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An Excursion into the Theory of Hankel Operators

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Abstract. This survey is an introduction to the theory of Hankel operators, a beautiful area of mathematical analysis that is also very important in applications. We start with classical results: Kronecker's theorem, Nehari's theorem, Hartman's theorem, Adamyan–Arov–Krein theorems. Then we describe the Hankel operators in the Schatten–von Neumann class $\boldsymbol{S}_p$ and consider numerous applications: Sarason's commutant lifting theorem, rational approximation, stationary processes, best approximation by analytic functions. We also present recent results on spectral properties of Hankel operators with lacunary symbols. Finally, we discuss briefly the most recent results involving Hankel operators: Pisier's solution of the problem of similarity to a contraction, self-adjoint operators unitarily equivalent to Hankel operators, and approximation by analytic matrix-valued functions.

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1. Introduction

I would like to invite the reader on an excursion into the theory of Hankel operators, a beautiful and rapidly developing domain of analysis that is important in numerous applications.

It was Hankel [1861] who began the study of finite matrices whose entries depend only on the sum of the coordinates, and therefore such objects are called *Hankel matrices*. One of the first theorems about infinite Hankel matrices was obtained by Kronecker [1881]; it characterizes Hankel matrices of finite rank. Hankel matrices played an important role in many classical problems of analysis, and in particular in the so-called moment problems; for example, Hamburger's moment problem is solvable if and only if the corresponding infinite Hankel matrix is positive semi-definite [Hamburger 1920; 1921].

Since the work of Nehari [1957] and Hartman [1958] it has become clear that Hankel operators are an important tool in function theory on the unit circle. Together with Toeplitz operators they form two of the most important classes of operators on Hardy spaces.

For the last three decades the theory of Hankel operators has been developing rapidly. A lot of applications in different domains of mathematics have been found: interpolation problems [Adamyan et al. 1968b; 1968a; 1971]; rational approximation [Peller 1980; 1983]; stationary processes [Peller and Khrushchëv 1982]; perturbation theory [Peller 1985]; Sz.-Nagy–Foiaş function model [Nikol'skiĭ 1986]. In the 1980s the theory of Hankel operators was fueled by the rapid development of H^∞ control theory and systems theory (see [Fuhrmann 1981; Glover 1984; Francis 1987]). It has become clear that it is especially important to develop the theory of Hankel operators with matrix-valued (and even operator-valued) symbols. I certainly cannot mention here all applications of Hankel operators. The latest application I would like to touch on here is Pisier's solution of the famous problem of similarity to a contraction; see Section 12 for more detail.

The development of the theory of Hankel operators led to different generalizations of the original concept. A lot of progress has taken place in the study of Hankel operators on Bergman spaces on the disk, Dirichlet type spaces, Bergman and Hardy spaces on the unit ball in $\mathbb{C}^n$, on symmetric domains; commutators, paracommutators, etc. This survey will not discuss such generalizations, but will concentrate on the classical Hankel operators on the Hardy class H^2—or, in other words, operators having Hankel matrices. Even under this constraint it is impossible in a survey to cover all important results and describe all applications. I have chosen several aspects of the theory and several applications to demonstrate the beauty of the theory and importance in applications.

So, if you accept my invitation, fasten seat belts and we shall be off!

In Section 2 we obtain the boundedness criterion and discuss symbols of Hankel operators of minimal L^∞ norm. As an application, we give in Section 3 a proof of Sarason's commutant lifting theorem, based on Nehari's theorem. Section 4 is devoted to the proof of Kronecker's theorem characterizing the Hankel operators of finite rank. In Section 5 we describe the compact Hankel operators. In Section 6 we prove the profound Adamyan–Arov–Krein theorem on finite-rank approximation of Hankel operators. Section 7 is devoted to membership of Hankel operators in Schatten–von Neumann classes $\boldsymbol{S}_p$. In Section 8 we consider applications of Hankel operators in the theory of rational approximation. Section 9 concerns hereditary properties of the operator of best uniform approximation by functions analytic in the unit disk. Section 10 deals with applications of Hankel operators in prediction theory. In Section 11 we study spectral properties of Hankel operators with lacunary symbols. We conclude the survey with Section 12 which briefly reviews some recent developments of Hankel operators and their applications; namely, we touch on the problem of unitary equivalent description of the self-adjoint Hankel operators, we discuss the problem of approximating a matrix function on the unit circle by bounded analytic functions, and conclude the section with Pisier's solution of the problem of similarity to a contraction.

Preliminaries. An infinite matrix A is called a *Hankel matrix* if it has the form

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\alpha = \{\alpha_j\}_{j\ge 0}$ is a sequence of complex numbers. In other words, a Hankel matrix is one whose entries depend only on the sum of the coordinates.

If $\alpha \in \ell^2$, we can consider the *Hankel operator* $\Gamma_\alpha : \ell^2 \to \ell^2$ with matrix A in the standard basis of ℓ^2 that is defined on the dense subset of finitely supported sequences.

Hankel operators admit the following important realizations as operators from the Hardy class H^2 of functions on the unit circle $\mathbb{T}$ to the space

$$H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2.$$

Let $\varphi \in L^2$. We define the Hankel operator H_φ on the dense subset of polynomials by

$$H_\varphi f \stackrel{\text{def}}{=} \mathbb{P}_- \varphi f,$$

where $\mathbb{P}_-$ is the orthogonal projection onto H^2_-. The function φ is called a *symbol* of the Hankel operator H_φ; there are infinitely many different symbols that produce the same Hankel operator. It is easy to see that H_φ has Hankel matrix $\{\hat\varphi(-j-k)\}_{j\ge 1, k\ge 0}$ in the bases $\{z^k\}_{k\ge 0}$ of H^2 and $\{\bar z^j\}_{j\ge 1}$ of H^2_-; here $\hat\varphi(m)$ is the m-th Fourier coefficient of φ.

We also need the notion of a Toeplitz operator on H^2. For $\varphi \in L^\infty$ we define the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ by

$$T_\varphi f = \mathbb{P}_+ \varphi f \quad \text{for } f \in H^2,$$

where $\mathbb{P}_+$ is the orthogonal projection of L^2 onto H^2. It is easy to see that $\|T_\varphi\| \le \|\varphi\|_{L^\infty}$. In fact, $\|T_\varphi\| = \|\varphi\|_{L^\infty}$; see, for example, [Douglas 1972; Sarason 1978].

Notation. The following notation is used throughout the survey:

- z stands for the identity function on a subset of $\mathbb{C}$.
- $\boldsymbol{m}$ is normalized Lebesgue measure on the unit circle $\mathbb{T}$.
- $\boldsymbol{m}_2$ is planar Lebesgue measure.
- $S : H^2 \to H^2$ is the shift operator; that is, $Sf \stackrel{\text{def}}{=} zf$ for $f \in H^2$.
- $\mathcal{S} : L^2 \to L^2$ is the bilateral shift operator; that is, $\mathcal{S}f \stackrel{\text{def}}{=} zf$ for $f \in L^2$.
- For a function f in $L^1(\mathbb{T})$ we denote by $\tilde{f}$ the harmonic conjugate of f.
- BMO is the space of functions φ on $\mathbb{T}$ of bounded mean oscillation:

$$\sup_{|I|} \frac{1}{|I|} \int_I |\varphi - \varphi_I| \, d\boldsymbol{m} < \infty,$$

  where the supremum is taken over all intervals I of $\mathbb{T}$ and $|I| \stackrel{\text{def}}{=} \boldsymbol{m}(I)$.
- VMO is the closed subspace of BMO consisting of functions φ satisfying

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |\varphi - \varphi_I| \, d\boldsymbol{m} = 0.$$

2. Boundedness

The following theorem of Nehari [1957] characterizes the bounded Hankel operators Γ_α on ℓ^2.

THEOREM 2.1. *The Hankel operator Γ_α is bounded on ℓ^2 if and only if there exists a function $\psi \in L^\infty$ on the unit circle $\mathbb{T}$ such that*

$$\alpha_m = \hat{\psi}(m) \quad \text{for } m \ge 0. \tag{2–1}$$

In this case

$$\|\Gamma_\alpha\| = \inf\{\|\psi\|_\infty : \hat{\psi}(n) = \alpha_n \text{ for } n \ge 0\}.$$

PROOF. Let $\psi \in L^\infty$ and set $\alpha_m = \hat{\psi}(m)$ for $m \ge 0$. Given two finitely supported sequences $a = \{a_n\}_{n\ge 0}$ and $b = \{b_k\}_{k\ge 0}$ in ℓ^2, we have

$$(\Gamma_\alpha a, b) = \sum_{j,k\ge 0} \alpha_{j+k} a_j \bar{b}_k. \tag{2–2}$$

Let

$$f = \sum_{j\ge 0} a_j z^j, \quad g = \sum_{k\ge 0} \bar{b}_k z^k.$$

Then f and g are polynomials in the Hardy class H^2. Put $q = fg$. It follows from (2–2) that

$$\begin{aligned}(\Gamma_\alpha a, b) &= \sum_{j,k\geq 0} \hat{\psi}(j+k) a_j \bar{b}_k = \sum_{m\geq 0} \hat{\psi}(m) \sum_{j=0}^{m} a_j \bar{b}_{m-j} \\ &= \sum_{m\geq 0} \hat{\psi}(m)\hat{q}(m) = \int_{\mathbb{T}} \psi(\zeta) q(\bar{\zeta})\, d\boldsymbol{m}(\zeta).\end{aligned}$$

Therefore

$$|(\Gamma_\alpha a, b)| \leq \|\psi\|_\infty \|q\|_{H^1} \leq \|\psi\|_\infty \|f\|_{H^2} \|g\|_{H^2} = \|\psi\|_{L^\infty} \|a\|_{\ell^2} \|b\|_{\ell^2}.$$

Conversely, suppose that Γ is bounded on ℓ^2. Let L_α be the linear functional defined on the set of polynomials in H^1 by

$$L_\alpha q = \sum_{n\geq 0} \alpha_n \hat{q}(n). \tag{2–3}$$

We show that L_α extends by continuity to a continuous functional on H^1 and its norm $\|L_\alpha\|$ on H^1 satisfies

$$\|L_\alpha\| \leq \|\Gamma_\alpha\|. \tag{2–4}$$

By the Hahn-Banach theorem this will imply the existence of some ψ in L^∞ that satisfies (2–1) and

$$\|\psi\|_\infty \leq \|\Gamma_\alpha\|.$$

Assume first that $\alpha \in \ell^1$. In this case the functional L_α defined by (2–3) is obviously continuous on H^1. We prove (2–4). Let $q \in H^1$ and $\|q\|_1 \leq 1$. Then q admits a representation $q = fg$, where f, $g \in H^2$ and $\|f\|_2 \leq 1$, $\|g\|_2 \leq 1$. We have

$$L_\alpha q = \sum_{m\geq 0} \alpha_m \hat{q}(m) = \sum_{m\geq 0} \alpha_m \sum_{j=0}^{m} \hat{f}(j)\hat{g}(m-j) = \sum_{j,k\geq 0} \alpha_{j+k} \hat{f}(j)\hat{g}(k) = (\Gamma_\alpha a, b),$$

where $a = \{a_j\}_{j\geq 0}$ with $a_j = \hat{f}(j)$ and $b = \{b_k\}_{k\geq 0}$ with $b_k = \overline{\hat{g}(k)}$. Therefore

$$|L_\alpha q| \leq \|\Gamma_\alpha\| \, \|f\|_2 \|g\|_2 \leq \|\Gamma_\alpha\|,$$

which proves (2–4) for $\alpha \in \ell^1$.

Now assume that α is an arbitrary sequence for which Γ_α is bounded. Let $0 < r < 1$. Consider the sequence $\alpha^{(r)}$ defined by

$$\alpha_j^{(r)} = r^j \alpha_j \quad \text{for } j \geq 0.$$

It is easy to see that $\Gamma_{\alpha^{(r)}} = D_r \Gamma_\alpha D_r$, where D_r is multiplication by $\{r^j\}_{j\geq 0}$ on ℓ^2. Since obviously $\|D_r\| \leq 1$, it follows that the operators $\Gamma_{\alpha^{(r)}}$ are bounded and

$$\|\Gamma_{\alpha^{(r)}}\| \leq \|\Gamma_\alpha\| \quad \text{for } 0 < r < 1.$$

Clearly $\alpha^{(r)} \in \ell^1$, so we have already proved that

$$\|L_{\alpha^{(r)}}\|_{H^1 \to \mathbb{C}} \le \|\Gamma_{\alpha^{(r)}}\| \le \|\Gamma_\alpha\|.$$

It is easy to see now that the functionals $L_{\alpha^{(r)}}$ being uniformly bounded converge strongly to L_α; that is, $L_{\alpha^{(r)}}\psi \to L_\alpha\psi$ for any $\psi \in H^1$. This proves that L_α is continuous and satisfies (2–4). □

Theorem 2.1 reduces the problem of whether a sequence α determines a bounded operator on ℓ^2 to the question of the existence of an extension of α to the sequence of Fourier coefficients of a bounded function. However, after the work of C. Fefferman on the space BMO of functions of bounded mean oscillation it has become possible to determine whether Γ_α is bounded in terms of the sequence α itself.

By C. Fefferman's theorem (see [Garnett 1981], for example), a function φ on the unit circle belongs to the space BMO if and only if it admits a representation

$$\varphi = \xi + \mathbb{P}_+\eta \quad \text{with } \xi, \eta \in L^\infty.$$

The space BMOA is by definition the space of BMO functions analytic in the unit disc $\mathbb{D}$:

$$\text{BMOA} = \text{BMO} \cap H^1.$$

It is easy to see that Nehari's and Fefferman's theorems imply the following result.

Theorem 2.2. *The operator* Γ_α *is bounded on* ℓ^2 *if and only if the function*

$$\varphi = \sum_{m \ge 0} \alpha_m z^m \tag{2–5}$$

belongs to BMOA.

Clearly Γ_α is a bounded operator if the function φ defined by (2–5) is bounded. However, the operator Γ_α can be bounded even with an unbounded φ. We consider an important example of such a Hankel matrix:

Example (the Hilbert matrix). Let $\alpha_n = 1/(n+1)$ for $n \ge 0$. The corresponding Hankel matrix Γ_α is called the *Hilbert matrix*. Clearly the function

$$\sum_{n \ge 0} \frac{1}{n+1} z^n$$

is unbounded in $\mathbb{D}$. However, Γ_α is bounded. Indeed, consider the function ψ on $\mathbb{T}$ defined by

$$\psi(e^{it}) = ie^{-it}(\pi - t) \quad \text{for } t \in [0, 2\pi).$$

It is easy to see that

$$\hat{\psi}(n) = \frac{1}{n+1} \quad \text{for } m \ge 0,$$

and that $\|\psi\|_{L^\infty} = \pi$. It follows from Theorem 2.1 that Γ_α is bounded and $\|\Gamma_\alpha\| \le \pi$. In fact, $\|\Gamma_\alpha\| = \pi$; see [Nikol'skiĭ 1986, Appendix 4, 165.21], for example.

Clearly, Theorem 2.1 admits the following reformulation.

THEOREM 2.3. *Let $\varphi \in L^2$. The following statements are equivalent*:

(a) *H_φ is bounded on H^2.*
(b) *There exists a function ψ in L^∞ such that*

$$\hat{\psi}(m) = \hat{\varphi}(m) \quad \textit{for } m < 0. \tag{2–6}$$

(c) $\mathbb{P}_-\varphi \in \mathrm{BMO}$.

If one of the conditions (a)–(c) *is satisfied*, *then*

$$\|H_\varphi\| = \inf\{\|\psi\|_{L^\infty} : \hat{\psi}(m) = \hat{\varphi}(m) \textit{ for } m < 0\}. \tag{2–7}$$

Equality (2–6) is equivalent to the fact that $H_\varphi = H_\psi$. Thus (b) means that H_φ is bounded if and only if it has a bounded symbol. So the operators H_φ with $\varphi \in L^\infty$ exhaust the class of bounded Hankel operators. If $\varphi \in L^\infty$, (2–7) can be rewritten in the following way:

$$\|H_\varphi\| = \inf\{\|\varphi - f\|_\infty : f \in H^\infty\}. \tag{2–8}$$

Let $\varphi \in L^\infty$. It follows easily from a compactness argument that the infimum on the right-hand side of (2–8) is attained for any $\varphi \in L^\infty$. A function f that realizes the minimum on the right-hand side of (2–8) is called *a best approximation of φ by analytic functions in the L^∞-norm.* The problem of finding, for a given $\varphi \in L^\infty$, a best approximation by analytic functions is called *Nehari's problem.* It plays a significant role in applications, particularly in control theory. If f realizes the minimum on the right-hand side of (2–8), then clearly, $\varphi - f$ is a symbol of H_φ of minimal L^∞-norm. A natural question arises of whether such a symbol of minimal norm is unique.

The first results in this direction were apparently obtained by Khavinson [1951] (see also [Rogosinski and Shapiro 1953]), where it was shown that for a continuous function φ on $\mathbb{T}$ there exists a unique best uniform approximation by analytic functions and that uniqueness fails in general; see also [Garnett 1981, Section IV.1]. However, in the case when the Hankel operator attains its norm on the unit ball of H^2, that is, when $\|H_\varphi g\|_2 = \|H_\varphi\|\,\|g\|_2$ for some nonzero $g \in H^2$, we do have uniqueness, as the following result shows [Adamyan et al. 1968b].

THEOREM 2.4. *Let φ be a function in L^∞ such that H_φ attains its norm on the unit ball of H^2. Then there exists a unique function f in H^∞ such that*

$$\|\varphi - f\|_\infty = \mathrm{dist}_{L^\infty}(\varphi, H^\infty).$$

Moreover, $\varphi - f$ *has constant modulus almost everywhere on* $\mathbb{T}$ *and admits a representation*

$$\varphi - f = \|H_\varphi\|\,\bar{z}\bar{\vartheta}\,\frac{\bar{h}}{h}, \tag{2–9}$$

where h *is an outer function in* H^2 *and* ϑ *is an inner function*.

PROOF. Without loss of generality we can assume that $\|H_\varphi\| = 1$. Let g be a function in H^2 such that $1 = \|g\|_2 = \|H_\varphi g\|_2$. Let $f \in H^\infty$ be a best approximation of φ, so that $\|\varphi - f\|_\infty = 1$. We have

$$1 = \|H_\varphi g\|_2 = \|\mathbb{P}_-(\varphi - f)g\|_2 \le \|(\varphi - f)g\|_2 \le \|g\|_2 = 1.$$

Therefore the inequalities in this chain are in fact equalities. The fact that

$$\|\mathbb{P}_-(\varphi - f)g\|_2 = \|(\varphi - f)\|_2$$

means that $(\varphi - f)g \in H^2_-$, so

$$H_\varphi g = H_{\varphi - f}\, g = (\varphi - f)g. \tag{2–10}$$

The function g, being in H^2, is nonzero almost everywhere on $\mathbb{T}$, so

$$f = \varphi - \frac{H_\varphi g}{g}.$$

Hence f is determined uniquely by H_φ: the ratio $(H_\varphi g)/g$ does not depend on the choice of g.

Since $\|\varphi - f\|_\infty = 1$, the equality

$$\|(\varphi - f)g\|_2 = \|g\|_2$$

means that $|\varphi(\zeta) - f(\zeta)| = 1$ a.e. on the set $\{\zeta : g(\zeta) \ne 0\}$, which is of full measure since $g \in H^2$. Thus $\varphi - f$ has modulus one almost everywhere on $\mathbb{T}$.

Consider the functions g and $\bar{z}\overline{H_\varphi g}$ in H^2. It follows from (2–10) that they have the same moduli. Therefore they admit factorizations

$$g = \vartheta_1 h, \quad \bar{z}\overline{H_\varphi g} = \vartheta_2 h,$$

where h is an outer function in H^2, and ϑ_1 and ϑ_2 are inner functions. Consequently,

$$\varphi - f = \frac{H_\varphi g}{g} = \frac{\bar{z}\bar{\vartheta}_2\bar{h}}{\vartheta_1 h} = \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\frac{\bar{h}}{h},$$

which proves (2–9) with $\vartheta = \vartheta_1\vartheta_2$. □

COROLLARY 2.5. *If* H_φ *is a compact Hankel operator*, *the conclusion of Theorem* 2.4 *holds*.

PROOF. Any compact operator attains its norm. □

We shall see in Section 5 that Hankel operators with continuous symbols are compact, so Corollary 2.5 implies Khavinson's theorem [1951] mentioned above.

Adamyan, Arov, and Krein proved in [Adamyan et al. 1968a] that, if there are at least two best approximations to φ, there exists a best approximation g such that $\varphi - g$ has constant modulus on $\mathbb{T}$. They found a formula that parametrizes all best approximations.

We now show that Hankel operators can be characterized as the operators that satisfy a certain commutation relation. Recall the S and $\mathcal{S}$ are the shift and bilateral shift operators, respectively.

THEOREM 2.6. *Let R be a bounded operator from H^2 to H^2_-. Then R is a Hankel operator if and only if it satisfies the commutation relation*

$$\mathbb{P}_-\mathcal{S}R = RS. \tag{2–11}$$

PROOF. Let $R = H_\varphi$, with $\varphi \in L^\infty$. Then

$$\mathbb{P}_-\mathcal{S}Rf = \mathbb{P}_- zH_\varphi f = \mathbb{P}_- z\mathbb{P}\varphi f = \mathbb{P}_- z\varphi f = H_\varphi zf.$$

Conversely, suppose that R satisfies (2–11). Let $n \ge 1$, $k \ge 1$. We have

$$(Rz^n, \bar{z}^k) = (RSz^{n-1}, \bar{z}^k) = (\mathbb{P}_-\mathcal{S}Rz^{n-1}, \bar{z}^k) = (\mathcal{S}Rz^{n-1}, \bar{z}^k) = (Rz^{n-1}, \bar{z}^{k+1}).$$

Therefore R has Hankel matrix in the bases $\{z^n\}_{n\ge 0}$ of H^2 and $\{\bar{z}^k\}_{k\ge 1}$ of H^2_-. It follows from Theorems 2.1 and 2.3 that $R = H_\varphi$ for some φ in L^∞. ☐

3. Sarason's Theorem

In this section we study the commutant of compressions of the shift operator on H^2 to its coinvariant subspaces, and we prove Sarason's commutant lifting theorem. We use an approach given in [Nikol'skiĭ 1986, Section VIII.1], based on Hankel operators and Nehari's theorem. Then we establish an important formula that relates functions of such a compression with Hankel operators.

Let ϑ be an inner function. Put

$$K_\vartheta = H^2 \ominus \vartheta H^2.$$

By Beurling's theorem (see [Hoffman 1962, Chapter 7] or [Nikol'skiĭ 1986, Section I.1], for example), any nontrivial invariant subspace of the backward shift operator S^* on H^2 coincides with K_ϑ for some inner function ϑ. Denote by S_ϑ the compression of the shift operator S to K_ϑ, defined by

$$S_\vartheta f = P_\vartheta zf \quad \text{for } f \in K_\vartheta, \tag{3–1}$$

where P_ϑ is the orthogonal projection from H^2 onto K_ϑ. Clearly, $S_\vartheta^* = S^*|K_\vartheta$.

It can easily be shown that

$$P_\vartheta f = f - \vartheta\mathbb{P}_+\bar{\vartheta}f = \vartheta\mathbb{P}_-\bar{\vartheta}f \quad \text{for } f \in H^2. \tag{3–2}$$

S_ϑ is the *model operator* in the Sz.-Nagy–Foiaş function model. Any *contraction* T (that is, a linear operator such that $\|T\| \le 1$) for which $\lim_{n\to\infty} T^{*n} = 0$ in the strong operator topology and $\operatorname{rank}(I - T^*T) = \operatorname{rank}(I - TT^*) = 1$ is unitarily equivalent to S_ϑ for some inner function ϑ; see [Sz.-Nagy and Foiaş 1967, Chapter 6; Nikol'skiĭ 1986, Lecture I].

The operator S_ϑ admits an H^∞ functional calculus. Indeed, given $\varphi \in H^\infty$, we define the operator $\varphi(S_\vartheta)$ by

$$\varphi(S_\vartheta)f = P_\vartheta \varphi f \quad \text{for } f \in K_\vartheta. \tag{3–3}$$

Clearly, this functional calculus is linear. It is also easy to verify that it is multiplicative. Hence, for any $\varphi \in H^\infty$, the operator $\varphi(S_\vartheta)$ commutes with S_ϑ, and it follows from (3–3) that

$$\|\varphi(S_\vartheta)\| \le \|\varphi\|_{H^\infty}.$$

This is known as *von Neumann's inequality*.

The following theorem of Sarason [1967] describes the commutant of S_ϑ. It is a partial case of the commutant lifting theorem of Sz.-Nagy and Foiaş [1967].

THEOREM 3.1. *Let T be an operator that commutes with S_ϑ. Then there exists a function φ in H^∞ such that $T = \varphi(S_\vartheta)$ and $\|T\| = \|\varphi\|_{H^\infty}$.*

LEMMA 3.2. *Let T be an operator on K_ϑ. Consider the operator $\tilde{T} : H^2 \to H^2_-$ defined by*

$$\tilde{T} f = \bar{\vartheta}\, T P_\vartheta\, f. \tag{3–4}$$

Then T commutes with S_ϑ if and only if $\tilde{T}$ is a Hankel operator.

PROOF. $\tilde{T}$ is a Hankel operator if and only if

$$\mathbb{P}_- z\tilde{T} f = \tilde{T} z f \quad \text{for } f \in H^2 \tag{3–5}$$

(see (2–11)), which means that

$$\mathbb{P}_- z\bar{\vartheta} T P_\vartheta f = \bar{\vartheta} T P_\vartheta z f \quad \text{for } f \in H^2,$$

which in turn is equivalent to

$$\vartheta \mathbb{P}_- \bar{\vartheta} z T P_\vartheta f = T P_\vartheta z f \quad \text{for } f \in H^2. \tag{3–6}$$

We have by (3–2)

$$\vartheta \mathbb{P}_- \bar{\vartheta} z T P_\vartheta f = P_\vartheta z T P_\vartheta f = S_\vartheta T P_\vartheta f.$$

Since obviously the left-hand side and the right-hand side of (3–6) are zero for $f \in \vartheta H^2$, it follows from (3–1) that (3–5) is equivalent to the equality

$$S_\vartheta T f = T S_\vartheta f \quad \text{for } f \in K_\vartheta. \qquad \square$$

PROOF OF THEOREM 3.1. By Lemma 3.2 the operator $\tilde{T}$ defined by (3–4) is a Hankel operator. By Nehari's theorem there exists a function ψ in L^∞ such that $\|\psi\|_\infty = \|\tilde{T}\|$ and $H_\psi = \tilde{T}$; that is,

$$\mathbb{P}_-\psi f = \bar{\vartheta}\, T P_\vartheta f \quad \text{for } f \in H^2.$$

It follows that $\mathbb{P}_-\psi f = 0$ for any $f \in \vartheta H^2$. That means that $H_{\psi\vartheta} = 0$. Put $\varphi = \psi\vartheta$. Clearly $\varphi \in H^\infty$ and $\psi = \bar{\vartheta}\varphi$. We have

$$\bar{\vartheta}\, T f = \mathbb{P}_-\bar{\vartheta}\varphi f \quad \text{for } f \in K_\vartheta,$$

so

$$T f = \vartheta\, \mathbb{P}_-\bar{\vartheta}\varphi f = P_\vartheta\, \varphi f = \varphi(S_\vartheta) f \quad \text{for } f \in K_\vartheta. \tag{3–7}$$

Obviously

$$\|T\| = \|\tilde{T}\| = \|\psi\|_\infty = \|\varphi\|_\infty,$$

which completes the proof. □

REMARK. Formula (3–7) implies a remarkable relation, due to Nikol'skiĭ [1986], between Hankel operators and functions of model operators: Let ϑ be an inner function, and let $\varphi \in H^\infty$. Then

$$\varphi(S_\vartheta) = \Theta H_{\bar{\vartheta}\varphi} | K_\vartheta, \tag{3–8}$$

where Θ is multiplication by ϑ. This formula shows that $\varphi(S_\vartheta)$ has the same metric properties as $H_{\bar{\vartheta}\varphi}$; compactness, nuclearity, etc.

Formula (3–8) relates the Hankel operators of the form $H_{\bar{\vartheta}\varphi}$ with functions of model operators. It can easily be shown that such Hankel operators are exactly the Hankel operators from H^2 to H^2_- that have a nontrivial kernel. It is worth mentioning that the set of functions of the form $\bar{\vartheta}\varphi$, where ϑ is inner and $\varphi \in H^\infty$, forms a dense subset of L^∞ [Douglas 1972, 6.32].

4. Finite Rank

One of the first results about Hankel matrices was a theorem of Kronecker [1881] that describes the Hankel matrices of finite rank.

Let $r = p/q$ be a rational function where p and q are polynomials. If p/q is in its lowest terms, the degree of r is, by definition,

$$\deg r = \max(\deg p,\, \deg q),$$

where $\deg p$ and $\deg q$ are the degrees of the polynomials p and q. It is easy to see that $\deg r$ is the sum of the multiplicities of the poles of r (including a possible pole at infinity).

We are going to describe the Hankel matrices of finite rank without any assumption on the boundedness of the matrix.

We identify sequences of complex numbers with the corresponding formal power series. If $a = \{a_j\}_{j\ge 0}$ is a sequence of complex numbers, we associate with it the formal power series

$$a(z) = \sum_{j\ge 0} a_j z^j.$$

The space of formal power series forms an algebra with respect to the multiplication

$$(ab)(z) = \sum_{m\ge 0}\left(\sum_{j=0}^{m} a_j b_{m-j}\right) z^m, \quad \text{with } a = \sum_{j\ge 0} a_j z^j \text{ and } b = \sum_{j\ge 0} b_j z^j.$$

Consider the shift operator S and the backward shift operator S^* defined on the space of formal power series in the following way:

$$(Sa)(z) = za(z), \qquad S^* \sum_{j\ge 0} a_j z^j = \sum_{j\ge 0} a_{j+1} z^j.$$

Let $\alpha = \{\alpha_j\}_{j\ge 0}$ be a sequence of complex numbers, which we identify with the corresponding formal power series

$$\alpha(z) = \sum_{j\ge 0} \alpha_j z^j. \tag{4–1}$$

Denote by Γ_α the Hankel matrix $\{\alpha_{j+k}\}_{j,k\ge 0}$.

THEOREM 4.1. *Γ_α has finite rank if and only if the power series* (4–1) *determines a rational function. In this case*

$$\operatorname{rank} \Gamma_\alpha = \deg z\alpha(z).$$

PROOF. Suppose that $\operatorname{rank}\Gamma_\alpha = n$. Then the first $n+1$ rows are linearly dependent. That means that there exists a nontrivial family $\{c_j\}_{0\le j\le n}$ of complex numbers (*nontrivial* means that not all the c_j are equal to zero) such that

$$c_0\alpha + c_1 S^*\alpha + \cdots + c_n S^{*n}\alpha = 0. \tag{4–2}$$

It is easy to see that

$$S^n S^{*k}\alpha = S^{n-k}\alpha - S^{n-k}\sum_{j=0}^{k-1}\alpha_j z^j \quad \text{for } k\le n. \tag{4–3}$$

It follows easily from (4–2) and (4–3) that

$$0 = S^n \sum_{k=0}^{n} c_k S^{*k}\alpha = \sum_{k=0}^{n} c_k S^n S^{*k}\alpha = \sum_{k=0}^{n} c_k S^{n-k}\alpha - p, \tag{4–4}$$

where p has the form

$$p(z) = \sum_{j=0}^{n-1} p_j z^j.$$

Put

$$q(z) = \sum_{j=0}^{n} c_{n-j} z^j. \tag{4–5}$$

Then p and q are polynomials and it follows from (4–4) that $q\alpha = p$, so $\alpha(z) = (p/q)(z)$ is a rational function. Clearly,

$$\deg z\alpha(z) \le \max\left(\deg zp(z),\, \deg q(z)\right) = n.$$

Conversely, suppose that $\alpha(z) = (p/q)(z)$ where p and q are polynomials such that $\deg p \le n-1$ and $\deg q \le n$. Consider the complex numbers c_j defined by (4–5). We have

$$\sum_{j=0}^{n} c_j S^{n-j}\alpha = p.$$

Therefore

$$S^{*n}\sum_{j=0}^{n} c_j S^{n-j}\alpha = \sum_{j=0}^{n} c_j S^{*j}\alpha = 0,$$

which means that the first $n+1$ rows of Γ_α are linearly dependent. Let $m \le n$ be the largest number for which $c_m \ne 0$. Then $S^{*m}\alpha$ is a linear combination of the $S^{*j}\alpha$ with $j \le m-1$:

$$S^{*m}\alpha = \sum_{j=0}^{m-1} d_j S^{*j}\alpha \quad \text{with } d_j \in \mathbb{C}.$$

We will show by induction that any row of Γ_α is a linear combination of the first m rows. Let $k > m$. We have

$$S^{*k}\alpha = (S^*)^{k-m} S^{*m}\alpha = \sum_{j=0}^{m-1} d_j (S^*)^{k-m+j}\alpha. \tag{4–6}$$

Since $k-m+j < k$ for $0 \le j \le m-1$, by the induction hypothesis each of the terms of the right-hand side of (4–6) is a linear combination of the first m rows. Therefore $\operatorname{rank}\Gamma_\alpha \le m$, which completes the proof. □

It is easy to see that Kronecker's theorem for Hankel operators H_φ on H^2 admits the following reformulation.

COROLLARY 4.2. *Let $\varphi \in L^\infty$. The Hankel operator H_φ has finite rank if and only if $\mathbb{P}_-\varphi$ is a rational function. In this case*

$$\operatorname{rank} H_\varphi = \deg \mathbb{P}_-\varphi.$$

COROLLARY 4.3. *H_φ has finite rank if and only if there exists a finite Blaschke product B such that $B\varphi \in H^\infty$.*

5. Compactness

In this section we establish Hartman's compactness criterion for Hankel operators. We also compute the essential norm of a Hankel operator and study the problem of approximation of a Hankel operator by compact Hankel operators.

Recall that the essential norm $\|T\|_e$ of an operator T from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is, by definition,

$$\|T\|_e = \inf\{\|T-K\| : K \text{ is compact}\}. \tag{5–1}$$

To compute the essential norm of a Hankel operator, we have to introduce the space $H^\infty + C$.

Definition. The space $H^\infty + C$ is the set of functions φ in L^∞ such that φ admits a representation $\varphi = f + g$, where $f \in H^\infty$ and $g \in C(\mathbb{T})$.

Theorem 5.1 [Sarason 1978]. *The set $H^\infty + C$ is a closed subalgebra of L^∞.*

To prove the theorem, we need the following elementary lemma, where

$$C_A \stackrel{\text{def}}{=} H^\infty \cap C(\mathbb{T}).$$

Lemma 5.2. *Let $\varphi \in C(\mathbb{T})$. Then*

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = \operatorname{dist}_{L^\infty}(\varphi, C_A). \tag{5–2}$$

Proof. The inequality $\operatorname{dist}(\varphi, H^\infty) \le \operatorname{dist}_{L^\infty}(\varphi, C_A)$ is trivial; we prove the opposite one. For $f \in L^\infty$ we consider its harmonic extension to the unit disc and keep the same notation for it. Put $f_r(\zeta) = f(r\zeta)$ for $\zeta \in \mathbb{D}$. Let $\varphi \in C(\mathbb{T})$, $h \in H^\infty$. We have

$$\begin{aligned}\|\varphi - h\|_\infty &\ge \lim_{r\to1} \|(\varphi-h)_r\|_\infty \ge \lim_{r\to1}(\|\varphi - h_r\|_\infty - \|\varphi-\varphi_r\|_\infty)\\ &= \lim_{r\to1}\|\varphi - h_r\|_\infty \ge \operatorname{dist}_{L^\infty}(\varphi, C_A),\end{aligned}$$

since $\|\varphi-\varphi_r\|_\infty \to 0$ for continuous φ. ☐

Proof of Theorem 5.1. Equality (5–2) means exactly that the natural imbedding of $C(\mathbb{T})/C_A$ in L^∞/H^∞ is isometric, so $C(\mathbb{T})/C_A$ can be considered as a closed subspace of L^∞/H^∞. Let $\rho : L^\infty \to L^\infty/H^\infty$ be the natural quotient map. It follows that $H^\infty + C = \rho^{-1}(C(\mathbb{T})/C_A)$ is closed in L^∞.

This implies that

$$H^\infty + C = \operatorname{clos}_{L^\infty}(\cup_{n\ge0}\bar{z}^n H^\infty). \tag{5–3}$$

It is easy to see that if f and g belong to the right-hand side of (5–3), then so does fg. Hence $H^\infty + C$ is an algebra. ☐

Now we are going to compute the essential norm of a Hankel operator. The following result was apparently discovered by Adamyan, Arov, and Krein [Adamyan et al. 1968b].

Theorem 5.3. *Let $\varphi \in L^\infty$. Then*

$$\|H_\varphi\|_e = \operatorname{dist}_{L^\infty}(\varphi,\, H^\infty{+}C).$$

Lemma 5.4. *Let K be a compact operator from H^2 to H^2_-. Then*

$$\lim_{n\to\infty} \|KS^n\| = 0.$$

Proof. Since any compact operator can be approximated by finite-rank operators, it is sufficient to prove the assertion for rank-one operators K. Let $Kf = (f,\xi)\eta$, $\xi \in H^2$, $\eta \in H^2_-$. We have $KS^nf = (f, S^{*n}\xi)\eta$, so

$$\|KS^n\| = \|S^{*n}\xi\|_2\|\eta\|_2 \to 0. \qquad \square$$

Proof of Theorem 5.3. By Corollary 4.2, H_f is compact for any trigonometric polynomial f. Therefore H_f is compact for any f in $C(\mathbb{T})$. Consequently,

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty + C) = \inf_{f\in C(\mathbb{T})} \|H_\varphi - H_f\| \geq \|H_\varphi\|_e.$$

On the other hand, for any compact operator K from H^2 to H^2_-,

$$\begin{aligned}\|H_\varphi - K\| &\geq \|(H_\varphi - K)S^n\| \geq \|H_\varphi S^n\| - \|KS^n\| = \|H_{z^n\varphi}\| - \|KS^n\|\\ &= \operatorname{dist}_{L^\infty}(\varphi, \bar{z}^n H^\infty) - \|KS^n\| \geq \operatorname{dist}_{L^\infty}(\varphi,\, H^\infty{+}C) - \|KS^n\|.\end{aligned}$$

Therefore, in view of Lemma 5.4,

$$\|H_\varphi\|_e \geq \operatorname{dist}_{L^\infty}(\varphi,\, H^\infty{+}C). \qquad \square$$

Remark. In Section 2 we studied the question of existence and uniqueness of a best H^∞ approximant in the L^∞-norm. The same question can be asked about approximation by $H^\infty{+}C$ functions; it was explicitly posed by Adamyan, Arov, and Krein in [Adamyan et al. 1984]. However, the situation here is quite different. It was shown in [Axler et al. 1979] that, for any $\varphi \in L^\infty \setminus H^\infty + C$, there are infinitely many best approximants in $H^\infty{+}C$. See also [Luecking 1980] for another proof.

We now obtain Hartman's compactness criterion.

Theorem 5.5. *Let $\varphi \in L^\infty$. The following statements are equivalent.*

(a) *H_φ is compact.*
(b) *$\varphi \in H^\infty + C$.*
(c) *There exists a function ψ in $C(\mathbb{T})$ such that $H_\varphi = H_\psi$.*

Proof. Obviously (b) and (c) are equivalent.

Suppose that $\varphi \in H^\infty + C$. Then $\|H_\varphi\|_e = 0$ by Theorem 5.3, which means that H_φ is compact. Thus (b) implies (a).

To show that (a) implies (b), assume H_φ is compact. Then Theorem 5.3 gives $\operatorname{dist}_{L^\infty}(\varphi,\, H^\infty + C) = 0$, which, in combination with Theorem 5.5, yields $\varphi \in H^\infty + C$. $\square$

Corollary 5.6. *Let* $\varphi \in L^\infty$. *Then*

$$\|H_\varphi\|_e = \inf\{\|H_\varphi - H_\psi\| : H_\psi \text{ is compact}\}. \tag{5–4}$$

In other words, to compute the essential norm of a Hankel operator we can consider on the right-hand side of (5–1) only compact Hankel operators.

Corollary 5.7. *Let* $\varphi \in H^\infty + C$. *Then for any* $\varepsilon > 0$ *there exists a function* ψ *in* $C(\mathbb{T})$ *such that* $H_\psi = H_\varphi$ *and* $\|\psi\|_\infty \le \|H_\varphi\| + \varepsilon$.

Proof. Without loss of generality we can assume that $\varphi \in C(\mathbb{T})$. By Theorem 5.3, $\|H_\varphi\| = \operatorname{dist}_{L^\infty}(\varphi, H^\infty)$. On the other hand, by Lemma 5.2,

$$\operatorname{dist}_{L^\infty}(\varphi, H^\infty) = \operatorname{dist}_{L^\infty}(\varphi, C_A).$$

This means that for any $\varepsilon > 0$ there exists a function $h \in C_A$ such that $\|\varphi - h\| \le \|H_\varphi\| + \varepsilon$. Thus $\psi = \varphi - h$ does the job. ☐

Example. For a compact Hankel operator, it is not always possible to find a continuous symbol whose L^∞-norm is equal to the norm of the operator. Indeed, let α be a real-valued function in $C(\mathbb{T})$ such that $\tilde{a} \notin C(\mathbb{T})$, where $\tilde{\alpha}$ is the harmonic conjugate of α. Put $\varphi = \bar{z}e^{i\tilde{\alpha}}$. Then $\varphi = \bar{z}e^{\alpha + i\tilde{\alpha}}e^{-\alpha}$. Clearly $e^{\alpha + i\tilde{\alpha}} \in H^\infty$ and $e^{-\alpha} \in C(\mathbb{T})$. It follows from Theorem 5.1 that $\varphi \in H^\infty + C$ and so H_φ is compact. Let us show that $\|H_\varphi\| = 1$. Put

$$h = \exp \tfrac{1}{2}(\tilde{\alpha} - i\alpha).$$

Clearly, h is an outer function. To prove that $h \in H^2$ we need the following theorem of Zygmund [Zygmund 1968, Chapter 7, Theorem 2.11]: If ξ is a bounded real function such that $\|\xi\|_{L^\infty} < \pi/(2p)$, then $\exp \tilde{\xi} \in L^p$. Indeed, approximating α by trigonometric polynomials, we can easily deduce from Zygmund's theorem that $h \in H^p$ for any $p < \infty$. Clearly $\|H_\varphi h\|_2 = \|\bar{z}\bar{h}\|_2 = \|h\|_2$. Hence $\|H_\varphi\| = \|\varphi\|_\infty = 1$. By Corollary 2.4, $\|\varphi + f\|_\infty > 1$ for any nonzero f in H^∞. It is also clear that $\varphi \notin C(\mathbb{T})$. This proves the result.

In Section 2 we gave a boundedness criterion for a Hankel operator H_φ in terms of $\mathbb{P}_- \varphi$. That criterion involves the condition $\mathbb{P}_- \varphi \in \mathrm{BMO}$. We can give a similar compactness criterion if we replace BMO by the space VMO of functions of vanishing mean oscillation.

Theorem 5.8. *Let* $\varphi \in L^2$. *Then* H_φ *is compact if and only if* $\mathbb{P}_- \varphi \in \mathrm{VMO}$.

This can be derived from Theorem 5.5 in the same way as it has been done in Section 2 if we use the following description of VMO due to Sarason:

$$\mathrm{VMO} = \{\xi + \mathbb{P}_+ \eta : \xi, \eta \in C(\mathbb{T})\}.$$

See [Garnett 1981], for example.

6. Approximation by Finite-Rank Operators

DEFINITION. For a bounded linear operator T from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, the *singular values* $s_m(T)$, for $m \in \mathbb{Z}_+$, are defined by

$$s_m(T) = \inf\{\|T - R\| : \operatorname{rank} R \le m\}. \tag{6–1}$$

Clearly, $s_0(T) = \|T\|$ and $s_{m+1}(T) \le s_m(T)$.

Adamyan, Arov, and Krein [Adamyan et al. 1971] proved that in order to find $s_m(T)$ for a Hankel operator T we can consider the infimum in (6–1) over only the Hankel operators of rank at most m. This is a deep and important result.

THEOREM 6.1. *Let Γ be a Hankel operator from H^2 to H^2_-, and let $m \ge 0$. Then there exists a Hankel operator Γ_m of rank at most m such that*

$$\|\Gamma - \Gamma_m\| = s_m(\Gamma). \tag{6–2}$$

By Kronecker's theorem, $\operatorname{rank}\Gamma_m$ is at most m if and only if Γ_m has a rational symbol of degree at most m, so Theorem 6.1 admits the following reformulation. Let $\tilde{\mathcal{R}}_m$ be the set of functions f in L^∞ such that $\mathbb{P}_- f$ is a rational function of degree at most m. Clearly, $\tilde{\mathcal{R}}_m$ can be identified with the set of meromorphic functions in $\mathbb{D}$ bounded near $\mathbb{T}$ and having at most m poles in $\mathbb{D}$ counted with multiplicities.

THEOREM 6.2. *Let $\varphi \in L^\infty$, $m \in \mathbb{Z}_+$. There exists a function ψ in $\tilde{\mathcal{R}}_m$ such that*

$$\|\varphi - \psi\|_\infty = s_m(H_\varphi). \tag{6–3}$$

We will prove Theorems 6.1 and 6.2 only for compact Hankel operators. For the general case see [Adamyan et al. 1971] or [Treil' 1985a], where an alternative proof is given. Another fact that we state without proof is that for a compact Hankel operator there exists a *unique* Hankel operator Γ_m of rank at most m that satisfies (6–2); see [Adamyan et al. 1971].

DEFINITION. Let T be a compact linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. If s is a singular value of T, consider the subspaces

$$E_s^{(+)} = \{x \in \mathcal{H}_1 : T^*Tx = s^2x\}, \qquad E_s^{(-)} = \{y \in \mathcal{H}_2 : TT^*y = s^2y\}.$$

Vectors in $E_s^{(+)}$ are called *Schmidt vectors* of T (or, more precisely, s-Schmidt vectors of T). Vectors in $E_s^{(-)}$ are called *Schmidt vectors* of T^* (or s-Schmidt vectors of T^*). Clearly, $x \in E_s^{(+)}$ if and only if $Tx \in E_s^{(-)}$. A pair $\{x, y\}$, with $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, is called a *Schmidt pair* of T (or s-Schmidt pair) if $Tx = sy$ and $T^*y = sx$.

Proof of Theorem 6.1 in the compact case. Put $s = s_m(\Gamma)$. If $s = \|\Gamma\|$, the result is trivial. Assume that $s < \|\Gamma\|$. Then there exist positive integers k and μ such that $k \le m \le k+\mu-1$ and

$$s_{k-1}(\Gamma) > s_k(\Gamma) = \cdots = s_{k+\mu-1}(\Gamma) > s_{k+\mu}(\Gamma). \tag{6–4}$$

Clearly, it suffices to consider the case $m = k$.

Lemma 6.3. *Let* $\{\xi_1, \eta_1\}$ *and* $\{\xi_2, \eta_2\}$ *be* s*-Schmidt pairs of* Γ. *Then* $\xi_1\bar{\xi}_2 = \eta_1\bar{\eta}_2$.

To prove the lemma we need the following identity, which is a consequence of (2–11):

$$\mathbb{P}_-(z^n\Gamma f) = \Gamma z^n f \quad \text{for } n \in \mathbb{Z}_+. \tag{6–5}$$

Proof of Lemma 6.3. Let $n \in \mathbb{Z}_+$. We have

$$\begin{aligned}\widehat{\xi_1\bar{\xi}_2}(-n) &= (z^n\xi_1, \xi_2) = s^{-1}(z^n\xi_1, \Gamma^*\eta_2) = s^{-1}(\Gamma z^n\xi_1, \eta_2)\\ &= s^{-1}(\mathbb{P}_- z^n\Gamma\xi_1, \eta_2) = (z^n\eta_1, \eta_2) = \widehat{\eta_1\bar{\eta}_2}(-n),\end{aligned}$$

by (6–5). Similarly, $\widehat{\xi_1\bar{\xi}_2}(n) = \widehat{\eta_1\bar{\eta}_2}(n)$, $n \in \mathbb{Z}_+$, which implies $\xi_1\bar{\xi}_2 = \eta_1\bar{\eta}_2$. □

Corollary 6.4. *Let* $\{\xi, \eta\}$ *be an* s*-Schmidt pair of* Γ. *Then the function*

$$\varphi_s = \frac{\eta}{\xi} \tag{6–6}$$

is unimodular and does not depend on the choice of $\{\xi, \eta\}$.

Proof. Let $\xi_1 = \xi_2 = \xi$ and $\eta_1 = \eta_2 = \eta$ in Lemma 6.3. It follows that $|\xi|^2 = |\eta|^2$ and so η/ξ is unimodular for any Schmidt pair $\{\xi, \eta\}$.

Let $\{\xi_1, \eta_1\}$ and $\{\xi_2, \eta_2\}$ be s-Schmidt pairs of Γ. By Lemma 6.3, $\eta_1/\xi_1 = \bar{\xi}_2/\bar{\eta}_2$. Since η_2/ξ_2 is unimodular, $\eta_1/\xi_1 = \eta_2/\xi_2$. □

We resume the proof of Theorem 6.1. Put

$$\Gamma_s = H_{s\varphi_s},$$

where φ_s is defined by (6–6). Clearly $\|\Gamma_s\| \le s$. The result will be established if we show that $\operatorname{rank}(\Gamma - \Gamma_s) \le k$.

Let $\{\xi, \eta\}$ be an s-Schmidt pair of Γ. We show that it is also an s-Schmidt pair of Γ_s. Indeed,

$$\Gamma_s\xi = s\mathbb{P}_-\frac{\eta}{\xi}\xi = s\eta, \quad \Gamma_s^*\eta = s\mathbb{P}_+\frac{\xi}{\eta}\eta = s\xi.$$

Set

$$E_+ = \{\xi \in H^2 : \Gamma^*\Gamma\xi = s^2\xi\} \quad \text{for } E_- = \{\eta \in H^2_- : \Gamma\Gamma^*\eta = s^2\eta\}$$

be the spaces of Schmidt vectors of Γ and Γ^*. Clearly, $\dim E_+ = \dim E_- = \mu$.

It follows easily from (6–5) that if $\Gamma\xi = \Gamma_s\xi$, then $\Gamma z^n\xi = \Gamma_s z^n\xi$ for any $n \in \mathbb{Z}_+$. Since $\Gamma_s|E_+ = \Gamma|E_+$, it follows that Γ and Γ_s coincide on the S-invariant subspace spanned by E_+, where S is multiplication by z on H^2. By Beurling's theorem this subspace has the form ϑH^2, where ϑ is an inner function (see [Nikol'skiĭ 1986, Lecture I, 1], for example). Denote by Θ multiplication by ϑ. We have $\Gamma\Theta = \Gamma_s\Theta$. The proof will be complete if we show that $\dim(H^2 \ominus \vartheta H^2) \le k$. Put $d = \dim(H^2 \ominus \vartheta H^2)$.

LEMMA 6.5. *The singular value s of the operator $\Gamma\Theta$ has multiplicity at least $d+\mu$.*

Note that $\Gamma\Theta$ is compact, so it will follow from Lemma 6.5 that $d < \infty$.

PROOF OF LEMMA 6.5. Let τ be an inner divisor of ϑ, which means that $\vartheta\tau^{-1} \in H^\infty$. We show that, for any $\xi \in E_+$,

$$(\Gamma_s\Theta)^*(\Gamma_s\Theta)\bar\tau\xi = s^2\bar\tau\xi \in E_+. \tag{6–7}$$

Indeed it is easy to see that $\Gamma^*\bar z\bar f = \bar z\overline{\Gamma f}$ for any $f \in H^2$. Let J be the transformation on L^2 defined by $Jf = \bar z\bar f$. It follows that J maps E_+ onto E_-. Since $E_+ \subset \vartheta H^2$, we have $E_- \subset \bar\vartheta H^2_-$.

Let $\xi \in E_+$ and set $\eta = s^{-1}\Gamma\xi \in E_-$. We can represent η as $\eta = \bar\vartheta\eta_*$, where $\eta_* \in H^2_-$. We have

$$\begin{aligned}(\Gamma_s\Theta)^*(\Gamma_s\Theta)\bar\tau\xi &= (\Gamma_s\Theta)^* s\mathbb{P}_-\frac{\eta}{\xi}\vartheta\bar\tau\xi = s(\Gamma_s\Theta)^*\mathbb{P}_-\eta_*\bar\tau\\ &= s(\Gamma_s\Theta)^*\eta_*\bar\tau = s^2\mathbb{P}_+\frac{\xi}{\eta}\bar\vartheta\eta_*\bar\tau = s^2\bar\tau\xi,\end{aligned}$$

which proves (6–7).

Since $d = \dim(H^2 \ominus \vartheta H^2)$, we can find for any $n < d$ inner divisors $\{\vartheta_j\}_{1\le j\le n+1}$ of ϑ such that $\vartheta_{n+1} = \vartheta$, $\vartheta_{j+1}\vartheta_j^{-1} \in H^\infty$, and ϑ_1 and the $\vartheta_{j+1}\vartheta_j^{-1}$ are not constants; see [Nikol'skiĭ 1986, Lecture II, 2], for example. Then it follows from (6–7) that the subspace

$$E_j = \operatorname{span}\{E_+, \bar\vartheta_1E_+, \dots, \bar\vartheta_jE_+\}, \quad \text{for } 1 \le j \le n+1,$$

consists of eigenvectors of $(\Gamma\Theta)^*(\Gamma\Theta)$ corresponding to the eigenvalue s^2. Clearly, $E_1 \setminus E_+ \ne \varnothing$ and $E_{j+1} \setminus E_j \ne \varnothing$ for $1 \le j \le n$. Therefore

$$\dim\operatorname{Ker}\big((\Gamma\Theta)^*\Gamma\Theta - s^2I\big) \ge \dim E_{n+1} \ge \mu + n + 1.$$

The left-hand side is equal to ∞ if $d = \infty$ and is at least $\mu + d$ if $d < \infty$. □

We can complete now the proof of Theorem 6.1. We have already observed that $s_j(\Gamma\Theta) \le s_j(\Gamma)$, so by Lemma 6.5 we have

$$s_{k+\mu}(\Gamma) < s_{k+\mu-1}(\Gamma) = \dots = s_k(\Gamma) = s_{d+\mu-1}(\Gamma\Theta) \le s_{d+\mu-1}(\Gamma).$$

Therefore $d+\mu-1 < k+\mu$ and so $d \le k$, which completes the proof of Theorem 6.1 in the compact case. □

7. Schatten–von Neumann Classes $\boldsymbol{S}_p$

In this section we study Hankel operators of Schatten–von Neumann class $\boldsymbol{S}_p$. We state the main result, which describes the Hankel operators of class $\boldsymbol{S}_p$, for $0 < p < \infty$, as those whose symbols belong to the Besov class $B_p^{1/p}$. However, we give the proof here only in the case $p = 1$.

DEFINITION. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Given p with $0 < p < \infty$, we say that $T \in \boldsymbol{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ (or simply $T \in \boldsymbol{S}_p$), if the sequence $\{s_j\}_{j\ge 0}$ of the singular values of T belongs to ℓ^p. We put

$$\|T\|_{\boldsymbol{S}_p} \stackrel{\text{def}}{=} \left(\sum_{j\ge 0} s_j^p\right)^{1/p}. \tag{7–1}$$

For $1 \le p < \infty$ the class $\boldsymbol{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ forms a Banach space with norm given by (7–1). If $\mathcal{H}$ is a Hilbert space and T is an operator on $\mathcal{H}$ of class $\boldsymbol{S}_1$, one can define the trace of T by

$$\operatorname{trace} T \stackrel{\text{def}}{=} \sum_{j\ge 0} (Te_j, e_j), \tag{7–2}$$

where $\{e_j\}_{j\ge 0}$ is an orthonormal basis of $\mathcal{H}$. The right-hand side of (7–2) does not depend on the choice of the orthonormal basis. The trace is a linear functional on $\mathcal{H}$, and $|\operatorname{trace} T| \le \|T\|_{\boldsymbol{S}_1}$. The dual space of $\boldsymbol{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ can be identified with the space $\mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ of bounded linear operators from $\mathcal{H}_2$ to $\mathcal{H}_1$ with respect to the pairing

$$\langle T, R\rangle \stackrel{\text{def}}{=} \operatorname{trace} TR \quad \text{for } T \in \boldsymbol{S}_1(\mathcal{H}_1, \mathcal{H}_2) \text{ and } R \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1).$$

We refer the reader to [Gohberg and Kreĭn 1965] for more detailed information about the classes $\boldsymbol{S}_p$.

We now define the Besov classes B_p^s of functions on $\mathbb{T}$. They admit many different equivalent definitions; see [Peetre 1976], for example. We need regularized de la Vallée Poussin type kernels V_n, which can be defined as follows. Let v be an infinitely differentiable function on $\mathbb{R}$ such that $\operatorname{supp} v = [\frac{1}{2}, 2]$, $v \ge 0$, and

$$\sum_{j\ge 0} v\left(\frac{x}{2^j}\right) = 1 \quad \text{for } x \ge 1.$$

It is very easy to construct such a function v. We can now define V_n by

$$V_n = \begin{cases} \sum_{k\in\mathbb{Z}} v\left(\dfrac{k}{2^n}\right) z^k & \text{for } n \ge 1,\\ \overline{V_{-n}} & \text{for } n < 0,\end{cases}$$

$$V_0(z) = \bar{z} + 1 + z.$$

DEFINITION. Let $0 < p < \infty$. The *Besov space* B_p^s consists of the distributions f on $\mathbb{T}$ satisfying

$$\sum_{n\in\mathbb{Z}} \left(2^{s|n|}\|f * V_n\|_{L^p}\right)^p < \infty.$$

If $p \geq 1$, the function v does not have to be infinitely smooth. In particular, in this case we can replace v by the piecewise linear function w satisfying $w(1) = 1$ and $\operatorname{supp} w = [\frac{1}{2}, 2]$, and replace the V_n by the trigonometric polynomials

$$W_n = \begin{cases} \sum_{k\in\mathbb{Z}} w\left(\dfrac{k}{2^n}\right) z^k & \text{for } n \geq 1, \\ \overline{W_{-n}} & \text{for } n < 0, \end{cases}$$
$$W_0(z) = \bar{z} + 1 + z.$$

It is clear from this definition that $\mathbb{P}_+ B_p^s \subset B_p^s$. We can identify in a natural way a function f in $\mathbb{P}_+ B_p^s$ with the function $\sum_{j\geq 0} \hat{f}(j) z^j$, analytic in $\mathbb{D}$. A function f analytic in $\mathbb{D}$ belongs to $\mathbb{P}_+ B_p^s$ if and only if

$$\int_{\mathbb{D}} |f^{(n)}(\zeta)|^p (1 - |\zeta|)^{(n-s)p-1}\, d\boldsymbol{m}_2(\zeta) < \infty,$$

where $n \in \mathbb{Z}_+$ and $n > s$.

For $s > \max\{1/p - 1,\, 0\}$, the class B_p^s consists of the functions f on $\mathbb{T}$ for which

$$\int_{\mathbb{T}} \frac{\|\mathbb{D}_\tau^n f\|_{L^p}^p}{|\tau - 1|^{1+sp}}\, d\boldsymbol{m}(\tau) < \infty,$$

where $n > s$ is a positive integer and $(\mathbb{D}_\tau f)(\zeta) \stackrel{\text{def}}{=} f(\tau\zeta) - f(\zeta)$.

THEOREM 7.1. *Let φ be a function on $\mathbb{T}$ of class* BMO *and let $0 < p < \infty$. Then $H_\varphi \in \boldsymbol{S}_p$ if and only if $\mathbb{P}_-\varphi \in B_p^{1/p}$.*

For technical reasons it is more convenient to work with Hankel matrices $\Gamma_\varphi = \{\hat{\varphi}(j+k)\}_{j,k\geq 0}$, where $\varphi = \sum_{j\geq 0} \hat{\varphi}(j) z^j$ is a function analytic in the unit disk. We shall identify Hankel matrices Γ_φ with operators on the space ℓ^2.

Clearly, the following statement is equivalent to Theorem 7.1.

THEOREM 7.2. *Let φ be a function analytic in the unit disk and let $0 < p < \infty$. Then the Hankel operator Γ_φ belongs to the class $\boldsymbol{S}_p$ if and only if $\varphi \in B_p^{1/p}$.*

Theorem 7.1 was proved in [Peller 1980] for $p \geq 1$, and in [Semmes 1984] and [Peller 1983] for $p < 1$ (the proofs are quite different). Pekarskii's theorem [1985] on rational approximation also gives another proof of Theorem 7.2. Later other proofs were found; see, for example, [Coifman and Rochberg 1980] for $p = 1$, and [Rochberg 1982; Peetre and Svensson 1984] for $1 < p < \infty$.

We prove Theorem 7.2 only for $p = 1$. We present the original proof from [Peller 1980], which gives rather sharp estimates from above and from below for the norms $\|\Gamma_\varphi\|_{\boldsymbol{S}_1}$.

Proof of Theorem 7.2 for $p=1$.. We first prove that $\Gamma_\varphi \in \boldsymbol{S}_1$ if $\varphi \in B_1^1$. It is easy to see that

$$\varphi = \sum_{n\ge 0} \varphi * W_n.$$

We have

$$\sum_{n\ge 0} 2^n \|\varphi * W_n\|_{L^1} < \infty.$$

Clearly, $\varphi * W_n$ is a polynomial of degree at most $2^{n+1}-1$. The following lemma gives sharp estimates of the trace norm of a Hankel operator with polynomial symbol.

Lemma 7.3. *Let f be an analytic polynomial of degree m. Then*

$$\|\Gamma_f\|_{\boldsymbol{S}_1} \le (m+1)\|f\|_1.$$

Proof. Given $\zeta \in \mathbb{T}$, we define elements x_ζ and y_ζ of ℓ^2 by

$$x_\zeta(j) = \begin{cases} \zeta^j & \text{if } 0 \le j \le m, \\ 0 & \text{if } j > m; \end{cases}$$

$$y_\zeta(k) = \begin{cases} f(\zeta)\bar{\zeta}^k & \text{if } 0 \le k \le m, \\ 0 & \text{if } k > m. \end{cases}$$

Define the rank-one operator A_ζ on ℓ^2 by setting $A_\zeta x = (x, x_\zeta)y_\zeta$ for $x \in \ell^2$. Then $A_\zeta \in \boldsymbol{S}_1$ and

$$\|A_\zeta\|_{\boldsymbol{S}_1} = \|x_\zeta\|_{\ell^2}\,\|y_\zeta\|_{\ell^2} = (m+1)|f(\zeta)|.$$

We prove that

$$\Gamma_f = \int_{\mathbb{T}} A_\zeta \, d\boldsymbol{m}(\zeta) \tag{7–3}$$

(the function $\zeta \mapsto A_\zeta$ being continuous, the integral can be understood as the limit of integral sums). We have

$$(\Gamma_f e_j, e_k) = \hat{f}(j+k) = \int_{\mathbb{T}} f(\zeta)\bar{\zeta}^{j+k}\, d\boldsymbol{m}(\zeta),$$
$$(A_\zeta, e_j, e_k) = f(\zeta)\bar{\zeta}^j\bar{\zeta}^k.$$

Therefore (7–3) holds and

$$\|\Gamma_f\|_{\boldsymbol{S}_1} \le \int_{\mathbb{T}} \|A_\zeta\|_{\boldsymbol{S}_1}\, d\boldsymbol{m}(\zeta) \le (m+1)\int_{\mathbb{T}} |f(\zeta)|\, d\boldsymbol{m}(\zeta). \qquad \square$$

We now complete the proof of the sufficiency of the condition $\varphi \in B_1^1$. It follows from Lemma 7.3 that

$$\|\Gamma_\varphi\|_{\boldsymbol{S}_1} \le \sum_{n\ge 0} \|\Gamma_{\varphi * W_n}\|_{\boldsymbol{S}_1} \le \sum_{n\ge 0} 2^{n+1}\|\varphi * W_n\|_{L^1}.$$

Now suppose that $\Gamma_\varphi \in \boldsymbol{S}_1$. Define polynomials Q_n and R_n, for $n \geq 1$, by

$$\hat{Q}_n(k) = \begin{cases} 0 & \text{if } k \leq 2^{n-1}, \\ 1 - |k - 2^n|/2^{n-1} & \text{if } 2^{n-1} \leq k \leq 2^n + 2^{n-1}, \\ 0 & \text{if } k \geq 2^n + 2^{n-1}; \end{cases}$$

$$\hat{R}_n(k) = \begin{cases} 0 & \text{if } k \leq 2^n, \\ 1 - |k - 2^n - 2^{n-1}|/2^{n-1} & \text{if } 2^n \leq k \leq 2^{n+1}, \\ 0 & \text{if } k \geq 2^{n+1}. \end{cases}$$

Clearly, $W_n = Q_n + \frac{1}{2} R_n$ for $n \geq 1$.

We now show that

$$\sum_{n\geq 0} 2^{2n+1} \|\varphi * Q_{2n+1}\|_{L^1} < \infty. \tag{7–4}$$

One proves in exactly the same way that

$$\sum_{n\geq 1} 2^{2n} \|\varphi * Q_{2n}\|_{L^1} < \infty,$$
$$\sum_{n\geq 0} 2^{2n+1} \|\varphi * R_{2n+1}\|_{L^1} < \infty,$$
$$\sum_{n\geq 1} 2^{2n} \|\varphi * R_{2n}\| < \infty.$$

To prove (7–4), we construct an operator B on ℓ^2 such that $\|B\| \leq 1$ and $\langle \Gamma_\varphi, B \rangle = \sum_{n\geq 0} 2^{2n} \|f * Q_{2n+1}\|_{L^1}$.

Consider the squares $S_n = [2^{2n-1}, 2^{2n-1} + 2^{2n} - 1] \times [2^{2n-1} + 1, 2^{2n-1} + 2^{2n}]$, for $n \geq 1$, on the plane.

Let $\{\psi_n\}_{n\geq 1}$ be a sequence of functions in L^∞ such that $\|\psi_n\|_{L^\infty} \leq 1$. We define the matrix $\{b_{jk}\}_{j,k\geq 0}$ of B by

$$b_{jk} = \begin{cases} \hat{\psi}_n(j+k) & \text{if } (j,k) \in S_n \text{ for } n \geq 1, \\ 0 & \text{if } (j,k) \notin \bigcup_{n\geq 1} S_n. \end{cases}$$

We show that $\|B\| \leq 1$. Consider the subspaces

$$\mathcal{H}_n = \operatorname{span}\{e_j : 2^{2n-1} \leq j \leq 2^{2n-1}{+}2^{2n}{-}1\},$$
$$\mathcal{H}'_n = \operatorname{span}\{e_j : 2^{2n-1}{+}1 \leq j \leq 2^{2n-1}{+}2^{2n}\}.$$

It is easy to see that

$$B = \sum_{n\geq 1} P'_n \Gamma_{\psi_n} P_n,$$

where P_n and P'_n are the orthogonal projection onto $\mathcal{H}_n$ and $\mathcal{H}'_n$. Since the spaces $\{\mathcal{H}_n\}_{n\geq 1}$ are pairwise orthogonal as well as the spaces $\{\mathcal{H}'_n\}_{n\geq 1}$, we have

$$\|B\| = \sup_n \|P'_n \Gamma_{\psi_n} P_n\| \leq \sup_n \|\Gamma_{\psi_n}\| \leq \sup_n \|\psi_n\|_{L^\infty} \leq 1.$$

We show that

$$\langle \Gamma_\varphi, B\rangle = \sum_{n\ge 0} 2^{2n}\langle Q_{2n+1} * \varphi,\, \psi_n\rangle,$$

where $\langle g,h\rangle \stackrel{\text{def}}{=} \int_{\mathbb{T}} f(\zeta)h(\bar\zeta)\,d\boldsymbol{m}(\zeta)$, for $g\in L^1$ and $h\in L^\infty$. We have

$$\begin{aligned}\langle \Gamma_\varphi, B\rangle &= \sum_{n\ge 1}\langle \Gamma_\varphi, P_n'\Gamma_{\psi_n}P_n\rangle\\ &= \sum_{n\ge 1}\sum_{j=2^{2n}}^{2^{2n}+2^{2n+1}} \bigl(2^{2n}-|j-2^{2n+1}|\bigr)\hat\varphi(j)\hat\psi_n(j)\\ &= \sum_{n\ge 1} 2^{2n}\langle Q_{2n+1}*\varphi,\,\psi_n\rangle.\end{aligned}$$

We can now pick a sequence $\{\psi_n\}_{n\ge 1}$ such that $\langle Q_{2n+1}*\varphi,\,\psi_n\rangle = \|Q_{2n+1}*\varphi\|_{L^1}$. Then

$$\langle \Gamma_\varphi, B\rangle = \sum_{n\ge 1} 2^{2n}\|Q_{2n+1}*\varphi\|_{L^1}.$$

Hence

$$\sum_{n\ge 1} 2^{2n+1}\|Q_{2n+1}*\varphi\|_{L^1} = 2\langle \Gamma_\varphi, B\rangle \le 2\|\Gamma_\varphi\|_{\boldsymbol{S}_1} < \infty. \qquad \square$$

Remark. This proof easily leads to the estimates

$$\frac{1}{6}\sum_{n\ge 1} 2^n\|\varphi * W_n\|_{L^1} \le \|\Gamma_\varphi\|_{\boldsymbol{S}_1} \le 2\sum_{n\ge 0} 2^n\|\varphi * W_n\|_{L^1}.$$

8. Rational Approximation

Classical theorems on polynomial approximation, as found in [Akhiezer 1965], for example, describe classes of smooth functions in terms of the rate of polynomial approximation in one norm or another. The smoother the function, the more rapidly its deviations relative to the set of polynomials of degree n decay. However, it turns out that in the case of rational approximation the corresponding problems are considerably more complicated. The first sharp result was obtained in [Peller 1980]; it concerned rational approximation in the BMO norm and was deduced from the $\boldsymbol{S}_p$ criterion for Hankel operators given in Theorem 7.1. There were also earlier results [Gonchar 1968; Dolženko 1977; Brudnyĭ 1979], but there were gaps between the "direct" and "inverse" theorems.

In this section we describe the Besov spaces $B_p^{1/p}$ in terms of the rate of rational approximation in the norm of BMO. Then we obtain an improvement of Grigoryan's theorem which estimates the L^∞ norm of $\mathbb{P}_- f$ in terms of $\|f\|_{L^\infty}$ for functions f such that $\mathbb{P}_- f$ is a rational function of degree n. As a consequence we obtain a sharp result about rational approximation in the L^∞ norm.

There are many different natural norms on BMO. We can use, for example,

$$\|f\|_{\text{BMO}} \stackrel{\text{def}}{=} \inf\{\|\xi\|_{L^\infty} + \|\eta\|_{L^\infty} : f = \xi + \mathbb{P}_+\eta \text{ for } \xi,\, \eta\in L^\infty\}.$$

Denote by $\mathcal{R}_n$, for $n \geq 0$, the set of rational functions of degree at most n with poles outside $\mathbb{T}$. For $f \in \mathrm{BMO}$ put

$$r_n(f) \stackrel{\text{def}}{=} \operatorname{dist}_{\mathrm{BMO}}\{f, \mathcal{R}_n\}.$$

The following theorem was proved in [Peller 1980] for $p \geq 1$ and in [Peller 1983; Semmes 1984; Pekarskiĭ 1985] for $p < 1$. Pekarskii [1985; 1987] also obtained similar results for rational approximation in the L^p norms. See also [Parfenov 1986] for other applications of Hankel operators in rational approximation.

THEOREM 8.1. *Let* $\varphi \in \mathrm{BMO}$ *and* $0 < p < \infty$. *Then* $\{r_n(\varphi)\}_{n \geq 0} \in \ell^p$ *if and only if* $\varphi \in B_p^{1/p}$.

PROOF. We have $\mathbb{P}_+\mathrm{BMO} \subset \mathrm{BMO}$ (see the introduction), $\mathbb{P}_+ B_p^{1/p} \subset B_p^{1/p}$ (Section 7), and $\mathbb{P}_+\mathcal{R}_n \subset \mathcal{R}_n$. Therefore it is sufficient to prove the theorem for $\mathbb{P}_-\varphi$ and $\mathbb{P}_+\varphi$. We do it for $\mathbb{P}_-\varphi$; the corresponding result for $\mathbb{P}_+\varphi$ follows by passing to complex conjugate.

It follows from Theorem 6.1 that

$$s_n(H_\varphi) = \inf\{\|H_\varphi - H_r\| : \operatorname{rank} H_r \leq n\}.$$

Without loss of generality we may assume that $r = \mathbb{P}_- r$. By Corollary 4.2, $\operatorname{rank} H_r \leq n$ if and only if $r \in \mathcal{R}_n$. Together with Theorem 2.3 this yields

$$c_1 s_n(H_\varphi) \leq \inf\{\|\varphi - r\|_{\mathrm{BMO}} : r \in \mathcal{R}_n\} \leq c_2 s_n(H_\varphi)$$

for some positive constants c_1 and c_2.

The result follows now from Theorem 7.1. □

Denote by $\mathcal{R}_n^+$ the set of rational functions of degree at most n with poles outside the closed unit disk, and put

$$r_n^+(f) \stackrel{\text{def}}{=} \operatorname{dist}_{\mathrm{BMOA}}\{f, \mathcal{R}_n^+\}.$$

COROLLARY 8.2. *Let* $\varphi \in \mathrm{BMOA}$ *and* $0 < p < \infty$. *Then* $\left\{r_n^+(\varphi)\right\}_{n \geq 0} \in \ell^p$ *if and only if* $\varphi \in \mathbb{P}_+ B_p^{1/p}$.

We now prove an improvement of a theorem of Grigoryan [1976], which estimates the $\|\mathbb{P}_-\varphi\|_{L^\infty}$ in terms of $\|\varphi\|_{L^\infty}$ in the case $\mathbb{P}_-\varphi \in \mathcal{R}_n$. Clearly, the last condition is equivalent to the fact that φ is a boundary value function of a meromorphic function in $\mathbb{D}$ bounded near $\mathbb{T}$ and having at most n poles, counted with multiplicities. It is not obvious that such an estimate exists. If we consider the same question in the case where $\mathbb{P}_-\varphi$ is a polynomial of degree n, it is well known that $\|\mathbb{P}_-\varphi\|_{L^\infty} \leq \operatorname{const}\log(1+n)$ (see [Zygmund 1968]; this follows immediately from the fact that $\|\sum_{j=0}^n z^j\|_{L^1} \leq \operatorname{const}\log(1+n)$). Grigoryan's theorem claims that, if $\mathbb{P}_-\varphi \in \mathcal{R}_n$, then

$$\|\mathbb{P}_-\varphi\|_{L^\infty} \leq \operatorname{const}\cdot n \|\varphi\|_{L^\infty}. \tag{8–1}$$

The following result, obtained in [Peller 1983], improves this estimate. The proof is based on the $\boldsymbol{S}_1$ criterion for Hankel operators given in Theorem 7.1.

THEOREM 8.3. *Let n be a positive integer and let φ be a function in L^∞ such that $\mathbb{P}_-\varphi \in \mathcal{R}_n$. Then*

$$\|\mathbb{P}_-\varphi\|_{B_1^1} \le \text{const}\cdot n\|\varphi\|_{L^\infty}. \tag{8–2}$$

Observe first that (8–2) implies (8–1). Indeed, if $f \in B_1^1$, then $\sum_{n\ge 0} 2^n\|f * W_n\|_{L^1} \le \text{const}\,\|f\|_{B_1^1}$ (see Section 7). It is easy to show that

$$\|\varphi\|_{L^\infty} \le \sum_{j\ge 0} |\hat{f}(j)| \le \text{const}\sum_{n\ge 0} 2^n\|f * W_n\|_{L^1},$$

which proves the claim.

PROOF OF THEOREM 8.3. Consider the Hankel operator H_φ. By Nehari's theorem, $\|H_\varphi\| \le \|\varphi\|_{L^\infty}$. By Kronecker's theorem, $\operatorname{rank} H_\varphi \le n$. Therefore $\|H_\varphi\|_{\boldsymbol{S}_1} \le n\|H_\varphi\|$. The result now follows from Theorem 7.1, which guarantees that $\|\mathbb{P}_-\varphi\|_{B_1^1} \le \text{const}\,\|H_\varphi\|_{\boldsymbol{S}_1}$. □

To conclude this section we obtain a result on rational approximation in the L^∞ norm [Peller 1983]. For $\varphi \in L^\infty$ we put

$$\rho_n(\varphi) \stackrel{\text{def}}{=} \operatorname{dist}_{L^\infty}\{\varphi, \mathcal{R}_n\} \quad \text{for } n \in \mathbb{Z}_+.$$

THEOREM 8.4. *Let $\varphi \in L^\infty$. Then the $\rho_n(\varphi)$ decay more rapidly than any power of n if and only if $\varphi \in \bigcap_{p>0} B_p^{1/p}$.*

Pekarskii [1987] obtained a result similar to Theorem 8.1 for rational approximation in L^∞ in the case $0 < p < 1$.

LEMMA 8.5. *Let $r \in \mathcal{R}_n$. Then*

$$\|r\|_{L^\infty} \le \text{const}\cdot n\,\|r\|_{\text{BMO}}.$$

PROOF. It suffices to prove the inequality for $\mathbb{P}_- r$ and $\mathbb{P}_+ r$; we do it for $\mathbb{P}_- r$. Let f be the symbol of H_r-minimal norm, that is, such that $\mathbb{P}_- r = \mathbb{P}_- f$ and $\|f\|_{L^\infty} = \|H_r\|$ (see Corollary 2.5). We have

$$\|\mathbb{P}_- r\|_{L^\infty} = \|\mathbb{P}_- f\|_{L^\infty} \le \text{const}\cdot n\|f\|_{L^\infty} = \text{const}\cdot n\|H_r\| \le \text{const}\cdot n\|\mathbb{P}_- r\|_{\text{BMO}},$$

by Theorems 8.3 and 2.3. □

Theorem 5.8 is an easy consequence of the following lemma.

LEMMA 8.6. *Let $\lambda > 1$ and let φ be a function in L^∞ such that $r_n(\varphi) \le \text{const}\cdot n^{-\lambda}$ for $n \ge 0$. Then*

$$\rho_n(\varphi) \le \text{const}\cdot n^{-\lambda+1} \quad \textit{for } n \ge 0.$$

PROOF. Suppose that $r_n \in \mathcal{R}_{2^n}$ and $\|\varphi - r_n\|_{BM0} \le \text{const}\, 2^{-n\lambda}$. We have

$$\varphi - r_n = \sum_{j\ge 0}\big((\varphi - r_{n+j}) - (\varphi - r_{n+j+1})\big) = \sum_{j\ge 0}(r_{n+j+1} - r_{n+j}).$$

Under the hypotheses of the lemma ,

$$\|r_{n+j+1} - r_{n+j}\|_{\text{BMO}} \le \text{const}\, 2^{-(n+j)\lambda},$$

and, since $r_{n+j+1} - r_{n+j} \in \mathcal{R}_{2^{n+j+2}}$, Lemma 8.5 gives

$$\|r_{n+j+1} - r_{n+j}\|_{L^\infty} \le \text{const}\, 2^{-(n+j)(\lambda-1)}.$$

Therefore

$$\rho_{2^n}(\varphi) \le \|\varphi - r_n\|_{L^\infty}\, \text{const}\, 2^{-n(\lambda-1)},$$

which implies the conclusion of the lemma. □

9. The Operator of Best Approximation by Analytic Functions

Let φ be a function in VMO. By Corollary 2.5, there exists a unique function f in BMOA such that $\varphi - f$ is bounded on $\mathbb{T}$ and

$$\|\varphi - f\|_{L^\infty} = \inf\{\|\varphi - g\|_{L^\infty} : g \in \text{BMOA with } \varphi - g \in L^\infty(\mathbb{T})\} = \|H_\varphi\|.$$

We define the nonlinear *operator of best approximation by analytic functions* on the space VMO by setting $\mathcal{A}\varphi \stackrel{\text{def}}{=} f$. This operator is very important in applications such as control theory and prediction theory.

We are going to study hereditary properties of $\mathcal{A}$. This means the following: Suppose that $X \subset \text{VMO}$ is a space of functions on $\mathbb{T}$. For which X does the operator $\mathcal{A}$ maps X into itself? Certainly not for arbitrary X: for example, $\mathcal{A}C(\mathbb{T}) \not\subset C(\mathbb{T})$, as follows from the remark after Corollary 5.7.

Shapiro [1952] showed that $\mathcal{A}X \subset X$ if X is the space of functions analytic in a neighbourhood of $\mathbb{T}$. Carleson and Jacobs [1972] proved that $\mathcal{A}\Lambda_\alpha \subset \Lambda_\alpha$ if $\alpha > 0$ and $\alpha \notin \mathbb{Z}$, where the $\Lambda_\alpha \stackrel{\text{def}}{=} B^\alpha_\infty$ are the Hölder–Zygmund classes (see Section 7).

In [Peller and Khrushchëv 1982] three big classes of function spaces X were found for which $\mathcal{A}X \subset X$. The first consists of the so-called $\mathcal{R}$-spaces, which are, roughly speaking, function spaces that can be described in terms of rational approximation in the BMO norm. The Besov spaces $B^{1/p}_p$, for $0 < p < \infty$, and the space VMO are examples of $\mathcal{R}$-spaces. I will not give a precise definition here.

The second class consists of function spaces X that satisfy the following axioms:

(A1) If $f \in X$, then $\bar{f} \in X$ and $\mathbb{P}_+ f \in X$.
(A2) X is a Banach algebra with respect to pointwise multiplication.
(A3) The trigonometric polynomials are dense in X.
(A4) The maximal ideal space of X can be identified naturally with $\mathbb{T}$.

Many classical spaces of functions on the unit circle satisfy these axioms: the space of functions with absolutely convergent Fourier series, the Besov spaces B_p^s for $1 \le p < \infty$, and many others (see [Peller and Khrushchëv 1982]). However, the Hölder–Zygmund classes Λ_α do not satisfy axiom (A3).

The third class of function spaces described in [Peller and Khrushchëv 1982] contains many nonseparable Banach spaces. In particular, it contains the classes Λ_α, for $\alpha > 0$. I will not define the third class here; see [Peller and Khrushchëv 1982] for the definition and other examples.

Other function spaces satisfying the property $\mathcal{A}X \subset X$ are described in [Vol'berg and Tolokonnikov 1985; Tolokonnikov 1991].

Another related question, also important in applications, is the continuity problem. Merino [1989] and Papadimitrakis [1993] showed that the operator $\mathcal{A}$ is discontinuous at any function $\varphi \in C(\mathbb{T}) \setminus H^\infty$ in the L^∞ norm. For function spaces satisfying Axioms (A1)–(A4), continuity points of $\mathcal{A}$ in the norm of X were described in [Peller 1990b]: if $\varphi \in X \setminus H^\infty$, then $\mathcal{A}$ is continuous at φ if and only if the singular value $s_0(H_\varphi)$ of the Hankel operator $H_\varphi : H^2 \to H^2_-$ has multiplicity one.

In this section we prove that $\mathcal{A}$ preserves the spaces $B_p^{1/p}$, for $0 < p < \infty$, and the space VMO. Moreover, it turns out that the operator $\mathcal{A}$ is *bounded* on such spaces; that is,

$$\|\mathcal{A}\varphi\|_X \le \text{const}\, \|\varphi\|_X, \tag{9–1}$$

for $X = B_p^{1/p}$ or $X = \text{VMO}$. Note, however, that this is a rather exceptional property. It was proved in [Peller 1992] that $\mathcal{A}$ is unbounded on X if $X = B_p^s$, with $s > 1/p$, and on Λ_a, with $\alpha > 0$. Then it was shown in [Papadimitrakis 1996] that $\mathcal{A}$ is unbounded on the space of functions with absolutely convergent Fourier series.

THEOREM 9.1. *Let $X = B_p^{1/p}$, with $0 < p < \infty$, or $X = \text{VMO}$. Then $\mathcal{A}X \subset X$ and* (9–1) *holds*.

To prove Theorem 9.1 we need a formula that relates the moduli of the Toeplitz operators T_u and $T_{\bar u}$ for a *unimodular* function u (one satisfying $|u(\zeta)| = 1$ a.e. on $\mathbb{T}$). This formula was found in [Peller and Khrushchëv 1982]:

$$H_{\bar u}^* H_{\bar u} T_u = T_u H_u^* H_u. \tag{9–2}$$

It is an immediate consequence of the definitions of the Toeplitz and Hankel operators. Nonetheless, it has many important applications.

Recall that each bounded linear operator T on a Hilbert space $\mathcal{H}$ admits a polar decomposition $T = U(T^*T)^{1/2}$, where U is an operator such that $\operatorname{Ker} U = \operatorname{Ker} T$ and $U|\mathcal{H} \ominus \operatorname{Ker} U$ is an isometry onto the closure of the range of T. The operator U is called the *partially isometric factor* of T.

We need the following well-known fact [Halmos 1967, Problem 152]. Let A and B be selfadjoint operators on Hilbert space and let T be an operator such that $AT = TB$. Then $AU = UB$, where U is the partially isometric factor of T.

We apply this to formula (9–2). Let u be a unimodular function on $\mathbb{T}$. Denote by $\boldsymbol{U}$ the partially isometric factor of T_u. Then

$$H_{\bar{u}}^* H_{\bar{u}} \boldsymbol{U} = \boldsymbol{U} H_u^* H_u. \tag{9–3}$$

The following theorem was proved in [Peller and Khrushchëv 1982].

THEOREM 9.2. *Let u be a unimodular function on $\mathbb{T}$ such that T_u has dense range in H^2. Then $H_{\bar{u}}^* H_{\bar{u}}$ is unitarily equivalent to $H_u^* H_u | H^2 \ominus E$, where*

$$E = \operatorname{Ker} T_u = \{f \in H^2 : H_u^* H_u f = f\}.$$

PROOF. Since $\boldsymbol{U}$ maps $H^2 \ominus E$ isometrically onto H^2, it follows from (9–3) that

$$H_{\bar{u}}^* H_{\bar{u}} = \boldsymbol{U} H_u^* H_u \boldsymbol{U}^* = \boldsymbol{U}(H_u^* H_u | H^2 \ominus E)\boldsymbol{U}^*,$$

which proves the result. □

To prove Theorem 9.1 we need one more elementary fact [Peller and Khrushchëv 1982].

LEMMA 9.3. *Let h be an outer function in H^2, τ an inner function, and let $u = \bar{\tau}\bar{h}/h$. Then T_u has dense range in H^2.*

PROOF. Assume that $f \perp T_u H^2$ is nonzero. Then $(f, ug) = 0$ for any $g \in H^2$. We have $f = f_{(o)} f_{(i)}$, where $f_{(o)}$ is outer and $f_{(i)}$ is inner. Put $g = \tau f_{(i)} h$. Then

$$(f, ug) = (f_{(i)} f_{(o)}, \bar{\tau}\tau f_{(i)} \bar{h}) = (f_{(o)}, \bar{h}) = f_{(o)}(0) h(0) = 0,$$

which is impossible since both h and $f_{(o)}$ are outer. □

PROOF OF THEOREM 9.1. We prove the theorem for $X = B_p^{1/p}$. The proof for $X = \mathrm{VMO}$ is exactly the same.

Without loss of generality we may assume that $\mathbb{P}_- \varphi \neq 0$. Multiplying φ, if necessary, by a suitable constant, we may also assume that $\|H_\varphi\| = 1$. Let $f = \mathcal{A}\varphi$. Put $u = \varphi - f$. By Corollary 2.5, u is unimodular and has the form $u = \bar{z}\bar{\vartheta}\bar{h}/h$, where ϑ is an inner function and h is an outer function in H^2. It follows from Lemma 9.3 that T_u has dense range in H^2.

Since $\mathbb{P}_- u = \mathbb{P}_- \varphi$, Theorem 7.1 implies that $H_u \in \boldsymbol{S}_p$ and $\|H_u\|_{\boldsymbol{S}_p}$ is equivalent to $\|\mathbb{P}_- \varphi\|_{B_p^{1/p}}$. We can now apply Theorem 9.2, which implies that

$$\|H_{\bar{u}}\|_{\boldsymbol{S}_p} \le \|H_u\|_{\boldsymbol{S}_p},$$

and so

$$\|\mathbb{P}_+ u\|_{B_p^{1/p}} \le \operatorname{const} \|\mathbb{P}_- u\|_{B_p^{1/p}}.$$

The result follows now from the obvious observation $f = \mathbb{P}_+ f = \mathbb{P}_+ \varphi - \mathbb{P}_+ u$. □

10. Hankel Operators and Prediction Theory

In this section we demonstrate how Hankel operators can be applied in prediction theory. By a discrete time stationary Gaussian process we mean a two-sided sequence $\{X_n\}_{n\in\mathbb{Z}}$ of random variables which belong to a Gaussian space (i.e., space of functions normally distributed) such that

$$\mathrm{E}X_n = 0$$

and

$$\mathrm{E}X_n X_k = c_{n-k}$$

for some sequence $\{c_n\}_{n\in\mathbb{Z}}$ of real numbers, where E is mathematical expectation.

It is easy to see that the sequence $\{c_n\}_{n\in\mathbb{Z}}$ is positive semi-definite, so by the Riesz–Herglotz theorem [Riesz and Sz.-Nagy 1965] there exists a finite positive measure μ on $\mathbb{T}$ such that $\hat{\mu}(n) = c_n$. The measure μ is called the *spectral measure of the process*.

We can now identify the closed linear span of $\{X_n\}_{n\in\mathbb{Z}}$ with the space $L^2(\mu)$ using the unitary map defined by

$$X_n \mapsto z^n \quad \text{for } n \in \mathbb{Z}.$$

This allows one to reduce problems of prediction theory to the corresponding problems in the space $L^2(\mu)$, and instead of the sequence $\{X_n\}_{n\in\mathbb{Z}}$ we can study the sequence $\{z^n\}_{n\in\mathbb{Z}}$. Note that if μ is the spectral measure of a stationary Gaussian process, its Fourier coefficients are real, so μ satisfies the condition

$$\mu(E) = \mu\{\zeta : \bar{\zeta} \in E\} \quad \text{for } E \in \mathbb{T}. \tag{10–1}$$

It can be shown that any finite positive measure satisfying (10–1) is the spectral measure of a stationary Gaussian process. However, to study regularity conditions in the space $L^2(\mu)$ we do not need (10–1). So from now on μ is an arbitrary positive finite Borel measure on $\mathbb{T}$, though if it does not satisfy (10–1), the results described below have no probabilistic interpretation.

With the process $\{z^n\}_{n\in\mathbb{Z}}$ we associate the following subspaces of $L^2(\mu)$:

$$\boldsymbol{G}^n \overset{\text{def}}{=} \underset{L^2(\mu)}{\operatorname{span}}\{z^m : m \ge n\} = z^n H^2(\mu)$$

("future starting at the moment n") and

$$\boldsymbol{G}_n \overset{\text{def}}{=} \underset{L^2(\mu)}{\operatorname{span}}\{z^m : m < n\} = z^n H^2_-(\mu)$$

("past till the moment n"). Here

$$H^2(\mu) \overset{\text{def}}{=} \underset{L^2(\mu)}{\operatorname{span}}\{z^m : m \ge 0\}, \qquad H^2_-(\mu) \overset{\text{def}}{=} \underset{L^2(\mu)}{\operatorname{span}}\{z^m : m < 0\},$$

and span means the closed linear span.

The process $\{z^n\}_{n\in\mathbb{Z}}$ is called *regular* if

$$\bigcap_{n\geq 0} \boldsymbol{G}^n = \{0\}.$$

We denote by $\mathcal{P}^n$ and $\mathcal{P}_n$ the orthogonal projections onto $\boldsymbol{G}^n$ and $\boldsymbol{G}_n$ respectively. It is easy to see that the process is regular if and only if $\lim_{n\to\infty} \mathcal{P}_0\mathcal{P}^n = 0$ in the strong operator topology.

By Szegö's theorem [Ibragimov and Rozanov 1970] the process is regular if an only if μ is absolutely continuous with respect to Lebesgue measure and its density w (called the *spectral density of the process*) satisfies $\log w \in L^1$.

In prediction theory it is important to study other regularity conditions (i.e., conditions expressing that the operators $\mathcal{P}_0\mathcal{P}^n$ are small in a certain sense) and characterize the processes satisfying such conditions in terms of the spectral densities.

A process $\{z^n\}_{n\in\mathbb{Z}}$ in $L^2(\mu)$ is called *completely regular* if

$$\rho_n \stackrel{\text{def}}{=} \lim_{n\to\infty} \|\mathcal{P}_0\mathcal{P}^n\| = 0;$$

this means that the spaces $\boldsymbol{G}^n$ and $\boldsymbol{G}_0$ become asymptotically orthogonal as $n \to \infty$, or the corresponding Gaussian subspaces become asymptotically independent.

The following results describe processes satisfying certain regularity conditions. See [Peller and Khrushchëv 1982] for other regularity conditions.

THEOREM 10.1. *The process* $\{z^n\}_{n\in\mathbb{Z}}$ *in* $L^2(w)$ *is completely regular if and only if* w *admits a representation*

$$w = |P|^2 e^{\varphi}, \tag{10–2}$$

where φ *is a real function in* VMO *and* P *is a polynomial with zeros on* $\mathbb{T}$.

Theorem 10.1 was proved in [Helson and Sarason 1967] and [Sarason 1972] (without mention of the space VMO, which was introduced later).

THEOREM 10.2. *The process* $\{z^n\}_{n\in\mathbb{Z}}$ *in* $L^2(w)$ *satisfies the condition*

$$\rho_n \leq \text{const}(1+n)^{-\alpha}, \quad \textit{for } \alpha > 0,$$

if and only if w *admits a representation of the form* (10–2), *where* φ *is a real function in* Λ^α *and* P *is a polynomial with zeros on* $\mathbb{T}$.

Theorem 10.2 was obtained by Ibragimov; see [Ibragimov and Rozanov 1970].

THEOREM 10.3. *The process* $\{z^n\}_{n\in\mathbb{Z}}$ *in* $L^2(w)$ *satisfies the condition*

$$\mathcal{P}_0\mathcal{P}^0 \in \boldsymbol{S}_p, \quad \textit{for } 0 < p < \infty,$$

if and only if w *admits a representation of the form* (10–2), *where* φ *is a real function in* $B_p^{1/p}$ *and* P *is a polynomial with zeros on* $\mathbb{T}$.

For $p = 2$ Theorem 10.3 was proved by Ibragimov and Solev; see [Ibragimov and Rozanov 1970]. It was generalized for $1 \le p < \infty$ in [Peller 1980; Peller and Khrushchëv 1982] and for $p < 1$ in [Peller 1983]. The Ibragimov–Solev proof works only for $p = 2$.

Original proofs were different for different regularity conditions; some of them (in particular, the original proof of Theorem 10.2) were technically very complicated. In [Peller and Khrushchëv 1982] a unified method was found that allowed one to prove all such results by the same method. The method involves Hankel and Toeplitz operators and it simplifies considerably many original proofs. In [Peller 1990a] the method was simplified further.

In this section we prove Theorem 10.1. The proofs of Theorems 10.2 and 10.3 are similar.

To prove Theorem 10.1 we need several well-known results from the theory of Toeplitz operators. We mention some elementary properties, which follow immediately from the definition:

$$\begin{aligned} T_\varphi^* &= T_{\bar\varphi} && \text{for } \varphi \in L^\infty, \\ T_{\bar\varphi f \psi} &= T_{\bar\varphi} T_f T_\psi && \text{for } f \in L^\infty \text{ and } \varphi, \psi \in H^\infty. \end{aligned}$$

An operator T on Hilbert space is called *Fredholm* if there exists an operator R such that $TR - I$ and $RT - I$ are compact. It is well-known that T is Fredholm if and only if $\dim \operatorname{Ker} T < \infty$, $\dim \operatorname{Ker} T^* < \infty$, and the range of T is closed. The *index* $\operatorname{ind} T$ of a Fredholm operator T is defined by

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^*.$$

If T_1 and T_2 are Fredholm, then $\operatorname{ind} T_1 T_2 = \operatorname{ind} T_1 + \operatorname{ind} T_2$. The proofs of these facts can be found in [Douglas 1972].

Clearly, a Fredholm operator with zero index is not necessarily invertible. However, the following result of Coburn (see [Sarason 1978; Nikol'skiĭ 1986, Appendix 4, 43], for example) shows that a Fredholm Toeplitz operator with zero index must be invertible.

LEMMA 10.4. *Let* $\varphi \in L^\infty$. *Then* $\operatorname{Ker} T_\varphi = \{0\}$ *or* $\operatorname{Ker} T_\varphi^* = \{0\}$.

PROOF. Let $f \in \operatorname{Ker} T_\varphi$ and $g \in \operatorname{Ker} T_\varphi^*$. Then $\varphi f \in H^2_-$ and $\bar\varphi g \in H^2_-$. Consequently, $\varphi f \bar g \in H^1_- \stackrel{\text{def}}{=} \{\psi \in L^1 : \hat\psi(n) = 0,\ n \le 0\}$ and $\bar\varphi \bar f g \in H^1_-$. Thus the Fourier coefficients of $\varphi f \bar g$ are identically equal to zero, and so $\varphi f \bar g = 0$. Therefore if φ is a nonzero function, then either f or g must vanish on a set of positive measure which implies that $f = 0$ or $g = 0$. □

We need one more well-known lemma of Devinatz and Widom; see, for example, [Douglas 1972; Nikol'skiĭ 1986, Appendix 4, 36].

LEMMA 10.5. *Let u be a unimodular function such that T_u is invertible. Then there exists an outer function η such that* $\|u - \eta\|_{L^\infty} < 1$.

PROOF. Clearly,

$$\|H_u f\|_{L^2}^2 + \|T_u f\|_{L^2}^2 = \|f\|_{L^2}^2 \quad \text{for } f \in H^2.$$

Since T_u is invertible, it follows that

$$\|H_u\| = \operatorname{dist}_{L^\infty}\{u, H^\infty\} < 1.$$

Let η be a function in H^∞ such that $\|u-\eta\|_{L^\infty} < 1$. We show that η is outer. We have

$$\|I - T_{\bar{u}\eta}\| = \|\mathbf{1} - \bar{u}\eta\|_{L^\infty} = \|u - \eta\|_{L^\infty} < 1$$

(here $\mathbf{1}$ is the function identically equal to 1). Thus $T_{\bar{u}\eta} = T_u^* T_\eta$ is invertible. Hence T_η is invertible. Clearly T_η is multiplication by η on H^2, and so η must be invertible in H^∞ which implies that η is outer. □

Finally, we prove the theorem of Sarason [1978] that describes the unimodular functions in VMO. We give the proof from [Peller and Khrushchëv 1982], which is based on Toeplitz operators.

THEOREM 10.6. *A unimodular function u belongs to* VMO *if an only if u admits a representation*

$$u = z^n \exp i(\tilde{q} + r), \tag{10–3}$$

where $n \in \mathbb{Z}$ and q and r are real functions in $C(\mathbb{T})$.

In other words, u belongs to VMO if and only if $u = z^n e^{i\kappa}$, where κ is a real function in VMO.

PROOF. Suppose that u is given by (10–3). Then

$$u = z^n \exp(q + i\tilde{q}) \exp(-q + ir) \in H^\infty + C,$$

since $H^\infty + C$ is an algebra (see Theorem 5.1). Hence H_u is compact, and so $\mathbb{P}_- u \in$ VMO (see Theorems 5.5 and 5.8). Similarly, $\mathbb{P}_- \bar{u} \in$ VMO, and so $u \in$ VMO.

Now suppose that $u \in$ VMO. It follows immediately from the definitions of Hankel and Toeplitz operators, that

$$I - T_u T_{\bar{u}} = H_{\bar{u}}^* H_{\bar{u}} \quad \text{for } I - T_{\bar{u}} T_u = H_u^* H_u.$$

Since the Hankel operators H_u and $H_{\bar{u}}$ are compact, the operator T_u is Fredholm. Put $u = z^n v$, where $n = \operatorname{ind} T_u$. If $n \ge 0$, then $T_u = T_v T_{z^n}$, whereas if $n \le 0$, then $T_u = T_{z^n} T_v$. Therefore $\operatorname{ind} T_u = \operatorname{ind} T_v + \operatorname{ind} T_{z^n} = \operatorname{ind} T_v - n = -n$. Hence $\operatorname{ind} T_v = 0$, and T_v is invertible by Lemma 10.4.

By Lemma 10.5 there exists an outer function η such that

$$\|v - \eta\|_{L^\infty} = \|\mathbf{1} - \bar{v}\eta\|_{L^\infty} < 1.$$

Hence $\bar{v}\eta$ has a logarithm in the Banach algebra $H^\infty + C$. Let $f \in C(\mathbb{T})$ and let $g \in H^\infty$ satisfy $(\bar{v}\eta)^{-1} = v/\eta = \exp(f+g)$. We have

$$v = \exp(ic + \log|\eta| + i\widetilde{\log|\eta|} + f + g),$$

where $c \in \mathbb{R}$. Since v is unimodular, it follows that $\log|\eta| + \operatorname{Re}(f+g) = 0$. Therefore, setting $q \stackrel{\text{def}}{=} \log|\eta| + \operatorname{Re} g$, we have $q \in C(\mathbb{T})$. Since $g \in H^\infty$, we get

$$\widetilde{\log|\eta|} + \operatorname{Im} g = \tilde{q} + \operatorname{Im} \hat{g}(0).$$

To complete the proof it remains to put $r \stackrel{\text{def}}{=} \operatorname{Im} f + c + \operatorname{Im} \hat{g}(0)$ and observe that u satisfies (10–3). □

Proof of Theorem 10.1. We first write the operator $\mathcal{P}_0\mathcal{P}^n$ in terms of a Hankel operator. Let h be an outer function in H^2 such that $|h|^2 = w$. Consider the unitary operators $\mathcal{U}$ and $\mathcal{V}$ from L^2 onto $L^2(w)$ defined by

$$\mathcal{U}f = f/h, \qquad \mathcal{V}f = f/\bar{h}, \qquad f \in L^2.$$

Since h is outer, it follows from Beurling's theorem (see [Nikol'skiĭ 1986], for example) that $\mathcal{U}H^2 = H^2(w)$ and $\mathcal{V}H^2_- = H^2_-(w)$. Therefore

$$\mathcal{P}_0 g = \mathcal{V}\mathbb{P}_-\mathcal{V}^{-1}g \quad \text{for } g \in L^2(w),$$

and

$$\mathcal{P}^n g = \mathcal{U}z^n\mathbb{P}_+\bar{z}^n\mathcal{U}^{-1}g \quad \text{for } g \in L^2(w).$$

Hence

$$\mathcal{P}_0\mathcal{P}^n g = \mathcal{V}\mathbb{P}_-\mathcal{V}^{-1}\mathcal{U}z^n\mathbb{P}_+\bar{z}^n\mathcal{U}^{-1}g = \mathcal{V}\mathbb{P}_-(\bar{h}/h)z^n\mathbb{P}_+\bar{z}^n\mathcal{U}^{-1}g \quad \text{for } g \in L^2(w).$$

It follows that

$$\rho_n = \|H_{z^n\bar{h}/h}\| \quad \text{for } n \ge 0. \tag{10–4}$$

Lemma 10.7. *The process* $\{z^n\}_{n\in\mathbb{Z}}$ *is completely regular if and only if* $\bar{h}/h \in$ VMO.

Proof. It follows from (10–4) that complete regularity is equivalent to the fact that $\|H_{z^n\bar{h}/h}\| \to 0$. We have

$$\begin{aligned}\|H_{z^n\bar{h}/h}\| &= \operatorname{dist}_{L^\infty}\{z^n\bar{h}/h, H^\infty\}\\ &= \operatorname{dist}_{L^\infty}\{\bar{h}/h, \bar{z}^n H^\infty\} \to \operatorname{dist}_{L^\infty}\{\bar{h}/h,\, H^\infty + C\},\end{aligned}$$

so $\lim_{n\to\infty}\rho_n = 0$ if and only if $\bar{h}/h \in H^\infty + C$. The last condition means that $H_{\bar{h}/h}$ is compact, which is equivalent to the fact that $\mathbb{P}_-\bar{h}/h \in$ VMO (see Theorem 5.8). It remains to show that this is equivalent to the inclusion $\bar{h}/h \in$ VMO.

Put $u = \bar{h}/h$. By Lemma 9.3 the Toeplitz operator T_u has dense range in H^2, so by Theorem 9.2 the Hankel operator $H_{\bar{u}}$ is compact. The result now follows from Theorem 5.8. □

We resume the proof of Theorem 10.1. It is easy to see that

$$\frac{\overline{z-\gamma}}{z-\gamma} = -\bar{\gamma}\bar{z} \quad \text{for } \gamma \in \mathbb{T}.$$

Therefore for a polynomial P of degree m with zeros on $\mathbb{T}$

$$\frac{\overline{P}}{P} = c\bar{z}^m$$

for $c \in \mathbb{T}$.

Suppose first that $w = |P|^2 e^{\varphi}$, where $\varphi \in \mathrm{VMO}$ and P is a polynomial of degree m. Consider the outer function $h_1 = \exp \frac{1}{2}(\varphi + i\tilde{f})$. Since Ph_1 is outer and $|Ph_1| = |h|$, we have $h = \omega P h_1$, where $\omega \in \mathbb{T}$. Hence

$$\frac{\bar{h}}{h} = \bar{\omega}^2 \frac{\overline{P}}{P}\frac{\bar{h}_1}{h_1} = \lambda \exp(-i\tilde{\varphi}), \quad \lambda \in \mathbb{T}.$$

By Theorem 10.6, $\bar{h}/h \in \mathrm{VMO}$, and so by Lemma 10.7 the process is completely regular.

Conversely, suppose that the process is completely regular. By Lemma 10.7, $\bar{h}/h \in \mathrm{VMO}$, so by Theorem 10.6

$$\frac{\bar{h}}{h} = \bar{z}^m e^{i\psi}$$

for some $m \in \mathbb{Z}$ and $\omega \in \mathrm{VMO}$. By Lemma 9.3, $T_{\bar{h}/h}$ has dense range, so $\operatorname{ind} T_{\bar{h}/h} \geq 0$. It follows from the proof of Theorem 10.6 that $T_{e^{i\psi}}$ is invertible, which implies that $m \geq 0$. Now consider the outer function

$$h_1 \stackrel{\text{def}}{=} \exp(-\tilde{\psi}/2 + i\psi/2).$$

As in the remark after Corollary 5.7 we can conclude that $h_1 \in H^2$.

Consider the Toeplitz operator $T_{\bar{z}\bar{h}/h} = T_{\bar{z}^{m+1}\bar{h}_1/h_1}$. Its index equals $m+1$, so it has $(m+1)$-dimensional kernel. Obviously, $z^j h_1 \in \operatorname{Ker} T_{\bar{z}^{m+1}\bar{h}_1/h_1}$ for $0 \leq j \leq m$, and so the functions $z^j h_1$, for $0 \leq j \leq m$, form a basis in $\operatorname{Ker} T_{\bar{z}\bar{h}/h}$. It is also obvious that $h \in \operatorname{Ker} T_{\bar{z}\bar{h}/h}$. Hence $h = Ph_1$ for some polynomial P of degree at most m. Since h and h_1 are outer, so is P, which implies that P has no zeros outside the closed unit disk.

We show that P has degree m and has no zeros in $\mathbb{D}$. Let $P = P_1 P_2$, where P_1 has zeros on $\mathbb{T}$ and P_2 has zeros outside the closed unit disk. Let $k = \deg P_1$. Then

$$\frac{\bar{h}}{h} = \frac{\overline{P}_1}{P_1}\frac{\overline{P}_2}{P_2}\frac{\bar{h}_1}{h_1} = \omega \bar{z}^k \overline{P}_2 \frac{\bar{h}_1}{h_1} P_2^{-1} \quad \text{for } \omega \in \mathbb{T}.$$

Consequently,

$$\operatorname{ind} T_{\bar{h}/h} = \operatorname{ind} T_{\bar{z}^k \overline{P}_2 (\bar{h}_1/h_1) P_2^{-1}} = k + \operatorname{ind} T_{\overline{P}_2} + \operatorname{ind} T_{\bar{h}_1/h_1} + \operatorname{ind} T_{P_2^{-1}} = k,$$

since the operators $T_{\overline{P}_2}$, $T_{\bar{h}_1/h_1}$, and $T_{P_2^{-1}}$ are clearly invertible. Hence $k = m$ which completes the proof. □

11. Spectral Properties of Hankel Operators with Lacunary Symbols

To speak about spectral properties we certainly have to realize Hankel operators as operators from a certain Hilbert space into itself. For $\varphi \in L^\infty$ we can consider the operator Γ_φ on ℓ^2 with Hankel matrix $\{\hat{\varphi}(j+k)\}_{j,k\ge 0}$ in the standard basis of ℓ^2. Not much is known about spectral properties of such operators in terms of φ. Power [1982] described the essential spectrum of Γ_φ for piecewise continuous functions φ. See also [Howland 1986; 1992a; 1992b], where spectral properties of self-adjoint operators Γ_φ with piecewise continuous φ are studied.

For a long time it was unknown whether there exists a nonzero quasinilpotent Hankel operator Γ, i.e., a Hankel operator Γ such that $\sigma(\Gamma) = \{0\}$ [Power 1984]. This question was answered affirmatively by Megretskii [1990], who considered Hankel operators with lacunary symbols and found an interesting approach to the description (in a sense) of their spectra. In particular, his method allows one to construct nonzero quasinilpotent Hankel operators.

In this section we describe the method of [Megretskii 1990]. In particular we prove that the operator with the following Hankel matrix is compact and quasinilpotent:

$$\Gamma_{\#} = \begin{pmatrix} i & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{8} & \cdots \\ 0 & 0 & 0 & \frac{1}{8} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{11–1}$$

We consider a more general situation of Hankel operators of the form

$$\Gamma = \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & \alpha_2 & 0 & \cdots \\ \alpha_1 & 0 & \alpha_2 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 & \cdots \\ \alpha_2 & 0 & 0 & 0 & \alpha_3 & \cdots \\ 0 & 0 & 0 & \alpha_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\{\alpha_k\}_{k\ge 0}$ is a sequence of complex numbers. In other words, we set $\Gamma = \{\gamma_{j+k}\}_{j,k\ge 0}$, where

$$\gamma_j = \begin{cases} \alpha_k & \text{if } j = 2^k - 1 \text{ with } k \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases}$$

We evaluate the norm of Γ and give a certain description of its spectrum.

Since BMOA $= (H^1)^*$ by Fefferman's theorem [Garnett 1981] with respect to the natural duality, it follows from Paley's theorem [Zygmund 1968] that $\sum_{k\ge 0} \alpha_k z^{2^k - 1} \in$ BMOA if and only if $\{\alpha_k\}_{k\ge 0} \in \ell^2$. Therefore, by Nehari's theorem, Γ is a matrix of a bounded operator if and only if $\{\alpha_k\}_{k\ge 0} \in \ell^2$;

moreover $\|\Gamma\|$ is equivalent to $\|\{\alpha_k\}_{k\geq 0}\|_{\ell^2}$. It is also clear that for $\{\alpha_k\}_{k\geq 0} \in \ell^2$ the function $\sum_{k\geq 0} \alpha_k z^{2^k-1}$ belongs to $VMOA$, so Γ is bounded if and only if it is compact.

We associate with Γ the sequence $\{\mu_k\}_{k\geq 0}$ defined by

$$\begin{aligned} \mu_0 &= 0, \\ \mu_{k+1} &= \tfrac{1}{2}\bigl(\mu_k + 2|\alpha_{k+1}|^2 + (\mu_k^2 + 4|\alpha_{k+1}|^2)^{1/2}\bigr) \quad \text{for } k \in \mathbb{Z}_+. \end{aligned} \tag{11–2}$$

The following theorem evaluates the norm of Γ.

THEOREM 11.1. *If* $\{\alpha_k\}_{k\geq 0} \in \ell^2$, *the sequence* $\{\mu_k\}_{k\geq 0}$ *converges and*

$$\|\Gamma\|^2 = \lim_{k\to\infty} \mu_k.$$

To describe the spectrum of Γ consider the class Λ of sequences of complex numbers $\{\lambda_j\}_{j\geq 0}$ satisfying

$$\begin{aligned} \lambda_0 &= \alpha_0, \\ (\lambda_j - \lambda_{j-1})\lambda_j &= \alpha_j^2 \quad \text{for } j \geq 1. \end{aligned} \tag{11–3}$$

THEOREM 11.2. *Suppose that* $\{\alpha_j\}_{j\geq 0} \in \ell^2$. *Any sequence* $\{\lambda_j\}_{j\geq 0}$ *in* Λ *converges. The spectrum* $\sigma(\Gamma)$ *consists of* 0 *and the limits of such sequences.*

To prove Theorems 11.1 and 11.2 we consider finite submatrices of Γ. Let $\mathfrak{L}_k$ be the linear span of the basis vectors e_j, for $j = 0, 1, \ldots, 2^k - 1$, and let P_k be the orthogonal projection from ℓ^2 onto $\mathfrak{L}_k$. Consider the operator $\Gamma_k \stackrel{\text{def}}{=} P_k\Gamma|\mathfrak{L}_k$ and identify it with its $2^k \times 2^k$ matrix. Put

$$\tilde{\Gamma}_k \stackrel{\text{def}}{=} \Gamma_k P_k = \begin{pmatrix} \Gamma_k & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\|\Gamma_k\| = \|\tilde{\Gamma}_k\|$ and $\sigma(\tilde{\Gamma}_k) = \sigma(\Gamma_k) \cup \{0\}$. Clearly,

$$\Gamma_{k+1} = \begin{pmatrix} \Gamma_k & \alpha_{k+1}J_k \\ \alpha_{k+1}J_k & 0 \end{pmatrix},$$

where J_k is the $2^k \times 2^k$ matrix given by

$$J_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We need a well-known fact from linear algebra: Let N be a block matrix of the form

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A and D are square matrices and D is invertible. Then

$$\det N = \det D \det(A - BD^{-1}C). \tag{11–4}$$

See [Gantmakher 1988, Chapter 2, § 5.3], for example.

PROOF OF THEOREM 11.1. Since $J_k^* = J_k$ and J_k^2 is the identity matrix of size $2^k \times 2^k$ (which we denote by I_k), we have

$$\Gamma_{k+1}^*\Gamma_{k+1} = \begin{pmatrix} \Gamma_k^*\Gamma_k + |\alpha_{k+1}|^2 I_k & \alpha_{k+1}\tilde{\Gamma}_k^* J_k \\ \bar{\alpha}_{k+1} J_k \tilde{\Gamma}_k & |\alpha_{k+1}|^2 I_k \end{pmatrix}. \tag{11–5}$$

Applying formula (11–4) to the matrix $\Gamma_{k+1}^*\Gamma_{k+1} - \lambda I_{k+1}$, where $\lambda \neq |\alpha_{k+1}|^2$, we obtain

$$\det(\Gamma_{k+1}^*\Gamma_{k+1} - \lambda I_{k+1}) = \rho^{2^k} \det\Bigl(-\frac{\lambda}{\rho}\Gamma_k^*\Gamma_k + \rho I_k\Bigr), \tag{11–6}$$

where $\rho \overset{\text{def}}{=} |\alpha_{k+1}|^2 - \lambda$.

Since Γ is a bounded operator, we have $\|\Gamma\| = \lim_{k\to\infty} \|\Gamma_k\|$. Therefore it is sufficient to show that $\mu_k = \|\Gamma_k\|^{1/2}$ or, which is the same, that μ_k is the largest eigenvalue of $\Gamma_k^*\Gamma_k$. We proceed by induction on k. For $k = 0$ the assertion is obvious.

If $\Gamma_k = 0$, the assertion is obvious. Otherwise, it follows easily from (11–5) that $\|\Gamma_{k+1}^*\Gamma_{k+1}\| > |\alpha_{k+1}|^2$.

It is easy to see from (11–6) that $\lambda \neq |\alpha_{k+1}|^2$ is an eigenvalue of $\Gamma_{k+1}^*\Gamma_{k+1}$ if and only if ρ^2/λ is an eigenvalue of $\Gamma_k^*\Gamma_k$. Put

$$\mu = \rho^2/\lambda = (|\alpha_{k+1}|^2 - \lambda)^2/\lambda.$$

If μ is an eigenvalue of $\Gamma_k^*\Gamma_k$, it generates two eigenvalues of $\Gamma_{k+1}^*\Gamma_{k+1}$:

$$\tfrac{1}{2}\bigl(\mu + 2\,|\alpha_{k+1}|^2 + (\mu^2 + 4\,|\alpha_{k+1}|^2)^{1/2}\bigr) \quad\text{and}\quad \tfrac{1}{2}\bigl(\mu + 2\,|\alpha_{k+1}|^2 - (\mu^2 + 4\,|\alpha_{k+1}|^2)^{1/2}\bigr).$$

Clearly, to get the largest eigenvalue of $\Gamma_{k+1}^*\Gamma_{k+1}$ we have to put $\mu = \mu_k$ and choose the first of the eigenvalues above. This proves that μ_{k+1} defined by (11–2) is the largest eigenvalue of $\Gamma_{k+1}^*\Gamma_{k+1}$. □

To prove Theorem 11.2 we need two lemmas.

LEMMA 11.3. *Let Λ_k be the set of k-th terms of sequences in Λ; that is,*

$$\Lambda_k = \{\lambda_k : \{\lambda_j\}_{j\geq 0} \in \Lambda\}.$$

If $\{\zeta_j\}_{j\geq 0}$ is an arbitrary sequence satisfying $\zeta_j \in \Lambda_j$, then it converges if and only if $\lim_{j\to\infty} \zeta_j = 0$ or there exists a sequence $\{\lambda_j\}_{j\geq 0} \in \Lambda$ such that $\zeta_j = \lambda_j$ for sufficiently large j.

LEMMA 11.4. *Let A be a compact operator on Hilbert space and let $\{A_j\}_{j\geq 0}$ be a sequence of bounded linear operators such that $\lim_{j\to\infty} \|A - A_j\| = 0$. Then the spectrum $\sigma(A)$ consists of the limits of all convergent sequences $\{\nu_j\}_{j\geq 0}$ such that $\nu_j \in \sigma(A_j)$.*

Lemma 11.4 is well known [Newburgh 1951] and we don't prove it here. Note that we need Lemma 11.4 for compact operators A_j, in which case it is proved in [Gohberg and Kreĭn 1965, Theorem 4.2].

Proof of Theorem 11.2 assuming Lemmas 11.3 and 11.4. Since Γ is compact, $0 \in \sigma(\Gamma)$.

For $\lambda \in \mathbb{C} \setminus \{0\}$ we apply formula (11–4) to the matrix $\Gamma_{k+1} - \lambda I_{k+1}$ and obtain

$$\det(\Gamma_{k+1} - \lambda I_{k+1}) = (-\lambda)^{2^k} \det\Bigl(\Gamma_k - \Bigl(\lambda - \frac{\alpha_{k+1}^2}{\lambda}\Bigr) I_k\Bigr). \tag{11–7}$$

Obviously,

$$0 \in \sigma(\Gamma_k) \quad \text{if and only if} \quad \alpha_k = 0.$$

Together with (11–7) this implies that $\lambda \in \sigma(\Gamma_k)$ if and only if there exists $\lambda' \in \sigma(\Gamma_{k-1})$ such that $(\lambda - \lambda')\lambda = \alpha_k^2$.

Let λ be a nonzero point in the spectrum of Γ. By Lemma 11.4, there exists a sequence $\{\nu_j\}_{j\ge 0}$ such that $\nu_j \to \lambda$ as $j \to \infty$ and $\nu_j \in \sigma(\tilde{\Gamma}_j)$. Since $\lambda \ne 0$ we may assume without loss of generality that $\nu_j \in \sigma(\Gamma_j)$. It follows now from Lemma 11.3 that there exists a sequence $\{\lambda_j\}_{j\ge 0}$ in Λ such that $\lambda_j = \nu_j$ for sufficiently large j, so $\lambda = \lim_{j\to\infty} \lambda_j$.

Conversely, let $\{\lambda_j\}_{j\ge 0} \in \Lambda$. By Lemma 11.3, $\{\lambda_j\}_{j\ge 0}$ converges to a point $\lambda \in \mathbb{C}$. As we have already observed, $\lambda_j \in \sigma(\Gamma_j)$, so Lemma 11.4 gives $\lambda \in \sigma(\Gamma)$. □

Proof of Lemma 11.3. Let $\{\lambda_j\}_{j\ge 0} \in \Lambda$. Then $|\lambda_j| \le |\lambda_{j-1}| + |\alpha_j|^2/|\lambda_j|$ for $j \ge 1$. It follows that

$$|\lambda_j| \le \max\{\varepsilon,\, |\lambda_{j-1}| + |\alpha_j|^2/\varepsilon\} \tag{11–8}$$

for any $\varepsilon > 0$. We show that either $\lambda_j \to 0$ as $j \to \infty$ or $|\lambda_j| \ge \delta$ for some $\delta > 0$ for sufficiently large j.

To do this, we show first that, if $\varepsilon > 0$ and $\liminf_{j\to\infty} |\lambda_j| < \varepsilon$, then $\limsup_{j\to\infty} |\lambda_j| \le 2\varepsilon$. Assume to the contrary that $\liminf_{j\to\infty} |\lambda_j| < \varepsilon$ and $\limsup_{j\to\infty} |\lambda_j| > 2\varepsilon$ for some $\varepsilon > 0$. It follows that for any $N \in \mathbb{Z}_+$ there exist positive integers m and n such that $N \le m < n$, $|\lambda_{m-1}| < \varepsilon$, $|\lambda_j| \ge \varepsilon$ for $m \le j \le n$, and $|\lambda_n| \ge 2\varepsilon$. It follows from (11–8) that $|\lambda_j| \le |\lambda_{j-1}| + |\alpha_j|^2/\varepsilon$ for $m \le j \le n$. Therefore

$$|\lambda_n| \le |\lambda_{m-1}| + \frac{\sum_{j=m}^n |\alpha_j|^2}{\varepsilon}.$$

Since $\{\alpha_k\}_{k\ge 0} \in \ell^2$, we can choose N so large that $(\sum_{j=m}^\infty |\alpha_j|^2)/\varepsilon < \varepsilon$, which contradicts the inequality $|\lambda_n| \ge 2\varepsilon$.

If $|\lambda_j| \ge \delta > 0$ for large values of j, then by (11–3)

$$|\lambda_j - \lambda_{j-1}| \le \frac{|\alpha_j|^2}{\delta}.$$

Therefore $\{\lambda_j\}_{j\ge 0}$ converges.

Now suppose that $\{\lambda_j\}_{j\ge0}$ and $\{\nu_j\}_{j\ge0}$ are sequences in Λ which have nonzero limits. Then, for sufficiently large j,

$$|\lambda_{j-1}-\nu_{j-1}| = \left|(\lambda_j-\nu_j)\Big(1+\frac{\alpha_j^2}{\lambda_j\nu_j}\Big)\right| \le |\lambda_j-\nu_j|(1+d|\alpha_j|^2)$$

for some $d>0$. Iterating this inequality, we obtain

$$|\lambda_{j-1}-\nu_{j-1}| \le \big|\lim_{j\to\infty}\lambda_j - \lim_{j\to\infty}\nu_j\big| \prod_{m=j}^{\infty}(1+d|\alpha_m|^2)$$

(the infinite product on the right-hand side converges since $\{\alpha_j\}_{j\ge0}\in\ell^2$). Therefore if $\lim_{j\to\infty}\lambda_j=\lim_{j\to\infty}\nu_j$, then $\lambda_j=\nu_j$ for sufficiently large j.

For $\varepsilon>0$ we consider the set of sequences $\{\lambda_j\}_{j\ge0}$ in Λ such that $\limsup\{|\lambda_j| : j\ge k\}\ge\varepsilon$ for any positive integer k. We show that the number of such sequences is finite. Suppose that $\{\lambda_j\}_{j\ge0}\in\Lambda$ is such a sequence. As we observed at the beginning of the proof, there exists $\delta>0$ and $j_0\in\mathbb{Z}_+$ such that $|\lambda_j|\ge\delta$ for sufficiently large j. Clearly, $|\alpha_j|<\delta$ for sufficiently large j. It follows that if j is sufficiently large, then λ_j is uniquely determined by λ_{j-1} and the conditions

$$(\lambda_j-\lambda_{j-1})\lambda_j=\alpha_j^2, \quad |\lambda_j|\ge\delta.$$

Hence there are only finitely many possibilities for such sequences.

Now let $\{\zeta_j\}_{j\ge0}$ be a converging sequence such that $\zeta_j\in\Lambda_j$ for $j\ge0$ and such that $\lim_{j\to\infty}\zeta_j\ne0$. As already proved, there are sequences $\{\lambda_j^{(s)}\}_{j\ge0}\in\Lambda$, for $s=1,\dots,m$, such that $\zeta_j\in\{\lambda_j^{(1)},\dots,\lambda_j^{(m)}\}$ for sufficiently large j and the sequences $\{\lambda_j^{(s)}\}_{j\ge0}$ have distinct limits. It follows that there exists an s in the range $1\le s\le m$ such that $\zeta_j=\lambda_j^{(s)}$ for sufficiently large j. □

We now proceed to the operator $\Gamma_\#$ defined by (11–1). In other words, we consider the operator Γ with

$$\alpha_0 = i,$$
$$\alpha_j = 2^{-j} \quad \text{for } j\ge1.$$

THEOREM 11.5. *$\Gamma_\#$ is a compact quasinilpotent operator.*

PROOF. It is easy to see by induction that if $\{\lambda_j\}_{j\ge0}$ satisfies (11–3), then $\lambda_j=2^{-j}i$, so Theorem 11.2 gives $\sigma(\Gamma_\#)=\{0\}$. We have already seen that bounded Hankel operators of this form are always compact. □

REMARK. We can consider a more general situation where $\Gamma=\{\gamma_{j+k}\}_{j,k\ge0}$, with

$$\gamma_j=\begin{cases}\alpha_k & \text{for } j=n_k-1 \text{ with } k\in\mathbb{Z}_+,\\ 0 & \text{otherwise,}\end{cases}$$

where $\{n_k\}_{k\ge0}$ is a sequence of natural numbers such that $n_{k+1}\ge2n_k$ for $k\ge0$ and $\{\alpha_k\}_{k\ge0}\in\ell^2$. It is easy to see that the same results hold and the same proofs also work in this situation, which allows one to construct other quasinilpotent Hankel operators. In particular, the Hankel operator

$$\begin{pmatrix} 0 & i & 0 & \frac{1}{2} & 0 & \cdots \\ i & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

is quasinilpotent.

It is still unknown whether there exist noncompact quasinilpotent Hankel operators.

12. Recent Developments

In this section we discuss briefly three recent developments in Hankel operators and their applications. We describe the results without proofs but give references.

Self-adjoint operators unitarily equivalent to Hankel operators. In Section 11 we discussed some spectral properties of Hankel operators. Here we consider the problem of describing all possible spectral types of self-adjoint Hankel operators.

The problem can also be described as follows. Let R be a (bounded) self-adjoint operator on a Hilbert space $\mathcal{H}$. When is R unitarily equivalent to a Hankel operator? In other words, is there an orthonormal basis $\{e_j\}_{j\geq 0}$ in $\mathcal{H}$ in which R is represented by a Hankel matrix?

Let me first say a few words about another related problem, posed in [Khrushchëv and Peller 1984], which appeared while we were studying geometric features of prediction theory. Let $\mathcal{K}$ and $\mathcal{L}$ be subspaces of a Hilbert space $\mathcal{H}$. The problem is to find out under which conditions there exists a stationary process $\{x_j\}_{j\in\mathbb{Z}}$ in $\mathcal{H}$ (i.e., the inner products $(x_j, x_k)_{\mathcal{H}}$ depend only on $j-k$) such that

$$\operatorname{span}\{x_j : j < 0\} = \mathcal{K} \quad \text{and} \quad \operatorname{span}\{x_j : j \geq 0\} = \mathcal{L}.$$

It was shown in [Khrushchëv and Peller 1984] that this problem is equivalent to the following one. Let K be a nonnegative self-adjoint operator on Hilbert space. Under which conditions does there exist a Hankel operator Γ whose modulus $|\Gamma| \stackrel{\text{def}}{=} (\Gamma^*\Gamma)^{1/2}$ is unitarily equivalent to K? In the same paper the following two simple necessary conditions were found:

(i) $\operatorname{Ker} K$ is either trivial or infinite-dimensional;
(ii) K is noninvertible.

We asked whether these conditions together are also sufficient.

Partial results in this direction were obtained in [Treil' 1985b; Vasyunin and Treil' 1989; Ober 1987; 1990], where the case of operators K with discrete spectrum was considered. The last two of these papers suggested a very interesting approach to the problem, based on linear systems with continuous time. Using Ober's approach, Treil [1990] gave in a complete solution by proving that conditions (i) and (ii) are sufficient. In the same article he showed that under these conditions there exists a *self-adjoint* Hankel operator whose modulus is unitarily equivalent to K.

Let me explain why the problem of describing the self-adjoint operators that are unitarily equivalent to Hankel operators is considerably more delicate. Recall that by von Neumann's spectral theory each self-adjoint operator R on Hilbert space is unitarily equivalent to multiplication by the independent variable on a direct integral of Hilbert spaces $\int \oplus \mathcal{K}(t)\, d\mu(t)$ that consists of measurable functions f such that $f(t) \in \mathcal{K}(t)$ and

$$\int \|f(t)\|^2_{\mathcal{K}(t)} d\mu(t) < \infty$$

(μ is a positive Borel measure on $\mathbb{R}$, called a *scalar spectral measure* of R). The *spectral multiplicity function* ν_R is defined μ-a.e. by $\nu_R(t) \stackrel{\text{def}}{=} \dim \mathcal{K}(t)$. Two self-adjoint operators are unitary equivalent if and only if their scalar spectral measures are mutually absolutely continuous and their spectral multiplicity functions coincide almost everywhere. See [Birman and Solomyak 1980] for the theory of spectral multiplicity.

Conditions (i) and (ii) describe the spectral multiplicity function $\nu_{|\Gamma|}$ of the moduli of self-adjoint Hankel operators. Namely, (i) means that $\nu(0) = 0$ or $\nu(0) = \infty$, while (ii) means that $0 \in \operatorname{supp} \nu$. Clearly, $\nu_{|\Gamma|}(t) = \nu_\Gamma(t) + \nu_\Gamma(-t)$, for $t > 0$. So the problem of describing the self-adjoint operators that are unitarily equivalent to Hankel operators is equivalent to the problem of investigating how $\nu_{|\Gamma|}(t)$ can be distributed between $\nu_\Gamma(t)$ and $\nu_\Gamma(-t)$.

The problem was solved recently in [Megretskii et al. 1995]. The main result of that paper is the following theorem. As usual, μ_a and μ_s are the absolutely continuous and the singular parts of a measure μ.

THEOREM 12.1. *Let R be a selfadjoint operator on Hilbert space*, μ *a scalar spectral measure of* R, *and* ν *its spectral multiplicity function*. *Then R is unitarily equivalent to a Hankel operator if and only if the following conditions hold*:

(i) *Either* $\operatorname{Ker} R = \{0\}$ *or* $\dim \operatorname{Ker} R = \infty$.
(ii) *R is noninvertible.*
(iii) $|\nu(t) - \nu(-t)| \le 1$, μ_a*-a.e., and* $|\nu(t) - \nu(-t)| \le 2$, μ_s*-a.e.*

The necessity of (i) and (ii) is almost obvious. The necessity of (iii) is more complicated. To prove that (iii) is necessary certain commutation relations between Hankel operators, the shift operator, and the backward shift were used in [Megretskii et al. 1995].

However, the most difficult problem is to prove sufficiency. It would be natural to try the method of linear systems with continuous time. Unfortunately (or perhaps fortunately), it does not work. To be more precise, it works if we replace (iii) by the stronger condition: $|\nu(t) - \nu(-t)| \le 1$, μ-a.e.

To prove sufficiency we used in [Megretskii et al. 1995] linear systems with discrete time (with scalar input and scalar output). Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $b, c \in \mathcal{H}$. Consider the linear system

$$\begin{cases} x_{n+1} = Ax_n + u_n b, \\ y_n \quad\; = (x_n, c), \end{cases} \tag{12–1}$$

for $n \in \mathbb{Z}$. Here $u_n \in \mathbb{C}$ is the input, $x_n \in \mathcal{H}$, and $y_n \in \mathbb{C}$ is the output. We assume that $\sup_{n\ge 0} \|A^n\| < \infty$.

We can associate with (12–1) the Hankel matrix $\Gamma_\alpha = \{\alpha_{j+k}\}_{j,k\ge 0}$, where $\alpha_j \stackrel{\text{def}}{=} (A^j b, c)$.

The Hankel operator Γ_α is related to the system (12–1) in the following way. We can associate with a sequence $v = \{v_n\}_{n\ge 0} \in \ell^2$ the input sequence $u = \{u_n\}_{n\in\mathbb{Z}}$ defined by

$$u_n = \begin{cases} v_{-1-n} & \text{if } n < 0, \\ 0 & \text{if } n \ge 0. \end{cases}$$

It is easy to see that under the initial condition $\lim_{n\to-\infty} x_n = 0$ the output $y = \{y_n\}_{n\ge 0}$ of the system (12–1) with input u satisfies $y = \Gamma_\alpha v$.

It was shown in [Megretskii et al. 1995] that under conditions (i)–(iii) of Theorem 12.1 there exists a triple $\{A, b, c\}$ such that the Hankel operator Γ_α is unitarily equivalent to R. The proof is very complicated. The triple $\{A, b, c\}$ is found as a solution of certain Lyapunov–type equations. In addition to that, A must satisfy the asymptotic stability condition

$$\|A^n x\| \to 0 \quad \text{for } x \in \mathcal{H}.$$

The most complicated part of the proof is to construct a solution satisfying the asymptotic stability condition above.

Approximation by analytic matrix functions. As mentioned in the introduction, Hankel operators play an important role in control theory, and it is especially important in control theory to consider Hankel operators whose symbols are matrix functions or even operator functions. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $\Phi \in L^\infty(\mathbb{B}(\mathcal{H}, \mathcal{K}))$, i.e., Φ is a bounded weakly measurable function taking values in the space $\mathbb{B}(\mathcal{H}, \mathcal{K})$ of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We can define the Hankel operator $H_\Phi : H^2(\mathcal{H}) \to H^2_-(\mathcal{K})$ by

$$H_\Phi f \stackrel{\text{def}}{=} \mathbb{P}_- \Phi f \quad \text{for } f \in H^2(\mathcal{H}),$$

where the spaces of vector functions $H^2(\mathcal{H})$ and $H^2_-(\mathcal{K})$ are defined as in the scalar case and $\mathbb{P}_-$ is the orthogonal projection onto $H^2_-(\mathcal{K})$. The analog of

Nehari's theorem says that

$$\|H_\Phi\| = \inf\{\|\Phi - F\|_{L^\infty(\mathbb{B}(\mathcal{H},\mathcal{K}))} : F \in H^\infty(\mathbb{B}(\mathcal{H},\mathcal{K}))\}. \tag{12–2}$$

The operator H_Φ is compact if and only if $\Phi \in H^\infty(\mathbb{B}(\mathcal{H},\mathcal{K})) + C(\mathfrak{K}(\mathcal{H},\mathcal{K}))$, where $C(\mathfrak{K}(\mathcal{H},\mathcal{K}))$ is the space of continuous functions that take values in the space $\mathfrak{K}(\mathcal{H},\mathcal{K})$ of compact operators from $\mathcal{H}$ to $\mathcal{K}$. The proofs of these facts can be found in [Page 1970]. As in the scalar case, Nehari's problem is to find, for a given $\Phi \in L^\infty(\mathbb{M}_{m,n})$, a function $F \in H^\infty(\mathbb{B}(\mathcal{H},\mathcal{K}))$ that minimizes the right-hand side of (12–2).

If $\dim \mathcal{H} = n < \infty$ and $\dim \mathcal{K} = m < \infty$, then $\mathcal{H}$ can be identified with $\mathbb{C}^n$, $\mathcal{K}$ with $\mathbb{C}^m$, and $\mathbb{B}(\mathcal{H},\mathcal{K})$ with the space $\mathbb{M}_{m,n}$ of $m \times n$ matrices.

It is important in applications to be able to solve Nehari's problem for matrix functions (and for operator functions). However, unlike the scalar case, it is only exceptionally that the problem has a unique solution. Consider the matrix function

$$\Phi = \begin{pmatrix} \bar z & 0 \\ 0 & \frac{1}{2}\bar z \end{pmatrix}.$$

Since $\|H_{\bar z}\| = 1$, it follows that $\operatorname{dist}_{L^\infty}\{\bar z, H^\infty\} = 1$, and since $\|\Phi\|_{L^\infty(\mathbb{M}_{2,2})} = 1$, we have $\operatorname{dist}_{L^\infty(\mathbb{M}_{2,2})}\{\Phi, H^\infty(\mathbb{M}_{2,2})\} = 1$. On the other hand, if f is a scalar function in H^∞ and $\|f\|_{H^\infty} \le \frac{1}{2}$, it is obvious that

$$\left\| \Phi - \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \right\|_{L^\infty(\mathbb{M}_{2,2})} = 1,$$

so Φ has infinitely many best uniform approximants by bounded analytic functions. Intuitively, however, it is clear that the "very best" approximation is the zero function, since a nonzero $f \in H^\infty$ increases the L^∞-norm of the lower right entry.

This suggests the idea of imposing additional constraints on $\Phi - F$. Given a matrix function Φ, we put

$$\Omega_0 = \{F \in H^\infty(\mathbb{M}_{m,n}) : F \text{ minimizes } t_0 = \sup_{\zeta\in\mathbb{T}} \|\Phi(\zeta) - F(\zeta)\|\};$$
$$\Omega_j = \{F \in \Omega_{j-1} : F \text{ minimizes } t_j = \sup_{\zeta\in\mathbb{T}} s_j(\Phi(\zeta) - F(\zeta))\}.$$

Here s_j is the j-th singular value.

Functions in $F \in \Omega_{\min\{m,n\}-1}$ are called *superoptimal approximations of* Φ *by analytic functions*, or superoptimal solutions of Nehari's problem. The numbers t_j are called *superoptimal singular values* of Φ. The notion of superoptimal approximation was introduced in [Young 1986]; it is important in H^∞ control theory.

The following uniqueness theorem was obtained in [Peller and Young 1994a].

THEOREM 12.2. *Let $\Phi \in H^\infty + C(\mathbb{M}_{m,n})$. Then there exists a unique superoptimal approximation $F \in H^\infty(\mathbb{M}_{m,n})$ by bounded analytic functions. It satisfies the equalities*

$$s_j(\Phi(\zeta) - F(\zeta)) = t_j \quad \textit{a.e. on } \mathbb{T} \quad \textit{for } 0 \le j \le \min\{m,n\} - 1.$$

Here is briefly the method of the proof. Let $v \in H^2(\mathbb{C}^n)$ be a maximizing vector of H_Φ (which exists since H_Φ is compact). Consider the vector function $w = H_\Phi v \in H^2(\mathbb{C}^m)$. It can be shown that v and $\bar{z}\bar{w}$ admit the factorizations

$$v = \vartheta_1 h \boldsymbol{v} \quad \text{and} \quad \bar{z}\bar{w} = \vartheta_2 h \boldsymbol{w},$$

where h is a scalar outer function, ϑ_1 and ϑ_2 are scalar inner functions, and $\boldsymbol{v}$ and $\boldsymbol{w}$ are column functions which are inner and co-outer (this means that $\|\boldsymbol{v}(\zeta)\|_{\mathbb{C}^n} = \|\boldsymbol{w}(\zeta)\|_{\mathbb{C}^m} = 1$ a.e. on $\mathbb{T}$, and both $\boldsymbol{v}$ and $\boldsymbol{w}$ have coprime entries, i.e., they do not have a common nonconstant inner divisor).

It is proved in [Peller and Young 1994a] that $\boldsymbol{v}$ and $\boldsymbol{w}$ admit *thematic completions*; that is, there exist matrix functions $V_c \in H^\infty(\mathbb{M}_{n,n-1})$ and $W_c \in H^\infty(\mathbb{M}_{m,m-1})$ such that the matrix functions $V \stackrel{\text{def}}{=} (\boldsymbol{v} \ \ \overline{V}_c)$ and $W \stackrel{\text{def}}{=} (\boldsymbol{w} \ \ \overline{W}_c)$ have the following properties:

(i) V and W take unitary values.
(ii) all minors of V and W on the first column are in H^∞.

Let Q be an arbitrary best approximant in $H^\infty(\mathbb{M}_{m,n})$. It is shown in [Peller and Young 1994a] that

$$W^t(\Phi - Q)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & \Phi^{(1)} \end{pmatrix}, \tag{12–3}$$

where $u_0 \stackrel{\text{def}}{=} \bar{z}\bar{\vartheta}_1\bar{\vartheta}_2\bar{h}/h$ and $\Phi^{(1)} \in H^\infty + C(\mathbb{M}_{m-1,n-1})$ (this inclusion is deduced in [Peller and Young 1994a] from the analyticity property (ii) of the minors). It is shown in the same article that the problem of finding a superoptimal approximation of Φ reduces to the problem of finding one for $\Phi^{(1)}$. Namely, if $F^{(1)}$ is a superoptimal approximation of $\Phi^{(1)}$, the formula

$$W^t(\Phi - F)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & \Phi^{(1)} - F^{(1)} \end{pmatrix}$$

determines a superoptimal approximation F to Φ. This allows us to reduce the size of the matrix function. Uniqueness now follows from the uniqueness result in the case $n = 1$, whose proof is the same as that of Theorem 2.4.

The proof of Theorem 12.2 given on [Peller and Young 1994a] is constructive. Another (less constructive) method for proving the same result was given in [Treil' 1995].

The proof obtained in [Peller and Young 1994a] gives interesting factorizations (*thematic factorizations*) of the error functions $\Phi - F$. To describe such factorizations, assume for simplicity that $m = n$. We denote by $\boldsymbol{I}_j$ the constant $j \times j$ identity matrix function.

THEOREM 12.3. *Let Φ be an $n \times n$ function satisfying the hypotheses of Theorem 12.2 and let F be the unique superoptimal approximation of Φ by bounded analytic functions. Then $\Phi - F$ admits a factorization*

$$\Phi - F = \overline{W}_0 \overline{W}_1 \cdots \overline{W}_{n-2} D V_{n-2}^* \cdots V_1^* V_0^*, \tag{12–4}$$

where

$$D = \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 \\ 0 & t_1 u_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n-1} u_{n-1} \end{pmatrix},$$

$u_0, \dots, u_{n-1}$ *are unimodular functions in* VMO *such that* $\operatorname{dist}_{L^\infty}\{u_j, H^\infty\} = 1$ *for* $0 \le j \le n-1$,

$$W_j = \begin{pmatrix} \boldsymbol{I}_j & 0 \\ 0 & \check{W}_j \end{pmatrix},$$
$$V_j = \begin{pmatrix} \boldsymbol{I}_j & 0 \\ 0 & \check{V}_j \end{pmatrix} \quad \textit{for } 0 \le j \le n-2,$$

and $\check{V}_j$ and $\check{W}_j$, for $0 \le j \le n-2$, are thematic matrix functions.

We can associate with the factorization (12–4) the indices $k_j \stackrel{\text{def}}{=} \dim \operatorname{Ker} T_{u_j}$, for $0 \le j \le n-1$. Since $\|H_{u_j}\| = 1$ and H_{u_j} is compact, it follows that $k_j \ge 1$. It was shown in [Peller and Young 1994a] that the indices are not determined uniquely by the function Φ: they can depend on the choice of a thematic factorization. However, combining our earlier methods with those of [Treil' 1995], we showed in [Peller and Young 1994b] that the sums of the indices that correspond to equal superoptimal singular values are uniquely determined by Φ.

Another result obtained in [Peller and Young 1994b] is an inequality between the singular values of the Hankel operator H_Φ and the terms of the *extended t-sequence*, which is defined as follows:

$$\tilde{t}_0 \stackrel{\text{def}}{=} t_0, \dots, \tilde{t}_{k_0-1} \stackrel{\text{def}}{=} t_0,\ \tilde{t}_{k_0} \stackrel{\text{def}}{=} t_1, \dots, \tilde{t}_{k_0+k_1-1} \stackrel{\text{def}}{=} t_1,\ \tilde{t}_{k_0+k_1} \stackrel{\text{def}}{=} t_2, \dots$$

(each term of the sequence $\{t_j\}_{0 \le j \le n-1}$ is repeated k_j times). The inequality is

$$\tilde{t}_j \le s_j(H_\Phi) \quad \text{for } 0 \le j \le k_0 + k_1 + \cdots + k_{n-1} - 1. \tag{12–5}$$

A similar result holds for infinite matrix functions (or operator functions) Φ under the condition that H_Φ is compact [Treil' 1995; Peller 1995; Peller and Treil' 1995].

In [Peller and Treil' 1997] the preceding results were shown to be true in a more general context, when the matrix function Φ does not necessarily belong to $H^\infty + C$. It is shown there also that these results generalize to the case when the essential norm of H_Φ is less than the smallest nonzero superoptimal singular value. In fact, the paper deals with the so-called four-block problem, which is more general than Nehari's problem. Another result obtained in it is the following inequality, which is stronger than (12–5):

$$s_j(H_{\Phi^{(1)}}) \le s_{j+k_0}(H_\Phi) \quad \text{for } j \ge 0,$$

where $\Phi^{(1)}$ is defined in (12–3).

It is also shown in [Peller and Young 1994a] that the nonlinear operator of superoptimal approximation has hereditary properties similar to those discussed in Section 9. Continuity properties of the operator of superoptimal approximation are studied in [Peller and Young 1997].

Similarity to a contraction. Here we consider one more application of Hankel operators, which has led recently to a solution of the famous problem of similarity to a contraction. Recall that operators T_1 and T_2 on Hilbert space are called *similar* if there exists an invertible linear operator V such that $T_2 = VT_1V^{-1}$. Clearly, similar operators have identical spectral properties. Sometimes one can prove that if an operator has the same properties as operators from a certain class, it is similar to an operator from that class. For example, it was proved in [Sz.-Nagy 1947] that if T is invertible and $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$, then T is similar to a unitary operator. However, if we know only that T satisfies $\sup_{n\ge 0} \|T^n\| < \infty$, it is not true that T must be similar to a contraction. The first example of such an operator was constructed in [Foguel 1964]; see [Davie 1974; Peller 1982; Bożejko 1987] for other examples).

It follows from von Neumann's inequality [von Neumann 1951] that any operator similar to a contraction is *polynomially bounded*, i.e.,

$$\|\varphi(T)\| \le \text{const}\cdot \max_{|\zeta|\le 1} |\varphi(\zeta)|$$

for any analytic polynomial φ.

The question of whether the converse is true was posed by Halmos [1970] and remained opened until recently.

Paulsen [1984] proved that T is similar to a contraction under the stronger condition of *complete polynomial boundedness*, which means that

$$\left\| \begin{pmatrix} \varphi_{11}(T) & \varphi_{12}(T) & \cdots & \varphi_{1n}(T) \\ \varphi_{21}(T) & \varphi_{22}(T) & \cdots & \varphi_{2n}(T) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}(T) & \varphi_{n2}(T) & \cdots & \varphi_{nn}(T) \end{pmatrix} \right\| \le c\cdot \max_{|\zeta|\le 1} \left\| \begin{pmatrix} \varphi_{11}(\zeta) & \varphi_{12}(\zeta) & \cdots & \varphi_{1n}(\zeta) \\ \varphi_{21}(\zeta) & \varphi_{22}(\zeta) & \cdots & \varphi_{2n}(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}(\zeta) & \varphi_{n2}(\zeta) & \cdots & \varphi_{nn}(\zeta) \end{pmatrix} \right\|$$

for any positive integer n and any polynomial matrix $\{\varphi_{jk}\}_{1\le j,k\le n}$; the constant c in the inequality does not depend on n.

Now let f be a function analytic in the unit disk $\mathbb{D}$. Consider the operator R_f on $\ell^2 \oplus \ell^2$ defined by

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix}, \tag{12–6}$$

where S is the shift operator on ℓ^2 and Γ_f is the Hankel operator on ℓ^2 with matrix $\{\hat{f}(j+k)\}_{j,k\geq 0}$ in the standard basis of ℓ^2. Such operators were introduced and used in [Peller 1982] to construct power bounded operators which are not similar to contractions. The operators R_f were considered independently by Foiaş and Williams; see [Carlson et al. 1994].

In [Peller 1984] the problem was posed of whether it is possible to find a function f for which R_f is polynomially bounded but not similar to a contraction. The reason why the operators R_f are convenient for this purpose is that functions of R_f can be evaluated explicitly: if φ is an analytic polynomial, then

$$\varphi(R_f) = \begin{pmatrix} \varphi(S^*) & \Gamma_{\varphi'(S^*)f} \\ 0 & \varphi(S) \end{pmatrix}. \tag{12–7}$$

It was shown in [Peller 1984] that, if $f' \in \text{BMOA}$ (see the definition on page 70), R_f is polynomially bounded. A stronger result was obtained later in by Bourgain [1986]: if $f' \in \text{BMOA}$, then R_f is completely polynomially bounded, and so it is similar to a contraction. Another proof of Bourgain's result was obtained later in [Stafney 1994].

Recently Paulsen has shown that R_f is similar to a contraction if and only if the matrix $\{(j-k)\hat{f}(j+k)\}_{j,k\geq 0}$ determines a bounded operator on ℓ^2. It follows from results of [Janson and Peetre 1988] that the last condition is equivalent to the fact that $f' \in \text{BMOA}$. (In the latter paper instead of matrices the authors study integral operators on $L^2(\mathbb{R})$, but their methods also work for matrices.) This implies that R_f is similar to a contraction if and only if $f' \in \text{BMOA}$.

There was hope of finding a function f with $f' \notin \text{BMOA}$ such that R_f is polynomially bounded, which would solve the problem negatively. However, this was recently shown to be is impossible, in [Aleksandrov and Peller 1996], the main result of the paper being the following:

THEOREM 12.4. *Let f be a function analytic in $\mathbb{D}$. The following statements are equivalent*:

(i) *R_f is polynomially bounded.*
(ii) *R_f is similar to a contraction.*
(iii) $f' \in \text{BMOA}$.

To prove Theorem 12.4 the following factorization result is established in [Aleksandrov and Peller 1996]. We denote by C_A the disk algebra of functions analytic in $\mathbb{D}$ and continuous in $\operatorname{clos}\mathbb{D}$.

THEOREM 12.5. *Let f be a function analytic in $\mathbb{D}$. Then $f \in H^1$ if and only if its derivative f' admits a representation*

$$f' = \sum_{j\geq 0} g_j' h_j,$$

where $g_j \in C_A$, $h_j \in H^1$, *and*

$$\sum_{j\geq 0} \|g_j\|_{L^\infty} \|h_j\|_{H^1} < \infty.$$

However, this was not the end of the story. Pisier [1997] considered the operator R_f on the space of vector functions $H^2(\mathcal{H}) \oplus H^2(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. The definition of R_f is exactly the same as given in (12–6). In this case S is the shift operator on $H^2(\mathcal{H})$, f is a function analytic in $\mathbb{D}$ and taking values in the space $\mathbb{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$, and Γ_f is the operator on $H^2(\mathcal{H})$ given by the block Hankel matrix $\{\hat{f}(j+k)\}_{j,k\geq 0}$. It is easy to see that (12–7) also holds in this setting for any scalar analytic polynomial φ. Pisier managed to construct a function f for which R_f is polynomially bounded but not similar to a contraction. To do that he used a sequence of operators $\{C_j\}_{j\geq 0}$ on $\mathcal{H}$ with the properties

$$\left\| \sum_{j\geq 0} \alpha_j C_j \right\| = \left(\sum_{j\geq 0} |\alpha_j|^2 \right)^{1/2} \quad \text{with } \alpha_j \in \mathbb{C}$$

and

$$\frac{1}{2} \sum_{j\geq 0} |\alpha_j| \leq \left\| \sum_{j\geq 0} \alpha_j C_j \otimes C_j \right\| \leq \sum_{j\geq 0} |\alpha_j| \quad \text{for } \alpha_j \in \mathbb{C}.$$

Such a sequence $\{C_j\}_{j\geq 0}$ always exists; see [Pisier 1996], for example.

The following result from [Pisier 1997] solves the problem of similarity to a contraction.

THEOREM 12.6. *Let* $\{\alpha_j\}_{j\geq 0}$ *be a sequence of complex numbers such that*

$$\sup_{k\geq 0} k^2 \sum_{j\geq k} |\alpha_j|^2 < \infty$$

and

$$\sum_{j\geq 1} j^2 |\alpha_j|^2 = \infty.$$

Then the operator R_f *with* $f = \sum_{j\geq 0} \alpha_j z^j C_j$ *is polynomially bounded but not similar to a contraction.*

It is very easy to construct such a sequence $\{\alpha_j\}_{j\geq 0}$. For example, one can put $\alpha_j = (j+1)^{-3/2}$.

Pisier's proof is rather complicated and involves martingales. Kislyakov [1996], Davidson and Paulsen [1997], and McCarthy [1996] have simplified the original argument and got rid of martingales.

References

[Adamyan et al. 1968a] V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, "Бесконечные ганкелевы матрицы и обобщённые задачи Каратеодори–Фейера и И. Шура", *Funkcional. Anal. i Priložen.* **2**:4 (1968), 1–17. Translated as "Infinite Hankel matrices and generalized Carathéodory–Fejér and I. Schur problems", *Functional Anal. Appl.* **2** (1968), 269–281.

[Adamyan et al. 1968b] V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, "О бесконечных ганкелевых матрицах и обобщённых задачах Каратеодори–Фейера и Ф. Рисса", *Funkcional. Anal. i Priložen.* **2**:1 (1968), 1–19. Translated as "Infinite Hankel matrices and generalized problems of Carathéodory–Fejér and F. Riesz", *Functional Anal. Appl.* **2** (1968), 1-18.

[Adamyan et al. 1971] V. M. Adamyan, V. Z. Arov, and M. G. Kreĭn, "Аналитические свойства пар Шмидта ганкелева оператора и обобщённая задача Шура–Такаги", *Mat. Sbornik* **86** (1971), 33–75. Translated as "Analytic properties of Schmidt pairs of a Hankel operator and generalized Schur–Takagi problem", *Math. USSR Sb.* **15** (1971), 31-73.

[Adamyan et al. 1984] V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, "Approximation of bounded functions by elements of $H^\infty + C$", pp. 254–258 in *Linear and complex analysis problem book: 199 research problems*, edited by V. P. Havin et al., Lecture Notes in Mathematics **1043**, Springer, Berlin, 1984. Russian original in Исследования по линеиным операторам и теорий функций: 99 нерешенных задач линейного и комплексного анализа, Записки научных семинаров ЛОМИ **81**, edited by N. K. Nikol'skii, V. P. Khavin, and S. V. Khrushchëv, Nauka, Leningrad, 1978.

[Akhiezer 1965] N. I. Akhiezer, Лекции по теории аппроксимаций, 2nd ed., Nauka, Moscow, 1965. Translated as *Theory of approximation*, Frederick Ungar Pub., New York, 1956; reprinted by Dover, New York, 1992.

[Aleksandrov and Peller 1996] A. B. Aleksandrov and V. V. Peller, "Hankel operators and similarity to a contraction", *Internat. Math. Res. Notices* **6** (1996), 263–275.

[Axler et al. 1979] S. Axler, I. D. Berg, N. Jewell, and A. Shields, "Approximation by compact operators and the space $H^\infty + C$", *Ann. of Math.* (2) **109**:3 (1979), 601–612.

[Birman and Solomyak 1980] M. S. Birman and M. Z. Solomyak, Спектральная теория самосопряжённых операторов в гильбертовом пространстве, 1980. Translated as *Spectral theory of selfadjoint operators in Hilbert space*, D. Reidel Pub., Dordrecht, 1987.

[Bourgain 1986] J. Bourgain, "On the similarity problem for polynomially bounded operators on Hilbert space", *Israel J. Math.* **54**:2 (1986), 227–241.

[Bożejko 1987] M. Bożejko, "Littlewood functions, Hankel multipliers and power bounded operators on a Hilbert space", *Colloq. Math.* **51** (1987), 35–42.

[Brudnyĭ 1979] J. A. Brudnyĭ, "Rational approximation and imbedding theorems", *Dokl. Akad. Nauk SSSR* **247**:2 (1979), 269–272. In Russian; translation in *Soviet Math. Dokl.* **20** (1979), 681-684.

[Carleson and Jacobs 1972] L. Carleson and S. Jacobs, "Best uniform approximation by analytic functions", *Ark. Mat.* **10** (1972), 219–229.

[Carlson et al. 1994] J. F. Carlson, D. N. Clark, C. Foiaş, and J. P. Williams, "Projective Hilbert **A**(**D**)-modules", *New York J. Math.* **1** (1994), 26–38.

[Coifman and Rochberg 1980] R. R. Coifman and R. Rochberg, "Representation theorems for holomorphic and harmonic functions in L^p", pp. 11–66 in *Representation theorems for Hardy spaces*, Astérisque **77**, Soc. Math. France, Paris, 1980.

[Davidson and Paulsen 1997] K. R. Davidson and V. I. Paulsen, "Polynomially bounded operators", *J. Reine Angew. Math.* **487** (1997), 153–170.

[Davie 1974] A. M. Davie, "Power-bounded elements in a Q-algebra", *Bull. London Math. Soc.* **6** (1974), 61–65.

[Dolženko 1977] E. P. Dolženko, "The dependence of the boundary properties of an analytic function on the rate of its approximation by rational functions", *Mat. Sb.* (*N.S.*) **103(145)**:1 (1977), 131–142, 144. In Russian.

[Douglas 1972] R. G. Douglas, *Banach algebra techniques in operator theory*, Pure and Applied Mathematics **49**, Academic Press, New York, 1972.

[Foguel 1964] S. R. Foguel, "A counterexample to a problem of Sz.-Nagy", *Proc. Amer. Math. Soc.* **15** (1964), 788–790.

[Francis 1987] B. A. Francis, *A course in H_∞ control theory*, Lecture Notes in Control and Information Sciences **88**, Springer, Berlin, 1987.

[Fuhrmann 1981] P. A. Fuhrmann, *Linear systems and operators in Hilbert space*, McGraw-Hill, New York, 1981.

[Gantmakher 1988] F. R. Gantmakher, Теория матриц, 4 ed., Nauka, Moscow, 1988. First edition translated as *The theory of matrices* (2 vols.), Chelsea, New York, 1959.

[Garnett 1981] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics **96**, Academic Press, New York, 1981.

[Glover 1984] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds", *Internat. J. Control* **39**:6 (1984), 1115–1193.

[Gohberg and Kreĭn 1965] I. C. Gohberg and M. G. Kreĭn, Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве, Nauka, Moscow, 1965. Translated as *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs **18**, Amer. Math. Soc., Providence, 1969.

[Gonchar 1968] A. A. Gonchar, "Скорость приближения рациональными дробями и свойства функций", pp. 329–346 in *Proc. Internat. Congr. Math.* (Moscow, 1966), Mir, Moscow, 1968. Translated as "Rate of approximation by rational functions and properties of functions", *Amer. Math. Soc. Transl.* (2) **91** (1969).

[Grigoryan 1976] L. D. Grigoryan, "Estimates of the norm of holomorphic components of meromorphic functions in domains with a smooth boundary", *Mat. Sb.* (*N.S.*) **100(142)**:1 (1976), 156–164. In Russian; translation in *Math. USSR Sb.* **29** (1976).

[Halmos 1967] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, NJ, and London, 1967. Second edition published by Springer, New York, 1982.

[Halmos 1970] P. R. Halmos, "Ten problems in Hilbert space", *Bull. Amer. Math. Soc.* **76** (1970), 887–933.

[Hamburger 1920] H. Hamburger, "Über eine Erweiterung des Stieltjesschen Momentproblems, I", *Math. Ann.* **81** (1920), 235–319.

[Hamburger 1921] H. Hamburger, "Über eine Erweiterung des Stieltjesschen Momentproblems, II and III", *Math. Ann.* **82** (1921), 120–164, 168–187.

[Hankel 1861] H. Hankel, *Über eine besondere Classe der symmetrischen Determinanten*, (Leipziger) dissertation, Göttingen, 1861.

[Hartman 1958] P. Hartman, "On completely continuous Hankel matrices", *Proc. Amer. Math. Soc.* **9** (1958), 862–866.

[Helson and Sarason 1967] H. Helson and D. Sarason, "Past and future", *Math. Scand.* **21** (1967), 5–16.

[Hoffman 1962] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962. Reprinted by Dover, New York, 1988.

[Howland 1986] J. S. Howland, "Spectral theory of selfadjoint Hankel matrices", *Michigan Math. J.* **33**:2 (1986), 145–153.

[Howland 1992a] J. S. Howland, "Spectral theory of operators of Hankel type, I", *Indiana Univ. Math. J.* **41**:2 (1992), 409–426.

[Howland 1992b] J. S. Howland, "Spectral theory of operators of Hankel type, II", *Indiana Univ. Math. J.* **41**:2 (1992), 427–434.

[Ibragimov and Rozanov 1970] I. A. Ibragimov and J. A. Rozanov, Гауссовские случайные процессы, Nauka, Moscow, 1970. Translated as *Gaussian random processes*, Applications of Mathematics, **9**, Springer, New York, 1978.

[Janson and Peetre 1988] S. Janson and J. Peetre, "Paracommutators—boundedness and Schatten – von Neumann properties", *Trans. Amer. Math. Soc.* **305**:2 (1988), 467–504.

[Khavinson 1951] S. Y. Khavinson, "О некоторых экстремальных задачах теории аналитических функций", *Moskov. Gos. Univ. Učenye Zapiski Matematika* **148(4)** (1951), 133–143. Translated as "On some extremal problems of the theory of analytic functions", *Amer. Math. Soc. Translations* (2) **32** (1963), 139-154.

[Khrushchëv and Peller 1984] S. V. Hruščëv and V. V. Peller, "Moduli of Hankel operators, past and future", pp. 92–97 in *Linear and complex analysis problem book: 199 research problems*, edited by V. P. Havin et al., Lecture Notes in Math. **1043**, Springer, Berlin, 1984.

[Kislyakov 1996] S. V. Kislyakov, "Operators (not) similar to a contraction: Pisier's counterexample via singular integrals", preprint 42, Université de Bordeaux, 1996.

[Kronecker 1881] L. Kronecker, "Zur Theorie der Elimination einer Variablen aus zwei algebraischen Gleichungen", *Monatsber. Königl. Preussischen Akad. Wies.* (*Berlin*) (1881), 535–600. Reprinted as pp. 113-192 in *Mathematische Werke*, vol. 2, B. G. Teubner, Leipzig, 1897 or Chelsea, New York, 1968.

[Luecking 1980] D. H. Luecking, "The compact Hankel operators form an M-ideal in the space of Hankel operators", *Proc. Amer. Math. Soc.* **79**:2 (1980), 222–224.

[Megretskii 1990] A. V. Megretskiĭ, "A quasinilpotent Hankel operator", *Algebra i Analiz* **2**:4 (1990), 201–212.

[Megretskii et al. 1995] A. V. Megretskiĭ, V. V. Peller, and S. R. Treil', "The inverse spectral problem for self-adjoint Hankel operators", *Acta Math.* **174**:2 (1995), 241–309.

[Merino 1989] O. Merino, "Stability of qualitative properties and continuity of solutions to problems of optimisation over spaces of analytic functions", preprint, 1989.

[McCarthy 1996] J. E. McCarthy, "On Pisier's construction of a polynomially bounded operator not similar to a contraction", preprint, MSRI, 1996. Available at http://www.msri.org/MSRI-preprints/online/1996-017.html.

[Nehari 1957] Z. Nehari, "On bounded bilinear forms", *Ann. of Math.* (2) **65** (1957), 153–162.

[von Neumann 1951] J. von Neumann, "Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes", *Math. Nachr.* **4** (1951), 258–281.

[Newburgh 1951] J. D. Newburgh, "The variation of spectra", *Duke Math. J.* **18** (1951), 165–176.

[Nikol'skiĭ 1986] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der mathematischen Wissenschaften **273**, Springer, Berlin, 1986. With an appendix by S. V. Hruščev and V. V. Peller. Translation of Лекции об операторе сдвига, Nauka, Moscow, 1980.

[Ober 1987] R. J. Ober, "A note on a system theoretic approach to a conjecture by Peller–Khrushchev", *Systems Control Lett.* **8**:4 (1987), 303–306.

[Ober 1990] R. Ober, "A note on a system-theoretic approach to a conjecture by Peller–Khrushchev: the general case", *IMA J. Math. Control Inform.* **7**:1 (1990), 35–45.

[Page 1970] L. B. Page, "Bounded and compact vectorial Hankel operators", *Trans. Amer. Math. Soc.* **150** (1970), 529–539.

[Papadimitrakis 1993] M. Papadimitrakis, "Continuity of the operator of best uniform approximation by bounded analytic functions", *Bull. London Math. Soc.* **25**:1 (1993), 44–48.

[Papadimitrakis 1996] M. Papadimitrakis, "On best uniform approximation by bounded analytic functions", *Bull. London Math. Soc.* **28**:1 (1996), 15–18.

[Parfenov 1986] O. G. Parfenov, "Estimates for singular numbers of the Carleson embedding operator", *Mat. Sb.* (*N.S.*) **131(173)**:4 (1986), 501–518. In Russian; translation in *Math. USSR Sb.* **59**:2 (1988), 497–514.

[Paulsen 1984] V. I. Paulsen, "Every completely polynomially bounded operator is similar to a contraction", *J. Funct. Anal.* **55**:1 (1984), 1–17.

[Peetre 1976] J. Peetre, *New thoughts on Besov spaces*, Duke University Mathematics Series **1**, Mathematics Department, Duke University, Durham, NC, 1976.

[Peetre and Svensson 1984] J. Peetre and E. Svensson, "On the generalized Hardy's inequality of McGehee, Pigno and Smith and the problem of interpolation between BMO and a Besov space", *Math. Scand.* **54**:2 (1984), 221–241.

[Pekarskiĭ 1985] A. A. Pekarskiĭ, "Classes of analytic functions defined by best rational approximations in H_p", *Mat. Sb.* (*N.S.*) **127(169)**:1 (1985), 3–20. In Russian; translation in *Math. USSR Sb.* **55** (1986), 1–18.

[Pekarskiĭ 1987] A. A. Pekarskiĭ, "Chebyshev rational approximation in a disk, on a circle and on a segment", *Mat. Sb.* (*N.S.*) **133(175)**:1 (1987), 86–102, 144. In Russian.

[Peller 1980] V. V. Peller, "Операторы Ганкеля класса $\mathfrak{S}_p$ и их приложения (рациональная аппроксимация, гауссовские процессы, проблема мажорации операторов)", *Mat. Sb.* (*N.S.*) **113(155)**:4 (1980), 538–581. Translated as "Hankel operators of class $\mathfrak{S}_p$ and their applications (rational approximation, Gaussian processes, the problem of majorization of operators)", *Math. USSR Sb.* **41** (1982), 443–479.

[Peller 1982] V. V. Peller, "Estimates of functions of power bounded operators on Hilbert spaces", *J. Operator Theory* **7**:2 (1982), 341–372.

[Peller 1983] V. V. Peller, "Описание операторов Ганкеля класса $\mathfrak{S}_p$ при $p > 0$, исследование скорости рациональной аппроксимации и другие приложения", *Mat. Sb.* (*N.S.*) **122(164)**:4 (1983), 481–510. Translated as "Description of Hankel operators of the class $\mathfrak{S}_p$ for $p > 0$, investigation of the rate of rational approximation and other applications", *Math. USSR Sb.* **50** (1985), 465-494.

[Peller 1984] V. V. Peller, "Estimates of functions of Hilbert space operators, similarity to a contraction and related function algebras", pp. 199–204 in *Linear and complex analysis problem book: 199 research problems*, edited by V. P. Havin et al., Lecture Notes in Math. **1043**, Springer, Berlin, 1984.

[Peller 1985] V. V. Peller, "Операторы Ганкеля в теории возмущений унитарных и самосопряжённых операторов", *Funktsional. Anal. i Prilozhen.* **19**:2 (1985), 37–51, 96. Translated as "Hankel operators in the theory of perturbations of unitary and selfadjoint operators", *Functional Anal. Appl.* **19**:2 (1985), 111–123.

[Peller 1990a] V. V. Peller, "Hankel operators and multivariate stationary processes", pp. 357–371 in *Operator theory: operator algebras and applications* (Durham, NH, 1988), vol. 1, edited by W. B. Arveson and R. G. Douglas, Proc. Sympos. Pure Math. **51**, Amer. Math. Soc., Providence, RI, 1990.

[Peller 1990b] V. V. Peller, "Операторы Ганкеля и свойства непрерывности операторов наилучшего приближения", *Algebra i Analiz* **2**:1 (1990), 163–189. Translated as "Hankel operators and continuity properties of best approximation operators", *Leningrad Math. J.*, **2** (1991), 139-160.

[Peller 1992] V. V. Peller, "Boundedness properties of the operators of best approximation by analytic and meromorphic functions", *Ark. Mat.* **30**:2 (1992), 331–343.

[Peller 1995] V. V. Peller, "Approximation by analytic operator-valued functions", pp. 431–448 in *Harmonic analysis and operator theory* (Caracas, 1994), Contemp. Math. **189**, Amer. Math. Soc., Providence, RI, 1995.

[Peller and Khrushchëv 1982] V. V. Peller and S. V. Khrushchëv, "Операторы Ганкеля, наилучшие приближения и стационарные гауссовские процессы", *Uspehi Mat. Nauk* (*N.S.*) **37**:1 (1982), 53–124. Translated as "Hankel operators, best approximations and stationary Gaussian processes", *Russian Math. Surveys* **37**:1 (1982), 61-144.

[Peller and Treil' 1995] V. V. Peller and S. R. Treil', "Superoptimal singular values and indices of infinite matrix functions", *Indiana Univ. Math. J.* **44**:1 (1995), 243–255.

[Peller and Treil' 1997] V. V. Peller and S. R. Treil', "Approximation by analytic matrix functions: the four block problem", *J. Funct. Anal.* **148**:1 (1997), 191–228.

[Peller and Young 1994a] V. V. Peller and N. J. Young, "Superoptimal analytic approximations of matrix functions", *J. Funct. Anal.* **120**:2 (1994), 300–343.

[Peller and Young 1994b] V. V. Peller and N. J. Young, "Superoptimal singular values and indices of matrix functions", *Integral Equations Operator Theory* **20**:3 (1994), 350–363.

[Peller and Young 1997] V. V. Peller and N. J. Young, "Continuity properties of best analytic approximation", *J. Reine Angew. Math.* **483** (1997), 1–22.

[Pisier 1996] G. Pisier, *Similarity problems and completely bounded maps*, Lecture Notes in Mathematics **1618**, Springer, Berlin, 1996.

[Pisier 1997] G. Pisier, "A polynomially bounded operator on Hilbert space which is not similar to a contraction", *J. Amer. Math. Soc.* **10**:2 (1997), 351–369.

[Power 1982] S. C. Power, *Hankel operators on Hilbert space*, Research Notes in Mathematics **64**, Pitman, Boston and London, 1982.

[Power 1984] S. C. Power, "Quasinilpotent Hankel operators", pp. 259–261 in *Linear and complex analysis problem book: 199 research problems*, edited by V. P. Havin et al., Lecture Notes in Mathematics **1043**, Springer, Berlin, 1984.

[Riesz and Sz.-Nagy 1965] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, 4th ed., Gauthier-Villars, Paris, 1965.

[Rochberg 1982] R. Rochberg, "Trace ideal criteria for Hankel operators and commutators", *Indiana Univ. Math. J.* **31**:6 (1982), 913–925.

[Rogosinski and Shapiro 1953] W. W. Rogosinski and H. S. Shapiro, "On certain extremum problems for analytic functions", *Acta Math.* **90** (1953), 287–318.

[Sarason 1967] D. Sarason, "Generalized interpolation in H^∞", *Trans. Amer. Math. Soc.* **127**:2 (1967), 179–203.

[Sarason 1972] D. Sarason, "An addendum to: 'Past and future' [Helson and Sarason 1967]", *Math. Scand.* **30** (1972), 62–64.

[Sarason 1978] D. Sarason, "Function theory on the unit circle", lecture notes, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, 1978.

[Semmes 1984] S. Semmes, "Trace ideal criteria for Hankel operators, and applications to Besov spaces", *Integral Equations Operator Theory* **7**:2 (1984), 241–281.

[Shapiro 1952] H. S. Shapiro, *Extremal problems for polynomials and power series*, Dissertation, MIT, Cambridge, MA, 1952.

[Stafney 1994] J. D. Stafney, "A class of operators and similarity to contractions", *Michigan Math. J.* **41**:3 (1994), 509–521.

[Sz.-Nagy 1947] B. de Sz. Nagy, "On uniformly bounded linear transformations in Hilbert space", *Acta Sci. Math.* (*Szeged*) **11** (1947), 152–157.

[Sz.-Nagy and Foiaş 1967] B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris, and Akadémiai Kiadó, Budapest, 1967. Translated as *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, and Akadémiai Kiadó, Budapest, 1970.

[Tolokonnikov 1991] V. A. Tolokonnikov, "Generalized Douglas algebras", *Algebra i Analiz* **3**:2 (1991), 231–252. In Russian; translation in *St. Petersburg Math. J.* **3**:2 (1992), 455–476.

[Treil' 1985a] S. R. Treil', "The Adamyan–Arov–Kreĭn theorem: a vector version", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **141** (1985), 56–71. In Russian.

[Treil' 1985b] S. R. Treil', "Moduli of Hankel operators and the V. V. Peller – S. V. Khrushchëv problem", *Dokl. Akad. Nauk SSSR* **283**:5 (1985), 1095–1099. In Russian; translation in *Soviet Math. Dokl.* **32** (1985), 293-297.

[Treil' 1990] S. R. Treil', "An inverse spectral problem for the modulus of the Hankel operator, and balanced realizations", *Algebra i Analiz* **2**:2 (1990), 158–182. In Russian; translation in *Leningrad Math. J.* **2**:2 (1991), 353–375.

[Treil' 1995] S. Treil', "On superoptimal approximation by analytic and meromorphic matrix-valued functions", *J. Funct. Anal.* **131**:2 (1995), 386–414.

[Vasyunin and Treil' 1989] V. I. Vasyunin and S. R. Treil', "The inverse spectral problem for the modulus of a Hankel operator", *Algebra i Analiz* **1**:4 (1989), 54–66. In Russian; translation in *Leningrad Math. J.* **1**:4 (1990), 859–870.

[Vol'berg and Tolokonnikov 1985] A. L. Vol'berg and V. A. Tolokonnikov, "Hankel operators and problems of best approximation of unbounded functions", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **141** (1985), 5–17, 188. In Russian.

[Young 1986] N. J. Young, "The Nevanlinna–Pick problem for matrix-valued functions", *J. Operator Theory* **15**:2 (1986), 239–265.

[Zygmund 1968] A. Zygmund, *Trigonometric series: I, II*, 2nd ed., Cambridge University Press, London, 1968.

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