) \le \lim_{r\to 1_-} \frac{1}{1-r}\sum_{n=1}^{\infty} a_n(1-r^n) \le \sum_{n=1}^{\infty} na_n.$$

This is enough. □

Recall the inner-outer factorization for a nonzero function $f \in \mathcal{H}^2$: $f(z) = I(z)F(z)$, where the inner and outer factors $I(z)$ and $F(z)$ satisfy $|I(e^{i\theta})| = 1$ a.e. on $\partial\mathbb{D}$ and

$$F(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|d\theta\right) \quad \text{for all } z \in \mathbb{D}.$$

One application of Lemma 1.3 is the following.

COROLLARY 1.4. *Let $f \in \mathcal{D}$ and let $F$ be an analytic function on $\mathbb{D}$ with $|F(z)| \ge |f(z)|$ in $\mathbb{D}$ and $|F(e^{i\theta})| = |f(e^{i\theta})|$ a.e. on $\partial\mathbb{D}$. Then $F \in \mathcal{D}$ and $\|F\| \le \|f\|$. In particular this estimate is true if $F$ is the outer factor of $f \in \mathcal{D}$.*

PROOF. Write $f = \sum f_n z^n$ and $F = \sum F_n z^n$. We have clearly

$$\sum |f_n|^2 = \|f\|_0^2 = \|F\|_0^2 = \sum |F_n|^2 \quad \text{and}$$

$$\sum |f_n|^2 r^{2n} = \|f(r\,\cdot\,)\|_0^2 \le \|F(r\,\cdot\,)\|_0^2 = \sum |F_n|^2 r^{2n} \quad \text{for all } r \in (0,1).$$

Since $D(f) = \sum n\,|f_n|^2 < \infty$, applying Lemma 1.3 to the inequality

$$\sum |F_n|^2(1-r^{2n}) = \|F\|_0^2 - \|F(r\,\cdot\,)\|_0^2 \le \|f\|_0^2 - \|f(r\,\cdot\,)\|_0^2 = \sum |f_n|^2(1-r^{2n}),$$

we get $D(F) \le D(f) < \infty$. This yields $F \in \mathcal{D}$ and $\|F\| \le \|f\|$. □

Let $(X,\mu)$ be a probability space and let $g \in \mathcal{L}^1(X,\mu)$ be positive $\mu$-a.e. on $X$ and satisfy $\log g \in \mathcal{L}^1(X,\mu)$. Jensen's inequality says that $\exp\bigl(\int_X \log g\, d\mu\bigr) \le \int_X g\, d\mu$. Set $E(g) = \int_X g\, d\mu - \exp\bigl(\int_X \log g\, d\mu\bigr)$.

Applying Jensen's inequality, Aleman [1992] proved the following inequality.

LEMMA 1.5. *Suppose that $g$ is positive $\mu$-a.e. and that $\log g \in \mathcal{L}^1(X,\mu)$. Then*

$$E(\min\{g,1\}) \le E(g) \quad \text{and} \quad E(\max\{g,1\}) \le E(g).$$

Let $F \in \mathcal{H}^2$ be a outer function. Define the outer functions $F_-$ and $F_+$ by

$$F_-(z) = \exp\Bigl(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\bigl(\min\{|F(e^{i\theta})|,\, 1\}\bigr)\, d\theta\Bigr)$$

$$F_+(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\bigl(\max\{|F(e^{i\theta})|,\, 1\}\bigr)\, d\theta\right).$$

It is clear that $F_-$ and $1/F_+$ are in $\mathcal{H}^\infty$, with norms bounded by 1, and that

$$F_-(z)F_+(z) = F(z), \qquad |F_-(z)| \le |F(z)|.$$

THEOREM 1.6. *Let $f \in \mathcal{H}^2$ and let $f = IF$ be the inner-outer factorization. If $f \in \mathcal{D}$, then $D(F_+) \le D(f)$ and $D(IF_-) \le D(f)$. Moreover $F_+$, $IF_-$, and $1/F_+$ are in $\mathcal{D}$, with*

$$\|F_+\| \le 1 + \|f\|, \qquad \|1/F_+\| \le 1 + \|f\|, \qquad \|IF_-\| \le \|f\|.$$

PROOF. We note first that $\|IF_-\|_0 \le \|f\|_0$. Using the fact $|1/F_+(z)| \le 1$, we get

$$|(1/F_+(z))'| = \left|F_+'(z)\right| / \left|F_+(z)\right|^2 \le \left|F_+'(z)\right|.$$

This implies $D(1/F_+) \le D(F_+)$. Therefore we only need to show $D(F_+) \le D(f)$ and $D(IF_-)$.

Applying Lemma 1.5 with $X = [0, 2\pi]$, $d\mu = 1/(2\pi) P_z(e^{i\theta})\, d\theta$, and $g(\theta) = |F(e^{i\theta})|^2$, we have

$$\int_0^{2\pi} P_z(e^{i\theta}) \left|F_\pm(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} - \left|F_\pm(z)\right|^2 \le \int_0^{2\pi} P_z(e^{i\theta}) \left|F(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} - \left|F(z)\right|^2. \tag{1–1}$$

Integrating both sides of the inequality for $F_+$ over $|z| = r \in (0, 1)$ with respect to the measure $d\theta/2\pi$, we obtain

$$\|F_+\|_0^2 - \left\|F_+(r\,\cdot\,)\right\|_0^2 \le \|F\|_0^2 - \left\|F(r\,\cdot\,)\right\|_0^2.$$

Applying Lemma 1.3 (as in the proof of Corollary 1.4), we obtain $D(F_+) \le D(F)$. Corollary 1.4 yields therefore $D(F_+) \le D(f)$.

Since $|F_-(z)| \le |F(z)|$, we have

$$\left(1 - |I(z)|^2\right) \left|F_-(z)\right|^2 \le \left|F(z)\right|^2 - \left|f(z)\right|^2.$$

Adding this inequality to (1–1) for $F_-$, we get

$$\int_0^{2\pi} P_z(e^{i\theta}) \left|I(e^{i\theta})F_-(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} - \left|I(z)F_-(z)\right|^2 \le \int_0^{2\pi} P_z(e^{i\theta}) \left|f(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} - \left|f(z)\right|^2.$$

Reasoning as above, we obtain $D(IF_-) \le D(f)$. □

REMARK 1.7. By Theorem 1.6, together with Lemma 1.2 and the identity $(1/F_+)f = IF_-$, we have $IF_- \in [f]$ if $f \in \mathcal{D}$.

PROOF. Proof of Theorem 1.1 Assume $f \ne 0$ and $f \in \mathcal{D}$. Let $f = IF$ be the inner-outer factorization. Since $F_-F_+ = F$, we have

$$f = \frac{IF_-}{1/F_+}.$$

Since $IF_-$, $1/F_+ \in \mathcal{H}^\infty$ and $IF_-$, $1/F_+ \in \mathcal{D}$ by Theorem 1.6, we obtain the desired decomposition by letting $\varphi = IF_-$ and $\psi = 1/F_+$. □

PROOF OF THEOREM 0.2. Let $\varphi \in \mathcal{N} \ominus z\mathcal{N}$ and $\|\varphi\| = 1$. Note that the polynomials are dense in $\mathcal{D}(m_\varphi)$. By the codimension-one property, the polynomial multiples of $\varphi$ are dense in $\mathcal{N}$ (see also [Richter 1988]). Thus, to see why $\mathcal{N} = [\varphi] = \varphi\mathcal{D}(m_\varphi)$, it is enough to show that $\|\varphi p\| = \|p\|_{m_\varphi}$ for every polynomial $p$. One can compute this directly by using the fact that $(M_z, \mathcal{D})$ is an analytic 2-isometry; see [Richter 1991] for details. To see why the function $\varphi$ is a multiplier of $\mathcal{D}$, note that $\mathcal{D} \subseteq \mathcal{D}(m_\varphi)$ if $\varphi$ is bounded. Hence $\varphi\mathcal{D} \subseteq \varphi\mathcal{D}(m_\varphi) = \mathcal{N} \subseteq \mathcal{D}$, and

$$\begin{aligned}\|\varphi f\| = \|f\|_{m_\varphi} &= \big(\|f\|_0^2 + D_{m_\varphi}(f)\big)^{1/2} \\ &\le \big(\|f\|_0 + \|\varphi\|^2_{\mathcal{H}^\infty}\, D(f)\big)^{1/2} \le \max\{1,\, \|\varphi\|_{\mathcal{H}^\infty}\}\, \|f\|\,.\end{aligned}$$

We show now $\varphi$ is bounded. Let $k$ be the order of the zero of $\varphi$ at the origin; thus $\varphi = z^k\psi$ with $\psi \in \mathcal{D}$ and $\psi(0) \ne 0$. It is well known that $\varphi = z^k\psi$ is a solution of the extremal problem

$$\inf\left\{\frac{\|z^k f\|}{|f(0)|} : z^k f \in \mathcal{N}\right\}. \tag{1–2}$$

We shall show that any unbounded function $z^k f \in \mathcal{N}$ with $f(0) \ne 0$ is not a solution of (1–2); that is, we can construct a bounded function $f_N$ so that $z^k f_N \in \mathcal{N}$ and $\|z^k f_N\| / |f_N(0)| < \|z^k f\| / |f(0)|$, or, equivalently,

$$\|z^k f\|^2 \left(1 - \frac{|f_N(0)|^2}{|f(0)|^2}\right) < \|z^k f\|^2 - \|z^k f_N\|^2 . \tag{1–3}$$

Let $f = IF$ be the inner-outer factorization of $f$. Consider $F_N = N(F/N)_-$, the outer function defined by

$$F_N(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\big(\min\{|F(e^{i\theta})|,\, N\}\big)\,d\theta\right).$$

Let $f_N = IF_N$. Then $|f_N(z)| \le N$ for every $z \in \mathbb{D}$ and $\|f_N\| \le \|f\|$ by Theorem 1.6; and $z^k f_N = (z^k f)_N \in [z^k f] \subseteq \mathcal{N}$ by Remark 1.7. Straightforward computation yields

$$1 - \frac{|f_N(0)|^2}{|f(0)|^2} = 1 - \frac{|F_N(0)|^2}{|F(0)|^2} = 1 - \exp\left(-\frac{1}{\pi}\int_0^{2\pi} \log\frac{\max\{|f(e^{i\theta})|, N\}}{N}\,d\theta\right).$$

Using the inequalities $\log(1+x) \le x$ and $1 - e^{-x} \le x$ for $x \ge 0$, we obtain

$$\begin{aligned}1 - \frac{|f_N(0)|^2}{|f(0)|^2} &\le 1 - \exp\left(-\frac{1}{\pi}\int_0^{2\pi} \frac{\max\{|f(e^{i\theta})|,\, N\} - N}{N}\,d\theta\right) \\ &\le \frac{1}{N\pi}\int_0^{2\pi} \big(\max\{|f(e^{i\theta})|,\, N\} - N\big)\,d\theta.\end{aligned}$$

Since $D(z^k f_N) = D((z^k f)_N) \le D(z^k f)$ by Theorem 1.6, we have

$$\begin{aligned}\|z^k f\|^2 - \|z^k f_N\|^2 &= \|f\|_0^2 - \|f_N\|_0^2 + D(z^k f) - D(z^k f_N) \ge \|F\|_0^2 - \|F_N\|_0^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} \big(|F(e^{i\theta})|^2 - \min\{|F(e^{i\theta})|^2, N^2\}\big)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \big(\max\{|f(e^{i\theta})|^2, N^2\} - N^2\big)\, d\theta \\ &\ge \frac{N}{2\pi}\int_0^{2\pi} \big(\max\{|f(e^{i\theta})|, N\} - N\big)\, d\theta.\end{aligned}$$

These estimates show that, if $f$ is unbounded, inequality (1–3) holds for large enough $N$. □

A set $Z \subset \mathbb{D}$ is a zero set of a space $\mathcal{H}$ of functions on $\mathbb{D}$ if $Z = \{z \in \mathbb{D} : f(z) = 0\}$ for some $f \in \mathcal{H}$. Theorem 0.2 implies the following result [Marshall and Sundberg 1993].

THEOREM 1.8. *A set $Z \subset \mathbb{D}$ is a zero set of $\mathcal{D}$ if and only if it is a zero set of $\mathcal{M}_{\mathcal{D}}$.*

PROOF. It is trivial that a zero set of $\mathcal{M}_{\mathcal{D}}$ is a zero set of $\mathcal{D}$. Assume $Z$ is a zero set of $\mathcal{D}$. Consider the set of functions $\mathcal{Z} = \{f \in \mathcal{D} : f(Z) = 0\}$. It is clear that $\mathcal{Z}$ is a nonzero invariant subspace for $(M_z, \mathcal{D})$. By Theorem 0.2, we have a function $\varphi \in \mathcal{Z} \ominus z\mathcal{Z}$ so that $\mathcal{Z} = [\varphi]$ and $\varphi$ is a multiplier of $\mathcal{D}$. Clearly this $\varphi$ has zero set $Z$. □

## 2. Interpolation

The connection between reproducing kernels and interpolation can be explained in terms of the adjoints of multiplication operators. If $\varphi \in \mathcal{M}_{\mathcal{D}}$ then for $f \in \mathcal{D}$ and $\zeta \in \mathbb{D}$, we have

$$M_\varphi^*(k_\zeta)(z) = \langle M_\varphi^*(k_\zeta), k_z\rangle = \overline{\langle \varphi k_z, k_\zeta\rangle} = \overline{\varphi(\zeta)}\, k_\zeta(z).$$

Suppose $\{z_j\}_1^n$ is a finite sequence of distinct points in $\mathbb{D}$. Let $\varphi \in \mathcal{M}_{\mathcal{D}}$ satisfy $\|\varphi\|_{\mathcal{M}} \le 1$ and

$$\varphi(z_j) = w_j, \quad j = 1, 2, \dots, n. \tag{2–1}$$

Then for any finite sequence $\{a_j\}_1^n$ of complex numbers we have, by straightforward computation,

$$0 \le \Big\|\sum a_j k_{z_j}\Big\|^2 - \Big\|M_\varphi^*\Big(\sum a_j k_{z_j}\Big)\Big\|^2 = \sum (1 - w_j \overline{w}_k) k(z_j, z_k) a_k \bar{a}_j. \tag{2–2}$$

This shows that the positive semidefiniteness of the $n \times n$ matrix

$$\big\{(1 - w_j \overline{w}_k) k(z_j, z_k)\big\}$$

is a necessary condition for the interpolation problem (2–1) to have a solution in $\mathcal{M}_{\mathcal{D}}$. This argument in fact works in any reproducing kernel Hilbert space. Agler [1986] proved that for the Dirichlet space the necessary condition is also sufficient.

THEOREM 2.1. *If* $\{z_j\}_1^n \subset \mathbb{D}$ *and* $\{w_j\}_1^n \subset \mathbb{C}$ *satisfy*

$$\{(1 - w_j\bar{w}_k)k(z_j, z_k)\} \geq 0,$$

*there exists* $\varphi \in \mathcal{M}_{\mathcal{D}}$ *with* $\|\varphi\|_{\mathcal{M}} \leq 1$ *and* $\varphi(z_i) = w_i$ *for* $i = 1, 2, \ldots, n$.

One says that the Dirichlet space has the Pick property, because Pick first established such a theorem for interpolation in $\mathcal{H}^\infty$. Note that, if the Pick property holds, it also applies to countable sequences.

REMARK 2.2. For $n = 2$, inequality (2–2) becomes

$$\frac{|k(z_1, z_2)|^2}{\|k_{z_1}\|^2 \, \|k_{z_2}\|^2} \leq \frac{(1 - |w_1|^2)(1 - |w_2|^2)}{|1 - w_1\bar{w}_2|^2}. \tag{2–3}$$

A sequence of vectors $\{x_n\}$ in a Hilbert space $\mathcal{H}$ is called independent if $x_n \notin \operatorname{Span}\{x_k : k \neq n\}$ for all $n$. A sequence of unit vectors $\{u_n\}$ in a Hilbert space $\mathcal{H}$ is called an interpolating sequence for $\mathcal{H}$ if the map $x \mapsto \big\{\langle x, u_n\rangle\big\}$ maps $\mathcal{H}$ onto $\ell^2$. We cite the Köthe–Toeplitz theorem here, which can be found in [Nikolskii 1986].

THEOREM 2.3. *Let* $\{u_n\}$ *be a sequence of unit vectors contained in a Hilbert space* $\mathcal{H}$. *Let* $\mathcal{K}$ *be the smallest closed subspace of* $\mathcal{H}$ *containing* $\{u_n\}$. *Then the following statements are equivalent.*

(1) *The sequence* $\{u_n\}$ *is an interpolating sequence for* $\mathcal{H}$.
(2) *For all* $x \in \mathcal{K}$ *satisfying* $\|x\|^2 \asymp \sum \|\langle x, u_n\rangle\|$, *and* $\{u_n\}$ *is independent*.
(3) $\big\|\sum a_n u_n\big\|^2 \asymp \sum |a_n|^2$ *for all sequences* $\{a_n\}$.
(4) $\big\|\sum b_n u_n\big\| \leq C\big\|\sum a_n u_n\big\|$ *for all sequences* $\{a_n\}$ *and* $\{b_n\}$ *such that* $|b_n| \leq |a_n|$ *for all* $n$.

A unit vector sequence $\{u_n\}$ with property (3) above is called a Riesz sequence in $\mathcal{H}$, and one with property (4) is called an unconditional basic sequence in $\mathcal{H}$. The following result ties the Pick property and Köthe–Toeplitz theorem together.

THEOREM 2.4. *Let* $\{z_n\} \subset \mathbb{D}$ *and let* $\tilde{k}_n = k_{z_n} / \|k_{z_n}\|$. *The following statements are equivalent*:

(1) $\{z_n\}$ *is an interpolating sequence for* $\mathcal{M}_{\mathcal{D}}$.
(2) $\{\tilde{k}_n\}$ *is an interpolating sequence for* $\mathcal{D}$.
(3) $\{\tilde{k}_n\}$ *is an unconditional basic sequence in* $\mathcal{D}$.
(4) $\{\tilde{k}_n\}$ *is a Riesz sequence in* $\mathcal{D}$.

PROOF. Theorem 2.3 implies the equivalence of (2), (3), and (4). To prove that (1) implies (3), suppose $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $|b_n| \le |a_n|$ for all $n$. Then there is a $\varphi \in \mathcal{M}_{\mathcal{D}}$ with $\varphi(z_n) = \overline{b_n/a_n}$ for $n = 1, 2, \ldots$. Let $C = \|\varphi\|_{\mathcal{M}}$. We have

$$\begin{aligned} 0 &\le \sum \bigl(C^2 - \varphi(z_j)\overline{\varphi(z_k)}\bigr)\, k(z_j, z_k) \frac{\bar{a}_j}{\|k_{z_j}\|} \frac{a_k}{\|k_{z_k}\|} \\ &= C^2 \Bigl\| \sum a_j \tilde{k}_j \Bigr\|^2 - \Bigl\| M_\varphi^* \Bigl( \sum a_j \tilde{k}_j \Bigr) \Bigr\|^2 \\ &= C^2 \Bigl\| \sum a_j \tilde{k}_j \Bigr\|^2 - \Bigl\| \sum b_j \tilde{k}_j \Bigr\|^2 . \end{aligned}$$

This proves that $\{\tilde{k}_n\}$ is an unconditional basic sequence in $\mathcal{D}$.

Conversely, suppose (3) holds. To prove (1), by weak convergence of operators of the form $M_\varphi$, it suffices to show that each finite subsequence of $\{z_n\}$ is an interpolating sequence with the solutions in $\mathcal{M}_{\mathcal{D}}$ having uniformly bounded norms. This follows from the Pick property and the following inequality, which is equivalent to (3):

$$\sum \Bigl( C^2 - \frac{\bar{b}_j}{\bar{a}_j} \frac{b_k}{a_k} \Bigr) k(z_j, z_k) \frac{\bar{a}_j}{\|k_{z_j}\|} \frac{a_k}{\|k_{z_k}\|} = C^2 \Bigl\| \sum a_j \tilde{k}_j \Bigr\|^2 - \Bigl\| \sum b_j \tilde{k}_j \Bigr\|^2 \ge 0,$$

where $|b_j| \le |a_j|$ for all $j$. □

SKETCH OF PROOF OF THEOREM 0.3. The proof is found in [Marshall and Sundberg 1993]. Suppose $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$; then $\{\tilde{k}_n\}$ is an interpolating sequence for $\mathcal{D}$, by Theorem 2.4. This is equivalent to

$$\sum |f(z_n)|^2 \Bigl( \log \frac{1}{1 - |z_n|^2} \Bigr)^{-1} = \sum \bigl| \langle f, \tilde{k}_n \rangle \bigr|^2 \le C \, \|f\|^2 \quad \text{for all } f \in \mathcal{D}.$$

Thus (0–4) holds, and hence so does (0–3).

Since $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$, we can find, for any distinct $m$ and $n$, a $\varphi \in \mathcal{M}_{\mathcal{D}}$ so that $\varphi(z_n) = 1$, $\varphi(z_m) = 0$, and $\|\varphi\|_{\mathcal{M}} \le C$. By Remark 2.2 we have

$$\frac{\Bigl| \log \dfrac{1}{1 - \bar{z}_m z_n} \Bigr|^2}{\log \dfrac{1}{1 - |z_n|^2} \log \dfrac{1}{1 - |z_m|^2}} = \frac{|k(z_n, z_m)|^2}{\|k_{z_n}\|^2 \, \|k_{z_m}\|^2} \le 1 - \frac{1}{C^2}.$$

Marshall and Sundberg [1993] proved that this condition is equivalent to (0–2).

Now suppose that (0–2) and (0–3) hold. To show that $\{z_n\}$ is an interpolating sequence, we use the Pick property of the Dirichlet space, which allows us to convert the interpolation problem to an "$\mathcal{L}^2$" problem. That is, we prove that $\{\tilde{k}_n\}$ is a Riesz sequence in $\mathcal{D}$, then use Theorem 2.4. It can be shown that the sequence $\{\tilde{k}_n\}$ will be a Riesz sequence if the sequence $\{K_n = \operatorname{Re}(\tilde{k}_n)\}$ is a Riesz sequence in the harmonic Dirichlet space $\mathcal{D}_h$, which consists of all harmonic functions in $\mathbb{D}$ with finite Dirichlet integral, the norm being given by $\|u\|^2 = \|u\|^2_{\mathcal{L}^2(\partial\mathbb{D})} + \|\nabla u\|^2_{\mathcal{L}^2(\mathbb{D})}$. By Theorem 2.3, the last condition holds if $\{K_n\}$ is an unconditional basic sequence in $\mathcal{D}_h$.

Suppose $|b_n| \le |a_n|$ and set $t_n = a_n/b_n$ for all $n$. Then $|t_n| \le 1$. To show that $\{K_n\}$ is an unconditional basic sequence in $\mathcal{D}_h$, we must show that

$$\left\|\sum t_n a_n K_n\right\| = \left\|\sum b_n K_n\right\| \le C \left\|\sum a_n K_n\right\|.$$

Set $T(K_n) = t_n K_n$. It suffices to show that $T$ can be extended to a bounded linear operator on the harmonic Dirichlet space $\mathcal{D}_h$. Note that

$$T^*(u)(z_n) = \big\langle T^*(u),\, \mathrm{Re}(k_{z_n})\big\rangle = \|k_{z_n}\| \,\big\langle u,\, T(K_n)\big\rangle = \bar{t}_n u(z_n).$$

Therefore it suffices to find a bounded linear map $u \mapsto v$ on $\mathcal{D}_h$ such that

$$v(z_n) = \bar{t}_n u(z_n) \quad \text{and} \quad \|v\| \le C\,\|u\|\,.$$

The construction of the desired linear map in [Marshall and Sundberg 1993] requires deep and elegant estimates involving Stegenga's capacity condition for Carleson measures for $\mathcal{D}$. We shall sketch the idea. First, one uses conditions (0–2) and (0–3) to construct a bounded function $\varphi$ on the disk $\mathbb{D}$ so that $\varphi(z_n) = t_n$ for $n = 1, 2, \ldots$ and such that

$$|\nabla\varphi(z)|^2\, dA(z)$$

is a Carleson measure for $\mathcal{D}$. This implies, by the Dirichlet principle,

$$\int_{\mathbb{D}} \big|\nabla P(\varphi^* u^*)\big|^2\, dA \le \int_{\mathbb{D}} \big|\nabla(\varphi u)\big|^2\, dA \le C\,\|u\|^2 \quad \text{for all } u \in \mathcal{D}_h,$$

where $P(\psi^*)$ is the Poisson integral of the boundary function $\psi^*$ of $\psi$. The desired linear map comes from correcting the function $P(\varphi^* u^*)$ by setting

$$v = P(\varphi^* u^*) + \sum \big(t_n u(z_n) - P(\varphi^* u^*)(z_n)\big) f_n,$$

where $f_n$ is a harmonic function in $\mathcal{D}_h$ with $f_n(z_m) = \delta_{n,m}$. The existence of such functions, which can even be chosen to be analytic, follows from a result proved in [Shapiro and Shields 1962], which requires the condition

$$\sum \Big(\log \frac{1}{1 - |z_n|^2}\Big)^{-1} \le M.$$

It is obvious that this condition follows from (0–3). Of course one needs to show that

$$\sum \big|P(\varphi^* u^*)(z_n) - t_n u(z_n)\big| \le C\,\|u\| \quad \text{for all } u \in \mathcal{D}_h.$$

This again requires deep estimates related to capacity. □

As noted in [Marshall and Sundberg 1993], the idea above provides also an easy proof of Carleson's interpolation theorem.

To end this section, we turn to another result proved in [Marshall and Sundberg 1993], which shows a different application of the Pick property.

Let $Z \subset \mathbb{D}$ be a zero set of $\mathcal{D}$ (or $\mathcal{M}_{\mathcal{D}}$), and take $z_0 \notin Z$. Consider the extremal problems

$$C_{\mathcal{D}} = \inf\{\|f\| : f(z_0) = 1 \text{ and } f(Z) = 0\} \tag{2–4}$$

and

$$C_{\mathcal{M}_{\mathcal{D}}} = \inf\{\|\varphi\|_{\mathcal{M}} : \varphi(z_0) = 1 \text{ and } \varphi(Z) = 0\}. \tag{2–5}$$

THEOREM 2.5. *The problems* (2–4) *and* (2–5) *have unique solutions* $f_0$ *and* $\varphi_0$, *and they satisfy*

$$\|f_0\| = \frac{\|\varphi_0\|_{\mathcal{M}}}{\|k_{z_0}\|} \quad \textit{and} \quad f_0 = \varphi_0 \frac{k_{z_0}}{\|k_{z_0}\|^2}.$$

PROOF. Standard reasoning shows that solutions exist. A little elementary work shows that the solution of (2–4) is unique. Indeed, if $f$ and $g$ are distinct solutions to (2–4), then $\|f\| = \|g\|$, $f(z_0) = g(z_0)$, and $f(Z) = g(Z) = 0$, where we have set $Z = \{z_j : j = 1, 2, \ldots\}$. We claim that $\operatorname{Re}\langle f, g\rangle = \|f\|^2$, and therefore that

$$\|f - g\|^2 = \|f\|^2 + \|g\|^2 - 2\operatorname{Re}\langle f, g\rangle = 0.$$

If this is not the case, then $h = \frac{1}{2}(f + g)$ satisfies $h(z_0) = 1$, $h(Z) = 0$, and

$$\|h\|^2 = \tfrac{1}{4}\big(\|f\| + \|g\|^2 + 2\operatorname{Re}\langle f, g\rangle\big) < \|f\|^2.$$

This is impossible.

Let $f_0$ be the unique solution of (2–4) and let $\varphi$ be any solution of (2–5). We show that

$$f_0 = \varphi \frac{k_{z_0}}{\|k_{z_0}\|^2},$$

so $\varphi$ is unique. It is easy to see that

$$0 \le \Big\|\sum a_j k_{z_j}\Big\|^2 - \frac{1}{\|f_0\|^2}\big|\big\langle \sum a_j k_{z_j}, f_0\big\rangle\big|^2 = \sum\left(1 - \frac{f_0(z_j)\overline{f_0(z_k)}}{\|f_0\|^2\,\|k_{z_0}\|^2}\right) k(z_j, z_k)\,\bar{a}_j a_k.$$

By the Pick property, there is a $\psi \in \mathcal{M}_{\mathcal{D}}$ with $\|\psi\|_{\mathcal{M}} \le 1$, and

$$\psi(z_0) = \frac{1}{\|f_0\|\,\|k_{z_0}\|}, \quad \text{and} \quad \psi(z_j) = 0 \text{ for } j = 1, 2, \ldots.$$

Since $\varphi$ is an extremal solution, we have

$$\|\varphi\|_{\mathcal{M}} \le \big\|\|f_0\|\,\|k_{z_0}\|\,\psi\big\|_{\mathcal{M}} \le \|f_0\|\,\|k_{z_0}\|.$$

On the other hand, the function

$$g = \varphi \frac{k_{z_0}}{\|k_{z_0}\|^2}$$

satisfies $g(z_0) = 1$, $g(Z) = 0$, and

$$\|g\| \le \|\varphi\|_{\mathcal{M}} \,/\, \|k_{z_0}\| \le \|f_0\| \,.$$

Therefore $f_0 = g$, as we wished to show. □

It is not hard to show that $f_0$ and $\varphi_0$ in Theorem 2.5 have zero set exactly $Z$ (counting multiplicities!).

## 3. Hankel Operators on the Dirichlet Space

In this section, we use the norm $\|f\| = \big(|f(0)|^2 + D(f)\big)^{1/2}$ for the Dirichlet space. In this case the reproducing kernel of $\mathcal{D}$ is

$$k_{\mathcal{D}}(z,w) = 1 + \log\Big(\frac{1}{1-\overline{w}z}\Big).$$

We note that

$$k_{\mathcal{B}}(z,w) = \partial_z \partial_{\overline{w}} k_{\mathcal{D}}(z,w) = \frac{1}{(1-\overline{w}z)^2}$$

is the reproducing kernel for the Bergman space $\mathcal{B}$.

Assume that the Hankel operator $\mathcal{H}_b$ is bounded on $\mathcal{D}$. For $g$ in $\mathcal{P}$, we have

$$\mathcal{H}_b(g)(z) = \overline{\big\langle b\overline{g}, k_{\mathcal{D}}(\,\cdot\,,z)\big\rangle} = \int_{\mathbb{D}} \overline{b} g \, dA + \overline{\big\langle b'\overline{g}, \partial_w k_{\mathcal{D}}(\,\cdot\,,z)\big\rangle}_{\mathcal{L}^2(\mathbb{D})}.$$

For simplicity, we remove the rank-one operator $g \mapsto \int_{\mathbb{D}} \overline{b} g \, dA$, and take

$$\mathcal{H}_b(g)(z) = \overline{\big\langle b'\overline{g}, \partial_w k_{\mathcal{D}}(\,\cdot\,,z)\big\rangle}_{\mathcal{L}^2(\mathbb{D})} \quad \text{for all } g \in \mathcal{P}, \tag{3–1}$$

as the definition for our Hankel operator on $\mathcal{D}$. It is easy to compute that

$$\partial_z \overline{\mathcal{H}_b(g)(z)} = \overline{\big\langle b'\overline{g}, k_{\mathcal{B}}(\,\cdot\,,z)\big\rangle}_{\mathcal{L}^2(\mathbb{D})} = \overline{P_{\mathcal{B}}(b'\overline{g})(z)},$$

where $P_{\mathcal{B}}$ is the orthogonal projection from $\mathcal{L}^2 = \mathcal{L}^2(\mathbb{D})$ onto $\mathcal{B}$. We have then

$$\big\langle h, \overline{\mathcal{H}_b(g)}\big\rangle = \big\langle h', P_{\mathcal{B}}(b'\overline{g})\big\rangle_{\mathcal{L}^2} = \big\langle h'g, b'\big\rangle_{\mathcal{L}^2} \quad \text{for all } h,\ g \in \mathcal{P}. \tag{3–2}$$

We see that $b \in \mathcal{D}$ is a necessary condition for the Hankel operator $\mathcal{H}_b$ to be bounded on $\mathcal{D}$. From (3–2), we get also that the boundedness or compactness of $\mathcal{H}_b$ on $\mathcal{D}$ are equivalent, respectively, to the boundedness or compactness of the operator

$$g \mapsto P_{\mathcal{B}}(b'\overline{g}) \tag{3–3}$$

from $\overline{\mathcal{D}} = \{\overline{g} : g \in \mathcal{D}\}$ to $\mathcal{B}$.

The following result is standard; see [Arazy et al. 1990], for example.

Lemma 3.1. *Suppose $\varphi$ is a $C^1$ function on a neighborhood of $\mathbb{D}$. Then*

$$\varphi(z) = P_{\mathcal{B}}(\varphi)(z) + \int_{\mathbb{D}} \frac{\partial_{\overline{w}}\varphi(w)(1-|w|^2)}{(z-w)(1-\overline{w}z)}\, dA(w) \quad \textit{for all } z \in \mathbb{D}.$$

LEMMA 3.2. *The linear operator*

$$f(z) \mapsto \int_{\mathbb{D}} \frac{f(w)(1-|w|^2)}{|z-w||1-\bar{w}z|}\, dA(w)$$

*is bounded on* $\mathcal{L}^2(\mathbb{D})$.

Lemma 3.2 can be proved easily by using Schur's test with the test function $u(z) = (1-|z|^2)^{-1/4}$.

PROOF OF THEOREM 0.4. Suppose the measure $|b'(z)|^2\, dA(z)$ is a Carleson measure for $\mathcal{D}$. Then $b'\bar{g}$ is in $\mathcal{L}^2$. This implies the map (3–3) is bounded.

Now suppose the Hankel operator $\mathcal{H}_b$ or, equivalently, the map (3–3) is bounded. We shall show that $b'\,\bar{g}$ is in $\mathcal{L}^2$, for every $g \in \mathcal{D}$, with norm bounded by $C\,\|g\|$. We note that $P_{\mathcal{B}}(b'\,\bar{g})$ is the best approximation to $b'\,\bar{g}$ in $\mathcal{B}$. Therefore by Lemmas 3.1 and 3.2, we have the following formula for the difference between $b'\bar{g}$ and $P_{\mathcal{B}}(b'\,\bar{g})$:

$$b'(z)\overline{g(z)} - P_{\mathcal{B}}(b'\bar{g})(z) = \int_{\mathbb{D}} \frac{b'(w)\overline{g'(w)}(1-|w|^2)}{(z-w)(1-\bar{w}z)}\, dA(w) \quad \text{for all } g \in \mathcal{P}. \tag{3–4}$$

Since $b \in \mathcal{D}$ and $\mathcal{D}$ is contained in the little Bloch space, we have

$$(1-|w|^2)\left|b'(w)\right| \le C\,\|b\|\,.$$

Thus by Lemma 3.2 and the assumption, we obtain

$$\|b'\bar{g}\|^2_{\mathcal{L}^2} = \left\|P_{\mathcal{B}}(b'\bar{g})\right\|^2_{\mathcal{L}^2} + \left\|b'\bar{g} - P_{\mathcal{B}}(b'\bar{g})\right\|^2_{\mathcal{L}^2} \le C\,\|b\|^2\,\|g\|^2 \quad \text{for all } g \in \mathcal{P}.$$

This proves part (a) of the theorem.

The operator defined by (3–4) is in fact compact from $\overline{\mathcal{D}}$ to $\mathcal{L}^2$, because $\mathcal{D}$ is a subset of the little Bloch space. Therefore the compactness of $\mathcal{H}_b$ is equivalent to the compactness of the multiplier $M_{b'} : \mathcal{D} \to \mathcal{B}$. A result in [Rochberg and Wu 1992] implies therefore that condition (0–5) is necessary and sufficient for the compactness of $\mathcal{H}_b$. □

The proof of Theorem 0.5 requires a general result about pairing of operators. Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. The trace class of linear operators from $\mathcal{H}$ to $\mathcal{K}$, denoted by $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{H},\mathcal{K})$, is the set of all compact operators $T$ from $\mathcal{H}$ to $\mathcal{K}$ for which the sequence of singular numbers

$$\left\{s_k(T) = \inf\{\|T-R\| : \operatorname{rank}(R) < k\}\right\}_1^\infty$$

belongs to $\ell^1$. The $\mathcal{S}_1$ norm of $T$ is defined by

$$\|T\|_{\mathcal{S}_1} = \|\{s_k(T)\}_1^\infty\|_{\ell^1}\,.$$

We will use $\mathcal{S}_0 = \mathcal{S}_0(\mathcal{H},\mathcal{K})$ and $\mathcal{S}_\infty = \mathcal{S}_\infty(\mathcal{H},\mathcal{K})$ for the sets of compact operators and bounded operators from $\mathcal{H}$ to $\mathcal{K}$, respectively. Let $T$ and $S$ be bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and from $\mathcal{K}$ to $\mathcal{H}$, respectively. The pairing of $T$ and $S$ is given by

$$\langle T, S \rangle = \operatorname{trace}(TS).$$

The following standard theorem can be found in [Zhu 1990], for example.

THEOREM 3.3. (a) $(\mathcal{S}_0)^* = \mathcal{S}_1$ *and* $(\mathcal{S}_1)^* = \mathcal{S}_\infty$.

(b) (*Schmidt decomposition*) *$\Phi$ is a compact operator from $\mathcal{H}$ to $\mathcal{K}$ if and only if $\Phi$ can be written as*

$$\Phi = \sum \lambda_j \langle \,\cdot\,, f_j \rangle_{\mathcal{H}} e_j,$$

*where $\{\lambda_j\}_1^\infty$ is a sequence of numbers tending to* 0, *and $\{f_j\}_1^\infty$ and $\{e_j\}_1^\infty$ are orthonormal sequences in $\mathcal{H}$ and $\mathcal{K}$, respectively. Moreover, if $\{\lambda_j\}_1^\infty$ is in $\ell^1$, then*

$$\|\Phi\|_{\mathcal{S}_1} = \sum |\lambda_j|\,.$$

PROOF OF THEOREM 0.5. It is easy to check that $\mathcal{W}_\mathcal{D} \subseteq (\mathcal{X}_\mathcal{D})^*$ and $\mathcal{X}_\mathcal{D} \subseteq (w_\mathcal{D})^*$ by using formula (3–2). We prove $(\mathcal{X}_\mathcal{D})^* \subseteq \mathcal{W}_\mathcal{D}$ next.

Suppose $T \in (\mathcal{X}_\mathcal{D})^*$. For any $g, h \in \mathcal{D}$, it is clear that $gh' \in \mathcal{X}_\mathcal{D}$ and $\|gh'\|_{\mathcal{X}_\mathcal{D}} \le \|g\|\|h\|$. Hence if in addition $h(0) = 0$, then

$$|T(gh')| \le \|T\|\|gh'\|_{\mathcal{X}_\mathcal{D}} \le \|T\|\,\|g\|\,\|h\| = \|T\|\,\|g\|\,\|h'\|_\mathcal{B}.$$

This inequality shows that for fixed $g \in \mathcal{D}$ the linear functional $h' \mapsto T(gh')$ on $\mathcal{B}$ is bounded. Hence by the Riesz–Fischer Theorem there is a $T_g \in \mathcal{B}$ such that

$$T(gh') = \langle h', T_g \rangle_{\mathcal{L}^2} \quad \text{for all } h' \in \mathcal{B}.$$

Clearly $T_g$ is uniquely determined by $g$ and the linear map $g \mapsto T_g$ from $\mathcal{D}$ to $\mathcal{B}$ is bounded with $\|T_g\|_\mathcal{B} \le \|T\|\,\|g\|$.

Let $b(z) = b_T(z) = \int_0^z T_1(\zeta)\,d\zeta \in \mathcal{D}$. For any $g \in \mathcal{D}$ we have

$$T_g(w) = \langle T_g, k_\mathcal{B}(\cdot\,, w) \rangle_{\mathcal{L}^2} = \overline{T(gk_\mathcal{B}(\cdot\,, w))}.$$

Since for fixed $w \in \mathbb{D}$, $gk_\mathcal{B}(\cdot\,, w)$ is always in $\mathcal{B}$, we have

$$T(gk_\mathcal{B}(\cdot\,, w)) = \langle gk_\mathcal{B}(\cdot\,, w), T_1 \rangle_{\mathcal{L}^2} = \langle gk_\mathcal{B}(\cdot\,, w), b' \rangle_{\mathcal{L}^2} = \partial_{\overline{w}} \langle g \partial_z k_\mathcal{D}(\cdot\,, w), b' \rangle_{\mathcal{L}^2}.$$

This implies that $T_g(w) = \partial_w \overline{\mathcal{H}_{b_T}(g)(w)}$, for any $g \in \mathcal{P}$. We conclude therefore

$$\|\mathcal{H}_b(g)\| = \|T_g\|_\mathcal{B} \le \|T\|\,\|g\|,$$

and hence $\|\mathcal{H}_b\| \le \|T\|$. By Theorem 0.4, we have $b \in \mathcal{W}_\mathcal{D}$ and then

$$T_g(w) = \partial_w \overline{\mathcal{H}_{b_T}(g)(w)} \quad \text{for all } g \in \mathcal{D}. \tag{3–5}$$

This discussion also yields (since $b \in \mathcal{W}_\mathcal{D}$)

$$T(gh') = \langle gh', T_1 \rangle_{\mathcal{L}^2} \quad \text{for all } g, h \in \mathcal{D}.$$

This implies that the map $T \mapsto b_T$ from $(\mathcal{X}_{\mathcal{D}})^*$ to $\mathcal{W}_{\mathcal{D}}$ is bounded and one-to-one. To complete the proof, it remains to verify that

$$T(f) = \langle f, b_T \rangle^* \quad \text{for all } f \in \mathcal{X}_{\mathcal{D}}.$$

This is easy to check by using (3–5) and (3–2).

To prove that $(w_{\mathcal{D}})^* \subseteq \mathcal{X}_{\mathcal{D}}$, we consider the map $b \mapsto \partial \overline{\mathcal{H}}_b$ from $w_{\mathcal{D}}$ to $\mathcal{S}_0(\overline{\mathcal{D}}, \mathcal{B})$. This map is clearly one-to-one and maps $w_{\mathcal{D}}$ onto a closed subspace of $\mathcal{S}_0$. Take $L \in (w_{\mathcal{D}})^*$. Extend $L$ to a bounded linear functional $\tilde{L}$ on $\mathcal{S}_0$ so that $\|\tilde{L}\| = \|L\|$. By Theorem3.3(a), there is a $\Phi$ in $\mathcal{S}_1(\mathcal{B}, \overline{\mathcal{D}})$ such that $\|\Phi\|_{\mathcal{S}_1} = \|\tilde{L}\|$ and $\tilde{L}(T) = \langle T, \Phi \rangle$, for any $T \in \mathcal{S}_0$. Suppose the Schmidt decomposition of $\Phi$ given by Theorem3.3(b) is

$$\Phi = \sum s_j \langle \,\cdot\,, f_j \rangle_{\mathcal{L}^2} \bar{g}_j,$$

where $\{s_j\}_1^\infty$ is the sequence of singular numbers of $\Phi$, and $\{f_j\}_1^\infty$ and $\{\bar{g}_j\}_1^\infty$ are orthonormal sequences in $\mathcal{B}$ and $\overline{\mathcal{D}}$, respectively.

It is clear that $\left\{h_j(z) = \int_0^z f_j(\zeta)\, d\zeta\right\}_0^\infty$ is an orthonormal sequence in $\mathcal{D}$. Set

$$f = f_L = \sum s_j g_j f_j = \sum s_j g_j h_j'. \tag{3–6}$$

Then clearly $f$ is in $\mathcal{X}_{\mathcal{D}}$, and

$$\|f\|_{\mathcal{X}_{\mathcal{D}}} \le \sum |s_j| = \|\Phi\|_{\mathcal{S}_1} = \|L\|.$$

For any $b \in w_{\mathcal{D}}$, we have

$$\begin{aligned} L(b) &= \tilde{L}\big(\partial \overline{\mathcal{H}}_b\big) = \big\langle \partial \overline{\mathcal{H}}_b, \Phi \big\rangle = \operatorname{trace}\big(\partial \overline{\mathcal{H}}_b \Phi\big) = \operatorname{trace}\big(\Phi \partial \overline{\mathcal{H}}_b\big) \\ &= \sum \big\langle \Phi \partial \overline{\mathcal{H}}_b(\bar{g}_j), \bar{g}_j \big\rangle = \sum s_j \big\langle \partial \overline{\mathcal{H}_b(g_j)}, f_j \big\rangle_{\mathcal{L}^2} = \sum s_j \big\langle b', g_j h_j' \big\rangle_{\mathcal{L}^2} \\ &= \big\langle b', f \big\rangle_{\mathcal{L}^2}; \end{aligned}$$

thus

$$L(b) = {}^*\langle b, f \rangle \quad \text{for all } b \in w_{\mathcal{D}}. \tag{3–7}$$

This implies $\|L\| \le \|f\|_{\mathcal{X}_{\mathcal{D}}}$, and hence $\|L\| = \|f\|_{\mathcal{X}_{\mathcal{D}}}$.

To complete the proof it remains to show that the map $L \mapsto f_L$ defined by (3–6) is well defined and one-to-one.

In fact for any $\zeta \in \mathbb{D}$, let $b_\zeta(z) = \partial_{\bar{\zeta}} k_{\mathcal{D}}(z, \zeta)$. By formula (3–1) and the equality $b_\zeta'(z) = k_{\mathcal{B}}(z, \zeta)$, we get

$$\mathcal{H}_{b_\zeta}(g)(w) = \big\langle g \partial_z k_{\mathcal{D}}(\,\cdot\,, w), b_\zeta' \big\rangle_{\mathcal{L}^2} = \partial_\zeta k_{\mathcal{D}}(\zeta, w) g(\zeta) \quad \text{for all } g \in \mathcal{P}.$$

This shows that $\mathcal{H}_{b_\zeta}$ is a compact operator (of rank one!). Thus $b_\zeta \in w_{\mathcal{D}}$ and

$$L(b_\zeta) = {}^*\langle b_\zeta, f_L \rangle = \langle b_\zeta', f_L \rangle_{\mathcal{L}^2} = \overline{f_L(\zeta)}.$$

The "one-to-one" part is then an immediate consequence of the identity (3–7). □

## References


[Agler 1986] J. Agler, "Interpolation", preprint, 1986.

[Aleman 1992] A. Aleman, "Hilbert spaces of analytic functions between the Hardy and the Dirichlet space", *Proc. Amer. Math. Soc.* **115**:1 (1992), 97–104.

[Arazy and Fisher 1985] J. Arazy and S. D. Fisher, "The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces", *Illinois J. Math.* **29**:3 (1985), 449–462.

[Arazy et al. 1988] J. Arazy, S. D. Fisher, and J. Peetre, "Hankel operators on weighted Bergman spaces", *Amer. J. Math.* **110**:6 (1988), 989–1053.

[Arazy et al. 1990] J. Arazy, S. D. Fisher, S. Janson, and J. Peetre, "An identity for reproducing kernels in a planar domain and Hilbert-Schmidt Hankel operators", *J. Reine Angew. Math.* **406** (1990), 179–199.

[Axler 1986] S. Axler, "The Bergman space, the Bloch space, and commutators of multiplication operators", *Duke Math. J.* **53**:2 (1986), 315–332.

[Axler 1992] S. Axler, "Interpolation by multipliers of the Dirichlet space", *Quart. J. Math. Oxford Ser.* (2) **43**:172 (1992), 409–419.

[Bercovici et al. 1985] H. Bercovici, C. Foiaş, and C. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series in Mathematics **56**, Amer. Math. Soc., Providence, 1985.

[Bourdon 1986] P. S. Bourdon, "Cellular-indecomposable operators and Beurling's theorem", *Michigan Math. J.* **33**:2 (1986), 187–193.

[Carleson 1958] L. Carleson, "An interpolation problem for bounded analytic functions", *Amer. J. Math.* **80** (1958), 921–930.

[Coifman et al. 1976] R. R. Coifman, R. Rochberg, and G. Weiss, "Factorization theorems for Hardy spaces in several variables", *Ann. of Math.* (2) **103**:3 (1976), 611–635.

[Horowitz 1974] C. Horowitz, "Zeros of functions in the Bergman spaces", *Duke Math. J.* **41** (1974), 693–710.

[Luecking 1992] D. H. Luecking, "Characterizations of certain classes of Hankel operators on the Bergman spaces of the unit disk", *J. Funct. Anal.* **110**:2 (1992), 247–271.

[Marshall and Sundberg 1993] D. E. Marshall and C. Sundberg, "Interpolating sequences for the multipliers of the Dirichlet space", preprint, 1993. Available at http://www.math.washington.edu/~marshall/preprints/preprints.html% posturl-hook.

[Nikolskii 1986] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der mathematischen Wissenschaften **273**, Springer, Berlin, 1986.

[Olin and Thomson 1984] R. F. Olin and J. E. Thomson, "Cellular-indecomposable subnormal operators", *Integral Equations Operator Theory* **7**:3 (1984), 392–430.

[Peller 1982] V. V. Peller, "Vectorial Hankel operators, commutators and related operators of the Schatten–von Neumann class $\gamma_p$", *Integral Equations Operator Theory* **5**:2 (1982), 244–272.

[Richter 1988] S. Richter, "Invariant subspaces of the Dirichlet shift", *J. Reine Angew. Math.* **386** (1988), 205–220.

[Richter 1991] S. Richter, "A representation theorem for cyclic analytic two-isometries", *Trans. Amer. Math. Soc.* **328**:1 (1991), 325–349.

[Richter and Shields 1988] S. Richter and A. Shields, "Bounded analytic functions in the Dirichlet space", *Math. Z.* **198**:2 (1988), 151–159.

[Richter and Sundberg 1991] S. Richter and C. Sundberg, "A formula for the local Dirichlet integral", *Michigan Math. J.* **38**:3 (1991), 355–379.

[Richter and Sundberg 1992] S. Richter and C. Sundberg, "Multipliers and invariant subspaces in the Dirichlet space", *J. Operator Theory* **28**:1 (1992), 167–186.

[Rochberg 1985] R. Rochberg, "Decomposition theorems for Bergman spaces and their applications", pp. 225–277 in *Operators and function theory* (Lancaster, 1984), edited by S. C. Power, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **153**, Reidel, Dordrecht, 1985.

[Rochberg and Wu 1992] R. Rochberg and Z. Wu, "Toeplitz operators on Dirichlet spaces", *Integral Equations Operator Theory* **15**:2 (1992), 325–342.

[Rochberg and Wu 1993] R. Rochberg and Z. Wu, "A new characterization of Dirichlet type spaces and applications", *Illinois J. Math.* **37**:1 (1993), 101–122.

[Seip 1993] K. Seip, "Beurling type density theorems in the unit disk", *Invent. Math.* **113**:1 (1993), 21–39.

[Shapiro and Shields 1962] H. S. Shapiro and A. L. Shields, "On the zeros of functions with finite Dirichlet integral and some related function spaces", *Math. Z.* **80** (1962), 217–229.

[Stegenga 1980] D. A. Stegenga, "Multipliers of the Dirichlet space", *Illinois J. Math.* **24**:1 (1980), 113–139.

[Wu 1993] Z. Wu, "The predual and second predual of $W_\alpha$", *J. Funct. Anal.* **116**:2 (1993), 314–334.

[Zhu 1990] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics **139**, Marcel Dekker Inc., New York, 1990.

Zhijian Wu
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487
United States
zwu@euler.math.ua.edu