Holomorphic Spaces
MSRI Publications
Volume **33**, 1998

Holomorphic Spaces: A Brief and Selective Survey

DONALD SARASON

ABSTRACT. This article traces several prominent trends in the development of the subject of holomorphic spaces, with emphasis on operator-theoretic aspects.

The term "Holomorphic Spaces," the title of a program held at the Mathematical Sciences Research Institute in the fall semester of 1995, is short for "Spaces of Holomorphic Functions." It refers not so much to a branch of mathematics as to a common thread running through much of modern analysis—through functional analysis, operator theory, harmonic analysis, and, of course, complex analysis. This article will briefly outline the development of the subject from its origins in the early 1900's to the present, with a bias toward operator-theoretic aspects, in keeping with the main emphasis of the MSRI program. I hope that the article will be accessible not only to workers in the field but to analysts in general.

Origins

The subject began with the thesis of P. Fatou [1906], a student of H. Lebesgue. The thesis is a study of the boundary behavior of certain harmonic functions in the unit disk (those representable as Poisson integrals). It contains a proof, for example, that a bounded holomorphic function in the disk has a nontangential limit at almost every point of the unit circle. This initial link between function theory on the circle (real analysis) and function theory in the disk (complex analysis) recurred continually in the ensuing years. Some of the highlights are the paper of F. Riesz and M. Riesz [1916] on the absolute continuity of analytic measures; F. Riesz's paper [1923] in which he christened the Hardy spaces, H^p, and introduced the technique of dividing out zeros (i.e., factoring by a Blaschke product); G. Szegö's investigations [1920; 1921] of Toeplitz forms; M. Riesz's proof [1924] of the L^p boundedness of the conjugation operator ($1 < p < \infty$);

A. N. Kolmogorov's proof [1925] of the weak-L^1 boundedness of the conjugation operator; G. H. Hardy and J. E. Littlewood's introduction of their maximal function [1930]; and R. Nevanlinna's development [1936] of his theory of functions of bounded characteristic. By the late 1930's the theory had expanded to the point where it could become the subject of a monograph. The well-known book of I. I. Privalov appeared in 1941, the year of the author's death, and was republished nine years later [Privalov 1950], followed by a German translation [Privalov 1956].

Beurling's Paper

Simultaneously, the general theory of Banach spaces and their operators had been developing. By mid-century, when A. Beurling published a seminal paper [1949], the time was ripe for mutual infusion. After posing two general questions about Hilbert space operators with complete sets of eigenvectors, Beurling's paper focuses on the (closed) invariant subspaces of the unilateral shift operator on the Hardy space H^2 of the unit disk (an operator whose adjoint is of the kind just mentioned).

For the benefit of readers who do not work in the field, here are a few of the basic definitions. For $p > 0$ the Hardy space H^p consists of the holomorphic functions f in the unit disk, $\mathbb{D}$, satisfying the growth condition $\sup_{0<r<1} \|f_r\|_p < \infty$, where f_r is the function on the unit circle defined by $f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$, and $\|f_r\|_p$ denotes the norm of f_r in the L^p space of normalized Lebesgue measure on the circle, hereafter denoted simply by L^p. As noted earlier, the spaces H^p were introduced by F. Riesz [1923]; they were named by him in honor of G. H. Hardy, who had proved [1915] that $\|f_r\|_p$ increases with r (unless f is constant). From the work of Fatou and his successors one knows that each function in H^p has an associated boundary function, defined almost everywhere on $\partial\mathbb{D}$ in terms of nontangential limits. Because of this, one can identify H^p with a subspace of L^p; in case $p \geq 1$, the subspace in question consists of the functions in L^p whose Fourier coefficients with negative indices vanish (i.e., the functions whose Fourier series are of power series type). A function in H^p, for $p \geq 1$, can be reconstructed from its boundary function by means of the Poisson integral, or the Cauchy integral. The space H^2 can be alternatively described as the space of holomorphic functions in $\mathbb{D}$ whose Taylor coefficients at the origin are square summable. In the obvious way it acquires a Hilbert space structure in which the functions z^n, for $n = 0, 1, 2, \ldots$, form an orthonormal basis.

The unilateral shift is the operator S on H^2 of multiplication by z, the identity function. It is an isometry, sending the n-th basis vector, z^n, to the $(n+1)$-st, z^{n+1}. It is, in fact, the simplest pure isometry. (A Hilbert space isometry is called pure if it has no unitary direct summand. Every pure isometry is a direct sum of copies of S.) Beurling showed that the invariant subspace structure of S mirrors the factorization theory of H^2 functions.

From the work of F. Riesz and Nevanlinna it is known that every nonzero function in H^p can be written as the product of what Beurling called an outer function and an inner function. The factors are unique to within unimodular multiplicative constants. An outer function is a nowhere vanishing holomorphic function f in $\mathbb{D}$ such that $\log|f|$ is the Poisson integral of its boundary function. Beurling showed that the outer functions in H^2 are the cyclic vectors of the operator S (i.e., the functions contained in no proper S-invariant subspaces). An inner function is a function in H^∞ whose boundary function has unit modulus almost everywhere. Beurling showed that if the H^2 function f has the factorization $f = uf_0$, with u an inner function and f_0 an outer function, then the S-invariant subspace generated by f is the same as that generated by u, and it equals uH^2. Finally, Beurling showed that every invariant subspace of S is generated by a single function, and hence by an inner function. Thus, understanding the invariant subspace structure of S is tantamount to understanding the structure of inner functions.

There are two basic kinds of inner functions: Blaschke products and singular functions. Only the constant inner functions are of both kinds, and every inner function is the product of a Blaschke product and a singular function, the factors being unique to within unimodular multiplicative constants. Blaschke products (products of Blaschke factors) are associated with zero sequences. The zero sequence of a function in H^2 (in fact, of a function in any H^p) is a so-called Blaschke sequence, a finite sequence in $\mathbb{D}$ or an infinite sequence $(z_n)_{n=1}^{\infty}$ satisfying $\sum(1-|z_n|) < \infty$ (the Blaschke condition). The Blaschke factor corresponding to a point w of $\mathbb{D}$ is, in case $w \neq 0$, the linear-fractional map of $\mathbb{D}$ onto $\mathbb{D}$ that sends w to 0 and 0 to the positive real axis, and in case $w = 0$ it is the identity function. A Blaschke product is the product of the Blaschke factors corresponding to the terms of a Blaschke sequence, or a unimodular constant times such a function. In the case of a finite sequence it is obviously an inner function, and in the case of an infinite sequence, the Blaschke condition is exactly what one needs to prove that the corresponding infinite product of Blaschke factors converges locally uniformly in $\mathbb{D}$ to an inner function. If the inner function associated with an S-invariant subspace is a Blaschke product, then the subspace is just the subspace of functions in H^2 that vanish at the points of the corresponding Blaschke sequence (with the appropriate multiplicities at repeated points).

A singular function is an inner function without zeroes in $\mathbb{D}$. The logarithm of the modulus of such a function, if the function is nonconstant, is a negative harmonic function in $\mathbb{D}$ having the nontangential limit 0 at almost every point of $\partial\mathbb{D}$. One can conclude on the basis of the theory of Poisson integrals that the logarithm of the modulus of a nonconstant singular function is the Poisson integral of a negative singular measure on $\partial\mathbb{D}$. The most general nonconstant singular function can thus be represented as

$$\lambda \exp\left(-\int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu\left(e^{i\theta}\right)\right) \qquad \text{for } z \in \mathbb{D},$$

where λ is a unimodular constant and μ is a positive singular measure on $\partial\mathbb{D}$. The simplest such function is

$$\exp\left(\frac{z+1}{z-1}\right),$$

corresponding to the case where $\lambda = 1$ and μ consists of a unit point mass at the point 1. If the inner function associated with an S-invariant subspace is a singular function, then the functions in the subspace have no common zeroes in $\mathbb{D}$, but the common singular inner factor they share forces them all to have the nontangential limit 0 almost everywhere on $\partial\mathbb{D}$ with respect to the associated singular measure.

Because of Beurling's theorem, the preceding description of inner functions translates into a description of the invariant subspaces of the operator S. The theorem is a splendid early example of how a natural question in operator theory can lead deeply into analysis.

Multiple Shifts and Operator Models

Beurling's work was extended to multiple shifts by P. D. Lax [1959] and P. R. Halmos [1961]. Here one naturally encounters vector-valued function theory.

For $1 \le n \le \aleph_0$, the unilateral shift of multiplicity n (that is, the direct sum of n copies of S) can be modeled as the operator of multiplication by z on a vector-valued version of H^2; the functions in this space have values belonging to an auxiliary Hilbert space $\mathcal{E}$ of dimension n. The space, usually denoted by $H^2(\mathcal{E})$, can be defined, analogously to the scalar case, as the space of holomorphic $\mathcal{E}$-valued functions in $\mathbb{D}$ whose Taylor coefficients at the origin are square summable. The shift-invariant subspaces of $H^2(\mathcal{E})$ have a description analogous to that in Beurling's theorem, the inner functions in that theorem being replaced by operator-valued analogues. Something is lost in the generalization, because the latter functions are not generally susceptible to a precise structural description like the one discussed above for scalar inner functions. (An exception is afforded by what are usually called matrix inner functions, bounded holomorphic matrix-valued functions in $\mathbb{D}$ having unitary nontangential limits almost everywhere on $\partial\mathbb{D}$. For this class of functions, and a symplectic analogue, V. P. Potapov [1955] has developed a beautiful structure theory.)

Multiple shifts play a prominent role in model theories for Hilbert space contractions. The prototypical theory of operator models is, of course, the classical spectral theorem, which in its various incarnations provides canonical models for self-adjoint operators, normal operators, one-parameter unitary groups, and commutative C^*-algebras. Model theories that go beyond the confines of the classical spectral theorem developed on several parallel fronts beginning in the 1950's. The theory originated by M. S. Livshitz [1952] and M. S. Brodskii [1956] focuses on operators that are "nearly" self-adjoint. The theories of B. Sz.-Nagy and C. Foiaş [1967] and L. de Branges and J. Rovnyak [1966, Appendix, pp. 347–392] apply to general contractions but are most effective for "nearly" unitary ones.

The spirit of these "nonclassical" model theories can be illustrated with the Volterra operator, the operator V on $L^2[0,1]$ of indefinite integration:

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad \text{for } 0 \le x \le 1.$$

The adjoint of V is given by

$$(V^*f)(x) = \int_x^1 f(t)\,dt \qquad \text{for } 0 \le x \le 1,$$

from which one sees that $V + V^*$ is a positive operator of rank one, and hence that the operator $(I-V)(I+V)^{-1}$ is a contraction and a rank-one perturbation of a unitary operator.

The invariant subspaces of V were determined, by different methods, by Brodskii [1957] and W. F. Donoghue [1957]. The result is also a corollary of earlier work of S. Agmon [1949]; it says that the only invariant subspaces of V are the obvious ones, the subspaces $L^2[a,1]$ for $0 \le a \le 1$. (Here, $L^2[a,1]$ is identified with the subspace of functions in $L^2[0,1]$ that vanish off $[a,1]$.) It was eventually recognized that the Agmon–Brodskii–Donoghue result is "contained" in Beurling's theorem [Sarason 1965].

To explain the last remark we consider, for $a > 0$, the singular inner function

$$u_a(z) = \exp\left(a\left(\frac{z+1}{z-1}\right)\right),$$

and the orthogonal complement of its corresponding invariant subspace, which we denote by K_a:

$$K_a = H^2 \ominus u_a H^2.$$

We look in particular at K_1, and on K_1 we consider the operator S_1 one obtains by compressing the shift S. Thus, to apply S_1 to a function in K_1, one first multiplies the function by z and then projects the result onto K_1. (The adjoint of S_1 is the restriction of S^* to K_1.)

There is a natural isometry, involving the Cayley transform, that maps L^2 (of $\partial\mathbb{D}$) onto $L^2(\mathbb{R})$. If one follows that isometry by the Fourier–Plancherel transformation, one obtains again an isometry of L^2 onto $L^2(\mathbb{R})$. The latter isometry maps H^2 onto $L^2[0,\infty)$ and maps K_a onto $L^2[0,a]$. And it transforms the operator S_1 on K_1 to the operator $(I-V)(I+V)^{-1}$ on $L^2(0,1)$. The operator S_1 is thus a "model" of the operator $(I-V)(I+V)^{-1}$.

By Beurling's theorem, the invariant subspaces of S_1 are exactly the subspaces $uH^2 \ominus u_1H^2$ with u an inner function that divides u_1 (divides, that is, in the algebra H^∞ of bounded holomorphic functions in $\mathbb{D}$). From the structure theory for inner functions described earlier one can see that the only inner functions that divide u_1 are the functions u_a with $0 \leq a \leq 1$ (and their multiples by unimodular constants). On the basis of the transformation described above, one concludes that the invariant subspaces of $(I - V)(I + V)^{-1}$ are the subspaces $L^2[a, 1]$ with $0 \leq a \leq 1$. Finally, each of the operators V and $(I-V)(I+V)^{-1}$ is easily seen to be approximable in norm by polynomials in the other, implying that these two operators have the same invariant subspaces. The Agmon–Brodskii–Donoghue result follows.

One can sum up the preceding remarks by saying that the Volterra operator, V, is "contained in" the shift operator, S. A simple and elegant observation of G. C. Rota [1960], which provides a hint of the Sz.-Nagy–Foiaş and de Branges–Rovnyak model theories, shows, startlingly, that all Hilbert space operators are "contained in" multiple shifts. Consider an operator T of spectral radius less than 1 on a Hilbert space $\mathcal{E}$. With each vector x in $\mathcal{E}$ we associate the $\mathcal{E}$-valued holomorphic function f_x given by the power series $\sum_{n=0}^{\infty} z^n T^{*n} x$. Because of the spectral condition imposed on T, the function f_x is holomorphic on $\overline{\mathbb{D}}$, so in particular it belongs to $H^2(\mathcal{E})$. The space of all such functions f_x is a subspace K of $H^2(\mathcal{E})$, invariant under the adjoint of the shift operator on $H^2(\mathcal{E})$. The map $x \to f_x$ from $\mathcal{E}$ onto K is a bounded, invertible operator that intertwines T^* with the adjoint of the shift operator. It follows that the operator T is similar to the compression of the shift operator to K. In a sense, then, multiple shifts provide replicas of all operators.

To be a bit more precise, Rota's observation provides a similarity model for every Hilbert space operator whose spectral radius is less than 1. The model space is the orthogonal complement of a shift-invariant subspace of a vector-valued H^2 space, and the model operator is the compression of the shift to the model space. The more powerful Sz.-Nagy–Foiaş and de Branges–Rovnyak theories provide unitarily equivalent models, not merely similarity models, for Hilbert space contractions. The theory of Sz.-Nagy and Foiaş springs from the subject of unitary dilations. Their model spaces include, among a wider class, the orthogonal complements of all shift-invariant subspaces of vector-valued H^2 spaces, the corresponding model operators being compressions of shifts. The model spaces of de Branges–Rovnyak are certain Hilbert spaces that live inside vector-valued H^2 spaces, not necessarily as subspaces but as contractively contained spaces, that is, spaces whose norms dominate the norms of the containing spaces.

The connection between the model theories of Sz.-Nagy–Foiaş and de Branges–Rovnyak has been explained by J. A. Ball and T. L. Kriete [1987]. Further insight was provided by N. K. Nikolskii and V. I. Vasyunin [1989], who developed what they term a coordinate-free model theory that contains, as particular cases, the Sz.-Nagy–Foiaş and de Branges–Rovnyak theories.

Interpolation

The operators S and S^* are themselves model operators in the Sz.-Nagy–Foiaş theory, and rather transparent ones. They model irreducible pure isometries and their adjoints, respectively. Next in simplicity are the compressions of S to proper S^*-invariant subspaces of H^2. For u a nonconstant inner function, let K_u denote the orthogonal complement in H^2 of the Beurling subspace uH^2, and let S_u denote the compression of S to K_u. (The action of S_u is thus multiplication by z followed by projection onto K_u.) J. W. Moeller [1962] showed that the spectrum of S_u consists of the zeros of u in $\mathbb{D}$ plus the points on $\partial\mathbb{D}$ where u has 0 as a cluster value. Moeller's paper and other considerations led the author to suspect that every operator commuting with S_u should be obtainable as the compression of an operator commuting with S. The operators of the latter kind are just the multiplication operators on H^2 induced by H^∞ functions. The result was eventually proved in a more precise form: an operator T that commutes with S_u is the compression of an operator that commutes with S and has the same norm as T [Sarason 1967]. There is a close link with two classical interpolation problems, the problems of Carathéodory–Fejér and Nevanlinna–Pick.

In the first of these problems [Carathéodory and Fejér 1911], one is given as data a finite sequence $c_0, c_1, \dots, c_{N-1}$ of complex numbers, and one wants to find a function in the unit ball of H^∞ having those numbers as its first N Taylor coefficients at the origin. To recast this as a problem about operators, let u be the inner function z^N. The functions $1, z, \dots, z^{N-1}$ form an orthonormal basis for the corresponding space K_u, and the matrix in this basis for the operator S_u has the entry 1 in each position immediately below the main diagonal and 0 elsewhere. Let $T = \sum_{j=0}^{N-1} c_j S_u^j$, so the matrix for T is lower triangular with the entry c_j in each position j steps below the main diagonal. Then T commutes with S_u, and the question of whether the Carathéodory–Fejér problem has a solution for the data $c_0, c_1, \dots, c_{N-1}$ is the same as the question of whether T is the compression of an operator commuting with S and having norm at most 1. According to the result from [Sarason 1967], T has such a compression if and only if its norm is at most 1. One recaptures in this way a solvability criterion for the Carathéodory–Fejér problem attributed by those authors to O. Toeplitz.

In the second classical interpolation problem [Nevanlinna 1919; Pick 1916], one is given as data a finite sequence $z_1, \dots, z_N$ of distinct points in $\mathbb{D}$ and a finite sequence $w_1, \dots, w_N$ of complex numbers. One wants to find a function in the unit ball of H^∞ taking the value w_j at z_j, for $j = 1, \dots, N$. For an operator reinterpretation, let u be the finite Blaschke product with zero sequence $z_1, \dots, z_N$. The space K_u is spanned by the kernel functions for the points $z_1, \dots, z_N$, the functions $k_j(z) = (1-\bar{z}_j z)^{-1}$, where $j = 1, \dots, N$. The distinctive property of k_j is that the linear functional it induces on H^2 is the functional of evaluation at z_j. From this one sees that $S^* k_j = \bar{z}_j k_j$, so the functions $k_1, \dots, k_N$ form an eigenbasis for S_u^*. Let the operator T on K_u be defined by $T^* k_j = \bar{w}_j k_j$.

Then T commutes with S_u, and the question of whether the Nevanlinna–Pick problem is solvable for the given data is the same as the question of whether T is the compression of an operator that commutes with S and has norm at most 1. By the result from [Sarason 1967], the latter happens if and only if the norm of T is at most 1, which is easily seen to coincide with Pick's solvability criterion, namely, the positive semidefiniteness of the matrix

$$\left(\frac{1-w_j\bar{w}_k}{1-z_j\bar{z}_k}\right)_{j,k=1}^{N}.$$

Subsequently, Sz.-Nagy and Foiaş established their famous commutant-lifting theorem [1968; 1967], according to which the result from [Sarason 1967] is a special case of a general property of unitary dilations. The commutant-lifting theorem provides an operator approach to a variety of interpolation problems. The books [Rosenblum and Rovnyak 1985; Foiaş and Frazho 1990], are good sources for this material. H. Dym's review of the latter book [Dym 1994] is also recommended.

The commutant-lifting approach is just one of several operator approaches to interpolation problems. In the same year that the commutant-lifting theorem appeared, V. M. Adamyan, D. Z. Arov and M. G. Krein published the first two of a remarkable series of papers on the Nehari interpolation problem [Adamyan et al. 1968b; 1968a]. In the Nehari problem one is given as data a sequence $(c_n)_{n=1}^{\infty}$ of complex numbers, and one wants to find a function f in the unit ball of L^∞ (on the unit circle) having these numbers as its negatively indexed Fourier coefficients (i.e., $\hat{f}(-n)=c_n$ for $n=1,2,\dots$). Z. Nehari [1957] proved that the problem is solvable if and only if the Hankel matrix $(c_{j+k+1})_{j,k=0}^{\infty}$ has norm at most 1 as an operator on l^2. Using a method akin to the operator approach to the Hamburger moment problem, Adamyan, Arov and Krein proved that finding a solution f, in case one exists, is tantamount to finding a unitary extension of a certain isometric operator constructed from the data. What is more, the family of all such solutions is in one-to-one correspondence with the family of all such unitary extensions (satisfying a minimality requirement), a connection that enabled them to derive a linear-fractional parameterization of the set of all solutions in case the problem is indeterminate. For the indeterminate Nevanlinna–Pick problem, a linear-fractional parameterization of the solution set was found by Nevanlinna [1919] on the basis of the Schur algorithm, a technique invented by I. Schur [1917] in connection with the Carathéodory–Fejér problem. Nevanlinna's parameterization, and the corresponding one implicit in Schur's paper, can be deduced from the one of Adamyan, Arov and Krein, because one can show that the Nehari problem embraces the Carathéodory–Fejér and Nevanlinna–Pick problems.

There is a close connection between the commutant-lifting theorem and the work of Nehari and Adamyan, Arov and Krein. Nehari's theorem is in fact a corollary of the commutant-lifting theorem; the details can be found, for example, in [Sarason 1991]. In the other direction, it was recognized by D. N. Clark (unpublished notes) and N. K. Nikolskii [1986] that the theorem from [Sarason 1967] can be deduced very simply from Nehari's theorem, and more recently R. Arocena [1989] has shown how to give a proof of the commutant-lifting theorem using the methods of Adamyan, Arov and Krein. Further discussion can be found in [Sarason 1987; 1991].

There follows a brief description of some other approaches to interpolation problems.

• The Abstract Interpolation Problem of V. E. Katsnelson, A. Ya. Kheifets and P. M. Yuditskii [Katsnelson et al. 1987; Kheifets and Yuditskii 1994] is based on the approach of V. P. Potapov and coworkers to problems of Nevanlinna–Pick type [Kovalishina and Potapov 1974; Kovalishina 1974; 1983]. It abstracts the key elements of Potapov's theory to an operator setting and yields, upon specialization, a wide variety of classical problems. The model spaces of de Branges–Rovnyak play an important role in this approach. As was the case with the Adamyan–Arov–Krein treatment of the Nehari problem, the solutions of the Abstract Interpolation Problem correspond to the unitary extensions of a certain isometric operator. There is a unified derivation of the linear-fractional parameterizations of the solution sets of indeterminate problems.

• The approach favored by H. Dym [1989] emphasizes reproducing kernel Hilbert spaces, especially certain de Branges–Rovnyak spaces, and J-inner matrix functions. (The J here refers to a signature matrix, a square, self-adjoint, unitary matrix. A meromorphic matrix function in $\mathbb{D}$, with values of the same size as J, is called J-inner if it is J-contractive at each point of $\mathbb{D}$ where it is holomorphic, and its boundary function is J-unitary almost everywhere on $\partial\mathbb{D}$. These are the symplectic analogues of inner functions that, as mentioned earlier, have been analyzed by Potapov [1955].)

• J. A. Ball and J. W. Helton [1983] have developed a Krein space approach to interpolation problems. In their approach an interpolation problem, rather than being reinterpreted as an operator extension problem, is reinterpreted as a subspace extension problem in a suitable Krein space. Shift-invariant subspaces of vector H^2 spaces that are endowed with a Krein space structure arise. One of the key results is a Beurling-type theorem for such subspaces. A treatment of the Nehari problem using this method can be found in [Sarason 1987].

• J. Agler [1989] has, in a sense, axiomatized the Nevanlinna–Pick problem and obtained the analogue of Pick's criterion in two new contexts, interpolation by multipliers of the Dirichlet space (the space of holomorphic functions in $\mathbb{D}$ whose derivatives are square integrable with respect to area), and interpolation by bounded holomorphic functions in the bidisk.

• M. Cotlar and C. Sadosky [1994] have used their theory of Hankel forms to attack problems of Nevanlinna–Pick and Nehari type in the polydisk.

• B. Cole, K. Lewis and J. Wermer have attacked problems of Nevanlinna–Pick type from the perspective of uniform algebras [Cole et al. 1992].

The foregoing list is but a partial sample of the enormous activity surrounding interpolation problems.

Systems Theory

In the early 1970's J. W. Helton became aware that there is a large overlap between the mathematics of linear systems theory and the operator theory that had grown around dilation theory and interpolation problems. In an April 17, 1972 letter to the author, he wrote: "I've spent the year learning engineering systems which at some levels is almost straight operator theory. Some of the best functional analysis (Krein, Livsic) has come from engineering institutes and I'm beginning to see why. Such collaboration does not exist in this country.... The [Sz.-]Nagy–Foiaş canonical model theory is precisely a study of infinite dimensional discrete time systems which lose and gain no energy."

Helton embarked upon a program to bridge the chasm between operator theorists and engineers in the United States. The result has been an enrichment of both mathematics and engineering. The systems theory viewpoint now permeates a large part of operator theory. On the engineering side, a new subject, H^∞ control, has sprung up [Francis 1987].

Bergman Spaces and the Bergman Shift

The mathematics discussed above flows, in large part, from Beurling's theorem via its generalization to vector H^2 spaces, in other words, to multiple shifts. Another natural direction for exploration unfolds when one replaces the shift, not by a multiple version of itself, but by its analogue (multiplication by z) on a holomorphic space of scalar functions other than H^2. There are countless possibilities for this other space; one that has turned out to be especially interesting is the Bergman space.

What was just referred to as "the" Bergman space is really just the most immediate member of a large family of spaces. Given a bounded domain in the complex plane and a positive number p, the Bergman space with exponent p for the domain consists of the holomorphic functions in the domain that are p-th power integrable with respect to area. These spaces are named for S. Bergman because the ones with exponent 2, which are Hilbert spaces, played a fundamental role in much of his work [Bergman 1970]. In the unit disk, the Bergman space with exponent p is denoted by A^p (or B^p, or L^p_a), and it is given the norm (or "norm," if $p < 1$) inherited from L^p of normalized area measure on the disk.

The powers of z form an orthogonal basis for the Hilbert space A^2, the norm of z^n being $1/\sqrt{n+1}$. Thus, a holomorphic function in $\mathbb{D}$ belongs to A^2 if and only if its Taylor coefficients at the origin are square summable when weighted against the sequence $\big(1/(n+1)\big)_{n=0}^{\infty}$. From this one can see that functions in A^2 need not possess boundary values in the usual sense of nontangential limits. For example, the function $\sum_{n=0}^{\infty}(n+1)^{-1/2}z^n$ is in A^2, but its coefficients are not square summable. A probabilistic argument shows that by randomly changing the signs of the coefficients of such a function, one almost surely obtains a function, obviously also in A^2, failing at almost every point of $\partial\mathbb{D}$ to have a radial limit (details are in [Duren 1970, p. 228]). With more work one can make a similar argument for any of the spaces A^p. Thus, in the study of Bergman spaces, one of the main techniques used to study Hardy spaces, the reliance on boundary functions, is lacking.

The Bergman shift is the operator on A^2 of multiplication by z. It acts on the orthonormal basis $\big(\sqrt{n+1}\ z^n\big)_{n=0}^{\infty}$ by sending the n-th basis vector to $\sqrt{(n+1)/(n+2)}$ times the $(n+1)$-st basis vector. It thus belongs to the class of weighted shifts, a seemingly restricted family of operators that exhibit surprisingly diverse behavior [Shields 1974]. A natural problem, in view of Beurling's theorem, is that of classifying the shift-invariant subspaces of A^2 (hereafter called just invariant subspaces of A^2). Part of that problem, and a natural starting place, is the problem of describing the zero sequences for A^2 functions, because associated with each zero sequence is the invariant subspace of functions in A^2 that vanish on it.

Significant progress in understanding A^p zero sequences was made by C. Horowitz [1974]. Among his results: (1) For $p<q$, there are A^p zero sequences that are not A^q zero sequences. This contrasts with the situation for the Hardy spaces: for every p, the H^p zero sequences are just the Blaschke sequences. (2) There exist two A^p zero sequences whose union is not an A^p zero sequence. Taking $p=2$, one obtains an example of a pair of nontrivial invariant subspaces of A^2 whose intersection is trivial, a phenomenon that does not occur in the space H^2. (3) Every subsequence of an A^p zero sequence is an A^p zero sequence. To elaborate, suppose $(z_k)_{k=1}^{\infty}$ is a subsequence of the zero sequence of the A^p function f. For each k, let b_k be the Blaschke factor for the point z_k. Horowitz showed that the infinite product $h=\prod b_k\,(2-b_k)$ converges, and that f/h is again in A^p. This furnishes an analogue of F. Riesz's Hardy space technique of dividing out zeros, but with a divisor h that need not itself be in A^p.

The Bergman shift belongs to a class, called $\mathbf{A}_{\aleph_0}$, studied by C. Apostol, H. Bercovici, C. Foiaş, C. Pearcy, and others [Apostol et al. 1985; Bercovici et al. 1985]. On the basis of the theory of what the preceding authors call dual algebras, they established certain properties of the lattice of invariant subspaces of A^2 that raise pessimism over the prospects of ever understanding that lattice well. Apostol, Bercovici, Foiaş and Pearcy showed that for each n between 1 and $\aleph_0$ there is an invariant subspace M of A^2 such that zM has codimension n in M; that the lattice of invariant subspaces of A^2 has a sublattice isomorphic to the lattice of all the subspaces of a Hilbert space of dimension $\aleph_0$; and, to strengthen one of Horowitz's results, that there is a family of invariant subspaces of A^2 with the cardinality of the continuum any two of which have a trivial intersection. The lattice of invariant subspaces of A^2 thus differs in striking ways from Beurling's lattice. It appears to be considerably "wilder."

Despite the preceding results, a substantially clearer picture of the invariant subspaces of A^2 has emerged in the past few years. The initial breakthrough came in a paper of H. Hedenmalm [1991]. In H^2, the normalized inner function (normalized in the sense that its first nonvanishing Taylor coefficient at 0 is positive) associated with a nontrivial invariant subspace solves a certain extremal problem: if N is the smallest integer such that some function in the subspace has a nonvanishing N-th derivative at the origin, then it maximizes $\operatorname{Re} f^{(n)}(0)$ among all functions f in the subspace having unit norm. Hedenmalm examined the corresponding functions associated with invariant subspaces of A^2. In case the invariant subspace consists of the functions that vanish along a given A^2 zero sequence, he showed that every function in the subspace is divisible in A^2 by the extremal function, the norm of the quotient being no larger than the norm of the original function. One can thus divide out the zeros of an A^2 function in such a way that both terms of the factored function are in A^2. For a general invariant subspace, Hedenmalm showed that the contractive divisibility property holds at least for functions in the invariant subspace generated by the extremal function. Hedenmalm's extremal functions appear to be natural A^2 analogues of inner functions.

Hedenmalm's results were quickly extended to general A^p spaces. This required new techniques and insights, provided by P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg [Duren et al. 1993; 1994]; see also [Hedenmalm 1994; Khavinson and Shapiro 1994]. An interesting aspect of this work is the role played by the biharmonic Green function, the Green function in $\mathbb{D}$ for the square of the Laplacian. The positivity of that function turns out to be the key to contractive divisibility. A. Aleman, S. Richter and Sundberg [Aleman et al. 1996], using related techniques, have shown that at least one vestige of Beurling's theorem carries over to A^2: if M is an invariant subspace of A^2, then M is generated as an invariant subspace by $M \ominus zM$.

By the index of an invariant subspace M of A^2 one means the dimension of $M \ominus zM$. As mentioned above, Apostol, Bercovici, Foiaş and Pearcy showed using the theory of dual algebras that this number can take any value between 1 and $\aleph_0$. Their argument, since it applies to a general class of operators, does not give insight into the mechanism behind the phenomenon for the Bergman shift. Hedenmalm [1993] responded by giving an explicit example of an invariant subspace of A^2 with index 2; his argument is based on K. Seip's characterization of sampling and interpolating sequences in A^2 [Seip 1993]. Subsequently, Hedenmalm, Richter and Seip [Hedenmalm et al. 1996] gave explicit examples of invariant subspaces of all indices in A^2. In the other direction, Aleman and Richter [1997] showed that an invariant subspace of A^2 will have index 1 if it contains a reasonably well-behaved function, for example, a function in the Nevanlinna class.

There has been much additional recent progress on invariant subspaces of A^2, and on other aspects of Bergman spaces, and many important contributions and contributors (B. I. Korenblum and K. Zhu, to name just two) are not mentioned in the discussion above, which is meant merely to present a sample of the work from this intensely active field.

Dirichlet Spaces and the Dirichlet Shift

The Dirichlet integral of a holomorphic function f in the unit disk is defined by

$$D(f) = \int_{\mathbb{D}} |f'|^2 \, dA ,$$

where A is normalized area measure on $\mathbb{D}$. The Dirichlet space, D, consists of those functions f for which $D(f)$ is finite. It is contained in H^2 and is a Hilbert space under the norm defined by

$$\|f\|_D^2 = \|f\|_2^2 + D(f) ,$$

(where $\|\cdot\|_2$ denotes the norm in H^2). The powers of z form an orthogonal basis for D, the norm of z^n being $\sqrt{n+1}$.

The Dirichlet shift is the operator on D of multiplication by z. It is another weighted shift, sending the n-th vector in the orthonormal basis $\bigl(z^n/\sqrt{n+1}\bigr)_{n=1}^{\infty}$ to $\sqrt{(n+2)/(n+1)}$ times the $(n+1)$-st vector. Because D is contained in the well-understood space H^2, one would expect the Dirichlet shift to be more manageable than the Bergman shift. Although that turns out to be the case, we are still a long way from a thorough understanding of the Dirichlet space and its invariant subspaces. For example, L. Carleson [1952] and later H. S. Shapiro and A. L. Shields [1962] long ago obtained information about the zero sequences of functions in D, but we still lack a characterization. More recently, L. Brown and Shields [1984] obtained information about the cyclic vectors of the Dirichlet shift, but, again, we still lack a characterization.

The Dirichlet shift belongs to a class of operators, called two-isometries, that arose in the work of J. Agler [1990]. A Hilbert space operator T is called a two-isometry if it satisfies $T^{*2}T^2 + I = 2T^*T$. S. Richter, in trying to understand the invariant subspaces of D, naturally began exploring general properties of two-isometries. From [Richter and Shields 1988] he knew that every nontrivial invariant subspace of D has index 1. In [Richter 1988] he proved a general result about two-isometries, which, together with the result just mentioned, implies that every nontrivial invariant subspace of D is cyclic; more precisely, if M is a nontrivial invariant subspace of D then M is generated as an invariant subspace by any nonzero vector in $M \ominus zM$. Thus, the restriction of the Dirichlet shift to any of its nontrivial invariant subspaces is a cyclic two-isometry. It is also what Richter calls analytic, meaning that the intersection of the ranges of its powers is trivial. A model theory for cyclic analytic two-isometries is developed in [Richter 1991].

Richter's model spaces are variants of the space D. Given a positive measure μ on $\partial\mathbb{D}$, let $P\mu$ denote its Poisson integral. For f a holomorphic function in $\mathbb{D}$, the Dirichlet integral of f with respect to μ is defined by

$$D_\mu(f) = \int_{\mathbb{D}} |f'|^2 \; P\mu \, dA \; .$$

The space $D(\mu)$ consists of all f such that $D_\mu(f)$ is finite. It is contained in H^2 and is a Hilbert space under the norm defined by

$$\|f\|_\mu^2 = \|f\|_2^2 + D_\mu(f) \; .$$

One obtains the space D by taking μ to be normalized Lebesgue measure on $\partial\mathbb{D}$. The space $D(0)$, corresponding to the zero measure, is defined to be just H^2. Richter's structure theorem states that S_μ, the operator of multiplication by z on $D(\mu)$, is a cyclic analytic two-isometry, and that any cyclic analytic two-isometry is unitarily equivalent to S_μ for a unique μ. The invariant spaces of D are thus modeled by certain of the spaces $D(\mu)$.

In collaboration with C. Sundberg, Richter continued the study of the spaces $D(\mu)$ [Richter and Sundberg 1991; 1992]. Among many other interesting results in those papers is a structure theorem for the invariant subspaces of $D(\mu)$: If M is a nontrivial invariant subspace of $D(\mu)$, then M has index 1, and if φ is a function of unit norm in $M \ominus zM$, then M is the isometric image of $D(|\varphi|^2\mu)$ under the operator of multiplication by φ. The functions φ in the preceding statement are the analogues in $D(\mu)$ of inner functions in H^2 and of Hedenmalm's extremal functions in A^2. The result of Richter and Sundberg reduces the problem of understanding the invariant subspace structure of $D(\mu)$ to the problem of understanding the structure of these extremal functions. This awaits further study, although interesting progress for the space D was made in [Richter and Sundberg 1994]. In particular, the authors showed that the extremal functions in D are contractive multipliers of D. This contrasts with Hedenmalm's extremal functions in A^2, which are expansive multipliers in the sense that multiplication by one of them does not decrease the A^2 norm of any polynomial. A study of S. M. Shimorin [1995] sheds further light on this phenomenon.

Hankel Operators

A Hankel matrix is a square matrix (finite or infinite) with constant cross diagonals, in other words, a matrix whose (j,k)-th entry depends only on the sum $j+k$. A famous example is the Hilbert matrix, $\big(1/(j+k+1)\big)_{j,k=0}^{\infty}$. According to [Hardy et al. 1952, p. 226], D. Hilbert proved in his lectures that this matrix induces a bounded operator on l^2. I. Schur [1911] showed that the sharp bound is π and W. Magnus [1950] that the spectrum is $[0,\pi]$ and purely continuous.

It was noted earlier that semi-infinite Hankel matrices arise in the Nehari interpolation problem. The theorem of Nehari (page) characterizes their boundedness as operators on l^2. Nehari's result will now be restated in slightly different language. By a Hankel operator we shall mean an operator on H^2, possibly unbounded, whose domain contains the vectors in the standard orthonormal basis, and whose matrix in that basis is a Hankel matrix. With each function φ in L^2 of the circle we associate such an operator, which we denote by Γ_φ; it is the one whose matrix has as its (j,k)-th entry the Fourier coefficient $\hat{\varphi}(-j-k-1)$, where $j,k=0,1,2,\dots$. Thus, Γ_φ depends only on $P_-\varphi$, the projection of φ onto $L^2\ominus H^2$. One calls φ a symbol of Γ_φ. Each Hankel operator Γ is Γ_φ for some φ; one can take for φ the image of $\Gamma 1$ under the "reflection" operator R, the operator on L^2 defined by $(Rf)\left(e^{i\theta}\right)=e^{-i\theta}f\left(e^{-i\theta}\right)$ (which sends H^2 onto $L^2\ominus H^2$, and vice versa). But as a symbol for Γ one can also take any function differing from $R\Gamma 1$ by a function in H^2. For example, the function $\varphi\left(e^{i\theta}\right)=i(\theta-\pi)$ $(0<\theta<2\pi)$ is a symbol for the operator whose matrix is the Hilbert matrix.

One easily sees that the action of Γ_φ, on polynomials, say, consists of multiplication by φ, followed by projection onto $L^2\ominus H^2$ (i.e., application of P_-), followed by reflection (i.e., application of R). In particular, if φ is bounded then so is Γ_φ, with norm at most $\|\varphi\|_\infty$. (Thus, because the operator corresponding to the Hilbert matrix has a symbol of supremum norm π, the norm of the Hilbert matrix is at most π, which is part of Schur's result.) Nehari's theorem, expressed qualitatively, states that a Hankel operator is bounded if and only if it has a bounded symbol. (The quantitative version adds that there exists a symbol whose supremum norm is the norm of the operator.) A companion theorem of P. Hartman [1958] states that a Hankel operator is compact if and only if it has a continuous symbol (of supremum norm arbitrarily close to the norm of the operator). Together with C. Fefferman's characterization of functions of bounded mean oscillation [Fefferman 1971; Garnett 1981] and a related characterization of functions of vanishing mean oscillation [Sarason 1975], these two theorems give the following boundedness and compactness criteria for Hankel operators: Γ_φ is bounded if and only if $P_-\varphi$ is in BMO, the space of functions of bounded mean oscillation on $\partial\mathbb{D}$, and Γ_φ is compact if and only if $P_-\varphi$ is in VMO, the space of functions of vanishing mean oscillation on $\partial\mathbb{D}$. The condition for a Hankel operator to have finite rank was found by L. Kronecker long ago [1881]: Γ_φ has finite rank if and only if $P_-\varphi$ is a rational function.

By the conjugate-analytic symbol of Γ_φ one means the function $P_-\varphi$, the unique symbol in $L^2\ominus H^2$. The results of Kronecker–Nehari–Hartman thus relate certain basic properties of a Hankel operator to the structure of its conjugate-analytic symbol. This illustrates a recurrent theme in concrete operator theory. Typically in this subject, one is given a natural class of operators induced in some way or other by certain functions, often referred to as the symbols of the operators they induce. One wants to understand how the properties of the operators are encoded by their inducing functions. The Kronecker–Nehari–Hartman results suggest the problem of characterizing the Hankel operators that belong to the Schatten class $\mathcal{S}_p$, the class of compact operators whose singular values are p-th-power summable, where $0 < p < \infty$.

The preceding question for $\mathcal{S}_2$, the Hilbert–Schmidt class, is easy, because an operator is in $\mathcal{S}_2$ if and only if its matrix entries (in any orthonormal basis) are square summable. It follows that a Hankel operator is in $\mathcal{S}_2$ if and only if its conjugate-analytic symbol is the image under the reflection operator R of (the boundary function of) a function in D, the Dirichlet space. The question for general $\mathcal{S}_p$ remained a mystery for a long time despite some suggestive work of M. Rosenblum and J. Howland [Howland 1971] pertaining to $\mathcal{S}_1$, the trace class. The breakthrough came in 1979 when V. V. Peller found the condition for a Hankel operator to be in $\mathcal{S}_1$. A short time later he handled the case of $\mathcal{S}_p$ for $p \geq 1$ [Peller 1980]. His result says that Γ_φ belongs to $\mathcal{S}_p$ if and only if $P_-\varphi$ belongs to a certain Besov space (the space $B_p^{1/p}$). This was extended to $0 < p < 1$ independently by Peller [1983] and S. Semmes [1984]. The results have interesting applications to prediction theory and to rational approximation, which can be found in [Peller and Khrushchëv 1982; Khrushchëv and Peller 1986].

Since Peller's work it has been recognized that results like his and those of his predecessors hold in many other settings. For Hankel operators on the Bergman space, A^2, for instance, boundedness and compactness criteria have been established by S. Axler [1986], K. Zhu [1987], and K. Stroethoff [1990], and Schatten class criteria by J. Arazy, S. Fisher and J. Peetre [Arazy et al. 1988]. More information on this and related matters can be found in [Zhu 1990].

The spectral theory of Hankel operators has been developed to an extent, notably by S. C. Power, whose book [1982] can be consulted for information and references. There have been some interesting developments since that book appeared. Power [1984] showed that there are no nontrivial nilpotent Hankel operators and raised the question whether there are any quasinilpotent ones. A. V. Megretskii [1990] used a clever construction to answer that question in the affirmative. In a very nice paper [Megretskii et al. 1995], Megretskii, Peller and S. R. Treil have given a spectral characterization of self-adjoint Hankel operators, that is, a set of necessary and sufficient conditions, expressed solely in terms of spectral data, for a self-adjoint operator to be unitarily equivalent to a Hankel operator. Ideas from systems theory, especially the notion of a balanced realization, play a prominent role in their analysis.

Toeplitz Operators

This vast subject cannot be adequately addressed in an article such as this one. Only a few highlights will be touched on. The excellent book of A. Böttcher and B. Silbermann [1990] can be consulted for further information.

A Toeplitz matrix is a square matrix with constant diagonals, in other words, a matrix whose (j,k)-th entry depends only on the difference $j-k$. Toeplitz operators can be introduced in many settings, but here the focus will be on the classical Toeplitz operators, which are the bounded operators on H^2 whose matrices with respect to the standard orthonormal basis are Toeplitz matrices. With each such operator is associated a unique symbol, namely, the function φ in L^2 such that the (j,k)-th entry of the corresponding Toeplitz matrix equals the Fourier coefficient $\hat{\varphi}(j-k)$. It is a result of P. Hartman and A. Wintner [1950] that φ is actually in L^∞, with supremum norm equal to the norm of the corresponding operator. The Toeplitz operator with symbol φ is denoted by T_φ. It is the compression to H^2 of the operator on L^2 of multiplication by φ; in other words, it acts on a function in H^2 as multiplication by φ followed by projection onto H^2.

Toeplitz operators are discrete versions of Wiener–Hopf operators, that is, integral operators on $L^2(0,\infty)$ whose kernels, which are functions on $(0,\infty)\times(0,\infty)$, depend only on the difference of the arguments (and so have the form $(x,y)\mapsto K(x-y)$, where K is a function on $\mathbb{R}$). There is in fact more than an analogy here, because the unitary map from H^2 to $L^2(0,\infty)$ mentioned earlier (in the discussion of Beurling's theorem) transforms Toeplitz operators to Wiener–Hopf operators (whose kernels, in general, can be distributions). The preceding observation was first made by M. Rosenblum [1965] and A. Devinatz [1967].

G. Szegö [1920; 1921] has already been mentioned as one of the pioneers of our subject. Among other things, he studied the asymptotic behavior of finite sections of Toeplitz matrices, a line of investigation that continues to the present [Böttcher and Silbermann 1990; Basor and Gohberg 1994].

Self-adjoint Toeplitz operators are well understood. The operator T_φ is self-adjoint if and only if φ is real valued. In that case, according to a theorem of Hartman and Wintner [1954], the spectrum of T_φ is the closed interval whose lower endpoint is the essential infimum of φ and whose upper endpoint is the essential supremum of φ. A concrete spectral representation of T_φ, for φ real, has been given in [Rosenblum 1965; Rosenblum and Rovnyak 1985]. If the essential range of φ is contained in a line, then T_φ is a linear function of a self-adjoint operator and so is described by Rosenblum's theorem, but that is the only time T_φ can be a normal operator, according to a result of A. Brown and P. R. Halmos [1963]. In investigating Toeplitz operators, therefore, one is largely beyond the scope of classical spectral theory.

There is no description of the spectrum of T_φ for general φ. Two general facts are known. One is the spectral inclusion theorem of Hartman and Wintner [1950], which states that the spectrum of T_φ always contains the essential range of φ. The other is a deep result of H. Widom [1964]: the spectrum of T_φ is always connected.

L. A. Coburn [1966] has observed that a nonzero Toeplitz operator cannot have both a trivial kernel and a trivial cokernel. Thus, a Toeplitz operator will be invertible if it is a Fredholm operator of index 0, a fact that has been useful in attempts to determine the spectrum of T_φ. A great many theorems have been obtained saying, roughly, that if φ belongs to such-and-such a class, then T_φ is a Fredholm operator if and only if the origin is not in the "range" of φ, in which case the index of T_φ is the negative of the "winding number" of φ about the origin. Here, "range" is usually interpreted in some generalized sense, and "winding number" is interpreted accordingly. The original and simplest version is where φ is continuous, in which case range and winding number have their usual meanings. That result goes back to I. C. Gohberg [1952], M. G. Krein [1958], Widom [1960], and Devinatz [1964]. The book of Böttcher and Silbermann [1990] contains many other versions.

Toeplitz operators have interacted strongly with the theory of operator algebras. The works of R. G. Douglas [1972; 1973] are early examples of the interaction. There has been much subsequent work, a discussion of which would be beyond this author's competence.

In another direction, a theory of similarity models for Toeplitz operators with rational symbols has been developed by D. N. Clark [1981; 1982] (see also earlier papers referenced there). The following theorem from [Clark and Morrel 1978] illustrates the kind of results obtained: Let φ be a rational function that is univalent in some annulus $\rho \leq |z| \leq 1$ and whose restriction to $\partial\mathbb{D}$ has winding number 1 about each point of the interior of $\varphi(\partial\mathbb{D})$. Then T_φ is similar to T_ψ, where ψ is a Riemann map of the unit disk onto the interior of $\varphi(\partial\mathbb{D})$. The theory has been extended by D. M. Wang [1984] and D. V. Yakubovich [1989; 1991] to encompass nonrational symbols satisfying certain smoothness and topological requirements.

It is unknown whether every Toeplitz operator has a nontrivial invariant subspace. The strongest result to date, due to V. V. Peller [1993], gives an affirmative answer for Toeplitz operators with piecewise continuous symbols satisfying certain extra conditions.

I hope that the scattered remarks above give at least an inkling of the richness of the subject of Toeplitz operators. There is much more that could be said. Toeplitz operators on vector H^p spaces have been studied extensively and are discussed in the book of Böttcher and Silbermann. Besides that, Toeplitz operators arise naturally in many other settings, and one is likely to find several papers on them reviewed in Section 47 of any recent issue of *Mathematical Reviews*.

Holomorphic Composition Operators

Associated with each holomorphic self-map φ of $\mathbb{D}$ is the corresponding composition operator, C_φ; it acts on any function f defined in $\mathbb{D}$ according to the formula $(C_\varphi f)(z) = f\big(\varphi(z)\big)$. A classical result, the surbordination principle of J. E. Littlewood [1925], guarantees that C_φ acts boundedly on the Hardy spaces H^p and on the Bergman spaces A^p.

The study of holomorphic composition operators, from an operator-theoretic viewpoint, does not date back as far as the study of Toeplitz operators or the study of Hankel operators. It is now thriving, thanks in large part to the influence of J. H. Shapiro, especially to a beautiful theorem of his characterizing when C_φ acts compactly on the spaces H^p. Shapiro began studying compact composition operators in [Shapiro and Taylor 1973/74], where, among other results, the authors proved that if C_φ is compact on one of the spaces H^p then it is compact on every H^p. They also obtained a necessary condition for C_φ to be compact on H^p, namely, that φ not possess an angular derivative in the sense of Carathéodory (an ADC, for short) at any point of $\partial\mathbb{D}$. (One says φ has an ADC at the point λ of $\partial\mathbb{D}$ if φ has a nontangential limit of unit modulus at λ, and the difference quotient $\big(\varphi(z) - \varphi(\lambda)\big)/(z - \lambda)$ has a nontangential limit at λ.) B. D. MacCluer and Shapiro [1986] showed that the angular derivative condition is not sufficient for the compactness of C_φ on the spaces H^p, although it is both necessary and sufficient in the spaces A^p. They also showed that if φ is univalent then the angular derivative condition does imply C_φ acts compactly on the spaces H^p.

Shapiro's characterization came a year later [1987]. It is quite easy to see that C_φ is compact on the spaces H^p if φ assumes no values near $\mathbb{D}$, in other words, if $\|\varphi\|_\infty < 1$. Shapiro's necessary and sufficient condition is, roughly, that φ not take too many values near $\mathbb{D}$ too often. This is quantitized by means of the Nevanlinna counting function, which is a device from Nevanlinna theory that gives a biased measure of the number of times φ assumes a given value w. The Nevanlinna counting function of φ is defined by

$$N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|} \qquad \text{for } w \in \mathbb{D}.$$

In the sum on the right side, the points z in $\varphi^{-1}(w)$ are counted with multiplicities, and the sum is interpreted to be 0 if w is not in $\varphi(\mathbb{D})$. The convergence of the series, in case $\varphi^{-1}(w)$ is infinite, follows by the Blaschke condition (assuming φ is not the constant function w). Shapiro's theorem: *C_φ is compact on the spaces H^p if and only if $N_\varphi(w)/\log\frac{1}{|w|} \to 0$ as $|w| \to 1$.* This remarkable theorem has inspired much additional interesting work. To mention just two results, D. H. Luecking and K. Zhu [1992] have characterized the functions φ for which C_φ belongs to one of the Schatten classes as an operator on H^2 or on A^2, and P. Poggi–Corradini [1997] has characterized those univalent φ for which C_φ is a Riesz operator (an operator with essential spectral radius 0) on H^2.

Shapiro's book [1993] is a delightful elementary account of his theorem and related issues. C. C. Cowen and MacCluer [1995] have written a comprehensive treatment of holomorphic composition operators.

Composition operators played a key role in L. de Branges's renowned proof of the Bieberbach–Robertson–Milin conjectures [de Branges 1985; 1987]. If the function φ is univalent and vanishes at the origin, the operator C_φ acts contractively in D, the Dirichlet space, and in certain closely related spaces (with indefinite metrics). De Branges recognized that the Robertson and Milin conjectures can be interpreted as norm inequalities involving composition operators, an observation that underlies his approach. Although the operator methodology that led de Branges to his proof has been discarded in many accounts, its appearance is really quite natural, at least to someone reared as a functional analyst. De Branges's ideas have been further developed in [Vasyunin and Nikolskii 1990; 1991].

Subnormal Operators

A Hilbert space operator is said to be subnormal if it has a normal extension, in other words, if there is a normal operator acting on a space containing the given one as a subspace and coinciding with the given operator in that subspace. (The given Hilbert space is thus an invariant subspace of the normal operator.) The notion was introduced by P. R. Halmos [1950].

Among subnormal operators are of course all normal ones, but also all isometries, and all analytic Toeplitz operators, i.e., Toeplitz operators whose symbols are in H^∞, on both H^2 and A^2. In particular, the unilateral shift and the Bergman shift are subnormal. These examples are all fairly evident, but some subnormal operators appear in disguised form. For instance, there are Toeplitz operators on H^2 that are subnormal yet neither normal nor analytic [Cowen and Long 1984; Cowen 1986]. There are composition operators on H^2 whose adjoints are subnormal, for nonobvious reasons [Cowen and Kriete 1988]. An unexpected and particularly striking example of a subnormal operator is the Cesàro operator, the operator on l^2 that sends the sequence $(x_n)_{n=0}^\infty$ to the sequence $(y_n)_{n=0}^\infty$ defined by $y_n = \frac{1}{n+1}\sum_{k=0}^n x_k$ [Kriete and Trutt 1971; Cowen 1984].

With each positive compactly supported measure μ in the complex plane, there is naturally associated a subnormal operator, the operator of multiplication by z on $P^2(\mu)$, the closure of the polynomials in $L^2(\mu)$. One easily sees on the basis of the spectral theorem that such operators model all cyclic subnormal operators. Thus, the study of subnormal operators quickly leads to questions about polynomial approximation and thence to questions about approximation by rational functions. A large portion of J. B. Conway's comprehensive account of subnormal operators [1991] is devoted to the subject of rational approximation.

The invariant subspace problem for subnormal operators, the question whether every subnormal operator has a nontrivial invariant subspace, because it has an obvious positive answer for noncyclic operators, is equivalent to the question whether, for every measure μ as above, the space $P^2(\mu)$ has a nontrivial invariant subspace (under multiplication by z). In case $P^2(\mu) = L^2(\mu)$ the answer is clear. In the contrary case, one would like to understand the mechanism behind the inequality $P^2(\mu) \neq L^2(\mu)$. For example, does it force the functions in $P^2(\mu)$ to behave, in some sense or other, like holomorphic functions, as happens when μ is Lebesgue measure on $\partial\mathbb{D}$ (in which case $P^2(\mu) = H^2$) and when μ is area measure on $\mathbb{D}$ (in which case $P^2(\mu) = A^2$)? One is thus led to the question: If $P^2(\mu) \neq L^2(\mu)$, does $P^2(\mu)$ carry some kind of analytic structure?

To settle the invariant subspace question, though, one could settle for much less than analytic structure. A point w in the plane is called a bounded point evaluation for $P^2(\mu)$ if the linear functional $p \mapsto p(w)$ on polynomials is bounded in the norm of $L^2(\mu)$. In that case the functional extends boundedly to $P^2(\mu)$; in other words, it is meaningful to evaluate the functions in $P^2(\mu)$ at w. Moreover, the subspace of functions in $P^2(\mu)$ that vanish at w is a nontrivial invariant subspace. (We are ignoring here the irrelevant case where μ is a point mass.) The question thus arises: If $P^2(\mu) \neq L^2(\mu)$, does $P^2(\mu)$ possess bounded point evaluations?

Until recently the preceding question was open, the strongest partial results being due to J. E. Brennan [1979a; 1979b] (and earlier papers). It is now known that the answer is positive—more on that presently. The invariant subspace problem for subnormal operators was eventually settled using a different tack in S. W. Brown's dissertation [1978a], surely one of the most influential dissertations in operator theory ever written. (The published version is [Brown 1978b].) Brown sidestepped $P^2(\mu)$ by working instead with $P^\infty(\mu)$, the weak-star closure of the polynomials in $L^\infty(\mu)$. The structure of $P^\infty(\mu)$ was well understood at the time of Brown's work, and in particular it was known that $P^\infty(\mu)$ possesses weak-star continuous point evaluations whenever it is not all of $L^\infty(\mu)$. By making various reductions, Brown was able to narrow the invariant subspace question for $P^2(\mu)$ to the case where $P^\infty(\mu)$ is just H^∞ of the unit disk, and $P^2(\mu)$ admits no bounded point evaluations. He showed in that case that the evaluation functionals on H^∞ at the points of $\mathbb{D}$ have spatial representations of a certain simple kind in $P^2(\mu)$, from which the existence of nontrivial invariant subspaces follows immediately.

The underlying concept guiding Brown's work was that of an algebra of operators on a Hilbert space H that is closed in the weak-star topology that the algebra of all operators on H acquires as the dual of $\mathcal{S}_1$, the space of trace-class operators on H. Such an algebra is then the dual of a certain quotient space of $\mathcal{S}_1$. Of particular interest is the unital weak-star-closed algebra $\mathcal{A}_T$ generated by a single operator T on H. For example, if T is the unilateral shift on H^2 then $\mathcal{A}_T$ consists of the algebra of analytic Toeplitz operators and so is a replica of

H^∞. Brown's philosophy was that enough information about the structure of $\mathcal{A}_T$ should enable one to find a nontrivial invariant subspace of T.

It was quickly realized that Brown's basic ideas, including his method for constructing spatial representations, apply far beyond the realm of subnormal operators. The outcome has been the theory of dual algebras [Bercovici et al. 1985], mentioned earlier in connection with the Bergman shift. Many existence theorems for invariant subspaces have resulted from this program. One striking example: A Hilbert space contraction whose spectrum contains the unit circle has a nontrivial invariant subspace, as shown by Brown, B. Chevreau and C. Pearcy [Brown et al. 1988]; see also H. Bercovici [1990].

After Brown's breakthrough, the existence question for bounded point evaluations on $P^2(\mu)$ remained open for over ten years. It was settled, with a vengeance, by J. E. Thomson [1991] (see also the last chapter of [Conway 1991]). A point in the plane is called an analytic bounded point evaluation of $P^2(\mu)$ if it belongs to an open set of bounded point evaluations on which the functions in $P^2(\mu)$ are holomorphic. Thomson showed that, if $P^2(\mu) \neq L^2(\mu)$, then $P^2(\mu)$ not only has bounded point evaluations, it has an abundance of analytic bounded point evaluations. He obtained a structure theorem for $P^2(\mu)$ saying, very roughly, that $P^2(\mu)$ can be decomposed into the direct sum of an L^2 space and a space of holomorphic functions. His proof is a tour de force involving powerful and delicate techniques from the theory of rational approximation plus a variant of Brown's basic construction.

Thomson's result shows, paradoxically, that the situation in which Brown originally applied his technique ($P^\infty(\mu) = H^\infty$, yet $P^2(\mu)$ has no bounded point evaluations) is in fact void. Even theorems about the empty set, it seems, can contain interesting ideas.

Another long-standing question about subnormal operators was recently settled. It concerns the relation between subnormality and a related concept, hyponormality, also introduced by Halmos [1950] (although the current terminology was fixed later). A Hilbert space operator T is called hyponormal if the self-adjoint operator $T^*T - TT^*$ is positive semidefinite. A simple argument shows that every subnormal operator is hyponormal.

Although the inequality $T^*T - TT^* \geq 0$ might seem at first glance a rather weak condition to impose on an operator, it has unexpectedly strong implications. A substantial and very interesting theory of hyponormal operators has grown over the years, which, however, will not be discussed here. See [Putnam 1967; Clancey 1979; Vol'berg et al. 1990; Martin and Putinar 1989].

If an operator is subnormal then so are all of its powers. Halmos [1950] gave an example of a hyponormal operator whose square is not hyponormal, thus showing that hyponormality does not imply subnormality. S. K. Berberian raised the question of whether an operator is subnormal if all of its powers are hyponormal. This was answered in the negative by J. G. Stampfli [1965]; his counterexample is a bilateral weighted shift. At about that time the question arose whether an operator T is subnormal if it is polynomially hyponormal, in other words, if $p(T)$ is hyponormal for every polynomial T. This question resisted attack for over 25 years, until R. Curto and M. Putinar [1993] obtained a strong negative answer. Their analysis shows a close relation between the question and classical moment problems.

Several Variables

The discussion so far has dealt almost exclusively with the one-dimensional theory. The theory in several variables, while less well developed, is being energetically pursued and is maturing. Because this author's knowledge is limited, the remarks to follow are brief and incomplete.

The basic theory of Hardy spaces in the polydisk and the ball of $\mathbb{C}^N$ can be found in two books of W. Rudin [1969; 1980]. Some properties from one dimension, such as existence of boundary values, extend nicely. Lacking, however, is a version of the inner-outer factorization. In fact, inner functions in several variables are hard to deal with. In the polydisk it is easy to produce examples, but the general inner function is not well understood; some information is in [Rudin 1969]. In the ball it was an open question for a long time whether there are any nonconstant inner functions. That there are was eventually proved by A. B. Aleksandrov [1982] and E. Løw [1982]. Although inner functions do not play the same central role in the ball that they do in the disk, the ideas needed to prove their existence have had interesting repercussions; further information can be found in [Rudin 1986].

By an invariant subspace of H^2 of the polydisk one means a subspace that is invariant under multiplication by all of the coordinate functions. Many studies of these invariant subspaces have been made, with still very incomplete results. This is part of the emerging theory of multivariable spectral theory, the study of commuting N-tuples of operators. An interesting approach, the theory of Hilbert modules, was initiated by R. G. Douglas and is developed in [Douglas and Paulsen 1989]. The recent book of J. Eschmeier and M. Putinar [1996] emphasizes sheaf-theoretic methods. See also the articles in [Curto et al. 1995].

Hankel, Toeplitz, and composition operators in several variables have received a great deal of attention. A recent study of Hankel operators is [Arazy 1996]. The book [Upmeier 1996] concerns Toeplitz operators. Information on holomorphic composition operators in several variables can be found in [Cowen and MacCluer 1995].

The corona problem, solved for the unit disk by L. Carleson [1962], is one of the basic open problems in several complex variables. The corona problem for a domain in the plane or in $\mathbb{C}^N$ is the problem of deciding whether the points of the domain (more accurately, the evaluation functionals at these points) are dense in the Gelfand space of the Banach algebra of bounded holomorphic functions in the domain. In more concrete terms, it asks whether a finite set of bounded holomorphic functions in the domain must generate the whole algebra of bounded holomorphic functions as an ideal, if the given functions satisfy the obvious necessary condition, namely, that they do not tend to 0 simultaneously on any sequence in the domain. Carleson's positive solution for the unit disk, despite not having been previously mentioned in this narrative, is a landmark of twentieth century function theory. It would be hard to overestimate the amount of mathematics that has flowed from Carleson's proof.

Carleson's theorem was quickly extended to finitely connected domains in the plane and to finite bordered Riemann surfaces by various people. The problem for general planar domains is still open, although a positive solution is known for some infinitely connected domains, notably domains whose complements lie on the real axis [Garnett and Jones 1985]. B. Cole has constructed a Riemann surface for which the solution is negative; his example can be found in [Gamelin 1978, Chapter IV].

A few years after Carleson's proof, L. Hörmander [1967] pointed out the connection between the corona problem and the $\bar{\partial}$-equation. A positive solution to a corona problem can be reduced to the existence of bounded solutions of certain $\bar{\partial}$-equations. Despite many advances in $\bar{\partial}$-technology, the corona problems for the polydisk and the ball in several variables remain open.

References

[Adamyan et al. 1968a] V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, "Бесконечные ганкелевы матрицы и обобщённые задачи Каратеодори–Фейера и И. Шура", *Funkcional. Anal. i Priložen.* **2**:4 (1968), 1–17. Translated as "Infinite Hankel matrices and generalized Carathéodory–Fejér and I. Schur problems", *Functional Anal. Appl.* **2** (1968), 269–281.

[Adamyan et al. 1968b] V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, "О бесконечных ганкелевых матрицах и обобщённых задачах Каратеодори–Фейера и Ф. Рисса", *Funkcional. Anal. i Priložen.* **2**:1 (1968), 1–19. Translated as "Infinite Hankel matrices and generalized problems of Carathéodory–Fejér and F. Riesz", *Functional Anal. Appl.* **2** (1968), 1-18.

[Agler 1989] J. Agler, "Some interpolation theorems of Nevanlinna–Pick type", preprint, 1989.

[Agler 1990] J. Agler, "A disconjugacy theorem for Toeplitz operators", *Amer. J. Math.* **112**:1 (1990), 1–14.

[Agmon 1949] S. Agmon, "Sur un problème de translations", *C. R. Acad. Sci. Paris* **229** (1949), 540–542.

[Aleksandrov 1982] A. B. Aleksandrov, "The existence of inner functions in the ball", *Mat. Sb.* **118(160)** (1982), 147–163. In Russian.

[Aleman and Richter 1997] A. Aleman and S. Richter, "Some sufficient conditions for the division property of invariant subspaces in weighted Bergman spaces", *J. Funct. Anal.* **144**:2 (1997), 542–556.

[Aleman et al. 1996] A. Aleman, S. Richter, and C. Sundberg, "Beurling's theorem for the Bergman space", *Acta Math.* **177**:2 (1996), 275–310.

[Apostol et al. 1985] C. Apostol, H. Bercovici, C. Foiaş, and C. Pearcy, "Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra, I", *J. Funct. Anal.* **63**:3 (1985), 369–404.

[Arazy 1996] J. Arazy, "Boundedness and compactness of generalized Hankel operators on bounded symmetric domains", *J. Funct. Anal.* **137**:1 (1996), 97–151.

[Arazy et al. 1988] J. Arazy, S. D. Fisher, and J. Peetre, "Hankel operators on weighted Bergman spaces", *Amer. J. Math.* **110**:6 (1988), 989–1053.

[Arocena 1989] R. Arocena, "Unitary extensions of isometries and contractive intertwining dilations", pp. 13–23 in *The Gohberg anniversary collection* (Calgary, AB, 1988), vol. 2, edited by H. Dym et al., Oper. Theory Adv. Appl. **41**, Birkhäuser, Basel, 1989.

[Axler 1986] S. Axler, "The Bergman space, the Bloch space, and commutators of multiplication operators", *Duke Math. J.* **53**:2 (1986), 315–332.

[Ball and Helton 1983] J. A. Ball and J. W. Helton, "A Beurling-Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory", *J. Operator Theory* **9** (1983), 107–142.

[Ball and Kriete 1987] J. A. Ball and T. L. Kriete, III, "Operator-valued Nevanlinna–Pick kernels and the functional models for contraction operators", *Integral Equations Operator Theory* **10**:1 (1987), 17–61.

[Basor and Gohberg 1994] E. L. Basor and I. Gohberg (editors), *Toeplitz operators and related topics* (Santa Cruz, CA, 1992), Oper. Theory Adv. Appl. **71**, Birkhäuser, Basel, 1994.

[Bercovici 1990] H. Bercovici, "Notes on invariant subspaces", *Bull. Amer. Math. Soc. (N.S.)* **23**:1 (1990), 1–36.

[Bercovici et al. 1985] H. Bercovici, C. Foiaş, and C. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series in Mathematics **56**, Amer. Math. Soc., Providence, 1985.

[Bergman 1970] S. Bergman, *The kernel function and conformal mapping*, revised ed., Mathematical Surveys **5**, American Mathematical Society, Providence, 1970.

[Beurling 1949] A. Beurling, "On two problems concerning linear transformations in Hilbert space", *Acta Math.* **81** (1949), 239–255.

[Böttcher and Silbermann 1990] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Springer, Berlin, 1990.

[de Branges 1985] L. de Branges, "A proof of the Bieberbach conjecture", *Acta Math.* **154**:1-2 (1985), 137–152.

[de Branges 1987] L. de Branges, "Underlying concepts in the proof of the Bieberbach conjecture", pp. 25–42 in *Proceedings of the International Congress of Mathematicians* (Berkeley, 1986), Amer. Math. Soc., Providence, 1987.

[de Branges and Rovnyak 1966] L. de Branges and J. Rovnyak, "Canonical models in quantum scattering theory", pp. 295–392 in *Perturbation theory and its application in quantum mechanics* (Madison, 1965), edited by C. H. Wilcox, Wiley, New York, 1966.

[Brennan 1979a] J. E. Brennan, "Invariant subspaces and subnormal operators", pp. 303–309 in *Harmonic analysis in Euclidean spaces* (Williamstown, 1978), Part 1, edited by G. Weiss and S. Wainger, Proc. Sympos. Pure Math. **35**, Amer. Math. Soc., Providence, 1979.

[Brennan 1979b] J. E. Brennan, "Point evaluations, invariant subspaces and approximation in the mean by polynomials", *J. Functional Anal.* **34** (1979), 407–420.

[Brodskii 1956] M. S. Brodskiĭ, "Characteristic matrix functions of linear operators", *Mat. Sb. N.S.* **39(81)** (1956), 179–200. In Russian.

[Brodskii 1957] M. S. Brodskiĭ, "On a problem of I. M. Gel'fand", *Uspehi Mat. Nauk* (*N.S.*) **12**:2 (=74) (1957), 129–132. In Russian.

[Brown 1978a] S. W. Brown, *Banach algebras that are dual spaces*, Doctoral dissertation, University of California at Santa Barbara, 1978.

[Brown 1978b] S. W. Brown, "Some invariant subspaces for subnormal operators", *Integral Equations Operator Theory* **1**:3 (1978), 310–333.

[Brown and Halmos 1963] A. Brown and P. R. Halmos, "Algebraic properties of Toeplitz operators", *J. Reine Angew. Math.* **213**:1-2 (1963), 89–102.

[Brown and Shields 1984] L. Brown and A. L. Shields, "Cyclic vectors in the Dirichlet space", *Trans. Amer. Math. Soc.* **285**:1 (1984), 269–303.

[Brown et al. 1988] S. W. Brown, B. Chevreau, and C. Pearcy, "On the structure of contraction operators, II", *J. Funct. Anal.* **76**:1 (1988), 30–55.

[Carathéodory and Fejér 1911] C. Carathéodory and L. Fejér, "Über den Zusammenhang der extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landau'schen Satz", *Rend. Circ. Mat. Palermo* **32** (1911), 218–239.

[Carleson 1952] L. Carleson, "On the zeros of functions with bounded Dirichlet integrals", *Math. Z.* **56** (1952), 289–295.

[Carleson 1962] L. Carleson, "Interpolations by bounded analytic functions and the corona problem", *Ann. of Math.* (2) **76**:3 (1962), 547–559.

[Clancey 1979] K. Clancey, *Seminormal operators*, Lecture Notes in Math. **742**, Springer, Berlin, 1979.

[Clark 1981] D. N. Clark, "On the structure of rational Toeplitz operators", pp. 63–72 in *Contributions to analysis and geometry* (Baltimore, 1980), edited by D. N. Clark et al., Johns Hopkins Univ. Press, Baltimore, 1981.

[Clark 1982] D. N. Clark, "On Toeplitz operators with loops, II", *J. Operator Theory* **7**:1 (1982), 109–123.

[Clark and Morrel 1978] D. N. Clark and J. H. Morrel, "On Toeplitz operators and similarity", *Amer. J. Math.* **100**:5 (1978), 973–986.

[Coburn 1966] L. A. Coburn, "Weyl's theorem for nonnormal operators", *Michigan Math. J.* **13** (1966), 285–288.

[Cole et al. 1992] B. Cole, K. Lewis, and J. Wermer, "Pick conditions on a uniform algebra and von Neumann inequalities", *J. Funct. Anal.* **107**:2 (1992), 235–254.

[Conway 1991] J. B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs **36**, Amer. Math. Soc., Providence, 1991.

[Cotlar and Sadosky 1994] M. Cotlar and C. Sadosky, "Nehari and Nevanlinna–Pick problems and holomorphic extensions in the polydisk in terms of restricted BMO", *J. Funct. Anal.* **124**:1 (1994), 205–210.

[Cowen 1984] C. C. Cowen, "Subnormality of the Cesàro operator and a semigroup of composition operators", *Indiana Univ. Math. J.* **33**:2 (1984), 305–318.

[Cowen 1986] C. C. Cowen, "More subnormal Toeplitz operators", *J. Reine Angew. Math.* **367** (1986), 215–219.

[Cowen and Kriete 1988] C. C. Cowen and T. L. Kriete, III, "Subnormality and composition operators on H^2", *J. Funct. Anal.* **81**:2 (1988), 298–319.

[Cowen and Long 1984] C. C. Cowen and J. J. Long, "Some subnormal Toeplitz operators", *J. Reine Angew. Math.* **351** (1984), 216–220.

[Cowen and MacCluer 1995] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[Curto and Putinar 1993] R. E. Curto and M. Putinar, "Nearly subnormal operators and moment problems", *J. Funct. Anal.* **115**:2 (1993), 480–497.

[Curto et al. 1995] R. E. Curto, R. G. Douglas, J. D. Pincus, and N. Salinas (editors), *Multivariable operator theory* (Seattle, 1993), Contemporary Mathematics **185**, American Mathematical Society, Providence, 1995.

[Devinatz 1964] A. Devinatz, "Toeplitz operators on H^2 spaces", *Trans. Amer. Math. Soc.* **112** (1964), 304–317.

[Devinatz 1967] A. Devinatz, "On Wiener-Hopf operators", pp. 81–118 in *Functional Analysis* (Irvine, CA, 1966), edited by B. R. Gelbaum, Academic Press, London, and Thompson Book Co., Washington, DC, 1967.

[Donoghue 1957] J. Donoghue, W. F., "The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation", *Pacific J. Math.* **7** (1957), 1031–1035.

[Douglas 1972] R. G. Douglas, *Banach algebra techniques in operator theory*, Pure and Applied Mathematics **49**, Academic Press, New York, 1972.

[Douglas 1973] R. G. Douglas, *Banach algebra techniques in the theory of Toeplitz operators* (Athens, GA, 1972), CBMS Regional Conference Series in Mathematics **15**, American Mathematical Society, Providence, 1973.

[Douglas and Paulsen 1989] R. G. Douglas and V. I. Paulsen, *Hilbert modules over function algebras*, Pitman Research Notes in Mathematics Series **217**, Longman, Harlow, 1989.

[Duren 1970] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics **38**, Academic Press, New York, 1970.

[Duren et al. 1993] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg, "Contractive zero-divisors in Bergman spaces", *Pacific J. Math.* **157**:1 (1993), 37–56.

[Duren et al. 1994] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg, "Invariant subspaces in Bergman spaces and the biharmonic equation", *Michigan Math. J.* **41**:2 (1994), 247–259.

[Dym 1989] H. Dym, *J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*, CBMS Regional Conference Series in Mathematics **71**, Amer. Math. Soc., Providence, 1989.

[Dym 1994] H. Dym, "Review of 'The commutant lifting approach to interpolation problems' by C. Foias and A. E. Frazho", *Bull. Amer. Math. Soc.* (*N.S.*) **31** (1994), 125–140.

[Eschmeier and Putinar 1996] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs (N.S.) **10**, Oxford University Press, New York, 1996.

[Fatou 1906] P. Fatou, "Séries trigonometriques et séries de Taylor", *Acta Math.* **30** (1906), 335–400.

[Fefferman 1971] C. Fefferman, "Characterizations of bounded mean oscillation", *Bull. Amer. Math. Soc.* **77** (1971), 587–588.

[Foiaş and Frazho 1990] C. Foiaş and A. E. Frazho, *The commutant lifting approach to interpolation problems*, Oper. Theory Adv. Appl. **44**, Birkhäuser, Basel, 1990.

[Francis 1987] B. A. Francis, *A course in H_∞ control theory*, Lecture Notes in Control and Information Sciences **88**, Springer, Berlin, 1987.

[Gamelin 1978] T. W. Gamelin, *Uniform algebras and Jensen measures*, London Mathematical Society Lecture Note Series **32**, Cambridge Univ. Press, Cambridge, 1978.

[Garnett 1981] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics **96**, Academic Press, New York, 1981.

[Garnett and Jones 1985] J. B. Garnett and P. W. Jones, "The corona theorem for Denjoy domains", *Acta Math.* **155**:1-2 (1985), 27–40.

[Gohberg 1952] I. C. Gohberg, "On an application of the theory of normed rings to singular integral equations", *Uspehi Matem. Nauk* (*N.S.*) **7**:2 (=48) (1952), 149–156. In Russian.

[Halmos 1950] P. R. Halmos, "Normal dilations and extensions of operators", *Summa Brasil. Math.* **2** (1950), 125–134.

[Halmos 1961] P. R. Halmos, "Shifts on Hilbert spaces", *J. Reine Angew. Math.* **208** (1961), 102–112.

[Hardy 1915] G. H. Hardy, "On the mean value of the modulus of an analytic function", *Proc. London Math. Soc.* **14** (1915), 269–277.

[Hardy and Littlewood 1930] G. H. Hardy and J. E. Littlewood, "A maximal theorem with function theoretic implications", *Acta Math.* **54** (1930), 81–116.

[Hardy et al. 1952] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1952.

[Hartman 1958] P. Hartman, "On completely continuous Hankel matrices", *Proc. Amer. Math. Soc.* **9** (1958), 862–866.

[Hartman and Wintner 1950] P. Hartman and A. Wintner, "On the spectra of Toeplitz's matrices", *Amer. J. Math.* **72** (1950), 359–366.

[Hartman and Wintner 1954] P. Hartman and A. Wintner, "The spectra of Toeplitz's matrices", *Amer. J. Math.* **76** (1954), 867–882.

[Hedenmalm 1991] H. Hedenmalm, "A factorization theorem for square area-integrable analytic functions", *J. Reine Angew. Math.* **422** (1991), 45–68.

[Hedenmalm 1993] P. J. H. Hedenmalm, "An invariant subspace of the Bergman space having the codimension two property", *J. Reine Angew. Math.* **443** (1993), 1–9.

[Hedenmalm 1994] H. Hedenmalm, "A factoring theorem for the Bergman space", *Bull. London Math. Soc.* **26**:2 (1994), 113–126.

[Hedenmalm et al. 1996] H. Hedenmalm, S. Richter, and K. Seip, "Interpolating sequences and invariant subspaces of given index in the Bergman spaces", *J. Reine Angew. Math.* **477** (1996), 13–30.

[Hörmander 1967] L. Hörmander, "Generators for some rings of analytic functions", *Bull. Amer. Math. Soc.* **73** (1967), 943–949.

[Horowitz 1974] C. Horowitz, "Zeros of functions in the Bergman spaces", *Duke Math. J.* **41** (1974), 693–710.

[Howland 1971] J. S. Howland, "Trace class Hankel operators", *Quart. J. Math. Oxford Ser.* (2) **22** (1971), 147–159.

[Katsnelson et al. 1987] V. È. Katsnel'son, A. Y. Kheĭfets, and P. M. Yuditskiĭ, "An abstract interpolation problem and the theory of extensions of isometric operators", pp. 83–96, 146 in *Operators in function spaces and problems in function theory*, edited by V. A. Marchenko, Naukova Dumka, Kiev, 1987. In Russian; translation in *Topics in interpolation theory*, edited by Harry Dym et al., Oper. Theory Adv. Appl. **95**, Birkhäuser, Basel, 1997, pp. 283–298.

[Khavinson and Shapiro 1994] D. Khavinson and H. S. Shapiro, "Invariant subspaces in Bergman spaces and Hedenmalm's boundary value problem", *Ark. Mat.* **32**:2 (1994), 309–321.

[Kheifets and Yuditskii 1994] A. Y. Kheĭfets and P. M. Yuditskiĭ, "An analysis and extension of V. P. Potapov's approach to interpolation problems with applications to the generalized bi-tangential Schur–Nevanlinna–Pick problem and J-inner-outer factorization", pp. 133–161 in *Matrix and operator valued functions*, edited by I. Gohberg and L. A. Sakhnovich, Oper. Theory Adv. Appl. **72**, Birkhäuser, Basel, 1994.

[Khrushchëv and Peller 1986] S. V. Hruščev and V. V. Peller, *Hankel operators of Schatten–von Neumann class and their application to stationary processes and best approximations*, Grundlehren der mathematischen Wissenschaften **273**, Springer, Berlin, 1986.

[Kolmogorov 1925] A. N. Kolmogorov, "Sur les fonctions harmoniques et les séries de Fourier", *Fund. Math.* **7** (1925), 24–29.

[Kovalishina 1974] I. V. Kovalishina, "J-expansive matrix functions in the problem of Carathéodory", *Akad. Nauk Armyan. SSR Dokl.* **59** (1974), 129–135. In Russian.

[Kovalishina 1983] I. V. Kovalishina, "Analytic theory of a class of interpolation problems", *Izv. Akad. Nauk SSSR Ser. Mat.* **47**:3 (1983), 455–497. In Russian.

[Kovalishina and Potapov 1974] I. V. Kovalishina and V. P. Potapov, "An indefinite metric in the Nevanlinna–Pick problem", *Akad. Nauk Armyan. SSR Dokl.* **59** (1974), 17–22. In Russian.

[Kreĭn 1958] M. G. Kreĭn, "Integral equations on the half-line with a kernel depending on the difference of the arguments", *Uspehi Mat. Nauk* (*N.S.*) **13**:5 (=83) (1958), 3–120.

[Kriete and Trutt 1971] T. L. Kriete, III and D. Trutt, "The Cesàro operator in l^2 is subnormal", *Amer. J. Math.* **93** (1971), 215–225.

[Kronecker 1881] L. Kronecker, "Zur Theorie der Elimination einer Variablen aus zwei algebraischen Gleichungen", *Monatsber. Königl. Preussischen Akad. Wies.* (*Berlin*) (1881), 535–600. Reprinted as pp. 113-192 in *Mathematische Werke*, vol. 2, B. G. Teubner, Leipzig, 1897 or Chelsea, New York, 1968.

[Lax 1959] P. D. Lax, "Translation invariant spaces", *Acta Math.* **101** (1959), 163–178.

[Littlewood 1925] J. E. Littlewood, "On inequalities in the theory of functions", *Proc. London Math. Soc.* **23** (1925), 481–519.

[Livshitz 1952] M. S. Livšic, "On the reduction of a linear non-Hermitian operator to 'triangular' form", *Dokl. Akad. Nauk SSSR* **84** (1952), 873–876.

[Løw 1982] E. Løw, "A construction of inner functions on the unit ball in $\mathbb{C}^p$", *Invent. Math.* **67**:2 (1982), 223–229.

[Luecking and Zhu 1992] D. H. Luecking and K. H. Zhu, "Composition operators belonging to the Schatten ideals", *Amer. J. Math.* **114**:5 (1992), 1127–1145.

[MacCluer and Shapiro 1986] B. D. MacCluer and J. H. Shapiro, "Angular derivatives and compact composition operators on the Hardy and Bergman spaces", *Canad. J. Math.* **38**:4 (1986), 878–906.

[Magnus 1950] W. Magnus, "On the spectrum of Hilbert's matrix", *Amer. J. Math.* **72** (1950), 699–704.

[Martin and Putinar 1989] M. Martin and M. Putinar, *Lectures on hyponormal operators*, Oper. Theory Adv. Appl. **39**, Birkhäuser, Basel, 1989.

[Megretskii 1990] A. V. Megretskiĭ, "A quasinilpotent Hankel operator", *Algebra i Analiz* **2**:4 (1990), 201–212.

[Megretskii et al. 1995] A. V. Megretskiĭ, V. V. Peller, and S. R. Treil', "The inverse spectral problem for self-adjoint Hankel operators", *Acta Math.* **174**:2 (1995), 241–309.

[Moeller 1962] J. W. Moeller, "On the spectra of some translation invariant spaces", *J. Math. Anal. Appl.* **4** (1962), 276–296.

[Nehari 1957] Z. Nehari, "On bounded bilinear forms", *Ann. of Math.* (2) **65** (1957), 153–162.

[Nevanlinna 1919] R. Nevanlinna, "Über beschränkte Funktionen die in gegebene Punkten vorgeschriebene Werte annehmen", *Ann. Acad. Sci. Fenn. Ser. A* **13**:1 (1919).

[Nevanlinna 1936] R. Nevanlinna, *Eindeutige analytische Funktionen*, Grundlehren der mathematischen Wissenschaften **46**, Springer, Berlin, 1936.

[Nikolskii 1986] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der mathematischen Wissenschaften **273**, Springer, Berlin, 1986.

[Nikolskii and Vasyunin 1989] N. K. Nikolskiĭ and V. I. Vasyunin, "A unified approach to function models, and the transcription problem", pp. 405–434 in *The Gohberg* *anniversary collection* (Calgary, AB, 1988), vol. 2, edited by H. Dym et al., Oper. Theory Adv. Appl. **41**, Birkhäuser, Basel, 1989.

[Peller 1980] V. V. Peller, "Операторы Ганкеля класса $\mathfrak{S}_p$ и их приложения (рациональная аппроксимация, гауссовские процессы, проблема мажорации операторов)", *Mat. Sb.* (*N.S.*) **113(155)**:4 (1980), 538–581. Translated as "Hankel operators of class $\mathfrak{S}_p$ and their applications (rational approximation, Gaussian processes, the problem of majorization of operators)", *Math. USSR Sb.* **41** (1982), 443–479.

[Peller 1983] V. V. Peller, "Описание операторов Ганкеля класса $\mathfrak{S}_p$ при $p > 0$, исследование скорости рациональной аппроксимации и другие приложения", *Mat. Sb.* (*N.S.*) **122(164)**:4 (1983), 481–510. Translated as "Description of Hankel operators of the class $\mathfrak{S}_p$ for $p > 0$, investigation of the rate of rational approximation and other applications", *Math. USSR Sb.* **50** (1985), 465-494.

[Peller 1993] V. V. Peller, "Invariant subspaces of Toeplitz operators with piecewise continuous symbols", *Proc. Amer. Math. Soc.* **119**:1 (1993), 171–178.

[Peller and Khrushchëv 1982] V. V. Peller and S. V. Khrushchëv, "Операторы Ганкеля, наилучшие приближения и стационарные гауссовские процессы", *Uspehi Mat. Nauk* (*N.S.*) **37**:1 (1982), 53–124. Translated as "Hankel operators, best approximations and stationary Gaussian processes", *Russian Math. Surveys* **37**:1 (1982), 61-144.

[Pick 1916] G. Pick, "Über die Beschränkungen analytischen Funktionen, welche durch vorgegebene Functionswerte bewirkt werden", *Math. Ann.* **77** (1916), 7–23.

[Poggi-Corradini 1997] P. Poggi-Corradini, "The Hardy class of geometric models and the essential spectral radius of composition operators", *J. Funct. Anal.* **143**:1 (1997), 129–156.

[Potapov 1955] V. P. Potapov, "The multiplicative structure of J-contractive matrix functions", *Trudy Moskov. Mat. Obšč.* **4** (1955), 125–236. In Russian; translation in *Amer. Math. Soc. Transl.* (2) **15** (1960), 131–243.

[Power 1982] S. C. Power, *Hankel operators on Hilbert space*, Research Notes in Mathematics **64**, Pitman, Boston, 1982.

[Power 1984] S. C. Power, "Quasinilpotent Hankel operators", pp. 259–261 in *Linear and complex analysis problem book: 199 research problems*, edited by V. P. Havin et al., Lecture Notes in Math. **1043**, Springer, Berlin, 1984.

[Privalov 1950] I. I. Privalov, Граничные свойства аналитических функций, 2nd ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950.

[Privalov 1956] I. I. Priwalow, *Randeigenschaften analytischer Funktionen*, VEB Deutscher Verlag, Berlin, 1956.

[Putnam 1967] C. R. Putnam, *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Mathematik und ihrer Grenzgebiete **36**, Springer, New York, 1967.

[Richter 1988] S. Richter, "Invariant subspaces of the Dirichlet shift", *J. Reine Angew. Math.* **386** (1988), 205–220.

[Richter 1991] S. Richter, "A representation theorem for cyclic analytic two-isometries", *Trans. Amer. Math. Soc.* **328**:1 (1991), 325–349.

[Richter and Shields 1988] S. Richter and A. Shields, "Bounded analytic functions in the Dirichlet space", *Math. Z.* **198**:2 (1988), 151–159.

[Richter and Sundberg 1991] S. Richter and C. Sundberg, "A formula for the local Dirichlet integral", *Michigan Math. J.* **38**:3 (1991), 355–379.

[Richter and Sundberg 1992] S. Richter and C. Sundberg, "Multipliers and invariant subspaces in the Dirichlet space", *J. Operator Theory* **28**:1 (1992), 167–186.

[Richter and Sundberg 1994] S. Richter and C. Sundberg, "Invariant subspaces of the Dirichlet shift and pseudocontinuations", *Trans. Amer. Math. Soc.* **341**:2 (1994), 863–879.

[Riesz 1923] F. Riesz, "Über die Randwerte einer analytischen Funktion", *Math. Z.* **18** (1923), 87–95. Reprinted in F. Riesz, *Oeuvres complètes*, Gauthier-Villars, Paris, 1960.

[Riesz 1924] M. Riesz, "Les fonctions conjugueés et les séries de Fourier", *C. R. Acad. Sci. Paris Ser. A-B* **178** (1924), 1464–1467.

[Riesz and Riesz 1916] F. Riesz and M. Riesz, "Über die Randwerte einer analytischen Funktion", pp. 27–44 in *Comptes Rendus du 4. Congr. des Math. Scand.* (Stockholm, 1916), 1916. Reprinted in F. Riesz, *Oeuvres complètes*, Gauthier-Villars, Paris, 1960.

[Rosenblum 1965] M. Rosenblum, "A concrete spectral theory for self-adjoint Toeplitz operators", *Amer. J. Math.* **87** (1965), 709–718.

[Rosenblum and Rovnyak 1985] M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Oxford Mathematical Monographs, Oxford University Press, New York, 1985. Reprinted with corrections by Dover, Mineola (NY), 1997.

[Rota 1960] G.-C. Rota, "On models for linear operators", *Comm. Pure Appl. Math.* **13** (1960), 469–472.

[Rudin 1969] W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, New York, 1969.

[Rudin 1980] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Grundlehren der Mathematischen Wissenschaften **241**, Springer, New York, 1980.

[Rudin 1986] W. Rudin, *New constructions of functions holomorphic in the unit ball of* $\mathbb{C}^n$, CBMS Regional Conference Series in Mathematics **63**, Amer. Math. Soc., Providence, 1986.

[Sarason 1965] D. Sarason, "A remark on the Volterra operator", *J. Math. Anal. Appl.* **12** (1965), 244–246.

[Sarason 1967] D. Sarason, "Generalized interpolation in H^∞", *Trans. Amer. Math. Soc.* **127**:2 (1967), 179–203.

[Sarason 1975] D. Sarason, "Functions of vanishing mean oscillation", *Trans. Amer. Math. Soc.* **207** (1975), 391–405.

[Sarason 1987] D. Sarason, "Moment problems and operators in Hilbert space", pp. 54–70 in *Moments in mathematics* (San Antonio, 1987), edited by H. J. Landau, Proc. Sympos. Appl. Math. **37**, Amer. Math. Soc., Providence, 1987.

[Sarason 1991] D. Sarason, "New Hilbert spaces from old", pp. 195–204 in *Paul Halmos*, edited by J. H. Ewing and F. W. Gehring, Springer, New York, 1991.

[Schur 1911] I. Schur, "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlicher", *J. Reine Angew. Math.* **140** (1911), 1–28.

[Schur 1917] I. Schur, “Über Potenzreihen die im Innern des Einheitskreises beschränkt sind”, *J. Reine Angew. Math.* **147** (1917), 205–232.

[Seip 1993] K. Seip, “Beurling type density theorems in the unit disk”, *Invent. Math.* **113**:1 (1993), 21–39.

[Semmes 1984] S. Semmes, “Trace ideal criteria for Hankel operators, and applications to Besov spaces”, *Integral Equations Operator Theory* **7**:2 (1984), 241–281.

[Shapiro 1987] J. H. Shapiro, “The essential norm of a composition operator”, *Ann. of Math.* (2) **125**:2 (1987), 375–404.

[Shapiro 1993] J. H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer, New York, 1993.

[Shapiro and Shields 1962] H. S. Shapiro and A. L. Shields, “On the zeros of functions with finite Dirichlet integral and some related function spaces”, *Math. Z.* **80** (1962), 217–229.

[Shapiro and Taylor 1973/74] J. H. Shapiro and P. D. Taylor, “Compact, nuclear, and Hilbert-Schmidt composition operators on H^2”, *Indiana Univ. Math. J.* **23** (1973/74), 471–496.

[Shields 1974] A. L. Shields, “Weighted shift operators and analytic function theory”, pp. 49–128 in *Topics in operator theory*, edited by C. Pearcy, Math. Surveys **13**, Amer. Math. Soc., Providence, 1974.

[Shimorin 1995] S. M. Shimorin, “On a family of conformally invariant operators”, *Algebra i Analiz* **7**:2 (1995), 133–158. In Russian; translation in *St. Petersburg Math. J.* **7**:2 (1996), 287–306.

[Stampfli 1965] J. G. Stampfli, “Hyponormal operators and spectral density”, *Trans. Amer. Math. Soc.* **117** (1965), 469–476. Errata in **115** (1965), 550.

[Stroethoff 1990] K. Stroethoff, “Compact Hankel operators on the Bergman space”, *Illinois J. Math.* **34**:1 (1990), 159–174.

[Sz.-Nagy and Foiaş 1967] B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris, and Akadémiai Kiadó, Budapest, 1967. Translated as *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, and Akadémiai Kiadó, Budapest, 1970.

[Sz.-Nagy and Foiaş 1968] B. Sz.-Nagy and C. Foiaş, “Dilatation des commutants d'opérateurs”, *C. R. Acad. Sci. Paris Sér. A-B* **266** (1968), A493–A495.

[Szegö 1920] G. Szegö, “Beiträge zur Theorie der Toeplitschen Formen I”, *Math. Z.* **6** (1920), 167–202.

[Szegö 1921] G. Szegö, “Beiträge zur Theorie der Toeplitschen Formen II”, *Math. Z.* **9** (1921), 167–190.

[Thomson 1991] J. E. Thomson, “Approximation in the mean by polynomials”, *Ann. of Math.* (2) **133**:3 (1991), 477–507.

[Upmeier 1996] H. Upmeier, *Toeplitz operators and index theory in several complex variables*, Operator Theory: Advances and Applications **81**, Birkhäuser, Basel, 1996.

[Vasyunin and Nikolskii 1990] V. I. Vasyunin and N. K. Nikol'skiĭ, “Quasi-orthogonal decompositions with respect to complementary metrics, and estimates for univalent functions”, *Algebra i Analiz* **2**:4 (1990), 1–81. In Russian; translation in *Leningrad Math. J.* **2**:4 (1991), 691–764.

[Vasyunin and Nikolskii 1991] V. I. Vasyunin and N. K. Nikol'skiĭ, "Operator-valued measures and coefficients of univalent functions", *Algebra i Analiz* **3**:6 (1991), 1–75 (1992). In Russian; translation in *St. Petersburg Math. J.* **3**:6 (1992), 1199–1270.

[Vol'berg et al. 1990] A. L. Vol'berg, V. V. Peller, and D. V. Yakubovich, "A brief excursion into the theory of hyponormal operators", *Algebra i Analiz* **2**:2 (1990), 1–38. In Russian; translation in *Leningrad Math. J.* **2**:2 (1991), 211–243.

[Wang 1984] D. M. Wang, "Similarity of smooth Toeplitz operators", *J. Operator Theory* **12**:2 (1984), 319–329.

[Widom 1960] H. Widom, "Inversion of Toeplitz matrices, II", *Illinois J. Math.* **4** (1960), 88–99.

[Widom 1964] H. Widom, "On the spectrum of a Toeplitz operator", *Pacific J. Math.* **14** (1964), 365–375.

[Yakubovich 1989] D. V. Yakubovich, "Riemann surface models of Toeplitz operators", pp. 305–415 in *Toeplitz operators and spectral function theory*, edited by N. K. Nikol'skiĭ, Oper. Theory: Adv. Appl. **42**, Birkhäuser, Basel, 1989.

[Yakubovich 1991] D. V. Yakubovich, "On the spectral theory of Toeplitz operators with a smooth symbol", *Algebra i Analiz* **3**:4 (1991), 208–226. In Russian.

[Zhu 1987] K. Zhu, "VMO, ESV, and Toeplitz operators on the Bergman space", *Trans. Amer. Math. Soc.* **302**:2 (1987), 617–646.

[Zhu 1990] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics **139**, Marcel Dekker Inc., New York, 1990.

Donald Sarason
Department of Mathematics
University of California
Berkeley, CA 94720
United States
sarason@math.berkeley.edu