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Hankel-Type Operators, Bourgain Algebras, and Uniform Algebras

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Abstract. Let $H^\infty(D)$ denote the algebra of bounded analytic functions on the open unit disc in the complex plane. For a function $g \in L^\infty(D)$, the Hankel-type operator S_g is defined by $S_g(f) = gf + H^\infty(D)$. We give here an overview of the study of the symbol of the Hankel-type operator, with emphasis on those symbols for which the operator is compact, weakly compact, or completely continuous. We conclude with a look at this operator on more general domains and several open questions.

We look at a uniform algebra A on a compact Hausdorff space X. We let $M(A)$ denote the maximal ideal space of A. We will consider the Hankel-type operator $S_g : A \to C(X)/A$ with symbol $g \in C(X)$ defined by $S_g(f) = fg + A$ for all $f \in A$.

Even though the space L^∞ does not look like an algebra of continuous functions, it is possible to identify it with the space of continuous functions on its maximal ideal space X as follows: for f in L^∞ define the Gelfand transform of f by $\hat{f}(x) = x(f)$ for all $x \in X$. Since the topology on X is given by saying that a net x_α converges to x in X if and only if $x_\alpha(f)$ converges to $x(f)$ for all $f \in L^\infty$, we see that the Gelfand transform defines a continuous function on X.

We will be most interested in the case in which $A = H^\infty(U)$, the algebra of bounded analytic functions on a bounded domain U in the complex plane, and $C(X) = L^\infty(U)$ with respect to area measure. When the domain does not matter or when we think no confusion should arise, we will write simply H^∞ or L^∞.

The purpose of this article is to indicate why we look at such operators, what one can do with these Hankel-type operators, and some of what remains to be done in this area.

1. The Relationship to Classical Hankel Operators

Why are these called Hankel-type operators? To answer this question, we first consider the Hankel-type operator S_f defined for $f \in L^\infty(\partial D)$. Let H^2 denote the usual Hardy space of functions on the circle ∂D. Let P denote the orthogonal projection of $L^2(\partial D)$ onto H^2 and let $f \in L^\infty(\partial D)$. Recall that the multiplication operator with symbol f is defined on $L^2(\partial D)$ by $M_f(g) = fg$ for $g \in L^2(\partial D)$. The Toeplitz operator T_f is defined by $T_f(g) = P(fg)$ and the (classical) Hankel operator H_f with symbol f is defined as the operator from H^2 into ${H^2}^\perp$ such that

$$H_f(g) = (I - P)(fg) \quad \text{for } g \in H^2.$$

These operators have been studied over the years and there exist many good references about them. See [Power 1982; Zhu 1990], as well as [Peller 1998] in this volume, for more information about classical Hankel operators.

Now suppose that we replace the *Hilbert space* L^2 above by the *uniform algebra* L^∞, and the Hardy space H^2 by the algebra H^∞ of boundary values of bounded analytic functions on the open unit disc D. What is the appropriate replacement for the Hankel operator? If we look closely at the Hankel operator, we see that it is a multiplication operator followed by an operator with kernel equal to the space H^2. Thus the replacement should ideally be a multiplication operator followed by a map that annihilates H^∞ functions. Our Hankel-type operators are multiplication operators followed by the quotient map, a map that takes functions in H^∞ to zero.

When we work on the unit disc rather than the unit circle, our Hankel-type operators are a generalization of the Hankel operator on the Bergman space $L^2_a(D)$, the space of square-integrable analytic functions on the disc. For $f \in L^\infty(D)$ we define the Hankel operator acting on the Bergman space as above, replacing the Szegő projection with the Bergman projection. We will return to these Hankel operators frequently for comparison.

2. Why Should We Look at Hankel-Type Operators?

One reason for studying Hankel-type operators is that they are a natural generalization of classical Hankel operators to uniform algebras. Multiplication operators, Hankel operators, and Toeplitz operators are important in the study of closed subalgebras of $L^\infty(\partial D)$ and many interesting results in this area were a consequence of careful study of these operators. In what follows, we will look at when these operators are compact (that is, when the norm closure of the image of the closed unit ball under S_g is compact), weakly compact (the weak closure of the image of the closed unit ball under S_g is weakly compact), and completely continuous (S_g takes weakly null sequences to norm null sequences). Complete continuity and compactness are equivalent in reflexive spaces; this was Hilbert's original definition of compactness. However, as we will see below, these three types of compactness need not be the same. See [Dunford and Schwartz 1958, Chapter 5] for elementary information on the subject.

There are two more good reasons for studying the operators S_g, both connected with the properties of compactness. The first has a long history and began with work of Sarason in 1967. As usual, $C(\partial D)$ denotes the algebra of continuous functions on the unit circle. Sarason [1976] looked at the linear space $H^\infty(\partial D) + C(\partial D)$ and showed the following:

Theorem 2.1. *The space $H^\infty(\partial D) + C(\partial D)$ is a closed subalgebra of $L^\infty(\partial D)$.*

In fact, $H^\infty(\partial D) + C(\partial D)$ is the closed algebra generated by H^∞ and the conjugate of the inner function z on the unit circle. Hartman's theorem [Power 1982] tells us that the classical Hankel operator H_g is compact if and only if the symbol g belongs to $H^\infty(\partial D) + C(\partial D)$. Douglas, in connection with the study of Toeplitz operators on the circle, asked whether every closed subalgebra B of $L^\infty(\partial D)$ containing $H^\infty(\partial D)$ was generated by $H^\infty(\partial D)$ together with the set of conjugates of inner functions invertible in B. Algebras with this property became known as Douglas algebras, and Sarason's theorem inaugurated the study of closed subalgebras of $L^\infty(\partial D)$ containing $H^\infty(\partial D)$. One of the most important theorems in this study is the Chang–Marshall theorem [Chang 1976; Marshall 1976], which answers Douglas's question affirmatively and gives a beautiful description of all closed subalgebras of $L^\infty(\partial D)$ containing $H^\infty(\partial D)$. This theory does not generalize well to spaces of bounded functions on other domains in the complex plane, but Sarason's theorem above does. Rudin [1975] showed that the same is true of algebras on the disc:

Theorem 2.2. *The space $H^\infty(D) + C(\overline{D})$ is a closed subalgebra of $L^\infty(D)$.*

It turns out that whenever U is a bounded open subset of the complex plane and σ is area measure on U, the closure of $H^\infty(U) + C(\overline{U})$ is a closed subalgebra of $L^\infty(\sigma)$ [Dudziak et al. $\geq$ 1998]. So the interesting part of the question really is: When is $H^\infty(U) + C(\overline{U})$ closed? Looking at the proofs of Theorems 2.1 and 2.2 (see [Axler and Shields 1987; Garnett 1981, p. 137], for example), one observes that they share a common ingredient: both use approximation of functions by a related harmonic extension of the function via the Poisson kernel.

Zalcman [1969] extended Sarason's theorem to algebras of analytic functions on certain infinitely connected domains (to be studied later in this paper). Davie, Gamelin and Garnett continued work along this line [Davie et al. 1973] and looked at algebras on a bounded open subset U of the complex plane for which the functions in $A(U) = H^\infty(U) \cap C(\overline{U})$ are pointwise boundedly dense in $H^\infty(U)$; that is, every function in $H^\infty(U)$ can be approximated pointwise on U by a bounded sequence in $A(U)$. They asked the following question: if $H^\infty(U)+C(\overline{U})$ is a closed subspace of $L^\infty(U)$, is $A(U)$ pointwise boundedly dense in $H^\infty(U)$?

Their work contains many related results and seems to suggest that this must be correct, yet this problem remains open.

Study of sums of closed algebras and investigation of when the sum is again a closed algebra continued. Aytuna and Chollet [1976] extended Sarason's result to strictly pseudoconvex domains in C^n. Cole and Gamelin [1982] continued this work in a natural way and studied the problem of when the double dual A^{**} of a uniform algebra A has the property that $A^{**} + C(X)$ is a closed subalgebra of $C(X)^{**}$. One of their main results is that S_g is weakly compact for all $g \in C(X)$ if and only if $A^{**} + C(X)$ is a closed subalgebra of $C(X)^{**}$. Thus, knowing when S_g is weakly compact is connected to the question of when sums of uniform algebras are closed algebras. If S_g is not weakly compact for all $g \in C(X)$, can we determine the space of functions for which S_g is weakly compact?

As it turns out, Cole and Gamelin showed that many of the algebras had the seemingly stronger property that S_g is compact for all $g \in C(X)$. In the same paper, they defined the notion of tightness: a uniform algebra A is said to be tight if S_g is weakly compact for all $g \in C(X)$. They showed that under certain conditions on the domain, one can use the fact that $A(U)$ is tight to show that $H^\infty(U) + C(\overline{U})$ is a closed algebra of continuous functions on U. Cole and Gamelin's results are quite general, and they gave plenty of examples of tight algebras. Their work concentrated on looking at algebras for which S_g is weakly compact for all $g \in C(X)$. Saccone [1995] continued studying tight algebras as well as strong tightness; a uniform algebra A is strongly tight if S_g is compact for all $g \in C(X)$. He discusses in some depth properties of tight and strongly tight algebras as well as the problem of tightness versus strong tightness.

In all cases that have been studied, the operators S_g are weakly compact if and only if they are compact. This brings us to one more question: For which $g \in C(X)$ is S_g compact?

It is not difficult to see that every compact operator is weakly compact and completely continuous. However, one can give examples to show that the three properties may be different in a space. (Saccone's thesis [1995] and Diestel's work [1984] are excellent references for some of these examples.)

It is an interesting problem to try to discover spaces in which these types of compactness or complete continuity actually coincide. A well-known problem in this direction is to characterize Banach spaces that have the *Dunford–Pettis property*, that is, those on which any weakly compact operator is completely continuous. This property was named after N. Dunford and B. J. Pettis, who first introduced it [1940] and showed that it holds for L^1 spaces. Grothendieck [1953] showed that $C(X)$ spaces have the Dunford–Pettis property. Bourgain [1984] showed that $H^\infty(D)$ has the same property, using the theory of ultra-products. In studying his work, Cima and Timoney [1987] noted that if the operators S_g are completely continuous for all $g \in C(X)$, then A has the Dunford–Pettis property. Using this approach, these authors were able to show that certain spaces from rational approximation theory have the property. It was their hope that they could do the same for $H^\infty(D)$ by characterizing the subalgebra of $L^\infty(\partial D)$ consisting of those symbols g for which S_g is completely continuous. They defined the Bourgain algebra of an algebra B to be

$$B_{\text{cc}} = \{f \in C(X) : S_f \text{ is completely continuous}\}.$$

Their work is another version of the question we looked at above in connection with Cole and Gamelin's work; that is, for which $g \in C(X)$ is S_g completely continuous?

Thus we are interested in knowing three things: for which $g \in C(X)$ is the Hankel-type operator S_g compact, weakly compact, or completely continuous.

3. Bourgain Algebras

Cima and Timoney [1987] showed that the Bourgain algebra of a uniform algebra B is a closed subalgebra of $C(X)$ and that $B \subset B_{\text{cc}}$. A great deal of work on Bourgain algebras followed; the reader is referred to the exposition in [Yale 1992] for a description of early work on the subject. One can change the domain of definition of the functions (see, for example, [Cima et al. 1993; Dudziak et al. $\geq$ 1998]), the subalgebra B (as in [Gorkin et al. 1992]), the superalgebra $C(X)$ [Izuchi et al. 1994], or the space on which the continuous functions act [Ghatage et al. 1992]. Finally, one can try to work in as general a context as possible [Izuchi 1992].

Many related questions arose. For example, Izuchi, Stroethoff and Yale [Izuchi et al. 1994] looked at the Bourgain algebra of closed linear subspaces rather than closed algebras. Tonev and Yale [1996] study invariance of Hankel-type operators under isomorphisms. One can also ask how the second Bourgain algebra of an algebra is related to the first [Cima et al. 1993; Gorkin et al. 1992; Izuchi 1992]; sometimes they are the same, sometimes not. Another interesting question is when the Bourgain algebras are monotonic; that is, if $A \subset B$, when is $A_{\text{cc}} \subset B_{\text{cc}}$?

4. Compactness of Hankel-Type Operators on the Circle

From here on, unless otherwise stated, we will only be concerned with algebras $H^\infty(U)$ of bounded analytic functions on a domain U in the complex plane as subalgebras of $L^\infty(U)$ with respect to area measure. One can also look at algebras on the boundary of these domains [Dudziak et al. $\geq$ 1998], but we will do so only for $H^\infty(\partial D)$ as a subalgebra of $L^\infty(\partial D)$ on the unit circle. We now return to the question of compactness of Hankel type operators.

As we mentioned above, for classical Hankel operators on the Hardy space we have this result:

THEOREM 4.1. *Let $f \in L^\infty(\partial D)$. The Hankel operator H_f defined on H^2 is compact if and only if $f \in H^\infty(\partial D) + C(\partial D)$.*

Cima, Janson and Yale [Cima et al. 1989] proved the following.

THEOREM 4.2. *Let $f \in L^\infty(\partial D)$. Then the Hankel-type operator S_f defined on H^∞ is completely continuous if and only if $f \in H^\infty(\partial D) + C(\partial D)$.*

Their proof uses a theorem of P. Beurling, as well as the theory of BMO and the Chang–Marshall theorem. It turns out to be relatively easy to eliminate the BMO theory, and this allows one to study Bourgain algebras of closed subalgebras of $L^\infty(\partial D)$ containing $H^\infty(\partial D)$ [Gorkin et al. 1992]. The proof of the theorem as stated above also does not use the full strength of the Chang–Marshall theorem. However, the theorem of P. Beurling, or ideas therein, turn out to be essential to the study of Hankel-type operators. As we shall see, once one has all these ideas in place it is not too difficult to show that complete continuity, compactness, and weak compactness are equivalent for these operators in this context. This seems to have first been noticed in [Dudziak et al. $\geq$ 1998].

In order to prove any result in this direction, one needs examples of weakly convergent sequences. Because we are working in the uniform algebra context, the Lebesgue dominated convergence theorem shows that a sequence of bounded analytic functions $\{f_n\}$ on D converges weakly to zero if and only if it is uniformly bounded and its Gelfand transform tends to zero pointwise on X. This makes it a bit easier to think about weakly null sequences, but Beurling's theorem helps us to construct many more weakly null sequences. Beurling's theorem is actually much more general than the version stated as Theorem 4.3 below; see [Garnett 1981].

Recall that if z_n are points in the disc satisfying $\sum(1-|z_n|) < \infty$ the Blaschke product with zeroes z_n is given by

$$B(z) = \prod \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - z\bar{z}_n}.$$

A sequence $\{z_n\}$ of points in D is called *interpolating* if it has the property that, whenever $\{w_n\}$ is a bounded sequence of complex numbers, there exists a function $f \in H^\infty$ with $f(z_n) = w_n$. While it is not clear that interpolating sequences exist in general, for the disc such sequences have been characterized; see, for example, [Garnett 1981, Chapter 7]. In fact, every sequence tending to the boundary of the disc has an interpolating subsequence. An infinite Blaschke product B is called an *interpolating Blaschke product* if the zero sequence of B forms an interpolating sequence. The following general result of P. Beurling [Garnett 1981, p. 298] is what we need to prove a version of Theorem 4.2; see Theorem 4.5 below.

THEOREM 4.3. *Let $\{z_n\}$ be an interpolating sequence in the disc. Then there are $H^\infty(D)$ functions $\{f_n\}$ such that*

$$f_n(z_n) = 1,$$
$$f_n(z_m) = 0 \quad \textit{for } n \neq m,$$

and a constant M such that $\sum |f_n(z)| < M$ for all $z \in D$.

Note that any such sequence of functions must converge to zero weakly, since, for any element φ in the dual of H^∞ and any positive integer N, if we let $a_n = \operatorname{sgn}\overline{(\varphi(f_n))}$, we have

$$\sum_1^N |\varphi(f_n)| = \sum_1^N a_n(\varphi(f_n)) = \varphi\left(\sum_1^N a_n f_n\right) \leq \|\varphi\| M.$$

Thus $\varphi(f_n) \to 0$ for all φ in the dual space of H^∞.

In the setting of this paper one can show [Dudziak et al. $\geq$ 1998] that, if S_g not compact, then it is an isomorphism (a bicontinuous operator onto its range) on a subspace J of $H^\infty(U)$ isomorphic to ℓ^∞. This implies that S_g cannot be completely continuous or weakly compact. That is why, in all situations presented here, these properties are all equivalent.

The following theorem, which is a special case of the Chang–Marshall theorem, can be used to give a quick proof of Theorem 4.5.

THEOREM 4.4. *Suppose that f is in $L^\infty(\partial D)$ but not in $H^\infty(\partial D) + C(\partial D)$. Then the closed subalgebra of $L^\infty(\partial D)$ generated by H^∞ and f contains the conjugate of an interpolating Blaschke product.*

This theorem has the advantage that it gives an easily understood proof of Cima, Janson and Yale's result containing the major ingredients of many of the proofs in this area. It has the disadvantage that it does not generalize easily to functions on general domains. Before turning to the proof, note that as long as $S_g(z^n)$ converges to zero in norm, we have $g \in H^\infty + C$, for

$$\operatorname{dist}(g, H^\infty + C) \leq \operatorname{dist}(g, \bar{z}^n H^\infty) = \operatorname{dist}(g z^n, H^\infty) = \|S_g z^n\| \to 0.$$

Thus, the strength of the following theorem really lies in the final assertion.

THEOREM 4.5. *Let $g \in L^\infty(\partial D)$. Then the Hankel-type operator S_g defined on H^∞ is compact if and only if $g \in H^\infty(\partial D) + C(\partial D)$. Furthermore, if g is not in $H^\infty(\partial D) + C(\partial D)$, then S_g is neither compact, completely continuous, nor weakly compact.*

PROOF. First we need to show that S_g is compact if $g \in H^\infty(\partial D) + C(\partial D)$. Since the result is clear for functions in $H^\infty(\partial D)$, we only have to show it for continuous functions. In addition, one can check that the space of symbols for which S_g is compact is a closed algebra, so it suffices to show that S_g is compact if $g(z) = \bar{z}$.

Suppose that $\{f_n\}$ is a bounded sequence of $H^\infty(\partial D)$ functions. By Montel's theorem there is a subsequence of $\{f_n\}$ converging uniformly on compacta to an $H^\infty(\partial D)$ function. Thus we may assume that $f_n \to 0$ uniformly on compacta. Note that, since $|z| = 1$, we have for $f \in H^\infty(\partial D)$

$$\bar{z} f = \frac{f - f(0)}{z} + \bar{z} f(0).$$

Now $\bar{z}f_n(0) \to 0$ and

$$\bar{z}f + H^\infty = \frac{f - f(0)}{z} + \bar{z}f(0) + H^\infty = \bar{z}f(0) + H^\infty,$$

so

$$\|S_{\bar{z}}(f_n)\| = \|\bar{z}f_n + H^\infty(\partial D)\| \leq \|\bar{z}f_n(0)\| \to 0.$$

Therefore, S_g is compact.

For the other direction, suppose that g is not in $H^\infty(\partial D) + C(\partial D)$. By Sarason's theorem (Theorem 4.4), there is an interpolating Blaschke product b with $\bar{b} \in H^\infty[g]$. Let $\{z_n\}$ denote the zero sequence of b. By Beurling's theorem, we can obtain a constant M and a sequence $\{f_n\}$ tending to zero weakly with $\sum |f_n| < M$ such that $f_n(z_m) = \delta_{nm}$. Let $\alpha = \{\alpha_n\}$ be an arbitrary sequence in ℓ^∞. Let $\tilde{g} = \sum \alpha_n f_n$. Then $\tilde{g} \in H^\infty$ and $\|\tilde{g}\| \leq \|\alpha\| M$. Now $\|S_{\bar{b}}\tilde{g}\| = \|\bar{b}\tilde{g} + H^\infty\|$. But $|b| = 1$ almost everywhere on ∂D, so we see that

$$\|S_{\bar{b}}\tilde{g}\| = \|\tilde{g} + bH^\infty\| \geq \sup |\tilde{g}(z_n)| \geq \sup |\alpha_n| \geq \|\tilde{g}\|/M.$$

Now the map defined for each $\alpha \in \ell^\infty$ by $\alpha \to \sum \alpha_n f_n$ is an embedding of ℓ^∞ into H^∞, and therefore $S_{\bar{b}}$ is an isomorphism on a subspace J of H^∞ isomorphic to ℓ^∞. This implies that $S_{\bar{b}}$ is not compact, weakly compact, or completely continuous. Since $\bar{b} \in H^\infty[g]$ and the spaces of symbols f for which S_f is compact, weakly compact, or completely continuous are uniformly closed algebras containing H^∞, we see that S_g cannot be compact, weakly compact, or completely continuous. □

One can also obtain the isomorphism statement directly for the operator S_g by using the full strength of the Chang–Marshall theorem rather than Sarason's theorem.

5. Compact Hankel-Type Operators on General Domains

The next result that appeared in this context was by Cima, Stroethoff and Yale. They replaced the domain above by the disc, but they used the result for the circle to obtain their result in this new situation.

Let's try to guess what the result might be. Obviously S_g is completely continuous for g in $H^\infty(D)$. We probably expect that it would be completely continuous for continuous symbols. Is there any other symbol that might make this operator completely continuous? We are looking for a symbol g such that $f_n \to 0$ weakly implies that $\|gf_n + H^\infty(D)\| \to 0$. Now we know from the uniform boundedness principle that the $\|f_n\|$ are uniformly bounded. Thus $f_n \to 0$ on compact subsets of D. So if we have a symbol for which f_n converging uniformly to zero on compacta implies $\|gf_n\| \to 0$, the corresponding Hankel type operator would be completely continuous. Of course any $L^\infty(D)$ function that vanishes outside a compact subset of D will have this property. Since such a function could be discontinuous on D, we see that it need not be in the algebra $H^\infty(D) + C(\overline{D})$.

Following Cima, Stroethoff, and Yale, we define the space L^∞_o to be the closure of the set of functions in L^∞ that vanish outside a compact set contained in D. This space of functions turns out to be precisely the space of functions for which the multiplication operator is compact. (See [Dudziak et al. $\geq$ 1998] for this result and a similar result for multiplication operators on general domains.)

Now we can state the result.

THEOREM 5.1. *Let $g \in L^\infty(D)$. Then S_g is completely continuous if and only if $g \in H^\infty(D) + C(\bar{D}) + L^\infty_o(D)$.*

The original proof of the above result used Cima, Janson, and Yale's theorem. The first obstacle to be overcome is that it is difficult to start with a function in L^∞ and pass in a natural way to a function on the boundary. Later proofs actually went the other way around, proving the result on an open subset of the plane rather than the boundary. Many of these proofs can be adapted to work on the boundary of the domain as well.

Cima, Stroethoff, and Yale's result appears to be quite different from the result on the circle. It was Izuchi who first noticed what the connection is. In order to state Izuchi's result here, we need to recall some definitions. Izuchi's result was stated for a general uniform algebra. However, we will continue to work on algebras of bounded analytic functions on domains in the plane.

Izuchi noticed that the results obtained depend on the circle having the property that every point of ∂D is what is called a peak point for the algebra $H^\infty(D)$. We can state this for more general domains (and it can be stated for more general algebras; see [Gamelin 1969]).

Let U be a domain in C and let $\lambda \in \bar{U}$. The fiber $M_\lambda(H^\infty)$ over λ is defined by

$$M_\lambda(H^\infty) = \{\varphi \in M(H^\infty) : \varphi(z) = \lambda\}.$$

We say that λ is a peak point for H^∞ if there exists a function $f \in H^\infty$ such that $f|M_\lambda(H^\infty) = 1$ (that is, $\varphi(f) = 1$ for all $\varphi \in M_\lambda(H^\infty)$) and $\left|f|M_\alpha(H^\infty)\right| < 1$ for all $\alpha \neq \lambda$. In the case of the unit circle, every point is a peak point, since a rotation of the function $(z+1)/2$ produces a peaking function. However, no point of the open unit disc can be a peak point of $H^\infty(D)$, for this would violate the maximum principle.

Note that when our domain is the unit disc or unit circle the function $\bar{z}$ is constant on every fiber, so if f is continuous, then f restricted to each fiber is constant as well. Therefore, in both results above we see if we let

$$B_{\mathrm{cc}} = \{g \in L^\infty : S_g \text{ is completely continuous}\},$$

then $B_{\mathrm{cc}}|M_\lambda(H^\infty) = (H^\infty + C)|M_\lambda(H^\infty) = H^\infty|M_\lambda(H^\infty)$ over every peak point λ. Thus it seems that our algebra does not change over a peak point, while it may change over a nonpeak point. Izuchi's result deals with general uniform algebras, but an inspection of the proof shows the following:

THEOREM 5.2. *Suppose that every point of ∂U is a peak point for $H^\infty(U)$. If $g \in L^\infty(U)$ and S_g is completely continuous, then $g|M_\lambda(H^\infty) \in H^\infty|M_\lambda(H^\infty)$ for all $\lambda \in \partial U$.*

Izuchi's proof uses the peak functions to construct a sequence of functions that have properties similar to the P. Beurling functions. His proof is like an earlier proof of Gamelin and Garnett [1970], which shows that a point $\lambda \in \overline{U}$ is a peak point if and only if every sequence of points in U tending to λ has an interpolating subsequence.

If we set

$$B_{\mathrm{c}} = \{g \in L^\infty : S_g \text{ is compact}\},$$

$$B_{\mathrm{wc}} = \{g \in L^\infty : S_g \text{ is weakly compact}\},$$

the next theorem tells us that what one now expects to be true is in fact so; see [Dudziak et al. $\geq$ 1998].

THEOREM 5.3. *Let U be a domain and $\lambda \in \overline{U}$ be a peak point for $H^\infty(U)$. Then*

$$B_{\mathrm{c}}|M_\lambda(H^\infty) = H^\infty|M_\lambda(H^\infty).$$

If, in addition, every point of ∂U is a peak point for $H^\infty(U)$, then S_g is compact if and only if g is in the uniform closure of

$$H^\infty(U) + C(\overline{U}) + L^\infty_{\mathrm{o}}(U).$$

Finally, if S_g is not compact, there is a subspace J of $H^\infty(U)$ isomorphic to ℓ^∞ on which S_g is an isomorphism.

6. Nonpeak Points and Hankel-Type Operators

The last result along these lines is one in which not every point of the boundary of the domain is a peak point. The so-called L-domains or roadrunner domains are well-known examples of this behavior. Denote an open disc of radius r_n and center c_n by Δ_n. The roadrunner domain that we will work on consists of the unit disc minus the union of the origin and a sequence of disjoint closed discs $\overline{\Delta}_n$. We require that the centers c_n be positive real numbers decreasing to 0, and that the discs accumulate only at zero. Such domains were first studied by Zalcman [1969], who showed that 0 is a peak point if and only if $\sum r_n/c_n = \infty$. He also showed that, if this sum is finite, one can define a homomorphism φ_0 by setting

$$\varphi_0(f) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)}{z}\,dz \quad \text{for } f \in H^\infty(D).$$

Such a homomorphism, called a distinguished homomorphism, is special in that it is the only weak-star continuous homomorphism in the fiber over 0. In fact, if we identify a point z with the linear functional that is evaluation at z, there exists a sequence of points $\{z_n\}$ of U converging to φ_0 *in the norm of the dual space of H^∞!* Now any point $z_0 \in U$ has a trivial fiber consisting only of such a homomorphism, and in some sense φ_0 thinks of itself as one of these homomorphisms, living in U. For this reason, the results and proofs mentioned above do not extend to the roadrunner set when 0 is not a peak point. One has to replace Beurling's result by something more general.

One well-known general result in this direction is the Rosenthal–Dor theorem [Diestel 1984, p. 201; Dor 1975]: In order that each bounded sequence in a Banach space X have a weakly Cauchy subsequence, it is necessary and sufficient that X contain no isomorphic copy of ℓ^1. The next theorem is the version of the P. Beurling theorem that one needs in our situation [Dudziak et al. ≥ 1998].

THEOREM 6.1. *Let $\{\mu_n\}$ be a sequence of measures on U converging weak-star in the dual of $C(\overline{U})$ to the point mass at $z_0 \in \partial U$. Then either $\{\mu_n\}$ converges in norm (in $(H^\infty(U))^*$) or for every $\varepsilon > 0$ there is a subsequence $\{\mu_{n_j}\}$, a constant M, and a sequence of $H^\infty(U)$ functions $\{f_k\}$ such that $\sum |f_k(z)| < M$ for every $z \in U$ and $\int f_k d\mu_{n_j} = \delta_{jk}$ for all j and k.*

More can be said in this setting; see [Dudziak et al. ≥ 1998] for this information and a proof of the preceding theorem. What is important for us is that, when the sequence converges in the norm of the dual space, it converges to the distinguished homomorphism. When this does not happen, we are in a situation in which every sequence has an interpolating subsequence.

Unfortunately, results obtained thus far require that the roadrunner have one more property. We need to require that there exist disjoint closed discs D_n with center c_n and radius R_n containing $\overline{\Delta}_n$ and satisfying $\sum r_n/R_n < \infty$. Note that there is always a distinguished homomorphism in this kind of domain, since

$$\sum \frac{r_n}{c_n} < \sum \frac{r_n}{R_n} < \infty.$$

Such domains were studied by Behrens [1970], who discovered that they have the following property.

THEOREM 6.2. *Given $\varepsilon > 0$ and $M > 0$ there exists an integer N such that, if $f_n \in H^\infty(\Delta_n^c)$ satisfies $f_n(\infty) = 0$ and $\|f_n\| < M$, then*

$$\sum_{m \ge N} |f_m(z)| < \varepsilon \quad \textit{for } z \notin \bigcup_{n \ge N} D_n,$$

$$\sum_{\substack{m \ge N \\ m \ne n}} |f_m(z)| < \varepsilon \quad \textit{for } z \in D_n.$$

This property and the use of certain projections allowed Behrens to prove the Corona theorem in such domains. The projections are defined as follows: for $f \in H^\infty(U)$ and an integer $n > 0$, define

$$P_n(f)(z) = \frac{1}{2\pi i} \int_{\partial \Delta_n} \frac{f(\zeta)}{z - \zeta} \, d\zeta \quad \text{for } z \notin \Delta_n.$$

This allows us to take functions from $H^\infty(U)$ to functions in $H^\infty(\Delta_n^c)$ vanishing at ∞. One can then use Theorem 6.2 to sum the resulting functions. These same properties are used to describe the Bourgain algebra $H^\infty(U)_{\mathrm{cc}}$, the algebra $H^\infty(U)_{\mathrm{c}}$ (consisting of symbols of compact Hankel-type operators), and the algebra $H^\infty(U)_{\mathrm{wc}}$ (consisting of symbols of weakly compact Hankel-type operators) for the algebra $H^\infty(U)$ when U is a Behrens roadrunner. A precise description of this result is given in [Dudziak et al. $\geq$ 1998]. We note here only that the three algebras turn out to be equal even in this general situation, but not equal to $H^\infty(U)+C(\overline{U})+L^\infty_{\mathrm{o}}(U)$.

7. Open Questions

We conclude by mentioning some open questions in this area, the first two of which arose in [Dudziak et al. $\geq$ 1998]. In order to help the reader, we write B_{cc} rather than the usual notation B_b for the Bourgain algebra.

QUESTION 7.1. *In every situation that we know of*, $B_{\mathrm{cc}} = B_{\mathrm{c}} = B_{\mathrm{wc}}$. *Are any or all of these equalities true in* $H^\infty(U)$ *for an arbitrary domain* U *in the plane*?

As mentioned above, in the case of the Behrens roadrunner the algebra $H^\infty(U)_c$ turns out to be different from $H^\infty(U)+C(\overline{U})+L^\infty_{\mathrm{o}}(U)$. In fact, any function $g \in L^\infty(U)$ that is the constant value 1 or the constant 0 on $D_n \setminus \Delta_n$ for every n is the symbol of a compact Hankel-type operator. If we take such a g and require that it be identically 1 on $D_{2n} \setminus \Delta_{2n}$ and identically 0 on $D_{2n+1} \setminus \Delta_{2n+1}$, the operator S_g will be compact, but $g \notin H^\infty(U)+C(\overline{U})+L^\infty_{\mathrm{o}}(U)$. At the time of this writing the following question remains open:

QUESTION 7.2. *Let* U *be a domain in* C. *For which* $g \in L^\infty(U)$ *is the Hankel-type operator* S_g *compact*? *weakly compact*? *completely continuous*?

The answer in the case of the Behrens roadrunner depends on a description of points tending to the distinguished homomorphism. In the general case, one would expect the distance in the pseudohyperbolic metric to the distinguished homomorphism to play an important role.

QUESTION 7.3. *The study of Bourgain algebras arose in an effort to find a simpler proof of the fact that* $H^\infty(D)$ *has the Dunford–Pettis property. Does* $H^\infty(U)$ *have the Dunford–Pettis property*?

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