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Reproducing Kernel Pontryagin Spaces

DANIEL ALPAY, AAD DIJKSMA, JAMES ROVNYAK,
AND HENDRIK S. V. DE SNOO

Abstract. The theory of reproducing kernel Pontryagin spaces is surveyed. A new proof is given of an abstract theorem that constructs contraction operators on Pontryagin spaces from densely defined relations. The theory is illustrated with examples from the theory of generalized Schur functions.

1. Introduction

We present here the main results of the theory of reproducing kernel Pontryagin spaces [Schwartz 1964; Sorjonen 1975] including some recent improvements [Alpay et al. 1997]. The paper is expository and is intended for nonspecialists in the indefinite theory. We presume knowledge of the Hilbert space case, that is, Aronszajn's theory [1950] of reproducing kernel Hilbert spaces. The main point is that much of the experience with the Hilbert space theory is transferable to Pontryagin spaces.

Section 2 presents background from operator theory. A key result here is a theorem to construct contraction operators by specifying their action on dense sets; we give a new proof that reduces the result to the isometric case. The main results on reproducing kernels are in Section 3. Examples in Section 4 illustrate the theory with kernels of the form $\big(1 - S(z)\overline{S(w)}\big)/(1 - z\bar{w})$ where $S(z)$ is a generalized Schur function.

Scalar-valued functions are assumed throughout. See [Alpay et al. 1997] for the extension to vector-valued functions and a detailed account of the theory of generalized Schur functions and associated colligations and reproducing kernel Pontryagin spaces.

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2. Contraction operators on Pontryagin spaces

Inner products are assumed to be linear and symmetric and defined on a complex vector space. Orthogonality and direct sum are defined for any inner product space as in the Hilbert space case. The *antispace* of an inner product space $(\mathfrak{H}, \langle \cdot , \cdot \rangle_{\mathfrak{H}})$ is $(\mathfrak{H}, -\langle \cdot , \cdot \rangle_{\mathfrak{H}})$. We write $\mathfrak{H}$ for $(\mathfrak{H}, \langle \cdot , \cdot \rangle_{\mathfrak{H}})$ when the inner product is understood.

Linear and symmetric inner product spaces are too general, and strong results can only be proved in special cases. A *Pontryagin space* is an inner product space $\mathfrak{H}$ which can be written as the orthogonal direct sum

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \tag{2–1}$$

of a Hilbert space $\mathfrak{H}_+$ and the antispace $\mathfrak{H}_-$ of a finite-dimensional Hilbert space. In a natural way, $\mathfrak{H}$ becomes a Hilbert space when $\mathfrak{H}_-$ is replaced by its antispace in (2–1). Such decompositions are not unique, but any two norms obtained in this way turn out to be equivalent. Every Pontryagin space thus has a unique *strong topology.* We write $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ for the space of continuous operators from $\mathfrak{H}$ into $\mathfrak{K}$ and $\mathfrak{L}(\mathfrak{H}) = \mathfrak{L}(\mathfrak{H}, \mathfrak{H})$. The dimensions of $\mathfrak{H}_\pm$ in (2–1) are independent of the choice of decomposition and called the *positive and negative indices* of $\mathfrak{H}$. The negative index κ of $\mathfrak{H}$ is the maximum dimension of a subspace $\mathfrak{H}_-$ which is the antispace of a Hilbert space in the inner product of $\mathfrak{H}$, and every such κ-dimensional subspace occurs in a decomposition (2–1).

Lemma 2.1 (Gram Matrices). *Let $g_1, \dots, g_n$ be vectors in a complex vector space $\mathfrak{H}$ with a linear and symmetric inner product $\langle \cdot , \cdot \rangle_{\mathfrak{H}}$. Then the number of negative eigenvalues of the Gram matrix $G = \big(\langle g_j, g_i \rangle_{\mathfrak{H}}\big)_{i,j=1}^n$ is equal to the maximum dimension of a subspace $\mathfrak{N}$ of the span of $g_1, \dots, g_n$ which is the antispace of a Hilbert space in the inner product of $\mathfrak{H}$.*

A proof is given in [Alpay et al. 1997, Lemma 1.1.1′].

The *adjoint* of an operator $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is the unique $A^* \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H})$ such that $\langle Ah, k \rangle_{\mathfrak{K}} = \langle h, A^* k \rangle_{\mathfrak{H}}$ for all $h \in \mathfrak{H}$ and $k \in \mathfrak{K}$. An operator in $\mathfrak{L}(\mathfrak{H})$ is *selfadjoint* if it is equal to its adjoint and a *projection* if it is selfadjoint and idempotent. As in the Hilbert space case, the set of selfadjoint operators is partially ordered by writing $A \le B$ to mean that $\langle Af, f \rangle_{\mathfrak{H}} \le \langle Bf, f \rangle_{\mathfrak{H}}$ for all $f \in \mathfrak{H}$ whenever $A, B \in \mathfrak{L}(\mathfrak{H})$ are selfadjoint operators. Closed subspaces $\mathfrak{M}$ of a Pontryagin space $\mathfrak{H}$ which are themselves Pontryagin spaces in the inner product of $\mathfrak{H}$ behave much like closed subspaces of Hilbert spaces. They coincide with ranges of projections, and the projection theorem holds: $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$. Such subspaces are called *regular.*

Identity operators are written as 1. We call $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ an *isometry* if $A^*A = 1$ and *unitary* if both A and A^* are isometric. More generally, $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a *partial isometry* if $AA^*A = A$. The properties of partial isometries are given in [Dritschel and Rovnyak 1996, Theorems 1.7, 1.8]. If $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a partial isometry, there exist regular subspaces $\mathfrak{M}$ of $\mathfrak{H}$ and $\mathfrak{N}$ of $\mathfrak{K}$ such that A maps $\mathfrak{M}$ isometrically onto $\mathfrak{N}$ and $\mathfrak{M}^\perp$ to the zero subspace. Conversely, any such operator $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a partial isometry, and A^*A and AA^* are projections onto $\mathfrak{M}$ and $\mathfrak{N}$. Unitary operators are isomorphisms and make the domain and range spaces abstractly indistinguishable.

We say that $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a *contraction* if

$$\langle Ah, Ah \rangle_{\mathfrak{K}} \le \langle h, h \rangle_{\mathfrak{H}}$$

for all $h \in \mathfrak{H}$. If both A and A^* are contractions, A is called a *bicontraction.* If $\mathfrak{H}$ and $\mathfrak{K}$ are Pontryagin spaces having the same negative index, every contraction $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a bicontraction [Dritschel and Rovnyak 1996, Corollary 2.5]. Without the index condition, this is not true (consider the identity operator acting on a Hilbert space to its antispace). See also [Azizov and Iokhvidov 1986; Bognár 1974; Iohvidov et al. 1982] for the basic properties of these notions.

In concrete situations, we define contraction operators by specifying their graphs. Typically, to start we have only partial information and have to deal with sets that are perhaps not graphs, that is, they may contain elements of the form $(0, k)$ with nontrivial $k \in \mathfrak{K}$. A *linear relation* from $\mathfrak{H}$ to $\mathfrak{K}$ is a subspace $\mathbf{R}$ of $\mathfrak{H} \times \mathfrak{K}$. Its *domain* is the set of first members of pairs in $\mathbf{R}$, and the *range* is the set of second members. We call $\mathbf{R}$ *contractive* if $\langle k, k \rangle_{\mathfrak{K}} \le \langle h, h \rangle_{\mathfrak{H}}$ for all (h, k) in $\mathbf{R}$, and *isometric* if equality holds in this inequality for all pairs.

The following result appears in [Alpay et al. 1997] with a proof based on a method of T. Ya. Azizov. An alternative proof is given below.

THEOREM 2.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Pontryagin spaces having the same negative index. Let $\mathbf{R}$ be a densely defined and contractive linear relation in $\mathfrak{H} \times \mathfrak{K}$. Then the closure of $\mathbf{R}$ is the graph of a contraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.*

The result is often applied when $\mathbf{R}$ is isometric, and then T is an isometry.

PROOF. *Step* 1: *Reduction to the isometric case.* Assume the result is known for densely defined isometric linear relations. Let $\mathbf{R}$ be any contractive linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ having dense domain. Define $\mathfrak{A}_0$ to be $\mathbf{R}$ as a vector space but considered in the inner product

$$\langle u_1, u_2 \rangle_{\mathfrak{A}_0} = \langle f_1, f_2 \rangle_{\mathfrak{H}} - \langle g_1, g_2 \rangle_{\mathfrak{K}}$$

for any $u_1 = (f_1, g_1)$ and $u_2 = (f_2, g_2)$ in $\mathbf{R}$. The inner product is nonnegative since $\mathbf{R}$ is contractive. Let $\mathfrak{N}$ be the set of elements u in $\mathfrak{A}_0$ such that $\langle u, v \rangle_{\mathfrak{A}_0} = 0$ for all v in $\mathfrak{A}_0$. A strictly positive inner product is induced in $\mathfrak{A}_0/\mathfrak{N}$, and this space can be completed to a Hilbert space $\mathfrak{A}$. If $u \in \mathfrak{A}_0$, let $[u] = u + \mathfrak{N}$ be the corresponding coset in $\mathfrak{A}$. Define an isometric linear relation from $\mathfrak{H}$ to $\mathfrak{K} \oplus \mathfrak{A}$ by

$$\mathbf{V} = \left\{ \left(f, \begin{pmatrix} g \\ [(f,g)] \end{pmatrix} \right) : (f,g) \in \mathbf{R} \right\}.$$

We verify the isometric property: for any $(f,g) \in \mathbf{R}$,

$$\left\langle \begin{pmatrix} g \\ [(f,g)] \end{pmatrix}, \begin{pmatrix} g \\ [(f,g)] \end{pmatrix} \right\rangle_{\mathfrak{K}\oplus\mathfrak{A}} = \langle g,g\rangle_{\mathfrak{K}} + \langle f,f\rangle_{\mathfrak{H}} - \langle g,g\rangle_{\mathfrak{K}} = \langle f,f\rangle_{\mathfrak{H}}.$$

If the result is known in the isometric case, $\bar{\mathbf{V}}$ is the graph of an isometry

$$V = \begin{pmatrix} T \\ A \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K}\oplus\mathfrak{A}).$$

We show that $\bar{\mathbf{R}}$ is the graph of T. If $(f,g) \in \bar{\mathbf{R}}$, then $(f_n,g_n) \to (f,g)$ for some sequence $\{(f_n,g_n)\}_1^\infty$ in $\mathbf{R}$. By the continuity of V,

$$\begin{pmatrix} g_n \\ [(f_n,g_n)] \end{pmatrix} = Vf_n \to Vf = \begin{pmatrix} Tf \\ Af \end{pmatrix};$$

hence $g_n \to Tf$ and so $(f,g) = (f,Tf)$ is in the graph of T. Conversely, if $(f,g) = (f,Tf)$ is in the graph of T, then (f,Vf) is in $\bar{\mathbf{V}}$. Hence there is a sequence $\{(f_n,g_n)\}_1^\infty$ in $\mathbf{R}$ such that

$$\left(f_n, \begin{pmatrix} g_n \\ [(f_n,g_n)] \end{pmatrix}\right) \to \left(f, \begin{pmatrix} Tf \\ Af \end{pmatrix}\right).$$

Then $f_n \to f$ and $g_n \to Tf$, so $(f,g) = (f,Tf)$ is in $\bar{\mathbf{R}}$. Thus the result holds in general if it is true in the isometric case.

Step 2: *Proof in the isometric case.* The argument uses Pontryagin's Theorem [Dritschel and Rovnyak 1996, Theorem 2.9]: For any dense subspace $\mathfrak{H}_0$ of a Pontryagin space $\mathfrak{H}$, there is a decomposition (2–1) such that $\mathfrak{H}_- \subseteq \mathfrak{H}_0$.

Assume that $\mathbf{R}$ is an isometric linear relation from $\mathfrak{H}$ to $\mathfrak{K}$ with dense domain. Let $\mathfrak{H}$ and $\mathfrak{K}$ have negative index κ. By the polarization identity,

$$\langle f_1,f_2\rangle_{\mathfrak{H}} = \langle g_1,g_2\rangle_{\mathfrak{K}} \quad \text{for } (f_1,g_1),\ (f_2,g_2) \in \mathbf{R}. \tag{2–2}$$

By Pontryagin's theorem, we can choose a fundamental decomposition (2–1) such that $\mathfrak{H}_-$ is contained in the domain of $\mathbf{R}$. Choose $f_1,\dots,f_\kappa \in \mathfrak{H}_-$ such that $\langle f_j,f_i\rangle_{\mathfrak{H}} = -\delta_{ij}$ for $i,j = 1,\dots,\kappa$, and let $(f_1,g_1),\dots,(f_\kappa,g_\kappa)$ be corresponding elements of $\mathbf{R}$. By (2–2), $\langle g_j,g_i\rangle_{\mathfrak{K}} = -\delta_{ij}$ for $i,j = 1,\dots,\kappa$. Hence $g_1,\dots,g_\kappa$ span a κ-dimensional subspace $\mathfrak{K}_-$ which is the antispace of a Hilbert space and part of a fundamental decomposition $\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$.

We show that $\mathbf{R}$ is a graph. If $(0,g)$ belongs to $\mathbf{R}$, then by (2–2), $\langle g,v\rangle_{\mathfrak{K}} = 0$ for all v in the range of $\mathbf{R}$ and hence for all v in $\mathfrak{K}_-$. Thus $g \in \mathfrak{K}_+$. Since g itself is in the range of $\mathbf{R}$, we have $\langle g,g\rangle_{\mathfrak{K}} = 0$ and so $g = 0$. It follows that $\mathbf{R}$ is the graph of a densely defined operator T_0 from $\mathfrak{H}$ into $\mathfrak{K}$.

The restriction of T_0 to $\mathfrak{H}_-$ maps $\mathfrak{H}_-$ isometrically onto $\mathfrak{K}_-$. The restriction of T_0 to $\mathfrak{H}_+$ is a densely defined isometry from $\mathfrak{H}_+$ to $\mathfrak{K}_+$. Since these are Hilbert spaces, T_0 has an extension by continuity to an isometry $T \in \mathfrak{L}(\mathfrak{H},\mathfrak{K})$. Then $\bar{\mathbf{R}}$ is the graph of T. □

The hypothesis in Theorem 2.2 that $\mathfrak{H}$ and $\mathfrak{K}$ have the same negative index is essential. Examples show what can go wrong when this condition is not met.

EXAMPLE 2.3. Let $\mathfrak{H} = \mathbb{C}$ be the complex numbers in the Euclidean norm. Let $\mathfrak{K}$ be $\mathbb{C}^3$ in the inner product

$$\langle a, b\rangle_{\mathfrak{K}} = a_1\bar{b}_1 + a_2\bar{b}_2 - a_3\bar{b}_3,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Let $\mathbf{R}$ be the set of pairs $(a, (a, c, c))$ with $a, c \in \mathbb{C}$. Then $\mathbf{R}$ is an isometric linear relation with domain $\mathfrak{H}$, but $\mathbf{R}$ contains elements $(0, g)$ with $g \neq 0$ and hence is not the graph of an operator.

EXAMPLE 2.4. Let $\mathfrak{K}$ be an infinite-dimensional Pontryagin space of negative index 1. Choose a nonclosed subspace $\mathfrak{H}$ of $\mathfrak{K}$ which is a Hilbert space in the inner product of $\mathfrak{K}$, that is, $\langle f, g\rangle_{\mathfrak{H}} = \langle f, g\rangle_{\mathfrak{K}}$ for $f, g \in \mathfrak{H}$. Such subspaces exist [Azizov and Iokhvidov 1986, Example 4.12, p. 27; Bognár 1974, Example 6.5, p. 111; Dritschel and Rovnyak 1996, Supplement]. The inclusion mapping V from $\mathfrak{H}$ into $\mathfrak{K}$ is thus an everywhere defined linear operator which preserves inner products, that is,

$$\langle Vf, Vg\rangle_{\mathfrak{K}} = \langle f, g\rangle_{\mathfrak{H}} \quad \text{for } f, g \in \mathfrak{H},$$

but V is not continuous relative to the strong topologies of $\mathfrak{H}$ and $\mathfrak{K}$ (if it were continuous, its range would be closed; see [Dritschel and Rovnyak 1996, p. 127]).

3. Reproducing Kernel Pontryagin Spaces

Let Ω be a nonempty set. By a *kernel* we mean a (complex-valued) function $K(s, t)$ on $\Omega \times \Omega$. Such a function is said to be *Hermitian* if $K(t, s) = \overline{K(s, t)}$ for $s, t \in \Omega$. In this case,

$$(K(s_k, s_j))_{j,k=1}^n = \begin{pmatrix} K(s_1, s_1) & K(s_2, s_1) & \dots & K(s_n, s_1) \\ K(s_1, s_2) & K(s_2, s_2) & \dots & K(s_n, s_2) \\ & & \dots & \\ K(s_1, s_n) & K(s_2, s_n) & \dots & K(s_n, s_n) \end{pmatrix}$$

is a selfadjoint matrix for any points $s_1, \dots, s_n$ in Ω ($n = 1, 2, 3, \dots$). We say that $K(s, t)$ has κ *negative squares*, κ a nonnegative integer, if every matrix of this form has at most κ negative eigenvalues and at least one such matrix has exactly κ negative eigenvalues (counted according to multiplicity). If this condition is satisfied with $\kappa = 0$, we call the kernel *nonnegative*. We remark that changing the matrix to $(K(s_k, s_j)c_k\bar{c}_j)_{j,k=1}^n$ for any complex numbers $c_1, \dots, c_n$ cannot increase the number of negative eigenvalues, and so the definition of κ negative squares used here is equivalent to that of [Alpay et al. 1997] in the scalar case.

Let $\mathfrak{H}$ be a Pontryagin space whose elements are functions on Ω. Linear operations are understood to be defined pointwise, hence all evaluation mappings $E(s) : f \to f(s)$, for $s \in \Omega$, are linear functionals on $\mathfrak{H}$. By a *reproducing kernel* for $\mathfrak{H}$ we mean a function $K(s,t)$ on $\Omega\times\Omega$ such that for each $s\in\Omega$ the function $K(s,\cdot)$ belongs to $\mathfrak{H}$, and

$$\langle f(\cdot), K(s,\cdot)\rangle_{\mathfrak{H}} = f(s)$$

for every function f in $\mathfrak{H}$. As in the Hilbert space case, it is an easy fact that a reproducing kernel exists if and only if all evaluation mappings are continuous. In this case, $E(s)\in\mathfrak{L}(\mathfrak{H},\mathbb{C})$ for every $s\in\Omega$. The reproducing kernel is unique and given by

$$K(s,t) = E(t)E(s)^* \quad \text{for } s,t\in\Omega,$$

where $E(s)^*\in\mathfrak{L}(\mathbb{C},\mathfrak{H})$ is the adjoint of the evaluation mapping $E(s)\in\mathfrak{L}(\mathfrak{H},\mathbb{C})$ for any fixed $s\in\Omega$. (In this equation, the right side is an operator on $\mathbb{C}$ to $\mathbb{C}$ and hence identified with a complex number. The equation can also be read to say that $K(s,\cdot)$ is the element of $\mathfrak{H}$ obtained from the element 1 of $\mathbb{C}$ under the action of $E(s)^*$.) Clearly, a reproducing kernel $K(s,t)$ is Hermitian. Applying Lemma 2.1 to the matrices

$$\big(\langle K(s_k,\cdot), K(s_j,\cdot)\rangle_{\mathfrak{H}}\big)_{j,k=1}^{n} = \left(K(s_k,s_j)\right)_{j,k=1}^{n}, \tag{3–1}$$

we see that $K(s,t)$ has at most κ negative squares, where κ is the negative index of $\mathfrak{H}$. A little further argument, using Pontryagin's Theorem on dense sets cited in the proof of Theorem 2.2, shows that $K(s,t)$ has exactly κ negative squares. Summarizing these properties, we have:

THEOREM 3.1. *A Pontryagin space $\mathfrak{H}$ of functions on Ω admits a reproducing kernel $K(s,t)$ if and only if all evaluation mappings are continuous. In this case, $K(s,t)$ is unique, and it is a Hermitian kernel having κ negative squares, where κ is the negative index of $\mathfrak{H}$.*

A converse result holds.

THEOREM 3.2. *If $K(s,t)$ is a Hermitian kernel on $\Omega\times\Omega$ having κ negative squares, there is a unique Pontryagin space $\mathfrak{H}$ of functions on Ω having $K(s,t)$ as reproducing kernel.*

PROOF. Let $\mathfrak{H}_0$ be the span of functions $K(s,\cdot)$ with $s\in\Omega$. By adding zero terms if necessary, any two functions f,g in $\mathfrak{H}_0$ can be represented in the form

$$f(\cdot) = \sum_{k=1}^{n} a_k K(s_k,\cdot), \qquad g(\cdot) = \sum_{j=1}^{n} b_j K(s_j,\cdot), \tag{3–2}$$

using the same points $s_1,\dots,s_n$ in Ω. Define an inner product by setting

$$\langle f,g\rangle_{\mathfrak{H}_0} = \sum_{j,k=1}^{n} a_k\bar{b}_j K(s_k,s_j).$$

Since $\langle f,g\rangle_{\mathfrak{H}_0} = \sum_{j=1}^n \bar{b}_j f(s_j) = \sum_{k=1}^n a_k \overline{g(s_k)}$, the inner product is well defined, linear, and symmetric. The identity $\langle f(\,\cdot\,), K(s,\cdot\,)\rangle_{\mathfrak{H}_0} = f(s)$ holds for all $s \in \Omega$ and all f in $\mathfrak{H}_0$.

Lemma 2.1, applied to the inner product space $\mathfrak{H}_0$ and the Gram matrices (3–1), implies that $\mathfrak{H}_0$ contains a subspace $\mathfrak{H}_-$ which is the antispace of a Hilbert space, and no subspace of higher dimension has this property. A reproducing kernel for $\mathfrak{H}_-$ can be exhibited in terms of any functions $u_1,\dots,u_\kappa$ in $\mathfrak{H}_-$ such that

$$\langle u_k, u_j\rangle_{\mathfrak{H}_0} = -\delta_{jk} \quad \text{for } j,k = 1,\dots,\kappa,$$

namely, $K_-(s,t) = -\sum_{l=1}^{\kappa} u_l(t)\overline{u_l(s)}$. Define

$$K_+(s,t) = K(s,t) - K_-(s,t)$$

on $\Omega\times\Omega$, and let $\mathfrak{H}_{0+}$ be the span of all functions $K_+(s,\cdot\,)$ with $s \in \Omega$. For any $s \in \Omega$ and $j = 1,\dots,\kappa$,

$$\begin{aligned}\langle K_+(s,\cdot\,), u_j(\,\cdot\,)\rangle_{\mathfrak{H}_0} &= \langle K(s,\cdot\,), u_j(\,\cdot\,)\rangle_{\mathfrak{H}_0} + \sum_{l=1}^{\kappa} \langle u_l(\,\cdot\,)\overline{u_l(s)}, u_j(\,\cdot\,)\rangle_{\mathfrak{H}_0} \\ &= \overline{u_j(s)} - \overline{u_j(s)} = 0,\end{aligned}$$

and hence $\mathfrak{H}_{0+} \perp \mathfrak{H}_-$ in the inner product of $\mathfrak{H}_0$. Next consider two functions f,g in $\mathfrak{H}_{0+}$. Representing f,g in the form (3–2) with $K(s,t)$ replaced by $K_+(s,t)$, we obtain

$$\begin{aligned}\langle f,g\rangle_{\mathfrak{H}_0} &= \left\langle \sum_{k=1}^n a_k K(s_k,\cdot\,) - \sum_{k=1}^n a_k K_-(s_k,\cdot\,), \sum_{j=1}^n b_j K(s_j,\cdot\,) - \sum_{j=1}^n b_j K_-(s_j,\cdot\,)\right\rangle_{\mathfrak{H}_0} \\ &= \sum_{j,k=1}^n a_k\bar{b}_j \bigl\{K(s_k,s_j) - K_-(s_k,s_j) - K_-(s_k,s_j) + K_-(s_k,s_j)\bigr\} \\ &= \sum_{j,k=1}^n a_k\bar{b}_j K_+(s_k,s_j).\end{aligned}$$

It follows that $K_+(s,t)$ is nonnegative, since otherwise we can find a $(\kappa+1)$-dimensional subspace of $\mathfrak{H}_0$ which is the antispace of a Hilbert space in the inner product of $\mathfrak{H}_0$. For if $K_+(s,t)$ is not nonnegative, another application of Lemma 2.1 implies that $\mathfrak{H}_{0+}$ contains an element $u_{\kappa+1}$ such that $\langle u_{\kappa+1}, u_{\kappa+1}\rangle_{\mathfrak{H}_0} < 0$, and then the span of $u_1,\dots,u_\kappa,u_{\kappa+1}$ has the stated properties.

The proof is completed using the known Hilbert space case [Aronszajn 1950]. Complete $\mathfrak{H}_{0+}$ to a functional Hilbert space $\mathfrak{H}_+$ with reproducing kernel $K_+(s,t)$. Define a Pontryagin space $\mathfrak{H}$ of functions on Ω by (2–1) with $\mathfrak{H}_\pm$ as above. We easily verify that $\mathfrak{H}$ has reproducing kernel $K(s,t)$. This establishes existence.

To prove uniqueness, consider a second Pontryagin space $\mathfrak{H}'$ of functions on Ω with reproducing kernel $K(s,t)$. Then $\mathfrak{H}'$ contains $\mathfrak{H}_0$ and $\mathfrak{H}_-$ isometrically, and so $\mathfrak{H}_{0+}$ is contained isometrically in $\mathfrak{H}' \ominus \mathfrak{H}_-$. Since $\mathfrak{H}' \ominus \mathfrak{H}_-$ and $\mathfrak{H}_+$ are two Hilbert space with reproducing kernel $K_+(s,t)$, they are equal isometrically by the Hilbert space version of the uniqueness result. Hence $\mathfrak{H}$ and $\mathfrak{H}'$ are equal isometrically. □

THEOREM 3.3. *Let $K(s,t) = K_+(s,t) + K_-(s,t)$ on $\Omega \times \Omega$, where $K_+(s,t)$ is nonnegative and $K_-(s,t) = -\sum_{l=1}^m u_l(t)\overline{u_l(s)}$ for some linearly independent functions $u_1, \dots, u_m$ on Ω. Then $K(s,t)$ has κ negative squares where $\kappa \le m$, and $\kappa = m$ if no nonzero function in the span of $u_1, \dots, u_m$ belongs to the Hilbert space with reproducing kernel $K_+(s,t)$.*

PROOF. Consider points $s_1, \dots, s_n$ in Ω. Write $C = A + B$, where

$$C = (K(s_k, s_j))_{j,k=1}^n\,, \qquad A = (K_+(s_k, s_j))_{j,k=1}^n\,.$$

Then $B = -FF^*$ where $F = (u_k(s_j))_{n\times m}$ has rank at most m, and so B has at most m negative eigenvalues. Now B and C are Gram matrices for the standard basis of $\mathbb{C}^n$ relative to the inner products $\langle a,b\rangle_C = \langle Ca,b\rangle_{\mathbb{C}^n}$ and $\langle a,b\rangle_B = \langle Ba,b\rangle_{\mathbb{C}^n}$. Since $C = A + B \ge B$, $\langle u,u\rangle_C \ge \langle u,u\rangle_B$ for all $u \in \mathbb{C}^n$. Hence, by Lemma 2.1, C has at most m negative eigenvalues. Thus $K(s,t)$ has $\kappa \le m$ negative squares.

Suppose that no nonzero function in the span $\mathfrak{H}_-$ of $u_1, \dots, u_m$ belongs to the Hilbert space $\mathfrak{H}_+$ with reproducing kernel $K_+(s,t)$. Then a Pontryagin space having reproducing kernel $K(s,t) = K_+(s,t) + K_-(s,t)$ is defined by forming a direct sum (2–1) with the given spaces $\mathfrak{H}_-$ and $\mathfrak{H}_+$. Since $\mathfrak{H}$ has negative index m by construction, $K(s,t)$ has m negative squares by Theorem 3.1. □

THEOREM 3.4. *Let $\mathfrak{H}$ be a Pontryagin space of functions on Ω with reproducing kernel $K(s,t)$. If $\mathfrak{G}$ is a closed subspace which is a Pontryagin space in the inner product of $\mathfrak{H}$, then $\mathfrak{G}$ has a reproducing kernel $K_{\mathfrak{G}}(s,t)$ such that, for each $s \in \Omega$, $K_{\mathfrak{G}}(s,\cdot\,)$ is the projection of $K(s,\cdot\,)$ into $\mathfrak{G}$.*

The proof is the same as in the Hilbert space case and omitted. We remark that a subspace $\mathfrak{G}$ as in Theorem 3.4 is the range of a projection [Dritschel and Rovnyak 1996, Theorem 1.3].

Recall the result on restrictions of reproducing kernels from the Hilbert space theory [Aronszajn 1950, p. 351]. Let $K(s,t)$ be the reproducing kernel for a Hilbert space $\mathfrak{H}$ of functions on Ω. Let $K_1(s,t)$ be the restriction of $K(s,t)$ to $\Omega_1 \times \Omega_1$ for some subset Ω_1 of Ω. Then $K_1(s,t)$ is the reproducing kernel for the Hilbert space $\mathfrak{H}_1$ whose elements consist of all restrictions h_1 of functions h in $\mathfrak{H}$ with

$$\|h_1\|_{\mathfrak{H}_1} = \inf\left\{\|h\|_{\mathfrak{H}} : h \in \mathfrak{H},\ h_1 = h|_{\Omega_1}\right\}.$$

In general, this fails for Pontryagin spaces.

EXAMPLE 3.5. The kernel $K(w,z) = 1 - z\bar{w}$ has one negative square on $\mathbb{C} \times \mathbb{C}$ by Theorem 3.3. By Theorem 3.2 it is the reproducing kernel for the two-dimensional Pontryagin space $\mathfrak{H}$ spanned by the functions $h_0(z) = 1$ and $h_1(z) =$

z, with

$$\langle h_0, h_0 \rangle_{\mathfrak{H}} = 1, \quad \langle h_1, h_1 \rangle_{\mathfrak{H}} = -1, \quad \langle h_0, h_1 \rangle_{\mathfrak{H}} = 0.$$

If $\Omega_1 = \{1\}$, the restriction of $K(w,z)$ to $\Omega_1 \times \Omega_1$ is $K_1(w,z) \equiv 0$. The set $\mathfrak{H}_1$ of restrictions of functions in $\mathfrak{H}$ to Ω_1 is a one-dimensional vector space, so there is no way to make $\mathfrak{H}_1$ a Pontryagin space with reproducing kernel $K_1(w,z)$. Note the decrease in number of negative squares: $K(w,z)$ has $\kappa = 1$ negative square, and $K_1(w,z)$ has $\kappa = 0$ negative squares.

THEOREM 3.6. *Let $\mathfrak{H}$ be a Pontryagin space of functions on Ω with reproducing kernel $K(s,t)$. If Ω_1 is a subset of Ω, the following conditions are equivalent*:

(a) *The set of functions in $\mathfrak{H}$ that vanish on Ω_1 forms a Hilbert space in the inner product of $\mathfrak{H}$.*
(b) *The restriction $K_1(s,t)$ of $K(s,t)$ to $\Omega_1 \times \Omega_1$ has the same number of negative squares as $K(s,t)$.*

In this case, $K_1(s,t)$ is the reproducing kernel for the Pontryagin space $\mathfrak{H}_1$ of functions on Ω_1 such that the restriction mapping $h \to h|_{\Omega_1}$ is a partial isometry from $\mathfrak{H}$ onto $\mathfrak{H}_1$.

Condition (a) is trivially satisfied if the only function in $\mathfrak{H}$ which vanishes on Ω_1 is the function identically zero.

PROOF. (b) $\Rightarrow$ (a) Let $\mathfrak{H}_1$ be the Pontryagin space with reproducing kernel $K_1(s,t)$. Our hypotheses imply (by Theorem 3.1) that $\mathfrak{H}_1$ and $\mathfrak{H}$ have the same negative index. Define a linear relation $\mathbf{R}$ in $\mathfrak{H}_1 \times \mathfrak{H}$ as the span of all pairs

$$(K_1(s,\cdot), K(s,\cdot)) \quad \text{for } s \in \Omega_1.$$

For any $s_1, \ldots, s_n \in \Omega_1$ and numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\sum_{j,k=1}^{n} a_k \bar{b}_j K_1(s_k, s_j) = \sum_{j,k=1}^{n} a_k \bar{b}_j K(s_k, s_j).$$

Thus $\mathbf{R}$ is isometric. By Theorem 2.2, the closure of $\mathbf{R}$ is the graph of an isometry $V \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H})$. Then $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{N}$, where $\mathfrak{K} = \operatorname{ran} V$ is a Pontryagin space having the same negative index as $\mathfrak{H}$. Hence $\mathfrak{N} = \ker V^*$ is a Hilbert space in the inner product of $\mathfrak{H}$. For any $s \in \Omega_1$ and $h \in \mathfrak{H}$,

$$(V^* h)(s) = \langle (V^* h)(\cdot), K_1(s,\cdot) \rangle_{\mathfrak{H}_1} = \langle h(\cdot), K(s,\cdot) \rangle_{\mathfrak{H}} = h(s),$$

so $\mathfrak{N} = \ker V^* = \{h : h \in \mathfrak{H},\ h|_{\Omega_1} \equiv 0\}$, and (a) follows.

(a) $\Rightarrow$ (b) Assume that $\mathfrak{N} = \{h : h \in \mathfrak{H},\ h|_{\Omega_1} \equiv 0\}$ is a Hilbert subspace of $\mathfrak{H}$. Then $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{N}$, where $\mathfrak{K}$ is a Pontryagin subspace of $\mathfrak{H}$ having the same negative index as $\mathfrak{H}$. Let $\mathfrak{H}_1$ be the Pontryagin space of all restrictions $h_1 = h|_{\Omega_1}$ of functions h in $\mathfrak{K}$ in the inner product such that the mapping $U : h \to h|_{\Omega_1}$ is an isometry from $\mathfrak{K}$ onto $\mathfrak{H}_1$. Then $\mathfrak{H}_1$ has the same negative index as $\mathfrak{H}$. If

$s \in \Omega_1$, then $K(s,\cdot\,)$ is orthogonal to $\mathfrak{N}$ and $U : K(s,\cdot\,) \to K_1(s,\cdot\,)$. Hence $K_1(s,\cdot\,)$ is in $\mathfrak{H}_1$, and for any h_1 of the form Uh, with $h \in \mathfrak{K}$, we have

$$\langle h_1(\,\cdot\,), K_1(s,\cdot\,)\rangle_{\mathfrak{H}_1} = \langle h(\,\cdot\,), K(s,\cdot\,)\rangle_{\mathfrak{H}} = h(s) = h_1(s).$$

Thus $\mathfrak{H}_1$ has reproducing kernel $K_1(s,t)$. Since $\mathfrak{H}_1$ and $\mathfrak{K}$ have the same negative index, $K_1(s,t)$ and $K(s,t)$ have the same number of negative squares by Theorem 3.1. This proves both (b) and the last statement in the theorem. □

In the case of holomorphic kernels, the number of negative squares is propagated to arbitrarily large domains. Let Ω be a region in the complex plane. A kernel $K(w,z)$ on $\Omega \times \Omega$ is called *holomorphic* if it is holomorphic in z for each fixed w and holomorphic in $\bar{w}$ for each fixed z. As in the Hilbert space case, a reproducing kernel $K(w,z)$ for a Pontryagin space $\mathfrak{H}$ of functions on Ω is holomorphic if and only if the elements of $\mathfrak{H}$ are holomorphic functions on Ω.

THEOREM 3.7. *Let $K(w,z)$ be a holomorphic Hermitian kernel on $\Omega \times \Omega$ for some region Ω in the complex plane. Let Ω_1 be a subregion of Ω, and assume that the restriction $K_1(w,z)$ of $K(w,z)$ to $\Omega_1 \times \Omega_1$ has κ negative squares. Then $K(w,z)$ has κ negative squares on $\Omega \times \Omega$.*

This is proved in [Alpay et al. 1997, Theorem 1.1.4], in a version for operator-valued functions, and we omit the argument here.

The Hilbert space theorem on sums of reproducing kernels is given in [Aronszajn 1950, p. 353]. Let $K(s,t) = K_1(s,t)+K_2(s,t)$, where $K(s,t)$, $K_1(s,t)$, and $K_2(s,t)$ are reproducing kernels for Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}_1$, and $\mathfrak{H}_2$. Then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are vector subspaces of $\mathfrak{H}$. Every element h of $\mathfrak{H}$ is of the form $h = h_1 + h_2$ with $h_1 \in \mathfrak{H}_1$ and $h_2 \in \mathfrak{H}_2$, and

$$\|h\|^2_{\mathfrak{H}} = \min\big(\|h_1\|^2_{\mathfrak{H}_1} + \|h_2\|^2_{\mathfrak{H}_2}\big),$$

where the minimum is over all such representations. The minimum is uniquely attained. These assertions fail in general for Pontryagin spaces.

EXAMPLE 3.8. Let $L(s,t) \not\equiv 0$ be the reproducing kernel for a finite-dimensional Hilbert space of functions on Ω. Put

$$K_1(s,t) = L(s,t), \qquad K_2(s,t) = -L(s,t).$$

Then $K(s,t) = K_1(s,t) + K_2(s,t)$ vanishes identically and is the reproducing kernel for $\mathfrak{H} = \{0\}$. Thus the reproducing kernel Pontryagin spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ corresponding to $K_1(s,t)$ and $K_2(s,t)$ are not contained in $\mathfrak{H}$.

Nevertheless, there are indefinite extensions of the Hilbert space results in this area. They are related to the complementation theory of de Branges [1988] for contractively contained spaces. A Pontryagin space $\mathfrak{H}_1$ is *contained contractively* in a Pontryagin space $\mathfrak{H}$ if $\mathfrak{H}_1$ is a vector subspace of $\mathfrak{H}$ and the inclusion mapping is a continuous and contractive operator from $\mathfrak{H}_1$ into $\mathfrak{H}$. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are two Pontryagin spaces which are contained contractively in $\mathfrak{H}$, they are called *complementary* if

(a) $(h_1, h_2) \to h_1 + h_2$ is a partial isometry from $\mathfrak{H}_1 \times \mathfrak{H}_2$ onto $\mathfrak{H}$, and
(b) the kernel of the partial isometry is a Hilbert space.

In this case, every $h \in \mathfrak{H}$ is of the form $h = h_1 + h_2$ with $h_1 \in \mathfrak{H}_1$, $h_2 \in \mathfrak{H}_2$, and

$$\langle h, h \rangle_{\mathfrak{H}} = \min \left(\langle h_1, h_1 \rangle_{\mathfrak{H}_1} + \langle h_2, h_2 \rangle_{\mathfrak{H}_2} \right),$$

where the minimum is over all such representations and uniquely attained, and $\kappa = \kappa_1 + \kappa_2$ where $\kappa, \kappa_1, \kappa_2$ are the negative indices of $\mathfrak{H}, \mathfrak{H}_1, \mathfrak{H}_2$.

THEOREM 3.9. *Suppose that $K_1(s,t), K_2(s,t)$ are reproducing kernels for Pontryagin spaces $\mathfrak{H}_1, \mathfrak{H}_2$ of functions on Ω with negative indices κ_1, κ_2. Then*

$$K(s,t) = K_1(s,t) + K_2(s,t) \tag{3–3}$$

is the reproducing kernel for a Pontryagin space $\mathfrak{H}$ having negative index $\kappa \le \kappa_1 + \kappa_2$. Equality holds if and only if $\mathfrak{R} = \mathfrak{H}_1 \cap \mathfrak{H}_2$ is a Hilbert space in the inner product

$$\langle h, k \rangle_{\mathfrak{R}} = \langle h, k \rangle_{\mathfrak{H}_1} + \langle h, k \rangle_{\mathfrak{H}_2} \quad \text{for } h, k \in \mathfrak{R}. \tag{3–4}$$

If $\kappa = \kappa_1 + \kappa_2$, then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are contained contractively in $\mathfrak{H}$ as complementary spaces.

A bit more is true. For example, the converse to the last statement holds: if $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are contained contractively in $\mathfrak{H}$ as complementary spaces, then $\kappa = \kappa_1 + \kappa_2$. See [Alpay et al. 1997, Chapter 1].

PROOF. We obtain the first assertion from Lemma 2.1 as in the proof of Theorem 3.3. (Hint. First write $K_j(s,t) = K_{j+}(s,t) + K_{j-}(s,t)$ for $j = 1, 2$, as in Theorem 3.3.)

Assume $\kappa = \kappa_1 + \kappa_2$. Let $\mathbf{R}$ be the linear relation in $\mathfrak{H} \times (\mathfrak{H}_1 \times \mathfrak{H}_2)$ spanned by all pairs $\big(K(s,\cdot), (K_1(s,\cdot), K_2(s,\cdot))\big)$ with $s \in \Omega$. Apply Theorem 2.2 to construct an isometry W^* from $\mathfrak{H}$ into $\mathfrak{H}_1 \times \mathfrak{H}_2$ such that

$$W^* : K(s,\cdot) \to (K_1(s,\cdot), K_2(s,\cdot)) \quad \text{for } s \in \Omega.$$

Then W is a partial isometry whose initial set is a Pontryagin subspace of $\mathfrak{H}_1 \times \mathfrak{H}_2$ having the negative index $\kappa = \kappa_1 + \kappa_2$, and whose kernel therefore is a Hilbert space. A short calculation shows that $W : (h_1, h_2) \to h_1 + h_2$. In fact, if $(h_1, h_2) \in \mathfrak{H}_1 \times \mathfrak{H}_2$ and $W : (h_1, h_2) \to h$, then

$$\begin{aligned} h(s) &= \langle h(\cdot), K(s,\cdot) \rangle_{\mathfrak{H}} = \langle (h_1(\cdot), h_2(\cdot)), (K_1(s,\cdot), K_2(s,\cdot)) \rangle_{\mathfrak{H}_1 \times \mathfrak{H}_2} \\ &= h_1(s) + h_2(s) \end{aligned}$$

for all $s \in \Omega$. The kernel of W is naturally isomorphic with $\mathfrak{R} = \mathfrak{H}_1 \cap \mathfrak{H}_2$ in the inner product (3–4). Thus $\mathfrak{R}$ is a Hilbert space in the inner product (3–4). We have also proved the last assertion of the theorem.

Conversely, assume $\mathfrak{R} = \mathfrak{H}_1 \cap \mathfrak{H}_2$ is a Hilbert space in the inner product (3–4). Then the elements $(h, -h)$ in $\mathfrak{H}_1 \times \mathfrak{H}_2$ with $h \in \mathfrak{R}$ form a Hilbert space in the inner product of $\mathfrak{H}_1 \times \mathfrak{H}_2$. Hence there is a Pontryagin space $\mathfrak{H}'$ such that the mapping

$$W : (h_1, h_2) \to h_1 + h_2$$

is a partial isometry from $\mathfrak{H}_1 \times \mathfrak{H}_2$ onto $\mathfrak{H}'$. We easily check that (3–3) is a reproducing kernel for $\mathfrak{H}'$, hence $\mathfrak{H}' = \mathfrak{H}$ isometrically. Since $\ker W$ is a Hilbert space, $\mathfrak{H}_1 \times \mathfrak{H}_2$ and $\mathfrak{H}$ have the same negative index, namely, $\kappa = \kappa_1 + \kappa_2$. □

Theorem 3.10. *Suppose that $K(s,t), K_1(s,t)$ are reproducing kernels for Pontryagin spaces $\mathfrak{H}, \mathfrak{H}_1$ of functions on Ω having negative indices κ, κ_1. If $\mathfrak{H}_1$ is contained contractively in $\mathfrak{H}$, then $\kappa \geq \kappa_1$ and*

$$K_2(s,t) = K(s,t) - K_1(s,t)$$

has $\kappa_2 = \kappa - \kappa_1$ negative squares and is the reproducing kernel for a Pontryagin space $\mathfrak{H}_2$ which is contained contractively in $\mathfrak{H}$ such that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are complementary.

Proof. By a theorem of de Branges [1988] (see also [Dritschel and Rovnyak 1991]), there is a unique Pontryagin space $\mathfrak{H}_2$ which is contained contractively in $\mathfrak{H}$ such that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are complementary. Evaluation mappings on $\mathfrak{H}_2$ are compositions of continuous mappings and thus continuous. Hence $\mathfrak{H}_2$ has a reproducing kernel $K_2(s,t)$. By the definition of complementary spaces, the mapping $W : (h_1, h_2) \to h_1 + h_2$ is a partial isometry from $\mathfrak{H}_1 \times \mathfrak{H}_2$ onto $\mathfrak{H}$, and $\ker W$ is a Hilbert space. Thus $\mathfrak{H}_2$ has negative index $\kappa - \kappa_1$. In particular, $\kappa \geq \kappa_1$.

For fixed $s \in \Omega$, $K_1(s,\cdot) + K_2(s,\cdot)$ belongs to $\mathfrak{H}$ and is the image under W of $(K_1(s,\cdot), K_2(s,\cdot))$, where the kernel function pair is in $\mathfrak{H}_1 \times \mathfrak{H}_2$ and orthogonal to $\ker W$. Every h in $\mathfrak{H}$ is the image under W of some pair (h_1, h_2) in $\mathfrak{H}_1 \times \mathfrak{H}_2$, and hence

$$\begin{aligned}\langle h(\cdot), K_1(s,\cdot) + K_2(s,\cdot)\rangle_{\mathfrak{H}} &= \langle (h_1(\cdot), h_2(\cdot)), (K_1(s,\cdot), K_2(s,\cdot))\rangle_{\mathfrak{H}_1 \times \mathfrak{H}_2} \\ &= h_1(s) + h_2(s) = h(s).\end{aligned}$$

Therefore $K_1(s,t) + K_2(s,t)$ is a reproducing kernel for $\mathfrak{H}$. Since a reproducing kernel is unique, $K_1(s,t) + K_2(s,t) = K(s,t)$, and so $K_2(s,t) = K(s,t) - K_1(s,t)$ is the reproducing kernel for $\mathfrak{H}_2$. The result follows. □

One more result may be mentioned.

Theorem 3.11. *Let $K_1(s,t)$ be the reproducing kernel for a Pontryagin space $\mathfrak{H}_1$ of functions on Ω having negative index κ_1. Define*

$$K_2(s,t) = A(t) K_1(s,t) \overline{A(s)} \quad \text{for } s, t \in \Omega,$$

where $A(t)$ is a fixed function on Ω. Then $K_2(s,t)$ is the reproducing kernel for a Pontryagin space $\mathfrak{H}_2$ of functions on Ω having negative index $\kappa_2 \leq \kappa_1$, and

$\kappa_1 = \kappa_2$ *if and only if the set of functions* $h(t)$ *in* $\mathfrak{H}_1$ *such that* $A(t)h(t) \equiv 0$ *is a Hilbert space in the inner product of* $\mathfrak{H}_1$. *In this case, multiplication by* $A(t)$ *is a partial isometry from* $\mathfrak{H}_1$ *onto* $\mathfrak{H}_2$ *whose kernel is a Hilbert space.*

The proof is another application of Theorem 2.2 and omitted (when $\kappa_1 = \kappa_2$, set up a linear relation $\mathbf{R}$ to define the adjoint of multiplication by $A(t)$).

We have presented the theory of reproducing kernel spaces in the context of Pontryagin spaces. The main definitions can be adapted to Kreĭn spaces, and some of the constructions carry over. However, new phenomena arise in this generality. A reproducing kernel is uniquely determined by the associated Kreĭn space, but, in contrast with Theorem 3.2, essentially different Kreĭn spaces can have the same reproducing kernel. For example, see [Alpay 1991; Schwartz 1964].

4. Examples Involving Generalized Schur Functions

The Schur class is the set of holomorphic functions $S(z)$ which are defined and satisfy $|S(z)| \le 1$ on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. For any such function, the associated kernel

$$K_S(w,z) = \frac{1 - S(z)\overline{S(w)}}{1 - z\bar{w}} \tag{4–1}$$

is nonnegative and therefore the reproducing kernel for a Hilbert space $\mathfrak{H}(S)$ of holomorphic functions on $\mathbb{D}$. These spaces have been extensively studied, and many properties are known. For example, the transformation

$$T : h(z) \to \frac{h(z) - h(0)}{z} \tag{4–2}$$

maps $\mathfrak{H}(S)$ into itself and satisfies the difference quotient inequality

$$\big\| \big(h(z) - h(0)\big)/z \big\|^2_{\mathfrak{H}(S)} \le \|h(z)\|^2_{\mathfrak{H}(S)} - |h(0)|^2 \tag{4–3}$$

for all elements $h(z)$ of the space. There are operators

$$\begin{cases} F : c \to \dfrac{S(z) - S(0)}{z}\, c & \text{on } \mathbb{C} \text{ to } \mathfrak{H}(S), \\ G : h(z) \to h(0) & \text{on } \mathfrak{H}(S) \text{ to } \mathbb{C}, \\ H : c \to S(0)c & \text{on } \mathbb{C} \text{ to } \mathbb{C}, \end{cases} \tag{4–4}$$

such that the colligation

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(S) \oplus \mathbb{C}) \tag{4–5}$$

is coisometric, that is, $VV^* = 1$. The characteristic function of the colligation is the given Schur function $S(z)$:

$$S(z)c = Hc + zG(1 - zT)^{-1}Fc \quad \text{for } c \in \mathbb{C}. \tag{4–6}$$

The colligation is unitary if and only if equality always holds in (4–3) and $S(z) \not\equiv 0$. An equivalent condition that V is unitary is that the function $S(z)$ itself does not belong to $\mathfrak{H}(S)$. If $S(z) = S_1(z)S_2(z)$ where $S_1(z)$ and $S_2(z)$ are Schur functions, then $\mathfrak{H}(S_1)$ is contained contractively in $\mathfrak{H}(S)$. If $\mathfrak{H}(S_1) \cap S_1\mathfrak{H}(S_2) = \{0\}$, the inclusion is isometric, and $\mathfrak{H}(S) = \mathfrak{H}(S_1) \oplus S_1\mathfrak{H}(S_2)$. The scalar case is treated in [de Branges and Rovnyak 1966; Sarason 1994].

We derive the preceding scalar results in a Pontryagin space setting by the methods of [Alpay et al. 1997].

DEFINITION 4.1. Let Ω be a region satisfying $0 \in \Omega \subseteq \mathbb{D}$. The *generalized Schur class* $\mathbf{S}_\kappa$ is the set of holomorphic functions $S(z)$ defined on Ω and such that the kernel (4–1) has κ negative squares on $\Omega \times \Omega$. For such a function, let $\mathfrak{H}(S)$ be the Pontryagin space of holomorphic functions with reproducing kernel (4–1).

The domain of a function $S(z) \in \mathbf{S}_\kappa$ will be denoted $\Omega = \Omega(S)$. By Theorem 3.7, this can be any region in $\mathbb{D}$ containing the origin on which $S(z)$ is holomorphic. Technically the spaces $\mathfrak{H}(S)$ are different for different regions, but any two such spaces can be identified in a natural way.

THEOREM 4.2. *For every $S(z)$ in $\mathbf{S}_\kappa$, the transformation* (4–2) *is everywhere defined on $\mathfrak{H}(S)$ and a bicontraction. The difference-quotient inequality*

$$\left\langle \frac{h(z)-h(0)}{z},\ \frac{h(z)-h(0)}{z} \right\rangle_{\mathfrak{H}(S)} \le \langle h(z), h(z)\rangle_{\mathfrak{H}(S)} - h(0)\overline{h(0)} \tag{4–7}$$

holds for every $h(z)$ in $\mathfrak{H}(S)$. There exist operators (4–4) *such that* (4–5) *is a coisometry. The identity* (4–6) *holds in a neighborhood of the origin.*

It is interesting to ask for a converse to Theorem 4.2, that is, to say when a Pontryagin space of holomorphic functions is of the form $\mathfrak{H}(S)$ for some function $S(z)$ in $\mathbf{S}_\kappa$. When $\kappa = 0$, such characterizations are known [Guyker 1991; Leech 1969], but it is an open problem to obtain a similar result in the general case. When more general spaces involving operator-valued functions are allowed, a complete characterization is possible and given in [Alpay et al. 1997, Theorem 3.1.2].

PROOF. Let the domain of $S(z)$ be Ω. Theorem 2.2 will be used to construct an isometry which turns out to be the adjoint of the operator V that we seek. The construction is based on knowing how V^* must act on special elements. This motivates the definition of a linear relation $\mathbf{R}$ in $\big(\mathfrak{H}(S) \oplus \mathbb{C}\big) \times \big(\mathfrak{H}(S) \oplus \mathbb{C}\big)$ as the span of all pairs

$$\left(\begin{pmatrix} K_S(\alpha,\cdot\,)a_1 \\ a_2 \end{pmatrix},\ \begin{pmatrix} \dfrac{K_S(\alpha,\cdot\,) - K_S(0,\cdot\,)}{\bar\alpha}\, a_1 + K_S(0,\cdot\,)a_2 \\ \dfrac{\overline{S(\alpha)} - \overline{S(0)}}{\bar\alpha}\, a_1 + \overline{S(0)}a_2 \end{pmatrix} \right)$$

with $0 \neq \alpha \in \Omega$ and $a_1, a_2 \in \mathbb{C}$. The inner product of the right members of any two such pairs with nonzero $\alpha, \beta \in \Omega$ and $a_1, a_2, b_1, b_2 \in \mathbb{C}$ is

$$\begin{aligned}
&\frac{K_S(\alpha,\beta) - K_S(0,\beta) - K_S(\alpha,0) + K_S(0,0)}{\bar\alpha\,\beta}\, a_1\bar b_1 + \frac{K_S(0,\beta) - K_S(0,0)}{\beta}\, a_2\bar b_1 \\
&\qquad + \frac{K_S(\alpha,0) - K_S(0,0)}{\bar\alpha\beta}\, a_1\bar b_2 + K_S(0,0) a_2\bar b_2 \\
&\qquad + \frac{S(\beta) - S(0)}{\beta}\,\frac{\overline{S(\alpha)} - \overline{S(0)}}{\bar\alpha\beta}\, a_1\bar b_1 + \frac{S(\beta) - S(0)}{\beta}\overline{S(0)}\, a_2\bar b_1 \\
&\qquad + S(0)\frac{\overline{S(\alpha)} - \overline{S(0)}}{\bar\alpha\beta}\, a_1\bar b_2 + S(0)\overline{S(0)}\, a_2\bar b_2 \\
&\quad = K_S(\alpha,\beta) a_1\bar b_1 + a_2\bar b_2 \\
&\quad = \left\langle \begin{pmatrix} K_S(\alpha,\cdot\,)a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} K_S(\beta,\cdot\,)b_1 \\ b_2 \end{pmatrix} \right\rangle_{\mathfrak{H}(S)\oplus\mathbb{C}}.
\end{aligned}$$

Therefore $\mathbf{R}$ is isometric, hence $\bar{\mathbf{R}}$ is the graph of an isometry

$$V^* = \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(S)\oplus\mathbb{C}).$$

The definition of $\mathbf{R}$ is made so that T, F, G, H are given by (4–2) and (4–4).

On solving the equation $(1 - wT)^{-1}h = g$, we see that

$$(1 - wT)^{-1} : h(z) \to \frac{zh(z) - wh(w)}{z - w}$$

for any $h(z)$ in $\mathfrak{H}(S)$ and w in a neighborhood of the origin. Therefore

$$\begin{aligned}
\bigl(H + wG(1 - wT)^{-1}F\bigr)c &= S(0)c + w\left\{\frac{z\,\dfrac{S(z)-S(0)}{z}\,c - w\,\dfrac{S(w)-S(0)}{w}\,c}{z - w}\right\}_{z=0} \\
&= S(0)c + w\,\frac{S(w)-S(0)}{w}\,c = S(w)c,
\end{aligned}$$

for w in a neighborhood of the origin and $c \in \mathbb{C}$, yielding (4–6).

Since V^* is an isometry and hence a contraction acting on a Pontryagin space into itself, it is a bicontraction. Hence $V^*V \le 1$ and $T^*T + G^*G \le 1$, which implies (4–7). By (4–7), T is a contraction. Since T maps a Pontryagin space into itself, it is a bicontraction. □

THEOREM 4.3. *In Theorem* 4.2, *the following conditions are equivalent*:

(a) *Equality always holds in* (4–7) *and* $S(z) \not\equiv 0$.
(b) $S(z)$ *does not belong to* $\mathfrak{H}(S)$.
(c) V *is unitary*.

PROOF. (a) ⇒ (b) Assume (a). Then $T^*T + G^*G = 1$ and so

$$1 - V^*V = \begin{pmatrix} 0 & -T^*F - G^*H \\ -F^*T - H^*G & 1 - F^*F - H^*H \end{pmatrix}.$$

Since V^* is an isometry and hence a bicontraction, V is a contraction. Hence $1 - V^*V \geq 0$. In view of the zero in the upper left entry of the operator matrix, this implies $T^*F + G^*H = 0$ by properties of nonnegative quadratic forms. The action of G^* is computed from the uniqueness of a reproducing kernel: $G^*c = K_S(0,\cdot)c$ for every $c \in \mathbb{C}$. Therefore the identity $T^*F + G^*H = 0$ yields

$$T^* : \frac{S(z) - S(0)}{z} \to -K_S(0,z)S(0).$$

If $S(z)$ belongs to $\mathfrak{H}(S)$, the last relation yields

$$(1 - G^*G)S(\,\cdot\,) = T^*TS(\,\cdot\,) = -K_S(0,\cdot\,)S(0).$$

Then $S(z) - K_S(0,z)S(0) \equiv -K_S(0,z)S(0)$ and $S(z) \equiv 0$, which is excluded in (a). Therefore $S(z)$ does not belong to $\mathfrak{H}(S)$.

(b) ⇒ (c) Assume (b). Since V^* is an isometry, to prove that V is unitary we need only show that $\ker V = \{0\}$. If

$$V\begin{pmatrix} f \\ c \end{pmatrix} = 0,$$

then

$$\frac{f(z) - f(0)}{z} + \frac{S(z) - S(0)}{z}\,c = 0,$$
$$f(0) + S(0)c = 0,$$

identically. Therefore $f(z) = -S(z)c$ belongs to $\mathfrak{H}(S)$. By (b), $c = 0$. Thus $\ker V = \{0\}$ and so V is unitary.

(c) ⇒ (a) The unitarity of V implies $V^*V = 1$. Hence $T^*T + G^*G = 1$, and therefore equality always holds in (4–7). If $S(z) \equiv 0$, then $F = 0$ and $H = 0$ by (4–4), contradicting the assumption that V is unitary. Therefore $S(z) \not\equiv 0$. □

In the case $\kappa = 0$, condition (b) in Theorem 4.3 has been much studied; see [de Branges and Rovnyak 1966; Sarason 1994]. The condition can also be pursued using Leech's theorem on the factorization of operator-valued functions. The result obtained in this way states that $S(z)$ belongs to $\mathfrak{H}(S)$ and $\|S(z)\|_{\mathfrak{H}(S)} \leq m$ if and only if

$$S(z) = \frac{mC_1(z)}{\sqrt{m^2+1} - zC_2(z)}$$

where $C_1(z)$ and $C_2(z)$ are holomorphic and satisfy $|C_1(z)|^2 + |C_2(z)|^2 \leq 1$ on $\mathbb{D}$. See [Alpay et al. 1997] for details and some of the results that are known in the indefinite case.

Theorem 4.4. *Suppose*

$$S = S_1 S_2,$$

where $S_1 \in \mathbf{S}_{\kappa_1}$, $S_2 \in \mathbf{S}_{\kappa_2}$. *Then* $S \in \mathbf{S}_\kappa$ *for some* $\kappa \leq \kappa_1 + \kappa_2$. *Equality holds if and only if the set* $\mathfrak{N}$ *of elements* $(h_1, h_2) \in \mathfrak{H}(S_1) \times \mathfrak{H}(S_2)$ *such that* $h_1 + S_1 h_2 \equiv 0$ *form a Hilbert space in the inner product of* $\mathfrak{H}(S_1) \times \mathfrak{H}(S_2)$.

The proof of the theorem and general results on reproducing kernels in § 3 yield additional conclusions. Let $S_1\mathfrak{H}(S_2)$ be the Pontryagin space with reproducing kernel $S_1(z)K_{S_2}(w,z)\overline{S_1(w)}$ (notice that when $S_1 \not\equiv 0$, multiplication by S_1 maps $\mathfrak{H}(S_2)$ isometrically onto $S_1\mathfrak{H}(S_2)$ by Theorem 3.11). Then, for example, when $\kappa = \kappa_1 + \kappa_2$, $\mathfrak{H}(S_1)$ and $S_1\mathfrak{H}(S_2)$ are contractively contained in $\mathfrak{H}(S)$ as complementary spaces.

Proof. We may suppose that the functions $S_1 \in \mathbf{S}_{\kappa_1}$, $S_2 \in \mathbf{S}_{\kappa_2}$, and $S = S_1 S_2$ are defined on a common region Ω with $0 \in \Omega \subseteq \mathbb{D}$. The result is easily checked when $S_1 \equiv 0$, so let us exclude this degenerate case. Then

$$K_S(w,z) = K_{S_1}(w,z) + S_1(z)K_{S_2}(w,z)\overline{S_1(w)},$$

where $K_{S_1}(w,z)$ has κ_1 negative squares and $S_1(z)K_{S_2}(w,z)\overline{S_1(w)}$ has κ_2 negative squares. By Theorem 3.9, $K_S(w,z)$ has $\kappa \leq \kappa_1 + \kappa_2$ negative squares. Define $S_1\mathfrak{H}(S_2)$ as in the remarks preceding the proof. Theorem 3.9 also says that $\kappa = \kappa_1 + \kappa_2$ if and only if $\mathfrak{R} = \mathfrak{H}(S_1) \cap S_1\mathfrak{H}(S_2)$ is a Hilbert space in the inner product defined by

$$\langle h,k \rangle_{\mathfrak{R}} = \langle h,k \rangle_{\mathfrak{H}(S_1)} + \langle h,k \rangle_{S_1\mathfrak{H}(S_2)} \quad \text{for } h,k \in \mathfrak{R}.$$

We show that $\mathfrak{R}$ is in one-to-one isometric correspondence with $\mathfrak{N}$ in the inner product of $\mathfrak{H}(S_1) \times \mathfrak{H}(S_2)$. In fact, the mapping $(h_1,h_2) \to h_1$ from $\mathfrak{N}$ onto $\mathfrak{R}$ is such a correspondence. For if $(h_1,h_2) \in \mathfrak{N}$, then $h_1 + S_1 h_2 \equiv 0$ and so $h_1 \in \mathfrak{H}(S_1) \cap S_1\mathfrak{H}(S_2) = \mathfrak{R}$. Clearly every element of $\mathfrak{R}$ arises in this way from a unique element of $\mathfrak{N}$, because as noted above multiplication by S_1 maps $\mathfrak{H}(S_2)$ isometrically onto $S_1\mathfrak{H}(S_2)$. Suppose $(k_1,k_2) \to k_1$ for a second pair in $\mathfrak{N}$. Then

$$\begin{aligned}
\langle (h_1,h_2),(k_1,k_2) \rangle_{\mathfrak{H}(S_1)\times\mathfrak{H}(S_2)} &= \langle h_1,k_1 \rangle_{\mathfrak{H}(S_1)} + \langle h_2,k_2 \rangle_{\mathfrak{H}(S_2)} \\
&= \langle h_1,k_1 \rangle_{\mathfrak{H}(S_1)} + \langle S_1h_2,S_1k_2 \rangle_{S_1\mathfrak{H}(S_2)} \\
&= \langle h_1,k_1 \rangle_{\mathfrak{H}(S_1)} + \langle h_1,k_1 \rangle_{S_1\mathfrak{H}(S_2)} = \langle h_1,k_1 \rangle_{\mathfrak{R}}.
\end{aligned}$$

Thus $\kappa = \kappa_1 + \kappa_2$ if and only if $\mathfrak{N}$ is a Hilbert space in the inner product of $\mathfrak{H}(S_1) \times \mathfrak{H}(S_2)$. □

We conclude with the answer to a question that will no doubt have occurred to the reader: what are the analytic continuation properties of functions in $\mathbf{S}_\kappa$? For the case $\kappa = 0$, it is well known that positivity of the kernel (4–1) for w,z in a neighborhood of the origin implies that $S(z)$ has an extension to a holomorphic function which is bounded by one on the unit disk, and so $\mathbf{S}_0$ is naturally identified with the usual Schur class of complex analysis. The general result is a well-known theorem of Kreĭn and Langer [1972].

Theorem 4.5. *Every function $S(z)$ in $\mathbf{S}_\kappa$ has a factorization*

$$S(z) = B(z)^{-1} S_0(z),$$

where $B(z)$ is a Blaschke product having κ factors and $S_0(z)$ belongs to the classical Schur class $\mathbf{S}_0$ and is nonvanishing at the zeros of $B(z)$. Conversely, every function of this form belongs to $\mathbf{S}_\kappa$.

To say that $B(z)$ is a Blaschke product of κ factors means that it has the form

$$B(z) = C \prod_{j=1}^{\kappa} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z},$$

where $\alpha_1, \dots, \alpha_\kappa$ are (not necessarily distinct) points of the unit disk and C is a constant of unit modulus. The case $\kappa = 0$ is included by interpreting an empty product as one. See [Alpay et al. 1997, § 4.2], for a proof of Theorem 4.5 in a version for operator-valued functions.

The generalized Schur class $\mathbf{S}_\kappa$ has deep connections with interpolation and operator theory. It has been studied by Kreĭn and Langer in a series of papers including [Kreĭn and Langer 1981]. However, a more complete account is beyond the scope of the present introductory survey. We refer the reader to [Alpay et al. 1997] for additional results and literature notes.

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Daniel Alpay
Department of Mathematics
Ben-Gurion University of the Negev
P. O. Box 653
84105 Beer-Sheva
Israel
dany@ivory.bgu.ac.il

Aad Dijksma
Department of Mathematics
University of Groningen
P. O. Box 800
9700 AV Groningen
The Netherlands
dijksma@math.rug.nl

James Rovnyak
Department of Mathematics
University of Virginia
Charlottesville, Virginia 22903-3199
United States
rovnyak@virginia.edu

Hendrik S. V. de Snoo
Department of Mathematics
University of Groningen
P. O. Box 800
9700 AV Groningen
The Netherlands
desnoo@math.rug.nl