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Some Open Problems in the Theory of Subnormal Operators

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Abstract. Subnormal operators arise naturally in complex function theory, differential geometry, potential theory, and approximation theory, and their study has rich applications in many areas of applied sciences as well as in pure mathematics. We discuss here some research problems concerning the structure of such operators: subnormal operators with finite-rank self-commutator, connections with quadrature domains, invariant subspace structure, and some approximation problems related to the theory. We also present some possible approaches for the solution of these problems.

Introduction

A bounded linear operator S on a separable Hilbert space $\mathcal{H}$ is called *subnormal* if there exists a normal operator N on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N|_{\mathcal{H}} = S$. The operator S is called *cyclic* if there exists an x in $\mathcal{H}$ such that

$$\mathcal{H} = \operatorname{clos}\{p(S)x : p \text{ is a polynomial}\},$$

and is called *rationally cyclic* if there exists an x such that

$$\mathcal{H} = \operatorname{clos}\{r(S)x : r \text{ is a rational function with poles off } \sigma(S)\}.$$

The operator S is *pure* if S has no normal summand and is *irreducible* if S is not unitarily equivalent to a direct sum of two nonzero operators.

The theory of subnormal operators provides rich applications in many areas, since many natural operators that arise in complex function theory, differential geometry, potential theory, and approximation theory are subnormal operators. Many deep results have been obtained since Halmos introduced the concept of a subnormal operator. In particular, Thomson's solution of the long-standing problem on the existence of bounded point evaluations reveals a structure theory of cyclic subnormal operators. Thomson's work answers many questions that had

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been open for a long time and promises to enable researchers to answer many more; see [Thomson 1991] or [Conway 1991]. The latter is a general reference for the theory of subnormal operators.

Here we will present some research problems on subnormal operators and discuss some possibilities for their solution.

1. Subnormal Operators with Finite-Rank Self-Commutator

Let A denote area measure in the complex plane $\mathbb{C}$. A bounded domain G is a *quadrature domain* if there exist points $z_1, \dots, z_N$ in G and constants $a_{m,n}$ such that

$$\int_G f(z)\, dA = \sum_{n=1}^{N} \sum_{m=0}^{N_n} a_{m,n} f^{(m)}(z_n)$$

for every function f analytic in G that is area-integrable. The theory of quadrature domains has been successfully studied by the techniques of compact Riemann surfaces, complex analysis and potential theory [Aharonov and Shapiro 1976; Gustafsson 1983; Sakai 1988].

The *self-commutator* of S is the operator $[S^*, S] = S^*S - SS^*$. The structure of subnormal operators with finite-rank self-commutator has been studied by many authors. Morrel [1973/74] showed that every subnormal operator with rank one self-commutator is a linear combination of the unilateral shift and the identity. Olin et al. [$\geq$ 1997] classified all cyclic subnormal operators with finite-rank self-commutator. D. Xia [1987a; 1987b] attempted to classify all subnormal operators with finite-rank self-commutator. His results, however, are incomplete. In [M^{c}Carthy and Yang 1997; 1995] a connection between a class of subnormal operators with finite-rank self-commutator and quadrature domains was established. However, the following problem still remains open.

PROBLEM 1.1. Classify all subnormal operators whose self-commutator has finite rank.

One can show that, if S is a pure subnormal operator with a finite rank self-commutator, the spectrum of the minimal normal extension N is contained in an algebraic curve. This is a modification of a result in [Xia 1987a]. If one assumes some additional conditions—for example, that the index of $S - \lambda$ is constant—then it turns out that S is unitarily equivalent to the direct sum of the bundle shifts over quadrature domains introduced in [Abrahamse and Douglas 1976]. (The argument is similar to that in [Putinar 1996], and uses the fact that the principle function is a constant on the spectrum minus the essential spectrum). For the general case, where the index of the subnormal operator $S - \lambda$ may change from one component to another, difficulties remain.

EXAMPLE 1.2. Set $r(z) = z(2z-1)/(z+2)$ and let $\Omega = r(\mathbb{D})$, where $\mathbb{D}$ is the open unit disk. If $\Gamma_1 = \partial\Omega$ and $\Gamma_2 = \operatorname{clos}\big(r(\partial\mathbb{D}) \setminus \partial\Omega\big)$, then Γ_2 is a closed simple curve. Let Ω_0 denote the region bounded by Γ_2 and let T_r be the Toeplitz operator on H^2 with symbol r. It is easy to show that T_r is a subnormal operator with a finite-rank self-commutator and that $\operatorname{ind}(T_r - \lambda) = -2$ for $\lambda \in \Omega_0$ and $\operatorname{ind}(T_r - \lambda) = -1$ for $\lambda \in \Omega \setminus \bar{\Omega}_0$. Obviously, neither Ω nor Ω_0 is a quadrature domain. The techniques in [M^cCarthy and Yang 1997] do not work for regions that have inner curves like Γ_2.

In [M^cCarthy and Yang 1997] one sees that the spectral pictures of rationally cyclic subnormal operators determine their structure. Therefore, in general, the difficulty consists of using the spectral picture of a subnormal operator with finite-rank self-commutator to obtain its structure.

EXAMPLE 1.3. Use the notation of Example 1.2. Let S be an irreducible subnormal operator satisfying these properties:

(1) $\sigma(S) = \bar{\Omega}$.
(2) $\sigma_e(S) = \Gamma_1 \cup \Gamma_2$.
(3) $\operatorname{ind}(S - \lambda) = -1$ for $\lambda \in \Omega \setminus \bar{\Omega}_0$ and $\operatorname{ind}(S - \lambda) = -3$ for $\lambda \in \Omega_0$.

The existence of such an irreducible subnormal operator is established by a method in [Thomson and Yang 1995]. Such an operator should not have a finite-rank self-commutator. However, how can one prove it? Understanding these examples will give some ideas on how to solve the problem.

Recently M. Putinar [1996] found another connection between operator theory and the theory of quadrature domains, by studying hyponormal operators with rank one self-commutator. It is of interest to see how operator-theoretical methods can be applied to the theory of quadrature domains.

Let G be a quadrature domain and let ω be the harmonic measure of G with respect to a fixed point in G. Let H_ω be the operator of multiplication by z on the Hardy space $H^2(G)$. It is easy to see that H_ω is irreducible with a finite-rank self-commutator and that the spectrum of the minimal normal extension N_ω is ∂G. Therefore, one sees that ∂G is a subset of an irreducible algebraic curve (the spectrum of N_ω is a subset of an algebraic curve). This was proved by A. Aharanov and H. Shapiro [1976] using function theory in 1976. B. Gustafsson [1983] obtained even better results: he showed that, except for possibly finitely many points, the boundary of a quadrature domain is an irreducible algebraic curve. In order to prove Gustafsson's theorem by using operator theory, one needs to prove the following operator-theoretical conjecture.

CONJECTURE 1.4. If S is an irreducible subnormal operator with finite-rank self-commutator, then except for possibly a finite number of points, the spectrum of the minimal normal extension N of S is equal to an irreducible algebraic curve.

A solution to this should indicate some ideas for obtaining the structure of subnormal operators with finite rank self-commutator.

Let G be a quadrature domain of connectivity $t+1$ and order n. Let W be a plane domain bounded by finitely many smooth curves and conformally equivalent to G. Let $\phi : W \to G$ be a conformal map. Define

$$C = \{z \in \partial G : z = \phi(w) \text{ for some } w \in \partial W \text{ with } \phi'(w) = 0\},$$
$$D = \{z \in \partial G : z = \phi(w_1) = \phi(w_2) \text{ for two different } w_j \in \partial W\}.$$

It was shown in [Aharonov and Shapiro 1976; Gustafsson 1983] that, if G is a quadrature domain, there is a meromorphic function $S(z)$ on G that is continuous on $\bar{G}$ except at its poles and such that $S(z) = \bar{z}$ on ∂G; moreover the equation $S(z) = \bar{z}$ has at most finitely many solutions in G. Let E denote the set of the solutions in G. Let c, d, e denote the cardinalities of C, D, E.

Theorem 1.5 [Gustafsson 1988]. $t + c + 2d + e \le (n-1)^2$.

Theorem 1.6 [Sakai 1988]. $c + e \ge t + n - 1$.

Using the model for rationally cyclic subnormal operators with finite rank self-commutator, it has only been possible to show that $t + e \le (n-1)^2$ [McCarthy and Yang 1997].

Problem 1.7. Use operator-theoretical methods to prove Gustafsson's theorem and Sakai's theorem.

In order to solve the problem, one has to understand deeply the operator-theoretical meanings of the numbers t, n, c, d, e. Discovering such methods may improve the results in the theory of quadrature domains and will help us understand the connections between these two areas better.

2. Invariant Subspaces of Subnormal Operators

Let S be a cyclic subnormal operator. A standard result [Conway 1991, p. 52] says that S is unitarily equivalent to the operator S_μ of multiplication by z on the space $P^2(\mu)$, the closure of the polynomials in $L^2(\mu)$.

A point $\lambda \in \mathbb{C}$ is a *bounded point evaluation* for $P^2(\mu)$ if there exists $C > 0$ such that $|p(\lambda)| \le C\|p\|$ for every polynomial p. In this section, we will assume that S_μ is an irreducible operator and that the set of bounded point evaluations for $P^2(\mu)$ is the open unit disk.

Apostol, Bercovici, Foiaş, and Pearcy showed in [Apostol et al. 1985] that, if the measure restricted to the unit circle is zero, there exists for each integer $1 \le n \le \infty$ an invariant subspace M of S_μ such that $\dim(M \ominus zM) = n$. It seems that the following problem has an affirmative answer.

Question 2.1. Suppose that the measure μ restricted to the unit circle is non-zero. Does every invariant subspace M of S_μ have the codimension-one property (which means that $\dim(M \ominus zM) = 1$)?

Partial answers are known. Miller [1989] showed that, if μ is area measure on the unit disk plus the Lebesgue measure on an arc, the question has an affirmative answer. In [Yang 1995] it is shown that, if μ is the area measure on the unit disk and Lebesgue measure restricted to a compact subset of the unit circle having positive Lebesgue measure, then each invariant subspace has the codimension-one property. The following is an analogous problem.

PROBLEM 2.2. Characterize all subnormal operators S_μ such that the restriction of S to any invariant subspace M is cyclic.

It seems that, if the measure μ restricted to the unit circle is nonzero, then S_μ restricted to each invariant subspace is cyclic. However, Problem 2.2 is harder than 2.1.

Say that an invariant subspace M of S_μ *has finite codimension* if the dimension of $P^2(\mu) \ominus M$ is finite. Here is an interesting problem concerning invariant subspaces of S_μ.

PROBLEM 2.3. Characterize the cyclic subnormal operators S_μ that have the property that every invariant subspace is an intersection of invariant subspaces having finite codimension.

This property implies that every invariant subspace is hyperinvariant (because every invariant subspace having finite codimension is hyperinvariant), and therefore that $P^\infty(S_\mu) = \{S_\mu\}'$, where $P^\infty(S_\mu)$ is the w^*-closure of

$$\{p(S_\mu) : p \text{ is a polynomial}\} \text{ and } \{S_\mu\}' = \{A : AS_\mu = S_\mu A\}.$$

3. Mean and Uniform Approximation

Let S_μ be irreducible and let G be the set of bounded point evaluations for $P^t(\mu)$. In [Olin and Yang 1995], it was shown that the algebra $P^2(\mu) \cap C(\operatorname{spt}\mu)$ equals $A(G)$, the algebra of continuous functions on $\bar{G}$ that are also analytic inside G. By a modification of the proof, one can actually show that $P^t(\mu) \cap C(\operatorname{spt}\mu) = A(G)$ for $1 < t < \infty$. However, for mean rational approximation, the situation is more complicated, since the closure of the set of bounded point evaluations may be strictly smaller than K; see [Conway 1991], for example.

Let Ω be the interior of the bounded point evaluations for $R^t(K,\mu)$ and let $A(K,\Omega)$ be the set of continuous functions on K that are also analytic on Ω. Using a result in [Conway and Elias 1993], it is easy to construct a space $R^t(K,\mu)$ with the set of bounded point evaluations Ω such that $R^t(K,\mu)$ does not contain $A(K,\Omega)$. Therefore, the correct question about the analogous property would be the following.

QUESTION 3.1. If $f \in R^t(K,\mu) \cap C(\operatorname{spt}\mu)$, is the function f continuous on K?

The answer is affirmative if we assume that the boundary of K contains no bounded point evaluations. The proof is similar to that in [Olin and Yang 1995].

Therefore, it only needs to be shown that the function f is continuous at each bounded point evaluation on the boundary.

Let $R(K)$ be the uniform closure of the rational functions with poles off K. It is well known that a smooth function f is in $R(K)$ if and only if $\bar{\partial} f = 0$ on the set of nonpeak points for $R(K)$; see [Browder 1969, p. 166, Theorem 3.2.9]. The situation for the mean rational approximation is different. For example, in [M$^{\mathrm{c}}$Carthy and Yang 1997] it is shown that there are a lot of nontrivial $R^t(K, \mu)$ spaces such that $\bar{z}p(z) \in R^t(K, \mu)$, where p is a polynomial. Notice that $\bar{\partial}\bar{z}p(z) = p(z) \neq 0$ except for finitely many points. Therefore, it is interesting to consider the following question.

Question 3.2. Let $R^t(K, \mu)$ be an irreducible space and let f be a smooth function on $\mathbb{C}$. If $f \in R^t(K, \mu)$, what is the relation between f, K, and μ?

If $f = \bar{z}p(z)$, where p is a polynomial, the relation is well understood by the theorem in [M$^{\mathrm{c}}$Carthy and Yang 1997]. That is, $K = \operatorname{clos}(\mathring{K})$, the set $\mathring{K}$ is a quadrature domain, and $\operatorname{spt} \mu$ is the union of the boundary of K with finitely many points in $\mathring{K}$. For an arbitrary smooth function f, one may guess that the support of μ inside the set $\{z : \bar{\partial} f \neq 0\}$ should not be big. On the other hand, the boundary of K should not be arbitrary. The boundary should have some "generalized quadrature domain" properties. This investigation may raise some interesting function-theoretical problems. Further study of those problems may have rich applications in complex analysis and potential theory.

Next we give a problem concerning uniform polynomial approximation. For a compact subset K of $\mathbb{C}$, let $P(K)$ be the uniform closures in $C(K)$ of polynomials.

Example 3.3. Set

$$K = \{z : |z| = 1 \text{ and } \operatorname{Im} z \geq 0\} \cup \{z : \operatorname{Im} z = 0 \text{ and } -1 \leq \operatorname{Re} z \leq 1\}.$$

Then $\bar{z}P(K) + P(K)$ is dense in $C(K)$, since $1 - |z|^2$ is in $\bar{z}P(K) + P(K)$; but $P(K) \neq C(K)$. On the other hand, if K is the unit circle, it is easy to show that $\bar{z}P(K) + P(K)$ is not dense in $C(K)$.

Naturally, we have the following problem.

Problem 3.4. Characterize the compact subsets K of the complex plane such that $\bar{z}P(K) + P(K)$ is dense in $C(K)$.

Example 3.3 indicates the hard part of the problem. That is, there are two toplogically equivalent compact subsets K_1 and K_2, for which $\bar{z}P(K_1) + P(K_1)$ is dense in $C(K_1)$ but $\bar{z}P(K_2) + P(K_2)$ is not dense in $C(K_2)$. However, there might be a geometric solution to the problem.

4. The Existence of Bounded Point Evaluations

Thomson's proof of the existence of bounded point evaluations for $P^t(\mu)$ uses Davie's deep estimation of analytic capacity, S. Brown's technique, and

Vitushkin's localization for uniform rational approximation. The proof is excellent but complicated. We believe that a simpler proof may exist. In this section, we will present one approach and point out the difficult part that is left unsolved.

THEOREM 4.1 [Luecking 1981]. *Let D be an open disk and let G be a subset of D. Then*

$$\int_D |p|^t \, dA \le C \int_G |p|^t \, dA$$

for all polynomials p if and only if there are positive numbers $\varepsilon, \delta > 0$ such that, for each disk O centered on the boundary of D with radius less than δ, we have

$$\operatorname{Area}(O \cap G) \ge \varepsilon \operatorname{Area}(O \cap D). \tag{4–1}$$

Equation (4–1) is called Luecking's condition.

Let ν be a complex Borel finite measure with $\int p \, d\nu = 0$ for each polynomial p. Let $\hat{\nu}$ be the Cauchy transform of ν, that is

$$\hat{\nu}(\lambda) = \int \frac{1}{z-\lambda} \, d\nu.$$

We say a point $\lambda \in \mathbb{C}$ is *heavy* if there exists a disk D centered at λ and there exists $\alpha > 0$ such that the set $G_\lambda = \{z : |\hat{\nu}(z)| \ge \alpha\} \cap D$ satisfies Luecking's condition.

PROPOSITION 4.2. *Suppose that λ is a heavy point. Then λ is a bounded point evaluation for $P^1(|\nu|)$.*

PROOF. From Luecking's theorem, we see that

$$\int_D |p| \, dA \le C \int_{G_\lambda} |p| \, dA \le \frac{C}{\alpha} \int_{G_\lambda} |p\hat{\nu}| \, dA = \frac{C}{\alpha} \int_{G_\lambda} |\widehat{p\nu}| \, dA \le C' \int |p| \, d|\nu|.$$

Since $|p(\lambda)| \le C \int_D |p| \, dA$, we see that λ is a bounded point evaluation for $P^1(|\nu|)$. □

Therefore, in order to prove the existence of bounded point evaluations, for each non-heavy point we need to construct some kind of "peak functions". Those "peak functions" enable us to show that the measure μ is zero on the complement of the closure of heavy points. Hence, the set of heavy points will not be empty since μ is a nonzero measure. It is possible to construct such functions because for each non-heavy point every disk centered at the point doesn't satisfy Luecking's condition. That means that the Cauchy transform will be small in some sense near the point.

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