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Convergence Theorems in Riemannian Geometry

PETER PETERSEN

Abstract. This is a survey on the convergence theory developed first by Cheeger and Gromov. In their theory one is concerned with the compactness of the class of riemannian manifolds with bounded curvature and lower bound on the injectivity radius. We explain and give proofs of almost all the major results, including Anderson's generalizations to the case where all one has is bounded Ricci curvature. The exposition is streamlined by the introduction of a norm for riemannian manifolds, which makes the theory more like that of Hölder and Sobolev spaces.

1. Introduction

This paper is an outgrowth of a talk given in October 1993 at MSRI and a graduate course offered in the Spring of 1994 at UCLA. The purpose is to introduce readers to the convergence theory of riemannian manifolds not so much through a traditional survey article, but by rigorously proving most of the key theorems in the subject. For a broader survey of this subject, and how it can be applied to various problems, we refer the reader to [Anderson 1993].

The prerequisites for this paper are some basic knowledge of riemannian geometry, Gromov–Hausdorff convergence and elliptic regularity theory. In particular, the reader should be familiar with the comparison geometry found in [Karcher 1989], for example. For Gromov–Hausdorff convergence, it suffices to read Section 6 in [Gromov 1981a] or Section 1 in [Petersen 1993]. In regard to elliptic theory, we have an appendix that contains all the results we need, together with proofs of those theorems that are not explicitly stated in [Gilbarg and Trudinger 1983].

In Section 2 we introduce the concept of (pointed) $C^{k+\alpha}$ convergence of riemannian manifolds. This introduces a natural topology on (pointed) riemannian manifolds and immediately raises the question of which "subsets" are precompact. To answer this, we use, for the first time in the literature, the idea that a riemannian manifold in a natural way has a $C^{k+\alpha}$ norm for each fixed scale

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$r > 0$. This norm is a quantitative version of the definition of a manifold. It is basically computed by finding atlases of charts from r-balls in $\mathbb{R}^n$ into the manifold and then, for each of these atlases, by computing the largest $C^{k+\alpha}$ norm of the metric coefficients in these charts, and for compatibility reasons, also the largest $C^{k+1+\alpha}$ norm of the transition functions. The scale r is an integral part of the definition, so that $\mathbb{R}^n$ becomes the only space that has zero norm on all scales, while flat manifolds have zero norm on small scales, and nonflat manifolds have the property that the norm goes to zero as the scale goes to zero. The norm concept is dual to the usual radii concepts in geometry, but has nicer properties.

We have found this norm concept quite natural to work with: It gives, for instance, a very elegant formulation of what we call The Fundamental Theorem of Convergence Theory. This theorem is completely analogous to the classical Arzela–Ascoli theorem, and says that, for each fixed scale r, the class of riemannian n-manifolds of $C^{k+\alpha}$ norm $\leq Q$ is compact in the $C^{k+\beta}$ topology for $\beta < \alpha$. The only place where we use Gromov–Hausdorff convergence is in the proof of this theorem. Aside from the new concept of norm and the use of Gromov–Hausdorff convergence, the proof of this fundamental theorem is essentially contained in [Cheeger 1970].

In Section 3 we use the Fundamental Theorem to prove the Compactness Theorem of Cheeger and Gromov for manifolds of bounded curvature, as it is stated in [Gromov 1981b]. In addition we give a new proof by contradiction of Cheeger's lemma on the injectivity radius, using convergence techniques. Finally, we give S.-H. Zhu's proof of the compactness of the class of n-manifolds with lower sectional curvature bounds and lower injectivity radius bounds.

The theory really picks up speed in Section 4, where we introduce $L^{p,k}$ norms on the scale of r, using harmonic coordinates. It is at this point that we need to use elliptic regularity theory. The idea of using harmonic coordinates, rather than just general coordinates, makes a big difference in the theory, since it makes the norm a locally realizable number. This is basically a consequence of the fact that we don't have to worry about the norms of transition functions. Our "harmonic" norm is, of course, dual to Anderson's harmonic radius, but we again find that the norm idea has some important technical advantages over the use of harmonic radius.

Most of Anderson's convergence results are proved in Section 5. They are all generalizations of the Cheeger–Gromov Compactness Theorem, but the proofs are, in contrast, all by contradiction.

The final Section 6 has some applications of convergence theory to pinching problems in riemannian geometry. Some of the results in this section are extensions of work in [Gao 1990].

2. The Fundamentals

For a function $f : \Omega \to \mathbb{R}^h$, where $\Omega \subset \mathbb{R}^n$ the Hölder C^α-constant $0 < \alpha \leq 1$ is defined as:

$$\|f\|_\alpha = \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

In other words, $\|f\|_\alpha$ is the smallest constant C such that $|f(x) - f(y)| \leq C\,|x-y|^\alpha$. Notice that $\|f\|_1$ is the best Lipschitz constant for f. If $\Omega \subset \mathbb{R}^n$ is open, $k \geq 0$ is an integer, and $0 < \alpha \leq 1$, we define the $C^{k+\alpha}$ norm of f as

$$\|f\|_{k+\alpha} = \|f\|_{C^k} + \max_{|j|=k} \|\partial^j f\|_\alpha.$$

Here $\|f\|_{C^k}$ is the usual C^k-norm and $\partial^j = \partial_1^{j_1} \cdots \partial_n^{j_n}$, where $\partial_i = \partial/\partial x^i$ and $j = (j_1, \ldots, j_n)$ is a multi-index. Note that the norm $\|f\|_{k+1}$ is not the same as $\|f\|_{C^{k+1}}$.

We denote by $C^{k+\alpha}(\Omega)$ the space of functions with finite $(k+\alpha)$-norm. This norm makes $C^{k+\alpha}(\Omega)$ into a Banach space. The classical Arzela–Ascoli Theorem says that for $k+\alpha > 0$, $0 < \alpha \leq 1$, and $l + \beta < k + \alpha$, any sequence satisfying $\|f_i\|_{k+\alpha} \leq \kappa$ has a subsequence that converges in the $l+\beta$-topology to a function f, with $\|f\|_{k+\alpha} \leq \kappa$. Note that if $\alpha = 0$ it is not necessarily true that f is C^k, but it will be C^{k-1+1}. We shall therefore always look at $(k+\alpha)$-topologies with $0 < \alpha \leq 1$.

We can now define $C^{k+\alpha}$ convergence of tensors on a given manifold M. Namely, a sequence T_i of tensors on M is said to converge to T in the $C^{k+\alpha}$-topology if we can find a covering $\varphi_s : U_s \to \mathbb{R}^n$ of coordinate charts so that the overlap maps are at least $C^{k+1+\alpha}$ and all the components of the tensors T_i converge in the $C^{k+\alpha}$ topology to the components of T in these coordinate charts, considered as functions on $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$. This convergence concept is clearly independent of our choice of coordinates. Note that it is necessary for the overlaps to be $C^{k+1+\alpha}$, since the components of tensors are computed by evaluating these tensors on the $C^{k+\alpha}$ fields $\partial/\partial x^i$ and dx^i.

In the sequel we shall restrict our attention to complete or closed riemannian manifolds. Some of the theory can, with modifications, be generalized to incomplete manifolds and manifolds with boundary. These generalizations, while useful, are not very deep and can be found in the literature.

A pointed sequence (M_i, g_i, p_i) of riemannian n-manifolds with metrics g_i, and $p_i \in M_i$, is said to converge to a riemannian manifold (M, g, p) in the pointed $C^{k+\alpha}$ topology if, for each $R > 0$, there is a domain Ω in M containing the open ball $B(p, R)$ and embeddings $f_i : \Omega \to M_i$, for large i, such that $f_i(\Omega) \supset B(p_i, R)$ and the $f_i^* g_i$ converge to g in the $C^{k+\alpha}$ topology on Ω. If we can choose $\Omega = M$ and $f_i(\Omega) = M_i$, we say that (M_i, g_i) converges to (M, g) in the $C^{k+\alpha}$-topology. There is obviously no significant difference between pointed and unpointed topologies when all manifolds in use are closed.

It is very important to realize that this concept of $C^{k+\alpha}$ convergence is not the same as the one we just defined for tensors on a given manifold M, even when we are considering a sequence of riemannian metrics g_i on the same space M. This is because one can have a sequence of metrics g_i and diffeomorphisms $f_i : M \to M$ such that $\{f_i^* g_i\}$ converges while $\{g_i\}$ does not converge.

We say that a collection of riemannian n-manifolds is precompact in the (pointed) $C^{k+\alpha}$ topology if any sequence in this collection has a subsequence that is convergent in the (pointed) $C^{k+\alpha}$-topology—in other words, if the (pointed) $C^{k+\alpha}$-closure is compact.

The rest of the paper is basically concerned with finding reasonable (geometric) conditions that imply precompactness in some of these topologies. In order to facilitate this task and ease the exposition a little, we need to introduce some auxiliary concepts.

For a riemannian n-manifold we define the $C^{k+\alpha}$-norm on the scale of $r > 0$, denoted $\|(M,g)\|_{k+\alpha,r}$, as the infimum over all numbers $Q \geq 0$ such that we can find coordinates charts $\varphi_s : B(0,r) \subset \mathbb{R}^n \to U_s \subset M$ with these properties:

(n1) Every ball of radius $\delta = \frac{1}{10} e^{-Q} r$ is contained in some U_s.
(n2) $|d\varphi_s^{-1}| \leq e^Q$ and $|d\varphi_s| \leq e^Q$ on $B(0,r)$.
(n3) $r^{|j|+\alpha} \|\partial^j g_{s\cdot\cdot}\|_\alpha \leq Q$ for all multi-indices j with $0 \leq |j| \leq k$.
(n4) $\|\varphi_s^{-1} \circ \varphi_t\|_{k+1+\alpha} \leq (10+r)\, e^Q$ on the domain of definition.

Here $g_{s\cdot\cdot}$ represents the metric components in the coordinates φ_s, considered as functions on $B(0,r) \subset \mathbb{R}^n$. The first condition is equivalent to saying that δ is a Lebesgue number for the covering $\{U_s\}$. The second condition can be rephrased as saying that the eigenvalues of $g_{s\cdot\cdot}$ with respect to the standard euclidean metric lie between e^{-Q} and e^Q. This gives both a C^0 bound for $g_{s\cdot\cdot}$ and a uniform positive definiteness for $g_{s\cdot\cdot}$.

If A is a subset of M, we can define $\|(A,g)\|_{k+\alpha,r}$ in a similar way, only changing (n1) to say that all δ-balls centered on A are contained in some U_s. In (n1) it would perhaps have been more natural to merely assume that the sets U_s cover M (or A). That, however, is not a desirable state of affairs, because this would put us in a situation where the norm $\|\ \|_{k+\alpha,r}$ wouldn't necessarily be realized by some Q. A fake but illustrative example is the sphere, covered by two balls of radius $r > \pi/2$ centered at antipodal points. As $r \to \pi/2$ the sets will approach a situation where they no longer cover the space.

From the definition it is clear that $\|(M,g)\|_{k+\alpha,r}$ must be finite for all r, when M is closed. For open manifolds it would be more natural to have some weight function $f : M \to \mathbb{R}$, which allows for Q to get bigger as we go farther and farther out on the manifold. We'll say a few more words about this later in this section.

EXAMPLE. If $M = \mathbb{R}^n$ with the canonical euclidean metric, then $\|M\|_{k+\alpha,r} = 0$ for all r. More generally, we can easily prove that $\|(M,g)\|_{k+\alpha,r} = 0$ for $r \leq$

$\operatorname{inj\,rad}(M,g)$ if (M,g) is a flat manifold. We shall see later in this section that these properties characterize $\mathbb{R}^n$ and flat manifolds in general.

With this example in mind, it is pretty clear that conditions (n1)–(n3) say that on a scale of r the metric on (M,g) is Q-close to the euclidean metric in the $C^{k+\alpha}$ topology. Condition (n4) is a compatibility condition that ensures that $C^{k+\alpha}$-closeness to euclidean space means the same in all coordinates.

For a given Q and r we can in the usual fashion consider maximal atlases of all charts satisfying (n1)–(n4), but we won't use this much. Let us now turn to some of the properties of this norm.

PROPOSITION 2.1. *Let (M,g) be a C^∞ riemannian n-manifold.*

(i) $\|(A,\lambda^2 g)\|_{k+\alpha,\lambda r} = \|(A,g)\|_{k+\alpha,r}$; *in other words the norm is scale invariant, for all $A \subset M$.*
(ii) *If M is compact, then $\|(M,g)\|_{k+\alpha,r}$ is finite for all r; moreover, this number depends continuously on r, and it tends to 0 as $r \to 0$.*
(iii) *If $(M_i,g_i,p_i) \to (M,g,p)$ in the pointed $C^{k+\alpha}$-topology, then for any bounded domain $B \subset M$ we can find domains $B_i \subset M_i$ such that*

$$\|(B_i,g_i)\|_{k+\alpha,r} \to \|(B,g)\|_{k+\alpha,r} \quad \text{for every } r > 0.$$

When all spaces involved are closed manifolds we can set $B_i = M_i$ and $B = M$.

PROOF. (i) If we change g to $\lambda^2 g$ we can change charts $\varphi^s : B(0,r) \to M$ to $\varphi^s_{\lambda^{-1}}(x) = \varphi(\lambda^{-1}x) : B(0,\lambda r) \to M$. Since we are also scaling the metric, this means that conditions (n1)–(n4) still hold with the same Q.

(ii) If, as in (i), we only change g to $\lambda^2 g$, but do not scale in euclidean space, we get

$$\begin{aligned} &\text{for } \lambda < 1, \quad \|(M,\lambda^2 g)\|_{k+\alpha,r} \le Q - \log\lambda \\ &\text{for } \lambda > 1, \quad \|(M,\lambda^2 g)\|_{k+\alpha,r} \le \max\{\lambda Q,\, Q + \log\lambda\}, \end{aligned}$$

where $\|(M,g)\|_{k+\alpha,r} < Q$. Thus

$$\|(M,g)\|_{k+\alpha,\lambda^{-1}r} \le \max\{\lambda Q,\, Q - \log\lambda,\, Q + \log\lambda\}.$$

If, therefore, the norm is finite for some r, it will be finite for all r. Furthermore, $f(r) = \|(M,g)\|_{k+\alpha,r}$ is a function with the property that, for each r,

$$f(\lambda^{-1}r) \le h(\lambda, f(r)) = \max\{\lambda f(r), f(r) - \log\lambda, f(r) + \log\lambda\},$$

and $f(r) = h(1, f(r))$. If, therefore, $r_i \to r$, we clearly have $\limsup f(r_i) \le f(r)$, since $f(r_i) = f\left(r\,(r_i/r)\right) \le g\left(r/r_i, f(r)\right)$. Conversely, $f(r) = f\left(r_i\, r/r_i\right) \le h\left(r_i/r, f(r_i)\right) = \max\left\{f(r_i)\, r_i/r,\, f(r_i) - \log(r_i/r),\, f(r_i) + \log(r_i/r)\right\} \le f(r_i) + \varepsilon$, for any $\varepsilon > 0$ as $r_i \to r$. Thus $f(r) \le \liminf f(r_i)$, and we have established continuity of $f(r)$.

To see that $\|(M,g)\|_{k+\alpha,r} \to 0$ as $r \to 0$ just use exponential coordinates $\exp_{p_i} : B(0,r) \to B(p_i,r)$.

(iii) Fix $r > 0$ and $B \subset M$. Then choose embeddings $f_i : \Omega \to M_i$ such that $f_i^* g_i$ converge to g and $B \subset \Omega$. Define $B_i = f_i(B)$.

For $Q > \|(B, g)\|_{k+\alpha,r}$, choose appropriate charts $\varphi^s : B(0, r) \to M$ satisfying (n1)–(n4), with $B = A$. Then $\varphi_i^s = f_i \circ \varphi^s : B(0, r) \to M_i$ will satisfy (n1) and (n4). Now, since $f_i^* g_i \to g$ in the $C^{k+\alpha}$-topology, (n2) and (n3) will hold with a Q_i satisfying $Q_i \to Q$ as $i \to \infty$. Thus $\limsup \|(B_i, g)\|_{k+\alpha,r} \le \|(B, g)\|_{k+\alpha,r}$.

Conversely, for large i and $Q > \liminf \|(B_i, g_i)\|_{k+\alpha,r}$. We can choose charts $\varphi_i^s : B(0, r) \to M_i$ satisfying (n1)–(n4) with $B_i = A$. Now consider $\varphi^s = f_i^{-1} \circ \varphi_i^s : B(0, r) \to M$. These charts will satisfy n1 and n4, and again n2 and n3 for some $Q_i \ge Q$, which can be chosen to converge to Q as $i \to \infty$. Thus $\liminf \|(B_i, g_i)\|_{k+\alpha,r} \ge \|(B, g)\|_{k+\alpha,r}$. □

A few comments about our definition of $\|(M, g)\|_{k+\alpha,r}$ are in order at this point. From a geometric point of view, it would probably have been nicer to assume that our charts $\varphi^s : B(p, r) \subset M \to \Omega \subset \mathbb{R}^n$, were defined on metric balls in M rather than on $\mathbb{R}^n$. The reason for not doing so is technical, but still worth pondering. With revised charts, $\|(M, g)\|_{k+\alpha,r}$ would definitely be ∞ for large r, unless (M, g) were euclidean space; even worse, our proof of (iii) would break down, as the charts $\varphi_i \circ f_i$ and $\varphi \circ f_i^{-1}$ would not be defined on metric r-balls but only on slightly smaller balls. We could partially avert these problems by maximizing r for fixed Q, rather than minimizing Q for fixed r, and thereby define a $(Q,\, k + \alpha)$ radius $r_{k+\alpha,Q}(M)$. This is exactly what was done in Anderson's work on convergence, and it follows more closely the standard terminology in riemannian geometry. But even with this definition, we still cannot conclude that this radius concept varies continuously in the $C^{k+\alpha}$ topology for fixed Q. All we get is that $r_{k+\alpha,Q_i}(M_i, g_i) \to r_{k+\alpha,Q}(M, g)$ for some sequence $Q_i \downarrow Q$, where Q is chosen in advance. Our concept of norm, therefore, seems to behave in a nicer manner from an analytic point of view. However, when it really matters, the two viewpoints are more or less equivalent. We also think that the norm concept is more natural, since it is a quantitative version of the qualitative coordinate definition of a manifold with a riemannian structure.

Another issue is exactly how to define the norm for noncompact manifolds. The most natural thing to do would be to consider pointed spaces (M, g, p), choose a nondecreasing weight function $f(R)$, and assume $\|B(p, R))\|_{k+\alpha,r} \le f(R)$. Our whole theory can easily be worked out in this context, but for the sake of brevity, we won't do this.

For noncompact spaces one can also use the norm concept to define what it should mean for a space to be asymptotically locally euclidean of order $k + \alpha$—namely, that it should satisfy $\|(M - B(p, R),\, g)\|_{k+\alpha,r} \to 0$ as $R \to \infty$. Here one can also refine this and assume that r should, in some way, depend on R.

Our next theorem is, in a way, as basic as our proposition, and in fact it uses none of our basic properties. The proof, however, is a bit of a mouthful. Nevertheless, the reader is urged to go through it carefully, because many of the subsequent corollaries are only true corollaries in the context of some of the constructions in the proof.

THEOREM 2.2 (FUNDAMENTAL THEOREM OF CONVERGENCE THEORY). *For given $Q > 0$, $n \geq 2$ integer, $k+\alpha > 0$, and $r > 0$, consider the class $\mathcal{H}^{k+\alpha}(n, Q, r)$ of complete, pointed riemannian n-manifolds (M, g, p) with $\|(M, g)\|_{k+\alpha,r} \leq Q$. Then $\mathcal{H}^{k+\alpha}(n, Q, r)$ is compact in the pointed $C^{k+\beta}$-topology for all $k+\beta < k+\alpha$.*

PROOF. We proceed in stages. First we make some general comments about the charts we use. We then show that $\mathcal{H} = \mathcal{H}^{k+\alpha}(n, Q, r)$ is precompact in the pointed Gromov–Hausdorff topology. Next we prove that $\mathcal{H}$ is compact in the Gromov–Hausdorff topology. Finally we consider the statement of the theorem.

Setup: First fix $K > Q$. Whenever we select an $M \in \mathcal{H}$ we shall assume that it comes equipped with an atlas of charts satisfying (n1)–(n4) with K in place of Q. Thus we implicitly assume that all charts under consideration belong to these atlases. We will, in consequence, only prove that limit spaces (M, g, p) satisfy $\|(M, g)\|_{k+\alpha,r} \leq K$, but since K was arbitrary we still get $(M, g, p) \in \mathcal{H}$.

We proceed by establishing several simple facts.

FACT 1. *Every chart $\varphi : B(0, r) \to U \subset M \in \mathcal{H}$ satisfies*

(a) $d(\varphi(x_1), \varphi(x_2)) \leq e^K |x_1 - x_2|$ *and*
(b) $d(\varphi(x_1), \varphi(x_2)) \geq \min\{e^{-K}|x_1 - x_2|, e^{-K}(2r - |x_1| - |x_2|)\}$,

where d is distance measured in M, and $|\cdot|$ is the usual euclidean norm. The condition $|d\varphi| \leq e^K$ together with convexity of $B(0, r)$ immediately implies (a). For (b), first observe that if any segment from x_1 to x_2 lies in U, then $|d\varphi| \geq e^{-K}$ implies that $d(\varphi(x_1), \varphi(x_2)) \geq e^{-K}|x_1 - x_2|$. So we may assume that $\varphi(x_1)$ and $\varphi(x_2)$ are joined by a segment $\sigma : [0, 1] \to M$ that leaves U. Split σ into $\sigma : [0, t_1) \to U$, and $\sigma : (t_2, 1) \to U$ such that $\sigma(t_i) \notin U$. Then we clearly have

$$\begin{aligned} d(\varphi(x_1), \varphi(x_2)) &= L(\sigma) \geq L(\sigma|_{[0,t_1)}) + L(\sigma|_{(t_2,1]}) \\ &\geq e^{-K}\big(L(\varphi^{-1} \circ \sigma|_{[0,t_1)}) + L(\varphi^{-1} \circ \sigma|_{(t_2,1]})\big) \\ &\geq e^{-K}(2r - |x_1| - |x_2|). \end{aligned}$$

The last inequality follows from the fact that $\varphi^{-1}\sigma(0) = x_1, \varphi^{-1} \circ \sigma(1) = x_2$ and that $\varphi^{-1} \circ \sigma(t)$ approaches the boundary of $B(0, r)$ as $t \nearrow t_1$, or $t \searrow t_2$.

FACT 2. *Every chart $\varphi : B(0, r) \to U \subset M \in \mathcal{H}$, and hence any δ-ball in M, where $\delta = \frac{1}{10}e^{-K}r$, can be covered by at most $N(\delta/4)$-balls, where N depends only on n, K, r.* Clearly there exists an $N(n, K, r)$ such that $B(0, r)$ can be covered by at most Ne^{-K} $(\delta/4)$-balls. Since $\varphi : B(0, r) \to U$ is a Lipschitz map with Lipschitz constant $\leq e^K$, we get the desired covering property.

FACT 3. *Every ball $B(x, l\delta/2) \subset M$ can be covered by $\leq N^l$ $\delta/4$-balls.* For $l = 1$ we just proved this. Suppose we know that $B(x, l\delta/2)$ is covered by $B(x_1, \delta/4)$, $\dots$, $B(x_{N^l}, \delta/4)$. Then $B(x,\, l\delta/2 + \delta/2) \subset \bigcup_{i=1}^{N} B(x_i, \delta)$. Now each $B(x_i, \delta)$

can be covered by at most $N\delta/4$-balls; hence $B(x,(l+1)\delta/2)$ can be covered by $\leq N\,N^l = N^{l+1}$ $\delta/4$-balls.

FACT 4. *$\mathcal{H}$ is precompact in the pointed Gromov–Hausdorff topology.* This is equivalent to asserting that, for each $R > 0$, the family of metric balls $B(p,R) \subset (M,g,p) \in \mathcal{H}$ is precompact in the Gromov–Hausdorff topology. This claim is equivalent to showing that we can find a function $N(\varepsilon) = N(\varepsilon, R, K, r, n)$ such that each $B(p,R)$ can contain at most $N(\varepsilon)$ disjoint ε-balls. To check this, let $B(x_1,\varepsilon),\ldots,B(x_s,\varepsilon)$ be a collection of disjoint balls in $B(p,R)$. Suppose that $l\delta/2 < R \leq (l+1)\delta/2$; then the volume of $B(p,R)$ is at most $N^{(l+1)}$ times the maximal volume of $(\delta/4)$-ball, therefore at most $N^{(l+1)}$ times the maximal volume of a chart, therefore at most $N^{(l+1)}e^{nK}\operatorname{vol}B(0,r) \leq F(R) = F(R,n,K,r)$. Conversely, each $B(x_i,\varepsilon)$ lies in some chart $\varphi : B(0,r) \to U \subset M$ whose preimage in $B(0,r)$ contains an $(e^{-K}\varepsilon)$-ball. Thus $\operatorname{vol}B(p_i,\varepsilon) \geq e^{-2nK}\operatorname{vol}B(0,\varepsilon)$. All in all we get $F(R) \geq \operatorname{vol}B(p,R) \geq \sum \operatorname{vol}B(p_i,\varepsilon) \geq s\,e^{-2nK}\operatorname{vol}B(0,\varepsilon)$. Thus $s \leq N(\varepsilon) = F(R)e^{2nK}(\operatorname{vol}B(0,\varepsilon))^{-1}$.

Now select a sequence (M_i,g_i,p_i) in $\mathcal{H}$. From the previous considerations we can assume that $(M_i,g_i,p_i) \to (X,d,p)$ converge to some metric space in the Gromov–Hausdorff topology. It will be necessary in many places to pass to subsequences of (M_i,g_i,p_i) using various diagonal processes. Whenever this happens, we shall not reindex the family, but merely assume that the sequence was chosen to have the desired properties from the beginning. For each (M_i,g_i,p_i) choose charts $\varphi_{is} : B(0,r) \to U_{is} \subset M_i$ satisfying (n1)–(n4). We can furthermore assume that the index set $\{s\} = \{1,2,3,4,\cdots\}$ is the same for all M_i, that $p_i \in U_{i1}$, and that the balls $B\,(p_i,\,(l/2)\delta)$ are covered by the first N^l charts. Note that these N^l charts will then be contained in $D\big(p_i,\,(l/2)\delta + \lfloor e^K+1\rfloor\delta\big)$. Finally, for each l, the sequence $D\,(p_i,\,(l/2)\delta) = \{x_i : d(x_i,p_i) \leq (l/2)\delta\}$ converges to $D\,(p,\,(l/2)\delta) \subset X$, so we can choose a metric on the disjoint union $Y_l = (D\,(p,\,(l/2)\delta) \amalg \coprod_{i=1}^{\infty} D\,(p_i,\,(l/2)\delta))$ such that $p_i \to p$ and $D\,(p_i,(l/2)\delta) \to D\,(p,\,(l/2)\delta)$ in the Hausdorff distance inside this metric space.

FACT 5. *(X,d,p) is a riemannian manifold of class $C^{k+1+\alpha}$ with norm $\leq K$.* Obviously we need to find bijections $\varphi_s : B(0,r) \to U_s \subset X$ satisfying (n1)–(n4). For each s consider the maps $\varphi_{is} : B(0,r) \to U_{is} \subset Y_{l+2\lfloor e^K+1\rfloor}$. Inequality (a) of Fact 1 implies that this is a family of equicontinuous maps into the compact space $Y_{l+2[e^K+1]}$. The Arzela–Ascoli Theorem shows that this sequence must subconverge (in C^0-topology) to a map $\varphi_s : B(0,r) \subset Y_{l+2\lfloor e^K+1\rfloor}$ which also has Lipschitz constant e^K. Furthermore, inequality (b) will also hold for this map as it holds for all the φ_{is} maps. In particular, φ_s is one-to-one. Finally, since $U_{is} \subset D\left(p_i,\,(l/2)\delta + \lfloor e^K+1\rfloor\right)$ and $D\left(p_i,\,(l/2)\delta + \lfloor e^K+1\rfloor\right)$ Hausdorff converges to $D\left(p,\,(l/2)\delta + \lfloor e^K+1\rfloor\right) \subset X$, we see that $\varphi_x(B(0,r)) = U_s \subset X$. A simple diagonal argument says that we can pass to a subsequence of (M_i,g_i,p_i) having the property that $\varphi_{is} \to \varphi_s$ for all s.

In this way, we have constructed (topological) charts $\varphi_s : B(0,r) \to U \subset X$, and we can easily check that they satisfy (n1). Since φ_s also satisfy inequalities (a) and (b), they would also satisfy (n2) if they were differentiable (this being equivalent to the transition functions being C^1). Now the transition functions $\varphi_{is}^{-1} \circ \varphi_{it}$ converge to $\varphi_s^{-1} \circ \varphi_t$, because the φ_{is} converge to φ_s. Note that these transition functions are not defined on the same domains; but we do know that the domain for $\varphi_s^{-1} \circ \varphi_t$ is the limit of the domains for $\varphi_{is}^{-1} \circ \varphi_{it}$, so the convergence makes sense on all compact subsets of the domain of $\varphi_s^{-1} \circ \varphi_t$. Now $\|\varphi_{is}^{-1} \circ \varphi_{it}\|_{C^{k+1+\alpha}} \leq K$, so a further application of Arzela–Ascoli, followed by passage to subsequences, tells us that $\|\varphi_s^{-1} \circ \varphi_t\|_{C^{k+1+\alpha}} \leq K$, and that we can assume that the $\varphi_{is}^{-1} \circ \varphi_{it}$ converge to $\varphi_s^{-1} \circ \varphi_t$ in the $C^{k+1+\beta}$-topology. This then establishes (n2) and (n4). We now construct a compatible riemannian metric on X that satisfies (n3). For each s, consider the metric $g_{is} = g_{is\cdot\cdot}$ written out in its components on $B(0,r)$ with respect to the chart φ_{is}. Since all of the $g_{is\cdot\cdot}$ satisfy (n3), we can again use Arzela–Ascoli to make these functions converge on $B(0,r)$ in the $C^{k+\beta}$-topology to functions $g_{s\cdot\cdot}$, which also satisfy (n3).

The local "tensors" $g_{s\cdot\cdot}$ thus obtained satisfy the change of variables formulae required to make them into global tensors on X. This is because all the $g_{is\cdot\cdot}$ satisfy these properties, and everything that needs to converge in order for these properties to be carried over to the limit also converges. Recall that the rephrasing of (n2) gives the necessary C^0 bounds, and also shows that $g_{s\cdot\cdot}$ is positive definite. We now have exhibited a riemannian structure on X such that $\varphi_s : B(0,r) \to U_s \subset X$ satisfy (n1)–(n4) with respect to this structure. This, however, does not guarantee that the metric generated by this structure is identical to the metric we got from X being the pointed Gromov–Hausdorff limit of (M_i, g_i, p_i). However, since Gromov–Hausdorff convergence implies that distances converge, and since we know at the same time that the riemannian metric converges locally in coordinates, it follows that the limit riemannian structure must generate the "correct" metric, at least locally, and therefore also globally.

FACT 6. *$(M_i, g_i, p_i) \to (X, d, p) = (X, g, p)$ in the pointed $C^{k+\beta}$-topology.* We assume the setup is as in Fact 5, where charts φ_{is}, transitions $\varphi_{is}^{-1} \circ \varphi_{it}$, and metrics $g_{is\cdot\cdot}$, converge to the same items in the limit space. Let's agree that two maps f, g between subsets in M_i and X are $C^{k+1+\beta}$ close if all the coordinate compositions $\varphi_s^{-1} \circ g \circ \varphi_{it}$ and $\varphi_s^{-1} \circ f \circ \varphi_{it}$ are $C^{k+1+\beta}$ close. Thus we have a well defined $C^{k+1+\beta}$ topology on maps from M_i to X. Our first observation is that $f_{is} = \varphi_{is} \circ \varphi_s^{-1} : U_s \to U_{is}$ and $f_{it} = \varphi_{it} \circ \varphi_t^{-1} : U_t \to U_{it}$ "converge to each other" in the $C^{k+1+\beta}$ topology. Furthermore, $(f_{is})^* g_i|_{U_{is}}$ converges to $g|_{U_s}$ in the $C^{k+\beta}$ topology. These are just restatements of what we already know. In order to finish the proof, we therefore only need to construct diffeomorphisms $F_{il} : \Omega_l = \bigcup_{s=1}^{l} U_s \to \Omega_{il} = \bigcup_{s=1}^{l} U_{is}$, which are closer and closer to the f_{is}, for $s = 1, \ldots, l$ maps (and therefore all f_{is}) as $i \to \infty$. We will construct F_{il} by induction on l and large i depending on l. For this purpose we shall need a partition of unity $\{\lambda_s\}$ on X subordinate to $\{U_s\}$. We can find such a partition since the covering $\{U_s\}$ is locally finite by choice, and we can furthermore assume that λ_s is $C^{k+1+\alpha}$.

For $l = 1$ simply define $F_{i1} = f_{i1}$.

Suppose we have maps $F_{il} : \Omega_l \to \Omega_{il}$, for large i, that get arbitrarily close to f_{is}, for $s = 1, \dots, l$, as $i \to \infty$. If $U_{l+1} \cap \Omega_l = \varnothing$, we just define $F_{il+1} = F_{il}$ on Ω_{il} and $F_{il+1} = f_{il+1}$ on U_{l+1}. In case $U_{l+1} \subset \Omega_l$, we simply let $F_{il+1} = F_{il}$. Otherwise we know that F_{il} and f_{il+1} are as close as we like in the $C^{k+1+\beta}$-topology when $i \to \infty$. So the natural thing to do is to average them on U_{l+1}. Define F_{il+1} on U_{l+1} as

$$\begin{aligned} F_{il+1}(x) &= \varphi_{il+1} \circ \left(\sum_{s=l+1}^{\infty} \lambda_s(x)\, \varphi_{il+1}^{-1} \circ f_{il+1}(x) + \sum_{s=1}^{l} \lambda_s(x)\, \varphi_{il+1}^{-1} \circ F_{il}(x) \right) \\ &= \varphi_{il+1} \circ (\mu_1(x)\varphi_{l+1}^{-1}(x) + \mu_2(x)\varphi_{il+1}^{-1} \circ F_{il}(x)). \end{aligned}$$

This map is clearly well defined on U_{l+1}, since $\mu_2(x) = 0$ on $U_{l+1} - \Omega_l$; since $\mu_1(x) = 0$ on Ω_l, the map is a smooth ($C^{k+1+\alpha}$) extension of F_{il}. Now consider this map in coordinates:

$$\begin{aligned} \varphi_{il+1}^{-1} \circ F_{il+1} \circ \varphi_{l+1}(y) &= \mu_1 \circ \varphi_{l+1}(y)\, y + \mu_2 \circ \varphi_{l+1}(y)\, \varphi_{il+1}^{-1} \circ F_{il} \circ \varphi_{l+1}(y) \\ &= \tilde{\mu}_1(y) F_1(y) + \tilde{\mu}_2(y) F_2(y). \end{aligned}$$

Now

$$\begin{aligned} \|\tilde{\mu}_1 F_1 + \tilde{\mu}_2 F_2 - F_1\|_{k+1+\beta} &= \|\tilde{\mu}_1(F_1 - F_1) + \tilde{\mu}_2(F_2 - F_1)\|_{k+1+\beta} \\ &\le \|\tilde{\mu}_2\|_{k+1+\beta}\, \|F_2 - F_1\|_{k+1+\beta}. \end{aligned}$$

This inequality is valid on all of $B(0, r)$, despite the fact that F_2 is not defined on all of $B(0, r)$, because $\tilde{\mu}_1 F_1 + \tilde{\mu}_2 F_2 = F_1$ on the region where F_2 is undefined. By assumption, $\|F_2 - F_1\|_{k+1+\beta}$ converges to 0 as $i \to \infty$, so F_{il+1} is $C^{k+1+\beta}$-close to f_{is}, for $s = 1, \dots, l+1$, as $i \to \infty$. It remains to be seen that F_{il+1} is a diffeomorphism. But we know that $\tilde{\mu}_1 F_1 + \tilde{\mu}_2 F_2$ is an embedding since F_1 is and the space of embeddings is open in the C^1-topology and $k + 1 + \beta > 1$. Also the map is one-to-one since the images $f_{il+1}/(U_{l+1} - \Omega_l)$ and $F_{il}(\Omega_l)$ don't intersect. □

Corollary 2.3. *The subclasses* $\mathcal{H}(D) \subset \mathcal{H} = \mathcal{H}^{k+\alpha}(n, Q, r)$ *and* $\mathcal{H}(V) \subset \mathcal{H}$, *where the elements in addition satisfy* diam $\le D$ *and* vol $\le V$, *respectively, are compact in the* $C^{k+\beta}$*-topology. In particular,* $\mathcal{H}(D)$ *and* $\mathcal{H}(V)$ *contain only finitely many diffeomorphism types.*

Proof. Use notation as in the Fundamental Theorem. If $\operatorname{diam}(M, g, p) \le D$, then clearly $M \subset B\left(p, \frac{1}{2}k\delta\right)$ for $k > (2/\delta)D$. Hence each element in $\mathcal{H}(D)$ can be covered by $\le N^k$ charts. Thus $C^{k+\beta}$-convergence is actually in the unpointed topology, as desired.

If instead $\operatorname{vol} M \le V$, we can use Fact 4 in the proof to see that we can never have more than $k = Ve^{2nk}(\operatorname{vol} B(0,\varepsilon))^{-1}$ disjoint ε-balls. In particular, $\operatorname{diam} \le 2\varepsilon k$, and we can use the above argument.

Clearly compactness in any $C^{k+\beta}$-topology implies that the class cannot contain infinitely many diffeomorphism types. □

This corollary is essentially contained in [Cheeger 1970]. Technically speaking, our proof comes pretty close to Cheeger's original proof, but there are some differences: notably, the use of Gromov–Hausdorff convergence which was not available to Cheeger at the time. Another detail is that our proof centers on the convergence of the charts themselves rather than the transition functions. This makes it possible for us to start out with the apparently weaker assumption that the covering $\{U_s\}$ has δ as a Lebesgue number, rather than assuming that M can be covered by chart images of euclidean $(r/2)$-balls that lie in transition domains $U_s \cap U_t$.

COROLLARY 2.4. *The norm* $\|(M,g)\|_{k+\alpha,r}$ *is always realized by some charts* $\varphi_s : B(0,r) \to U_s$ *satisfying* (n1)–(n4) *with* $\|(M,g)\|_{k+\alpha,r}$ *in place of* Q.

PROOF. Choose appropriate charts $\varphi_s^Q : B(0,r) \to U_s^Q \subset M$ for each $Q > \|(M,g)\|_{k+\alpha,r}$, and let $Q \to \|(M,g)\|_{k+\alpha,r}$. If the charts are chosen to conform with the proof of the fundamental theorem, we will obviously get some limit charts with the wanted properties. □

COROLLARY 2.5. *M is a flat manifold if* $\|(M,g)\|_{k+\alpha,r} = 0$ *for some* r, *and M is euclidean space with the canonical metric if* $\|(M,g)\|_{k+\alpha,r} = 0$ *for all* $r > 0$.

PROOF. Using the previous corollary, M can be covered by charts $\varphi_s : B(0,r) \to U_s \subset M$ satisfying $|d\varphi_s| \equiv 1$. This clearly makes M locally euclidean, and hence flat. If M is not euclidean space, the same reasoning clearly shows that $\|(M,g)\|_{k+\alpha,r} > 0$ for $r > \operatorname{inj\,rad}(M,g)$. □

Finally we should mention that all properties of this norm concept would not change if we changed (n1)–(n4) to, say,

(n1′) U_s has Lebesgue number $f_1(n,Q,r)$,
(n2′) $(f_2(n,Q))^{-1} \le |d\varphi_s| \le f_2(n,Q)$,
(n3′) $r^{|j|+\alpha}\|\partial^j g_{s\cdot\cdot}\|_\alpha \le f_3(n,Q)$ for $0 \le |j| \le k$,
(n4′) $\|\varphi_s^{-1} \circ \varphi_t\|_{k+\alpha} \le f_4(n,Q,r)$,

for continuous functions $f_1,\dots,f_4$ with $f_1(n,0,r) = 0$ and $f_2(n,0) = 1$. The key properties we want to preserve are the continuity of $\|(M,g)\|_{k+\alpha,r}$ with respect to r, the Fundamental Theorem, and the characterization of flat manifolds and euclidean space.

We should also mention that we could develop an L^p-norm concept, but we shall delay this till Section 4, in order to do it in the context of harmonic coordinates.

Another interesting thing happens if in the definition of $\|(M,g)\|_{k+\alpha,r}$ we let $k=\alpha=0$. Then (n3) no longer makes sense, because $\alpha=0$, but aside from that we still have a C^0-norm concept. The class $\mathcal{H}^0(n,Q,r)$ is now only compact in the pointed Gromov–Hausdorff topology, but the characterization of flat manifolds is still valid. The subclasses $\mathcal{H}(D)$ and $\mathcal{H}(V)$ are also only compact with respect to the Gromov–Hausdorff topology, and the finiteness of diffeomorphism types apparently fails.

It is, however, possible to say more. If we investigate the proof of the Fundamental Theorem we see that the problem lies in constructing the maps $F_{ik}:\Omega_k\to\Omega_{ik}$, because we now only have convergence of the coordinates in the C^0 (or C^α, for $\alpha<1$) topology, so the averaging process fails as it is described. We can, however, use a deep theorem from topology about local contractibility of homeomorphism groups [Edwards and Kirby 1971] to conclude that two C^0-close topological embeddings can be "glued" together in some way without altering them too much in the C^0-topology. This makes it possible to exhibit topological embeddings $F_{ik}:\Omega\hookrightarrow M_i$ such that the pullback metrics (not riemannian metrics) converge. As a consequence, we see that $\mathcal{H}(D)$ and $\mathcal{H}(V)$ contain only finitely many homeomorphism types. This is exactly the content of the original version of Cheeger's Finiteness Theorem, including the proof as we have outlined it. But, as we have pointed out earlier, he also considered the easier to prove finiteness theorem for diffeomorphism types given better bounds on the coordinates.

3. The Cheeger–Gromov Compactness Results

The focus of this section is the relationship between volume, injectivity radius, sectional curvature and the norm concept from Section 2.

First let's see what exponential coordinates can do for us. If (M,g) is a riemannian manifold with $|\sec M|\le K$ and $\operatorname{inj rad} M\ge i_0$, we know from the Rauch comparison theorems that there is a continuous function $f(r,i_0,K)$, for $r<i_0$, such that $f(0,i_0,K)=1$, $f(r,i_0,0)=1$, and $\exp_p: B(0,r)\subset T_pM\to B(p,r)\subset M$ satisfies $(f(r,i_0,K))^{-1}\le|d\exp_p|\le f(r,i_0,K)$. In particular we have:

THEOREM 3.1. *For every $Q>0$ there exists $r>0$ depending only on i_0,K such that any complete (M,g) with $|\sec M|\le K$ and $\operatorname{inj rad} M\ge i_0$ satisfies $\|(M,g)\|_{0,r}\le Q$. Furthermore, if (M_i,g_i,p_i) satisfy $\operatorname{inj rad} M_i\ge i_0$ and $|\sec M_i|\le K_i\to 0$, then a subsequence will converge in the pointed Gromov–Hausdorff topology to a flat manifold with $\operatorname{inj rad}\ge i_0$.*

The proof follows immediately from our constructions in Section 2.

This theorem does not seem very satisfactory, because even though we have assumed a C^2 bound on the riemannian metric, locally we only get a C^0 bound.

To get better bounds under the same circumstances, we must look for different coordinates. Our first choice for alternate coordinates is distance coordinates.

LEMMA 3.2. *Given a riemannian manifold* (M, g) *with* $\operatorname{inj\,rad} \geq i_0$ *and* $|\sec| \leq K$, *and given* $p \in M$, *the Hessians* $H(v) = \operatorname{Hess} r(x)(v) = D_v \operatorname{grad} r$ *of the distance function* $r(x) = d(x, p)$ *has* $\operatorname{grad} r$ *as an eigenvector with eigenvalue* 0, *all other eigenvalues lie in the interval*

$$\left[\sqrt{K}\cot(r(x)\sqrt{K}),\ \sqrt{K}\coth(r(x)\sqrt{K})\right].$$

PROOF. We know that the operator H satisfies the Riccati equation $\partial H/\partial r + H^2 = -R$, where $R(v) = R(v, \partial/\partial r)(\partial/\partial r) = R(v, \operatorname{grad} r)\operatorname{grad} r$. The eigenvalues for $R(v)$ are assumed to lie in the interval $[-K, K]$, so elementary differential equation theory implies our estimate. □

Now fix $(M, g), p \in M$ as in the lemma, and choose an orthonormal basis $e_1, \dots, e_n$ for T_pM. Then consider the geodesics $\gamma_i(t)$ with $\gamma_i(0) = p$, $\dot\gamma_i(0) = e_i$ and, together with them, the distance functions $r_i(x) = d\big(x, \gamma_i(i_0/(4\sqrt{K}))\big)$. These distance functions will then have uniformly bounded Hessians on $B(p, \delta)$, for $\delta = i_0/(8\sqrt{K})$. Set $\varphi_p(x) = (r_1(x), \dots, r_n(x))$ and $g_{pij} = \langle \partial/\partial r_i, \partial/\partial r_j\rangle$. Note that the inverse of g_{pij} is $g_p^{ij} = \langle \operatorname{grad} r_i, \operatorname{grad} r_j\rangle$.

THEOREM 3.3. *Given* i_0 *and* $K > 0$ *there exist* $Q, r > 0$ *such that any* (M, g) *with* $\operatorname{inj\,rad} \geq i_0$ *and* $|\sec| \leq K$ *satisfies* $\|(M, g)\|_{0+1,r} \leq Q$.

PROOF. The inverses of the φ_p are our potential charts. First observe that $g_{pij}(p) = \delta_{ij}$, so the uniform Hessian estimate shows that $|d\varphi_p| \leq e^Q$ and $|d\varphi_p^{-1}| \leq e^Q$ on $B(p, \varepsilon)$, where Q, ε depend only on i_0, K. The proof of the inverse function theorem then tells us that there is $\hat\varepsilon > 0$ depending only on Q, n such that $\varphi_p : B(0, \hat\varepsilon) \to \mathbb{R}^n$ is one-to-one. We can then easily find r such that $\varphi_p^{-1} : B(0, r) \to U_p \subset B(p, \varepsilon)$ satisfies (n2). The conditions (n3), (n4) now immediately follow from the Hessian estimates, except we might have to increase Q somewhat. Finally (n1) holds since we have coordinates centered at every $p \in M$. □

Notice that Q cannot be chosen arbitrarily small, because our Hessian estimates cannot be improved by going to smaller balls. This will be taken care of in the next section, by using even better coordinates. Theorem 3 was first proved in [Gromov 1981b] as stated. The reader should be aware that what Gromov refers to as a $C^{1,1}$-manifold is in our terminology a manifold with $\|(M, h)\|_{0+1,r} < \infty$, i.e., C^{0+1}-bounds on the riemannian metric.

Without going much deeper into the theory we can easily prove:

LEMMA 3.4 (CHEEGER'S LEMMA). *Given a compact* n*-manifold* (M, g) *with* $|\sec| \leq K$ *and* $\operatorname{vol} B(p, 1) \geq v > 0$ *for all* $p \in M$, *we have* $\operatorname{inj\,rad} M \geq i_0$, *where* i_0 *depends only on* n, K *and* v.

Proof. The proof goes by contradiction, using Theorem 3.3. Assume we have (M_i, g_i) with $\operatorname{inj\,rad} M_i \to 0$ and satisfying the assumptions of the lemma. Find $p_i \in M_i$ such that $\operatorname{inj\,rad}_{p_i} = \operatorname{inj\,rad} M_i$ and consider the pointed sequence (M_i, h_i, p_i), where $h_i = (\operatorname{inj\,rad} M_i)^{-2} g_i$ is rescaled so that $\operatorname{inj\,rad}(M_i, h_i) = 1$ and $|\sec(M_i, h_i)| \leq \operatorname{inj\,rad}(M_i, g_i)\, K = K_i \to 0$. Theorem 3.3, together with the Fundamental Theorem 2.2, then implies that some subsequence (M_i, h_i, p_i) will converge in the pointed $C^{0+\alpha}$ topology ($\alpha < 1$) to a flat manifold (M, g, p).

The first observation about (M, g, p) is that $\operatorname{inj\,rad}(p) \leq 1$. This follows because the conjugate radius for (M_i, h_i) is at least $\pi/\sqrt{K_i}$, which goes to infinity, so Klingenberg's estimate for the injectivity radius implies that there must be a geodesic loop of length 2 at $p_i \in M_i$. Since (M_i, h_i, p_i) converges to (M, g, p) in the pointed $C^{0+\alpha}$-topology, the geodesic loops must converge to a geodesic loop in M based at p, having length 2. Hence $\operatorname{inj\,rad}(M) \leq 1$.

The other contradictory observation is that (M, g) is $\mathbb{R}^n$ with the canonical metric. Recall that $\operatorname{vol} B(p_i, 1) \geq v$ in (M_i, g_i), so relative volume comparison shows that there is a $v'(n, K, v)$ such that $\operatorname{vol} B(p_i, r) \geq v' r^n$, for $r \leq 1$. The rescaled manifold (M_i, h_i) therefore satisfies $\operatorname{vol} B(p_i, r) \geq v' r^n$, for $r \leq (\operatorname{inj\,rad}(M_i, g_i))^{-1}$. Using again the convergence of (M_i, h_i, p_i) to (M, g, p) in the pointed $C^{0+\alpha}$-topology, we get $\operatorname{vol} B(p, r) \geq v' r^n$ for all r. Since (M, g) is flat, this shows that it must be euclidean space. □

This lemma was proved by a more direct method in [Cheeger 1970], but we have included this perhaps more convoluted proof in order to show how our convergence theory can be used. The lemma also shows that Theorem 3.3 remains true if the injectivity radius bound is replaced by a lower bound on the volume of balls of radius 1.

Theorem 3.3 can be generalized in another interesting direction.

Theorem 3.5. *Given* $i_0, k > 0$ *there exist* Q, r *depending on* i_0, k *such that any manifold* (M, g) *with* $\operatorname{inj\,rad} \geq i_0$ *and* $\sec \geq -k^2$ *satisfies* $\|(M, g)\|_{0+1, r} \leq Q$.

Proof. It suffices to get some Hessian estimate for distance functions $r(x) = d(x, p)$. As before, $\operatorname{Hess} r(x)$ has eigenvalues $\leq k \coth(k r(x))$. Conversely, if $r(x_0) < i_0$, then $r(x)$ is supported from below by $f(x) = i_0 - d(x, y_0)$, where $y_0 = \gamma(i_0)$, and γ is the unique unit speed geodesic which minimizes the distance from p to x_0. Thus $\operatorname{Hess} r(x) \geq \operatorname{Hess} f(x)$ at x_0. But $\operatorname{Hess} f(x)$ has eigenvalues at least $-k \coth(d(x_0, y_0) k) = -k \coth(k(i_0 - r(x_0)))$ at x_0. Hence we have two-sided bounds for $\operatorname{Hess} r(x)$ on appropriate sets. The proof can then be finished as for Theorem 3.3. □

Interestingly enough, this theorem is optimal in two ways. Consider rotationally symmetric metrics $dr^2 + f_\varepsilon^2(r)\, d\theta^2$, where f_ε is concave and satisfies $f_\varepsilon(r) = r$ for $0 \leq r \leq 1 - \varepsilon$ and $f_\varepsilon(r) = \frac{3}{4} r$, with $1 + \varepsilon \leq r$. These metrics have $\sec \geq 0$ and $\operatorname{inj\,rad} = \infty$. As $\varepsilon \to 0$ we get a C^{1+1} manifold with a C^{0+1} riemannian metric (M, g). In particular, $\|(M, g)\|_{0+1, r} < \infty$ for all r. Limit spaces of sequences with $\operatorname{inj\,rad} \geq i_0$ and $\sec \geq k$ can therefore not in general be assumed to be smoother than the above example.

With a more careful construction, we can also find g_ε with $g_\varepsilon(r) = \sin(r)$ for $0 \leq r \leq \pi/2 - \varepsilon$ and $g_\varepsilon(r) = 1$ for $r \geq 1 + \varepsilon$, having the property that the metric $dr^2 + g_\varepsilon^2(r)\, d\theta^2$ satisfies $|\sec| \leq 4$ and $\operatorname{inj\,rad} \geq \frac{1}{4}$. As $\varepsilon \to 0$ we get a limit metric of class C^{1+1}. So while we may suspect (this is still unknown) that limit metrics from Theorem 3.3 are C^{1+1}, we only prove that $\|(M,g)\|_{1+\alpha,r} \leq f(r)$ with $f(r) \to 0$ as $r \to 0$, where $f(r)$ depends only on $n = \dim M$, i_0 ($\leq \operatorname{inj\,rad} M$), and K ($\geq |\sec M|$); see Theorems 4.1 and 4.2.

4. The Best Possible of All Possible . . .

We are now ready to introduce and use harmonic coordinates in our convergence theory. We will use Einstein summation convention whenever convenient.

The Laplace–Beltrami operator Δ on (M,g) is defined as $\Delta = \operatorname{trace}(\operatorname{Hess})$. In coordinates one can compute

$$\begin{aligned}\Delta &= \frac{1}{g}\sum \frac{\partial}{\partial x^i}\Big(g\, g^{ij}\frac{\partial}{\partial x^j}\Big) = \sum g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \sum \frac{1}{g}\frac{\partial(g\, g^{ij})}{\partial x^i}\frac{\partial}{\partial x^j}\\ &= \frac{1}{g}\partial_i(g\, g^{ij}\partial_j) = g^{ij}\partial_i\partial_j + \frac{1}{g}\partial_i(g\, g^{ij})\partial_j,\end{aligned}$$

where g_{ij} are the metric components, (g^{ij}) is the inverse of (g_{ij}), and $g = \sqrt{\det g_{ij}}$. A function u is said to be harmonic (on (M,g)) if $\Delta u = 0$. Notice that if we scale g to $\lambda^2 g$ then Δ changes to $\lambda^{-2}\Delta$; hence the concept of harmonicity doesn't change. A coordinate system $\varphi : U \to \mathbb{R}^n$ is said to be harmonic if each of the coordinate functions are harmonic. In harmonic coordinates, the Laplace operator has the form $\Delta = g^{ij}\partial_i\partial_j$, since

$$0 = \Delta x^k = g^{ij}\partial_i\partial_j x^k + \frac{1}{g}\partial_i(g\, g^{ij})\partial_j x^k = 0 + \frac{1}{g}\partial_i(g\, g^{ij}\delta_j^k) = \frac{1}{g}\partial_i(g\, g^{ij}).$$

The nicest possible coordinates that one could ask for would be linear coordinates ($\operatorname{Hess} \equiv 0$). But such coordinates clearly only exist on flat manifolds. In contrast, any riemannian manifold admits harmonic coordinates around every point. It turns out that these coordinates, while harder to construct, have much nicer properties than both exponential and distance coordinates. First of all we have the important equation $\Delta g_{ij} + Q(g_{ij}, \partial g_{ij}) = \operatorname{Ric}_{ij}$ for the Ricci tensor in harmonic coordinates, where Q is a term quadratic in ∂g_{ij}. We shall often write this equation in symbolic form as $\Delta g + Q(g, \partial g) = \operatorname{Ric}$, and merely imagine the appropriate indices.

This equation was used in [DeTurck and Kazdan 1981] to prove that the metric always has maximal regularity in harmonic coordinates (if it is not C^∞). And in a very important paper [Jost and Karcher 1982], Theorem 3.3 was improved as follows:

Theorem 4.1. *Given (M,g) as in Theorem 3.3, then for every $\alpha < 1$ and $Q > 0$ there exists r depending on i_0, K, n, α, Q such that $\|(M,g)\|_{1+\alpha,r} \le Q$.*

The coordinates used to give this improved bound were harmonic coordinates. We should point out their original theorem has been rephrased into our terminology. With these improved coordinates, the following result was then proved in [Peters 1987; Greene and Wu 1988]:

Theorem 4.2. *The class of riemannian n-manifolds with $|\sec| \le K$, $\operatorname{diam} \le D$, $\operatorname{vol} \ge v$, for fixed but arbitrary $K, D, v > 0$, is precompact in the $C^{1,\alpha}$-topology for any $\alpha < 1$, and contains only finitely many diffeomorphism types.*

This is, of course, a direct consequence of Corollary 2.3, Lemma 3.4, and Theorem 4.1.

It is the purpose of this section to set up the theory of harmonic coordinates so that we can prove Anderson's generalization of these two results. For this purpose, let us define for a riemannian manifold (M,g) the (harmonic) $L^{p,k}$ norm at the scale of r, denoted $\|(M,g)\|_{p,k,r}$, as the infimum of all $Q \ge 0$ such that there are charts $\varphi_s : B(0,r) \to U_s \subset M$ satisfying:

(h1) $\delta = \frac{1}{10} e^{-Q} r$ is a Lebesgue number for the covering $\{U_s\}$.
(h2) $|d\varphi_s^{-1}| \le e^Q$ and $|d\varphi_s| \le e^Q$ on $B(0,r)$.
(h3) $r^{|j|-n/p} \|\partial^j g_{s\cdot\cdot}\|_p \le Q$ for $1 \le |j| \le k$.
(h4) $\varphi_s^{-1} : U_s \to B(0,r)$ are harmonic coordinates.

Here $\|f\|_p = (\int f^p)^{1/p}$ in the usual L^p-norm on euclidean space.

Several comments are in order. All riemannian manifolds in this section are C^∞, so our harmonic coordinates will automatically also be C^∞. Thus we are only concerned with how the metric is bounded, and not with how smooth it might be. When $k = 0$ condition (h3) is obviously vacuous, so we will always assume $k \ge 1$; for analytical reasons we will in addition assume that $p > n$ if $k = 1$ and $p > n/2$ if $k \ge 2$. The transition condition (n4) has been replaced with the somewhat different harmonicity condition. To recapture (n4), we need to use the L^p elliptic estimates from Appendix A together with Appendix B. In harmonic coordinates we have $\Delta = g^{ij}\partial_i\partial_j$, and (h2), (h3) imply that the eigenvalues for g^{ij} are in $[e^{-Q}, e^Q]$ and that $\|g^{ij}\|_{p,k} \le \tilde{Q}$, where $\tilde{Q}$ depends on n, Q. Thus the L^p estimates say that $\|u\|_{p,k+1} \le C(\|\Delta u\|_{p,k-1} + \|u\|_{L^p})$. In particular we get $L^{p,k+1}$ bounds on transition functions on compact subsets of domain of definition, since they satisfy $\Delta = 0$.

In the case where $k - n/p > 0$ is not an integer, we have a continuous embedding $L^{p,k} \subset C^{k-n/p}$; we can therefore bound $\|(M,g)\|_{k-n/p,\tilde{r}}$, for $\tilde{r} < r$, in terms of $\|(M,g)\|_{p,k,r}$. This, together with the Fundamental Theorem 2.2, immediately implies:

Theorem 4.3. *Let Q, $r > 0$, $p > 1$ and $k \in \mathbb{N}$ be given. The class $\mathcal{H}^{p,k}(n,Q,r)$ of n-dimensional riemannian manifolds with $\|(M,g)\|_{p,k,r} \le Q$ is precompact in the pointed $C^{l+\alpha}$ topology for $l + \alpha < k - n/p$.*

The concept of $L^{p,k}$-convergence for riemannian manifolds is defined as for $C^{l+\alpha}$-convergence, using convergence in appropriate fixed coordinates for the pullback metrics.

Before discussing how to achieve $L^{p,k}$-convergence, we warm up with some elementary properties of our new norm concept. Note that if $A \subset M$, then $\|(A,g)\|_{p,k,r}$ is defined just like $\|(A,g)\|_{l+\alpha,r}$.

PROPOSITION 4.4. *Let (M,g) be a C^∞ riemannian manifold.*

(i) $\|(A,\lambda^2 g)\|_{p,k,\lambda r} = \|(A,g)\|_{p,k,r}$.
(ii) $\|(A,g)\|_{p,k,r}$ *is finite for some r, then it is finite for all r, and in this case it will be a continuous function of r that approaches 0 as $r \to 0$.*
(iii) *For any $D > 0$, we have*

$$\|(M,g)\|_{p,k,r} = \sup\{\|(A,g)\|_{p,k,r} : A \subset M, \operatorname{diam} A \le D\}.$$

(iv) *If the sequence (M_i, g_i, p_i) converges to (M,g,p) in the pointed $L^{p,k}$-topology, and all spaces are C^∞ riemannian manifolds, then for each bounded $B \subset M$ there are bounded sets $B_i \subset M_i$ such that*

$$\|(B_i, g_i)\|_{p,k,r} \to \|(B,g)\|_{p,k,r}.$$

PROOF. Parts (i) and (ii) are proved as before, except for the statement that $\|(M,g)\|_{p,k,r}$ converges to 0 as $r \to 0$. To see this, observe that we know that $\|(M,g)\|_{2k+\alpha,r} \to 0$ as $r \to 0$. Then approximate these coordinates by harmonic maps by solving Dirichlet boundary value problems. This will then give the right type of harmonic coordinates.

Property (iii) is unique to norms that come from harmonic coordinates. What fails in the general case is the condition (n4). Clearly $K = \sup\{\|(A,g)\|_{p,k,r} : A \subset M, \operatorname{diam} A \le D\} \le \|(M,g)\|_{p,k,r}$. Conversely, choose $Q > K$. If every set of diameter $\le D$ can be covered with coordinates satisfying (h1)–(h4), then M can obviously be covered by similar coordinates.

To prove (iv), suppose that (M_i, g_i, p_i) converges to (M,g,p) in the pointed $L^{k,p}$-topology and that B is a bounded subset of M. If $B \subset B(p,R) \subset \Omega$ and we have $F_i : \Omega \to \Omega_i \supset B(p_i,R)$ such that $F_i^* g_i \to g$ in the $L^{k,p}$-topology, we can just let $B_i = F_i(B)$.

Let's first prove the inequality $\limsup \|(B_i,g)\|_{p,k,r} \le \|(B,g)\|_{p,k,r}$. For this purpose, choose $Q > \|(B,g)\|_{p,k,r}$; and then using (i) (see also the proof part (i) of Proposition 2.1), choose $\varepsilon > 0$ such that $\|(B,g)\|_{p,k,(r+\varepsilon)} < Q$.

Then select a finite collection of charts $\varphi_s : B(0, r+\varepsilon) \to U_s \subset M$ that realizes this inequality. Define $U_{is} = F_i(\varphi_s(D(0, r+\varepsilon/2)))$; then U_{is} is a closed disk with boundary $\partial U_{is} = F_i(\varphi_s(S(0, r+\varepsilon/2)))$. On each U_{is}, solve the Dirichlet boundary value problem

$$\psi_{is} : U_{is} \to \mathbb{R}^n, \text{ with } \Delta_i \psi_{is} = 0 \text{ and } \psi_{is}|_{\partial U_{is}} = \varphi_s^{-1} \circ F_i^{-1}|_{\partial U_{is}}.$$

Then on each $\varphi_s\left(D\left(0,\, r+\varepsilon/2\right)\right)$ we have two maps: φ_s^{-1}, which satisfies $\Delta\varphi_s^{-1}\equiv 0$, and $\psi_{is}\circ F_i$, which satisfies $\Delta_i\psi_{is}\circ F_i\equiv 0$, where Δ_i also denotes the Laplacian of the pullback metric $F_i^*g_i$. We know that $F_i^*g_i\to g$ in $L^{p,k}$ with respect to the fixed coordinate system φ_s on $B(0,\, r+\varepsilon)$. What we want to show is that the metric $F_i^*g_i$ in the harmonic coordinates ψ_{is} also converges to g in the $\mathrm{L}^{p,k}$ topology on $B(0,r)$, or equivalently on $\varphi_s(B(0,r))$. This will clearly be true if we can prove that $\|\varphi_s^{-1}-\psi_{is}\circ F_i\|_{p,k+1}$ converges to as $i\to\infty$. If we write Δ_i in the fixed coordinate system φ_s, then the elliptic estimates for divergence operators (see Theorem A.3) tell us that

$$\begin{aligned}\|\varphi_s^{-1}-\psi_{is}\circ F_i\|_{p,2,B(0,r+\varepsilon/2)} &\le C\,\|\Delta_i(\varphi_s^{-1}-\psi_{is}\circ F_i)\|_{p,B(0,r+\varepsilon/2)}\\ &= C\,\|\Delta_i\varphi_s^{-1}\|_{p,B(0,r+\varepsilon/2)}\end{aligned}\tag{4.1}$$

when $k=1$ and $p>n$, while

$$\begin{aligned}&\|\varphi_s^{-1}-\psi_{is}\circ F_i\|_{p,k+1,B(0,r)}\\ &\qquad\le C\left(\|\Delta_i\varphi_s^{-1}\|_{p,k-1,B(0,r+\varepsilon/2)}+\|\varphi_s^{-1}-\psi_{is}\circ F_i\|_{p,B(0,r+\varepsilon/2)}\right)\end{aligned}\tag{4.2}$$

when $k\ge 2$ and $p>n/2$.

To use these inequalities, observe that $\|\Delta_i\varphi_s^{-1}\|_{p,k-1,B(0,r+\varepsilon/2)}$ approaches 0 as $i\to\infty$ since the coefficients of $\Delta_i=a^{ij}\partial_i\partial_j+b^j\partial_j$ converge to those of Δ in $L^{p,k-1}$, and $\Delta\varphi_s^{-1}=0$ (see also Appendix B). Inequality 4.1 therefore takes care of the cases when $k=1$ and $p>n$.

For the remaining cases, when $k\ge 2$ and $p>n/2$, recall that we have a Sobolev embedding $L^{2p,1}\supset L^{p,k}$. Thus we can again use 4.1 to see that $\|\varphi_s^{-1}-\psi_{is}\circ F_i\|_{p,B(0,r+\varepsilon/2)}$ approaches 0 as $i\to\infty$. Inequality 4.2 then shows that $\|\varphi_s^{-1}-\psi_{is}\circ F_i\|_{p,k+1,B(0,r)}\to 0$ as $i\to\infty$.

To check the reverse inequality $\liminf\|(B_i,g)\|_{p,k,r}\ge\|(B,g)\|_{p,k,r}$, fix some $Q>\liminf\|(B_i,g)\|_{p,k,r}$. For some subsequence of (M_i,g_i,p_i) we can then choose charts $\varphi_{is}:B(0,r)\to U_{is}\subset M_i$ satisfying (h1)–(h4) that converge to limit charts $\varphi_s:B(0,r)\to U_s\subset M$ satisfying (h1)–(h3). Since the metrics converge in the $L^{k,p}$-topology, $k\ge 1$, the Laplacians must also converge, and so we can easily check that the charts φ_s^{-1} are harmonic and therefore give charts showing that $\|(B,g)\|_{p,k,r}\le Q$. □

One of the important consequences of the Fundamental Theorem is that convergence of the metric components in appropriate charts implies global convergence of the manifolds. The key step in getting these global diffeomorphisms is to glue together compositions of charts using a center of mass construction. It is clearly desirable to have a similar local-to-global convergence construction for $L^{p,k}$ convergence, but the method as we described fails. This is because products of $L^{p,k}$ functions are not necessarily $L^{p,k}$. If, however, one of the functions in the product or composition has universal C^∞ bounds, there won't be any problems. So we arrive at the following result:

LEMMA 4.5. *Suppose (M_i, g_i, p_i) and (M, g, p) are pointed C^∞ riemannian manifolds and that we have charts $\varphi_{is} : B(0,r) \to U_{is} \subset M_i$ as in the proof of the Fundamental Theorem 2.2 satisfying $(h1)$, $(h2)$, and $(h4)$, and such that the metric components $g_{is\cdot\cdot}$ converge in the $L^{p,k}$-topology, to the limit metric (M, g). Then a subsequence of (M_i, g_i, p_i) will converge to (M, g, p) in the pointed $L^{p,k}$ topology.*

PROOF. The proof is almost word for word the same as that of the Fundamental Theorem. The important observation is that the limit coordinates $\varphi_s : B(0,r) \to U_s \subset M$ satisfy (h4) and are therefore C^∞, since (M, g) was assumed to be C^∞. The partition of unity on (M, g) is therefore also C^∞. Thus the compositions $\varphi_{is} \circ \varphi_s^{-1}$ are all close in the $L^{p,k+1}$ topology, and we also have

$$\|\mu_1 F_1 + \mu_2 F_2 - F_1\|_{p,k+1} \le \|\mu_2\|_{C^\infty} \|F_2 - F_1\|_{p,k+1}.$$

Finally, $\mu_1 F_1 + \mu_2 F_2$ is also C^1 close to F_1, since $k+1-n/p > 1$ and $L^{p,k+1} \subset C^1$ is continuous. □

5. Generalized Convergence Results

Many of the results in this section have also been considered by Gao and D. Yang, but we will follow the approach taken by Anderson. The survey paper [Anderson 1993] has many good references for further applications and to the papers where many of these things were first considered.

The machinery developed in the previous section makes it possible to state and prove the most general results immediately. Rather than assuming pointwise bounds on curvature, we will use L^p bounds. Our notation is $\|\operatorname{Ric}\|_p = \left(\int_M |\operatorname{Ric}|^p d\operatorname{vol}\right)^{1/p}$ and $\|R\|_p = \left(\int_M |R|^p d\operatorname{vol}\right)^{1/p}$, where $|\operatorname{Ric}|$ and $|R|$ are the pointwise bound on the Ricci tensor and curvature tensor. Note that if we scale the metric g to $\lambda^2 g$, the curvature tensors are multiplied by λ^{-2} and the volume form by λ^n, so the new L^p norms on curvature are $\|\operatorname{Ric}_{\text{new}}\|_p = \lambda^{(n/p)-2}\|\operatorname{Ric}_{\text{old}}\|_p$ and $\|R_{\text{new}}\|_p = \lambda^{(n/p)-2}\|R_{\text{old}}\|_p$. So if $p > n/2$ the L^p bounds scale just as the $C^0 = L^\infty$ bounds, while if $p = n/2$ they remain unchanged. We shall, therefore, mostly be concerned with L^p bounds on curvature, for $p > n/2$. Another heuristic reason for doing so is that if $f \in L^{p,2}$ for $p \in (n/2, n)$, then $f \in C^{2-n/p} \cap L^{1,q}$, where $1/q = 1/p - 1/n$. In particular we have C^0 control on f. So L^p bounds $p > n/2$ on curvature should somehow control the geometry.

With this philosophy behind us, we can state and prove Anderson's convergence results.

THEOREM 5.1. *Let $p > n/2$, $\Lambda \ge 0$, and $i_0 > 0$ be given. For every $Q > 0$ there is an $r = r(n, p, \Lambda, i_0)$ such that any riemannian n-manifold (M, g) with $\|\operatorname{Ric}\|_p \le \Lambda$ and $\operatorname{inj rad} \ge i_0$ satisfies $\|(M, g)\|_{p,2,r} \le Q$.*

PROOF. Suppose this were not true. Then for some $Q > 0$ we could find (M_i, g_i) with $\|(M_i, g_i)\|_{p,2,r_i} \ge Q$ for some sequence $r_i \to 0$. By further decreasing r_i we can actually assume $\|(M_i, g_i)\|_{p,2,r_i} = Q$ for $r_i \to 0$. Now define $h_i = r_i^{-2} g_i$; then $\|(M_i, h_i)\|_{p,2,1} = Q$. Next select $p_i \in M_i$ such that $\|(A_i, h_i)\|_{p,2,1} \geq Q/2$, for all $A_i \ni p_i$. After possibly passing to a subsequence, we can assume that (M_i, h_i, p_i) converges to (M, g, p) in the pointed $C^{l+\alpha}$, topology, where $l + \alpha < 2 - n/p$. Together with this convergence we will select charts $\varphi_{is} : B(0, r) \to U_{is} \subset M_i$ and $\varphi_s : B(0, r) \to U_s \subset M$ such that the φ_{is} satisfy (h1)–(h4) for some $\tilde{Q} > Q$, $r > 1$, and converging to φ_s in $C^{l+1+\alpha}$. Denote the metric components of h_i in φ_{is} by $h_{is\cdot\cdot}$, and similarly denote by $g_{s\cdot\cdot}$ the components for g in φ_s. By assumption the $h_{is\cdot\cdot}$ are bounded in $L^{p,2}$. If $p > n$ then $2 - n/p > 1$, so $l = 1$; hence we can assume that the $h_{is\cdot\cdot}$ converge to $g_{s\cdot\cdot}$ in $C^{1+\alpha}$, with $\alpha = 1 - n/p$. If $p < n$ then $1/(2p) > 1/p - 1/n$, and we can assume that the $h_{is\cdot\cdot}$ converge to $g_{s\cdot\cdot}$ in $L^{2p,1}$. In particular, g has a well defined Laplacian Δ on M, which maps $L^{p,2}$ into L^p, and the Laplacians Δ_i for h_i on M_i converge to Δ in $L^{2p,1}$. So if $u \in L^{p,2}$ then $\Delta_i u \to \Delta u$ in L^p.

We will now establish simultaneously two contradictory statements, namely that (M, g) is $\mathbb{R}^n$ with the canonical metric, and that the (M_i, h_i, p_i) converge to (M, g, p) in the pointed $L^{p,2}$ topology. These statements are contradictory since we know that $\|(\mathbb{R}^n, \mathrm{can})\|_{p,2,1} = 0$, and therefore Proposition 4.4(iv) implies that $\|(B(p_i, \delta), h_i)\|_{p,2,1} \to 0$ as $i \to \infty$, which goes against our assumption that

$$\|(B(p_i, \delta), h_i)\|_{p,2,1} \geq Q/2.$$

Since we are using harmonic coordinates, $h_{is\cdot\cdot}$ satisfies

$$\Delta_i h_{is\cdot\cdot} + Q(h_{is\cdot\cdot}, \partial h_{is\cdot\cdot}) = \mathrm{Ric}_{is\cdot\cdot},$$

where $\mathrm{Ric}_{is\cdot\cdot}$ represents the components of the Ricci tensor for h_i in the φ_{is} coordinates. Since $h_i = r_i^{-2} g_i$ and $r_i \to 0$ we know that $\|\mathrm{Ric}_{is\cdot\cdot}\|_p \to 0$ as $i \to 0$. Also, $h_{is\cdot\cdot} \to g_{s\cdot\cdot}$ in C^0 and $\partial h_{is\cdot\cdot} \to \partial g_{s\cdot\cdot}$ in L^{2p}. Therefore $Q(h_{is\cdot\cdot}, \partial h_{is\cdot\cdot}) \to Q(g_{s\cdot\cdot}, \partial g_{s\cdot\cdot})$ in L^p, since Q is a universal function of class C^∞ in its arguments and quadratic in ∂g. Thus we conclude that $\Delta_i h_{is\cdot\cdot} \to \Delta g_{s\cdot\cdot}$ in L^p. We can then use the L^p estimates from Appendix A (Theorem A.2) to see that

$$\begin{aligned}
&\|h_{is\cdot\cdot} - g_{s\cdot\cdot}\|_{p,2} \\
&\qquad \leq C_i \|\Delta_i (h_{is\cdot\cdot} - g_{s\cdot\cdot})\|_{L^p} + C_i \|h_{is\cdot\cdot} - g_{s\cdot\cdot}\|_{L^p} \\
&\qquad \leq C_i \|\Delta_i (h_{is\cdot\cdot}) - \Delta(g_{s\cdot\cdot})\|_{L^p} + C_i \|(\Delta - \Delta_i) g_{s\cdot\cdot}\|_{L^p} + C_i \|h_{is\cdot\cdot} - g_{s\cdot\cdot}\|_{L^p},
\end{aligned}$$

which converges to 0 as $i \to \infty$. As usual, C_i is bounded by the coefficients of Δ_i, which we know are universally bounded. There is also the slight detail about the left-hand side being measured on a compact subset of $B(0, r)$. But this can be finessed because $\tilde{Q} > Q = \|(M_i, h_i, p_i)\|_{p,2,1}$, so we can find a universal $r = 1 + \varepsilon$ by the proof of Proposition 4.4(i) such that $\|(M_i, h_i)\|_{p,2,r} < \tilde{Q}$. Thus we have the convergence taking place on $B(0, 1)$, and we conclude using Proposition 4.4(iv) that $(M_i, h_i, p_i) \to (M, g, p)$ in the pointed $L^{p,2}$ topology. (We will show below that (M, g) is C^∞.)

The limit metric (M,g) now satisfies $\Delta g = -Q(g,\partial g)$ in our weak $L^{p,2}$ harmonic coordinates. If $p > n$, we have $g \in C^{1+\alpha}$ as already observed; thus the leading coefficients for Δ are in C^1, and $Q(g,\partial g)$ is in C^α. Then standard regularity theory implies $g \in C^{2+1+\alpha}$. A bootstrap argument then shows $g \in C^\infty$. The equation $\Delta g + Q = 0$ then shows that M is Ricci-flat. We must now recall that $\operatorname{inj\,rad}(M_i,h_i) = r_i^{-1} i_0$, which goes to ∞, so $\operatorname{inj\,rad}(M,g) = \infty$. The Cheeger–Gromoll splitting theorem then implies that $(M,g) = (\mathbb{R}^n, \mathrm{can})$.

We must now deal with the case where $p \in (n/2, n)$. Define $\tilde p = (2/p - 2/n)^{-1}$, so $f \in L^{p,2}$ implies $f \in L^{2\tilde p,1} \cap C^{2-n/p}$. Since $g \in L^{p,2}$, we must therefore have $Q(g,\partial g) \in L^{\tilde p}$. But then $\Delta g \in L^{\tilde p}$, and $g \in L^{\tilde p,2}$ since the coefficients for Δ are C^0. Now $\tilde p > n/2 + 2\,(p - n/2) = 2p - n/2 = p + (p - n/2)$. Iterating this procedure k times, we get $g \in L^{2,q}$ for $q > p + k\,(p - n/2)$. If k is big enough, we clearly have $q > n$, and we get $g \in C^{1+\beta}$, where $\beta = 1 - n/q > 0$. We can then proceed as above. □

This theorem has some immediate corollaries.

Corollary 5.2. *Given $\Lambda \geq 0$, $i_0, Q > 0$, and $p > 1$, there is an $r = r(n,p,Q,\Lambda,i_0)$ such that any riemannian n-manifold (M,g) with $|\mathrm{Ric}| \leq \Lambda$ and $\operatorname{inj\,rad} \geq i_0$ satisfies $\|(M,g)\|_{p,2,r} \leq Q$.*

This corollary clearly implies that we have $C^{1+\alpha}$ control ($\alpha < 1$) on the metric, given C^0 bounds on $|\mathrm{Ric}|$ and lower bounds on injectivity radius.

Corollary 5.3. *Given Λ, $i_0, V > 0$, and $p > n/2$, the class of n-manifolds with: $\|\mathrm{Ric}\|_p \leq \Lambda$, $\operatorname{inj\,rad} \geq i_0$, and $\operatorname{vol} \leq V$ is precompact in the C^α-topology, where $\alpha < 2 - n/p$.*

Recall that, when we had bounded sectional curvature, we could replace the injectivity radius bound by a lower volume bound on balls of radius 1. This is no longer possible when we only bound the Ricci curvature, because there are many nonflat Ricci-flat metrics with volume growth of order r^n—for example, the Eguchi–Hanson metric described in Appendix C. Also, if we assume $\|R\|_p \leq \Lambda$, we can construct examples with lower volume bounds on balls of radius 1, but without any control on the metric. This is partly due to the fact that we have no way of showing that these assumptions imply the volume growth condition $\operatorname{vol} B(p,r) \geq v'\,r^n$ for $r \leq 1$. To get such a condition, one must have at least a lower bound on Ricci curvature. The best we can do is therefore this:

Theorem 5.4. *Given $p > n/2$, $\Lambda \geq 0$, $v > 0$, and $Q > 0$, there is an $r = r(n,p,Q,\Lambda,v)$ such that any n-manifold (M,g) with $\|R\|_p \leq \Lambda$ and $\operatorname{vol} B(p,s) \geq v\,s^n$ for all $p \in M$ and $s \leq 1$ satisfies $\|(M,g)\|_{p,2,r} \leq Q$.*

Proof. The setup is identical to the proof of Theorem 5.1. This time the limit space will be flat, since $\|R\|_p$ varies continuously in the $L^{p,2}$ topology, and have $\operatorname{vol} B(p,r) \geq v\,r^n$ for all r. Thus $(M,g) = (\mathbb{R}^n, \mathrm{can})$. □

Considering that the proofs of Theorems 5.1 and 5.4 only differ in how we characterize euclidean space, it would seem reasonable to conjecture that any appropriate characterization of euclidean space should yield some kind of theorem of this type. We give some examples of this now:

EXAMPLE 5.5. $(\mathbb{R}^n, \text{can})$ is the only space with $\text{Ric} \geq 0$ and $\text{inj rad} = \infty$, by the Cheeger–Gromoll splitting theorem. In [Anderson and Cheeger 1992], it is proved by a slightly different method from what we have described above that, for any $p > n$, the $L^{1,p}$ norm at some scale r is controlled by dimension, lower Ricci curvature bounds, and lower injectivity radius bounds.

EXAMPLE 5.6. If in Theorems 5.1 and 5.4 we also assume that the covariant derivatives of Ric up to order k are bounded in L^p, where $p > n/2$, we can control the $L^{p,k+2}$ norm. The proof of this is exactly like the proof of Theorems 5.1 and 5.4: we just use the stronger assumption that Ric goes to zero in the $L^{p,k}$ topology.

EXAMPLE 5.7. Let ω_n be the volume of the unit ball in euclidean space. Then $(\mathbb{R}^n, \text{can})$ is characterized as the only Ricci-flat manifold with $\text{vol}\, B(p, r) \geq (\omega_n - \varepsilon_n) r^k$ for all r, where $\varepsilon_n > 0$ is a universal constant only depending on dimension. The existence of such a constant was established in [Anderson 1990b]. Note that relative volume comparison shows that the volume growth need only be checked for one $p \in M$. Also if (M, g, p) satisfies these conditions, then $(M, \lambda^2 g, p)$ satisfies these conditions for any $\lambda > 0$.

It is actually an interesting application of Theorem 5.1 to prove that this really gives a characterization of $(\mathbb{R}^n, \text{can})$. We proceed by contradiction on the existence of such an $\varepsilon_n > 0$. Using scale invariance of norm and the conditions, we can then find a sequence (M_i, g_i, p_i) of Ricci-flat manifolds such that $\text{vol}\, B(p_i, r) \geq (\omega_n - i^{-1}) r^n$ for all r, and a sequence $R_i \to \infty$ such that, say, $Q \geq \|(A_i, g_i)\|_{2n,2,1} \geq Q/2$, where $p_i \in A_i \subset B(p_i, R_i)$, for some $Q > 0$. The proof of Theorem 5.1 then shows that $(B(p_i, R_i), p_i) \to (M, g, p)$ in the $L^{2n,2}$-topology. But this limit space is now Ricci-flat and satisfies $\text{vol}\, B(p, r) = \omega_n r^n$ for all r, and must, therefore, be $(\mathbb{R}^n, \text{can})$. This contradicts continuity of the $L^{2n,2}$ norm in the $L^{2n,2}$ topology.

We can then control the $L^{p,2}$ norm in terms of r_0 and Λ, where $\|\text{Ric}\|_p \leq \Lambda$ and $\text{vol}\, B(p, r) \geq (\omega_n - \varepsilon_n) r^n$, for $r \leq r_0$. If we impose the stronger curvature bound $|\text{Ric}| \leq \Lambda$, then relative volume comparison shows that we need only check that $\text{vol}\, B(p, r_0) \geq (\omega_n - \varepsilon_n)(v(n, -\Lambda, 1))^{-1} v(n, -\Lambda, r_0)$, where $v(n, -\Lambda, r)$ is the volume of an r-ball in constant curvature $-\Lambda$ and dimension n.

As already pointed out, the Eguchi–Hanson metric shows that one cannot expect to get as nice control on the metric with Ricci curvature and general volume bounds as one gets in the presence of sectional curvature bounds. But if we also assume $L^{n/2}$ bounds on R, then we can do better than in Example 5.7.

EXAMPLE 5.8. In [Bando et al. 1989] it is proved that any complete n-manifold with $\operatorname{Ric} \equiv 0, \|R\|_{n/2} < \infty$ and $\operatorname{vol} B(p,r) \geq v r^n$, for all r and some $v > 0$, is an asymptotically locally euclidean space. This implies, in particular, that $\lim \operatorname{vol} B(p,r)/r^n = \omega_n/k$ for some integer k (in fact, k is the order of the fundamental group at ∞). If, therefore, $v > \frac{1}{2}\omega_n$, then $k = 1$. But then relative volume comparison and nonnegative Ricci curvature implies that the space is euclidean space.

This again leads to two results. First: The $L^{2,p}$ norm for any p is controlled by Λ, r_0, and ε, where $|\operatorname{Ric}| \leq \Lambda$, $\|R\|_{n/2} \leq \Lambda$, and

$$\operatorname{vol} B(p, r_0) \geq \left(\tfrac{1}{2}\omega_n + \varepsilon\right) (v(n, -\Lambda, 1))^{-1} v(n, -\Lambda, r_0).$$

Second: The $L^{2,p}$ norm is controlled by Λ, r_0 and ε where $\|\operatorname{Ric}\|_p \leq \Lambda$, $\|R\|_{n/2} \leq \Lambda$ and $\operatorname{vol} B(p,r) \geq \left(\frac{1}{2}\omega_n + \varepsilon\right) r^n$, for $r \leq r_0$.

For even more general volume bounds, we have:

EXAMPLE 5.9. With a contradiction argument similar to the one used in Example 5.7, one can easily see that there is an $\varepsilon(n,v) > 0$ such that any complete n-manifold with $\operatorname{Ric} \equiv 0, \|R\|_{n/2} \leq \varepsilon(n,v)$ and $\operatorname{vol} B(p,r) \geq v r^n$ for all r is $(\mathbb{R}^n, \mathrm{can})$. Hence given $v, r_0, \Lambda > 0$ with $|\operatorname{Ric}| \leq \Lambda$, $\operatorname{vol} B(p, r_0) \geq v$, and the $(n/2)$-norm of R on balls of radius r_0 satisfying $\|R|_{B(p,r_0)}\|_{n/2} \leq \varepsilon(n,v)$ we can control the $L^{p,2}$-norm for any p. We can, as before, modify this argument if we only wish to assume L^p bounds on Ric (see also [Yang 1992] for a different approach to this problem).

This result seems quite promising for getting our hands on manifolds with given upper bounds on $|\operatorname{Ric}|$, $\|R\|_{n/2}$, and lower bounds on $\operatorname{vol} B(p,1)$. The Eguchi–Hanson metric will, however, still give us trouble. This metric has $\operatorname{Ric} \equiv 0$, $\|R\|_{n/2} < \infty$, and $\operatorname{vol} B(p,r) \geq \frac{1}{2}\omega_n r^n$ for all p. All these quantities are scale-invariant, but if we multiply by larger and larger constants, the space will converge in the Gromov–Hausdorff topology to the euclidean cone over $\mathbb{RP}^3$, which is not even a manifold. Away from the vertex of the cone, the convergence is actually in any topology we like, since $\operatorname{Ric} \equiv 0$. What happens is that $\|R\|_{n/2}$, while staying finite, will concentrate around the singularity that develops and go to zero elsewhere. Questions related to this are investigated in [Anderson and Cheeger 1991], where the authors prove that the class of n-manifolds with $|\operatorname{Ric}| \leq \Lambda$, $\|R\|_{n/2} \leq \Lambda$, $\operatorname{diam} \leq D$, and $\operatorname{vol} \geq v$ contains only finitely many diffeomorphism types. The idea is that these bounds give $L^{p,2}$-control on each of the spaces away from at most $N(n, \Lambda, D, v)$ points, and that each of these bad points can only degenerate in a very special fashion that looks like the degeneration of the Eguchi–Hanson metric.

Recall that the injectivity radius of a closed manifold is computed as either half the length of the shortest closed geodesic (abbreviated $\operatorname{scg}(M,g)$) or the distance to the first conjugate point. Thus the injectivity radius of a manifold with bounds on $|R|$ is completely controlled by $\mathrm{scg}(M)$. In the next example, we shall see how at least the $L^{p,2}$ norm is controlled by $\|R\|_p$ and $\mathrm{scg}(M)$. Some auxiliary comments are needed before we proceed. Let us denote by $\mathrm{sgl}(M,g)$ the shortest geodesic loop. (A closed geodesic is a smooth map $\gamma : S^1 \to M$ with $\ddot{\gamma} \equiv 0$, while a geodesic loop is a smooth map $\gamma : [0,l] \to M$ with $\gamma(0) = \gamma(l)$ and $\ddot{\gamma} \equiv 0$. The base point for a loop is $\gamma(0)$.) On a closed manifold M we have $\mathrm{scg}(M,g) \geq \mathrm{sgl}(M,g)$. On a complete flat manifold, $\mathrm{scg}(M,g) = \mathrm{sgl}(M,g)$, and each geodesic loop realizing $\mathrm{sgl}(M,g)$ is a noncontractible closed geodesic.

Example 5.10. (See also [Anderson 1991]). Our characterization of $(\mathbb{R}^n, \mathrm{can})$ is that it is flat and contains no closed geodesics. First we claim that given $p > n/2$ and $l, \Lambda, Q > 0$ there is an $r(n,p,l,\Lambda,Q)$ such that any closed manifold with $\|R\|_p \leq \Lambda$ and $\mathrm{sgl} \geq l$ satisfies $\|(M,g)\|_{p,2,r} \leq Q$. This is established by our usual contradiction argument. If the statement were not true, we could find a sequence (M_i, g_i) with $\|R(g_i)\|_p \to 0$, $\mathrm{sgl}(M_i,g_i) \to \infty$, and $\|(M_i,g_i)\|_{p,2,1} = Q > 0$ for all i. For each (M_i,g_i) select $p_i \in M_i$ such that $\|(A_i,g_i)\|_{p,2,1} = Q$ for any $A_i \ni p_i$. Then the pointed sequence (M_i,g_i,p_i) will subconverge to a flat manifold (M,g,p) in the pointed $L^{p,2}$ topology. Thus (M,g,p) also has the property that $\|(A,g)\|_{p,2,1} = Q$ for all $A \ni p$. This implies that $\mathrm{inj\,rad}(M,g) < 1$ at p; thus we have a geodesic loop $\gamma : [0,l] \to M$ of length $l < 2$ and $\gamma(0) = \gamma(l) = p$. Using the embeddings $F_{ik} : \Omega_k \to M_i$, we can then construct a loop $c_i : [0,l_i] \to M_i$ of length $l_i \to l$ with $c_i(0) = c_i(l_i) = p_i$. Since γ is not contractible, the loops c_i will not be contractible in $B(p_i,1)$ for large i. We can, therefore, shorten c_i through loops based at p_i until we get a nontrivial geodesic loop $\gamma_i : [0,\tilde{l}_i] \to M_i$ of length $\tilde{l}_i \leq l_i$ based at p_i. This, however, contradicts our assumption that $\mathrm{sgl}(M_i,g_i) \to \infty$.

We can now use this to establish the following statement: Given $p > n/2$ and $\Lambda, l, Q > 0$, there is an $r = r(n,p,Q,\Lambda,l)$ such that any closed manifold with $\|R\|_p \leq \Lambda$ and $\mathrm{scg} \geq l$ satisfies $\|(M,g)\|_{p,2,r} \leq Q$. We contend that there is an $\tilde{l} = \tilde{l}(n,p,\Lambda,l)$ such that any such manifold also satisfies $\mathrm{sgl} \geq \tilde{l}$. Otherwise, we could find a pointed sequence (M_i,g_i,p_i) with $\|R(g_i)\|_p \to 0$, $\mathrm{scg}(M_i,g_i) \to \infty$, and $\mathrm{sgl}(M_i,g_i) = 2$. Let $\gamma_i : [0,2] \to M_i$ be a geodesic loop at p_i of length 2. From our previous considerations, it follows that (M_i,g_i,p_i) subconverges to a flat manifold (M,g,p) in the pointed $L^{p,2}$ topology. The loops γ_i will converge to a geodesic loop $\gamma : [0,2] \to M$ of length 2 and $\gamma(0) = \gamma(2) = p$. If $\mathrm{sgl}(M,g) = \mathrm{scg}(M,g) < 2$ we can, as before, find geodesic loops in M_i of length less than 2 for large i. Hence γ is actually a closed geodesic of length 2 which is noncontractible. This implies that the geodesic loops γ_i are noncontractible inside $B(p_i,2)$ for large i. By assumption γ_i cannot be a closed geodesic, and is therefore not smooth at p_i. Thus the closed curve γ_i, when based at $\gamma_i(\varepsilon)$ for some fixed but small $\varepsilon > 0$, can be shortened through curves based at $\gamma_i(\varepsilon)$ to a nontrivial geodesic loop of length less than 2. This again contradicts the assumption that $\mathrm{sgl}(M_i,g_i) = 2$.

Our proof of this last result deviates somewhat from that of [Anderson 1991]. This is required by our insisting on using norms rather than radii to measure smoothness properties of manifolds.

EXAMPLE 5.11. If a complete manifold satisfies $\mathrm{Ric} \geq -\Lambda$, $\mathrm{vol}\, B(p,1) \geq v$ for all p, and the conjugate radius is at least r_0, then the injectivity radius has a lower bound, $\mathrm{inj\,rad} \geq i_0(n,\Lambda,v,r_0)$. This result was observed by Zhu and the author, and independently by Dai and Wei. It follows from the proof of the injectivity radius estimate in [Cheeger et al. 1982]: the above bounds are all one needs in order for the proof to work. Thus we conclude that the class of n-manifolds with $|\mathrm{Ric}| \leq \Lambda$, $\mathrm{vol} \geq v$, $\mathrm{diam} \leq D$, and $\mathrm{conj\,rad} \geq r_0$ is precompact in the $C^{1+\alpha}$-topology for all $\alpha < 1$.

The local model for a class of manifolds is the type of space characterized by the inequalities and equations one gets by multiplying the metrics in the class by constants going to infinity and observing how the quantities defining the class changes.

Some examples are: The class of manifolds with $\|\mathrm{Ric}\|_p \leq \Lambda$ and $\mathrm{inj\,rad} \geq i_0$ has local model $\mathrm{Ric} \equiv 0$, $\mathrm{inj\,rad} \equiv \infty$, which, as we know, is $(\mathbb{R}^n, \mathrm{can})$. The class of manifolds with $\|R\|_p \leq \Lambda$ has local model $R \equiv 0$, i.e., flat manifolds. The class with $|\mathrm{Ric}| \leq \Lambda$, $\|R\|_{n/2} \leq M$, and $\mathrm{vol}\, B(p,r_0) \geq v$ has local model $\mathrm{Ric} \equiv 0$, $\|R\|_{n/2} < \infty$, $\mathrm{vol}\, B(p,r) \geq v' r^n$ for all r; these are the nice asymptotically locally euclidean spaces.

In all our previous results, the local model has been $(\mathbb{R}^n, \mathrm{can})$. It is, therefore, reasonable to expect any class that has $(\mathbb{R}^n, \mathrm{can})$ as local model to satisfy some kind of precompactness condition. If the local model is not unique, the situation is obviously somewhat more complicated, but it is often possible to say something intelligent. It is beyond the scope of this article to go into this. Instead we refer the reader to [Anderson 1993; Fukaya 1990; Cheeger et al. 1992; Anderson and Cheeger 1991].

As a finale, Table 1 lists schematically some of the results we have proved.

6. Applications of Convergence Theory to Pinching Problems

In this section we will present some pinching results inspired by the work in [Gao 1990]. Our results are more general and the proofs rely only on the convergence results and their proofs given in previous sections, not on the work in [Gao 1990].

For $p > n/2 \geq 1$ and $Q, r > 0$, we have the class $\mathcal{H}(n,p,Q,r)$ of pointed C^∞ riemannian n-manifolds with $\|\cdot\|_{p,2,r} \leq Q$. We will be concerned with the subclass $\mathcal{H}(V) = \mathcal{H}(n,p,Q,r,V)$ of compact manifolds that in addition have volume at most V. If $p > n$ we know that $\mathcal{H}(V)$ is precompact in the $C^{1+\alpha}$ topology, for $\alpha < 1 - n/p$, while if $n/2 < p < n$ the class $\mathcal{H}(V)$ is precompact in both the C^α and $L^{1,q}$, topologies, for $\alpha < 2 - n/p$ and $1/q > 1/p - 1/n$.

	Class	Local model $= \mathbb{R}^n$	Precompact in C^β-topology
1	$\lvert R\rvert \le \Lambda, (\lvert\nabla^k \mathrm{Ric}\rvert \le M)$ $\mathrm{vol} \ge v$ $\mathrm{diam} \le D$	$R \equiv 0$ $\mathrm{vol}\, B(p,r) \ge v' \cdot r^n$	$\beta < 2(+k)$
2	$\lVert\mathrm{Ric}\rVert_p \le \Lambda, (\lVert\nabla^k \mathrm{Ric}\rVert_p \le M)$ $\mathrm{inj} \ge i_0$ $\mathrm{vol} \le V$ or $\mathrm{diam} \le D$	$\mathrm{ric} \equiv 0$ $\mathrm{inj} \equiv \infty$	$\beta < 2(+k) - \frac{n}{p}$, where $p > \frac{n}{2}$
3	$\lVert R\rVert_p \le \Lambda, (\lVert\nabla^k \mathrm{Ric}\rVert_p \le M)$ $\mathrm{scg} \ge l_0$ $\mathrm{vol} \le V$ or $\mathrm{diam} \le D$	$R \equiv 0$ $\mathrm{scg} \equiv \infty$	$\beta < 2(+k) - \frac{n}{p}$, where $p > \frac{n}{2}$
4	$\lvert\mathrm{Ric}\rvert \le \Lambda$ $\mathrm{vol} \ge v$ $\mathrm{diam} \le D$ $\lVert R\vert_{B(p,r_0)}\rVert_{\frac{n}{2}} \le \varepsilon(n, v \cdot D^{-n})$	$\mathrm{Ric} \equiv 0$ $\mathrm{vol}\, B(p,r) \ge v' \cdot r^n$ $\lVert R\rVert_{\frac{n}{2}} \le \varepsilon(n, v')$	$\beta < 2$
5	$\lvert\mathrm{Ric}\rvert \le \Lambda$ $\mathrm{vol} \ge v$ $\mathrm{diam} \le D$ $\mathrm{conj} \ge r_0$	$\mathrm{Ric} \equiv 0$ $\mathrm{vol}\, B(p,r) \ge v' \cdot r^n$ $\mathrm{conj} \equiv \infty$	$\beta < 2$

Table 1. Comparison of results. All classes consist of connected, closed riemannian n-manifolds.

Note that the volume bound tells us that any convergence takes place in the unpointed sense, so we are basically reducing ourselves to studying convergence of riemannian metrics on a fixed manifold.

Let $g \circ g$ be the Kulkarni–Nomizu product of g with itself.

THEOREM 6.1. *Fix $l \in (1, p]$ and $c \in \mathbb{R}$. There exists $\varepsilon = \varepsilon(l, c, n, p, Q, r, v)$ such that any $(M, g) \in \mathcal{H}(V)$ with $\lVert \mathrm{Ric}_g - cg\rVert_l = \left(\int_M \lvert \mathrm{Ric}_g - cg\rvert^l\right)^{1/l} \le \varepsilon$ is C^α, close to an Einstein metric on M, where $\alpha < 2 - n/p$. If $\lVert R_g - c\, g \circ g\rVert_l < \varepsilon$, then (M, g) is C^α close to a constant curvature metric.*

PROOF. Note that we think of $\lvert \mathrm{Ric}_g - c\, g\rvert$ as a function on M whose value at $p \in M$ is $\sup\{\lvert \mathrm{Ric}(v, v) - c\rvert : p \in T_pM,\ g(v, v) = 1\}$. A similar definition is given for $\lVert R - c\, g \circ g\rVert_l$.

The proof is by contradiction on the existence of such an ε. For the sake of concreteness suppose $n/2 < p < n$. Using all of our previous work, we can therefore suppose that we we have a sequence of metrics g_i on a fixed manifold M such that $(M, g_i) \in \mathcal{H}(V)$ and $\|\mathrm{Ric}_{g_i} - c\, g_i\|_l$ converges to 0. We can furthermore assume that for each i, we have harmonic charts $\psi_{is} : B(0,r) \to M$, for $s = 1, \ldots, N$, so that $g_{is\cdot\cdot}$ converges on $B(0,r)$ to some limit riemannian metric $g_{s\cdot\cdot}$ in the $\psi_{is\cdot\cdot}$ coordinates. Note that the coordinates vary, but that they will themselves converge to limit charts ψ_s. As in the preliminaries of Theorem 5.1, we know that the limit coordinates are harmonic.

Now fix s and consider, in the ψ_{is} coordinates, the equation $\Delta_i g_i + Q(g_i, \partial g_i) = \mathrm{Ric}_{g_i}$. Here we know that $Q(g_i, \partial g_i)$ converges to $Q(g, \partial g)$ in $L^{q/2}$ where $1/q > 1/p - 1/n$, and g is the limit metric. Moreover, $\|\mathrm{Ric}_{g_i} - c\, g_i\|_l \to 0$. So we can conclude that, in the limit harmonic coordinates ψ_s, we have the equation $\Delta g + Q(g, \partial g) = c\, g$ or $\Delta g = -Q(g, \partial g) + c\, g$. Here the right-hand side is in $L^{q/2}$, with $q/2 > p$. So we can argue as in the proof of Theorem 5.1 that g is a C^∞ metric that satisfies the equation $\Delta g + Q(g, \partial g) = c\, g$ in harmonic coordinates. But this implies that the metric is Einstein, which contradicts our assumption that the g_i were not C^α close to an Einstein metric.

If we assume that $\|R - c\, g \circ g\|_l \to 0$, we have in particular

$$\|\,\mathrm{Ric}_g - (n-1)\, c\, g\|_l \to 0.$$

So the limit metric is again C^∞ and Einstein. We will, in addition, have $\|W\|_l \to 0$, where W is the Weyl tensor. Since $\|W\|_l$ varies semicontinuously in the C^α topology, the limit space must satisfy $\|W\|_l = 0$. But then the metric will also be conformally flat, and this makes it a constant curvature metric. □

It is worthwhile pointing out that if we fix $c \in \mathbb{R}$ and assume that $\|\,\mathrm{Ric} - c\, g\|_p$ or $\|R - c\, g \circ g\|_p$ is small, where $p > n/2$, then we have L^p bounds on curvature. It is, therefore, possible to get some better pinching results in this case. Let's list some examples.

THEOREM 6.2. *Let $p > n/2 \geq 1$, $l, V > 0$, and $c \in \mathbb{R}$ be given. There exists $\varepsilon = \varepsilon(n, p, l, V, c)$ such that any (M^n, g) with* $\mathrm{scg} \geq l$, $\mathrm{vol} \leq V$, *and $\|R_g - c\, g \circ g\|_p \leq \varepsilon$ is $L^{p,2}$ close to a constant curvature metric on M.*

PROOF. The proof, of course, uses Example 5.9 and proceeds as Theorem 1 with the added finesse that, since $\|R - c\, g \circ g\|_p \to 0$, we can achieve convergence in $L^{p,2}$ rather than in the weaker C^α topology, $\alpha < 2 - n/p$. □

THEOREM 6.3. *Let $p > n/2 \geq 1$, $i_0, D > 0$, and $c \in \mathbb{R}$ be given. There exists $\varepsilon = \varepsilon(n, p, i_0, D, c)$ such that any (M^n, g) with* $\mathrm{inj\,rad} \geq i_0, \mathrm{diam} \leq D$ *and $\|\,\mathrm{Ric} - c\, g\|_p \leq \varepsilon$ is $L^{p,2}$ close to an Einstein metric.*

PROOF. Same as before. □

Finally, we can also improve Gao's result on $L^{n/2}$ curvature pinching.

THEOREM 6.4. *Let $n \geq 2$, $\Lambda, v, D > 0$, and $c \in \mathbb{R}$ be given. There exists $\varepsilon = \varepsilon(n, \Lambda, v, D)$ such that any (M^n, g) satisfying* $|\operatorname{Ric}| \leq \Lambda$, $\operatorname{vol} \geq v$, $\operatorname{diam} \leq D$, *and $\|R - c\, g \circ g\|_{n/2} \leq \varepsilon$ is $C^{1+\alpha}$ close to a constant curvature metric for any $\alpha < 1$.*

PROOF. Notice that the smallness of $\|R - c\, g \circ g\|_{n/2}$ implies that $\|R|_{B(p,r_0)}\|$ will be sufficiently small for some small r. Thus, the class is precompact in the $C^{1+\alpha}$ topology for any $\alpha < 1$. We can then use Theorem 6.1 to finish the proof. □

When $\dim \geq 3$ we know from Schur's lemma and the identity $d \operatorname{Scal} = 2 \operatorname{div} \operatorname{Ric}$ that pointwise constant sectional curvature implies constant curvature and that pointwise constant Ricci curvature implies constant Ricci curvature. This can be used to establish further pinching results, where instead of assuming that the curvature is close to a constant, we assume smallness of the tensors

$$R - (n(n-1))^{-1} \operatorname{Scal} g \circ g$$

or $\operatorname{Ric} - n^{-1} \operatorname{Scal} g$. With the background we have established so far, it is not hard to find the appropriate pinching theorems. Further details are left to the energetic reader.

Appendix A: Useful Results from [Gilbarg and Trudinger 1983]

Fix a nice bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. For $f \in C^\infty(\Omega)$ and $0 < \alpha \leq 1$, define

$$\|f\|_{k+\alpha,\Omega} = \max\left\{ \sup_{x \in \Omega} |\partial^j f(x)| : |j| \leq k \right\} + \max\left\{ \sup_{x,y \in \Omega} \frac{|\partial^j f(x) - \partial^j f(y)|}{|x-y|^\alpha} : |j| = k \right\}.$$

Since we don't allow $\alpha = 0$, there should be no ambiguity about what $\|f\|_{k+1}$ is. Also define

$$\|f\|_{p,k,\Omega} = \sum_{0 \leq |j| \leq k} \left(\int_\Omega |\partial^j f(x)|^p dx \right)^{1/p}.$$

Then define $C^{k+\alpha}(\Omega)$ and $L^{p,k}(\Omega)$ as the completions of $C^\infty(\Omega)$ with respect to the norms $\|\cdot\|_{k+\alpha,\Omega}$ and $\|\cdot\|_{p,k,\Omega}$, respectively.

Sobolev embeddings. Any closed and bounded subset of $L^{p,k}$ is compact with respect to the weaker norms $\|\cdot\|_{l+\alpha}$ (or $\|\cdot\|_{q,l}$) provided $l + \alpha < k - n/p$ (or $l < k$ and $1/q > 1/p - (k-l)/n$). In other words, we have compact inclusions $C^{l+\alpha} \supset L^{p,k}$ for $l + \alpha < k - n/p$ and $L^{q,l} \supset L^{p,k}$ for $l < k$ and $1/q > 1/p - (k-l)/n$.

Elliptic estimates. Fix again a nice bounded $\Omega \subset \mathbb{R}^n$ and a compact $\Omega' \subset \mathring{\Omega}$. We shall work exclusively with functions $u \in C^\infty(\Omega)$ and operators whose coefficients are in $C^\infty(\Omega)$. The issues here are, therefore, quantitative rather than qualitative. Fix a second-order operator $L = a^{ij}\partial_i\partial_j$, where $\{a^{ij}\} : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ is a positive definite symmetric matrix with eigenvalues in $[e^{-Q}, e^Q]$ whose entries satisfy either $\|a^{ij}\|_{k+\alpha,\Omega} \le Q$ or $\|a^{ij}\|_{p,l,\Omega} \le Q$.

THEOREM A.1 (SCHAUDER ESTIMATES). *If $\|a^{ij}\|_{k+\alpha} \le Q$, there exists $C = C(k+\alpha,\, n,\, Q,\, \Omega' \subset \Omega)$ such that*

$$\|u\|_{k+2+\alpha,\Omega'} \le C\left(\|Lu\|_{k+\alpha,\Omega} + \|u\|_{C^0,\Omega}\right).$$

PROOF. For $k = 0$, this is [Gilbarg and Trudinger 1983, Theorem 6.2]. For $k \ge 1$, we can proceed by induction. Suppose we have established the inequality for some k. Fix ∂_j and use

$$\begin{aligned}
\|\partial_j u\|_{k+2+\alpha} &\le C(k)(\|L\partial_j u\|_{k+\alpha} + \|\partial_j u\|_{C^0})\\
&\le C(k)(\|\partial_j Lu\|_{k+\alpha} + \|[L,\partial_j]a\|_{k+\alpha} + \|\partial_j u\|_{C^0})\\
&\le C(k)(\|Lu\|_{k+1+\alpha} + \|a^{lm}\|_{k+\alpha}\|u\|_{k+2+\alpha} + \|\partial_j u\|_{C^0})\\
&\le C(k)(\|Lu\|_{k+1+\alpha} + (Q+1)\|u\|_{k+2+\alpha})\\
&\le C(k)(\|Lu\|_{k+1+\alpha} + (Q+1)C(k)(\|Lu\|_{k+\alpha} + \|u\|_{C^0}))\\
&\le C(k+1)(\|Lu\|_{k+1+\alpha} + \|u\|_{C^0}).
\end{aligned}$$

Since $\|u\|_{k+1+2+\alpha} \le \|u\|_{k+2+\alpha} + \sum_j \|\partial_j u\|_{k+2+\alpha}$, we get the desired inequality. □

THEOREM A.2 (L^p ESTIMATES). *If $\operatorname{vol}\Omega \le V$ and $\|a^{ij}\|_{p,k} \le Q$, where $p > n$ if $k = 1$ and $p > n/2$ if $k \ge 2$, there exists $C = C(p,k,n,Q,\Omega' \subset \Omega, V)$ such that*

$$\|u\|_{p,k+1,\Omega'} \le C\left(\|Lu\|_{p,k-1,\Omega} + \|u\|_{L^p,\Omega}\right).$$

Furthermore, if $\Omega = B(0,r)$ is a euclidean ball and $u = 0$ on $\partial B(0,r) = S(0,r)$, then we can find $C = C(p,n,Q,r)$ such that

$$\|u\|_{p,2,B(0,r)} \le C\left(\|Lu\|_{p,B(0,r)} + \|u\|_{p,B(0,r)}\right).$$

PROOF. The condition $p - n/k > 0$ implies by the Sobolev imbedding results that $\|a^{ij}\|_{l+\alpha,\Omega} \le K(n,p,k,l,\alpha)\,Q$ for $l + \alpha < p - n/k$. In particular, the functions a^{ij} have moduli of continuity bounded on Ω in terms of Q, n, p, k. When $k = 1$, we can, therefore, use Theorem 9.11 (or 9.13 for the second inequality) in [Gilbarg and Trudinger 1983] to get the estimate. When $k \ge 2$, we use induction again. Assume the inequality holds for some k and that $\|a^{ij}\|_{p,k+1,\Omega} \le Q$. Since $p > n/2$, we have $1/(2p) > 1/q - 1/n$, so $\|a^{ij}\|_{2p,k,\Omega} \le K(n,p,k)\,Q$. Also, recall that $\|f\|_{p,\Omega} \le C(p,q,\operatorname{vol}\Omega)\|f\|_{q,\Omega}$ for $q > p$ by Hölder's inequality. Now we compute as with the Schauder estimates:

$$\begin{aligned}\|\partial_l u\|_{p,k+1,\Omega'} &\leq C(k,p)(\|L\partial_l u\|_{p,k-1} + \|\partial_l u\|_{L^p}) \\ &\leq C(k,p)(\|Lu\|_{p,k} + \|[L,\partial_l]u\|_{p,k-1} + \|\partial_l u\|_{L^p}) \\ &\leq C(k,p)(\|Lu\|_{p,k} + \|\partial_l a^{ij}\|_{2p,k-1}\, \|u\|_{2p,k-1} + \|\partial_l u\|_{L^p}) \\ &\leq C(k,p)(\|Lu\|_{p,k} + (KQ+1)\|u\|_{2p,k-1}) \\ &\leq C(k+1,p)(\|Lu\|_{p,k} + \|u\|_p).\end{aligned}$$

By adding up, we therefore have

$$\|u\|_{p,k+2,\Omega'} \leq C(n,p)\left(\|Lu\|_{p,k,\Omega} + \|u\|_{p,\Omega}\right). \qquad \square$$

Now suppose our operator is given to us in divergence form, $L = \partial_i(a^{ij}\partial_j)$, where a^{ij} is a positive definite symmetric matrix with eigenvalues in $[e^{-Q}, e^{Q}]$. All Laplacians are written in this form unless we use harmonic coordinates, so it is obviously important for our theory to understand this case as well.

THEOREM A.3 (L^p ESTIMATES FOR DIVERGENCE OPERATORS). *Assume $\Omega' \subset \int \Omega$ is compact, Ω is bounded with $\operatorname{vol}\Omega \leq V$ and $\|a^{ij}\|_{p,k} \leq Q$, where $p > n$ if $k = 1$ and $p > n/2$ if $k \geq 2$. There exists $C = C(p,k,n,Q,\Omega' \subset \Omega, V)$ such that*

$$\|u\|_{p,k+1,\Omega'} \leq C\left(\|Lu\|_{p,k-1,\Omega} + \|u\|_{p,\Omega}\right).$$

Furthermore, if $\Omega = B(0,r)$ and $u = 0$ on $\partial\Omega$, there is $C = C(p,n,r,Q)$ such that

$$\|u\|_{p,2,B(0,r)} \leq C\left(\|Lu\|_{p,B(0,r)}\right).$$

PROOF. We will concentrate on the second inequality. The proof of the first inequality is similar but simpler for the case where $k = 1$. For general k one can just use induction as before. The proof proceeds on two steps. First we show that there is a C such that $\|u\|_{p,2} \leq C\left(\|Lu\|_p + \|u\|_p\right)$. Then we show by a simple contradiction argument that there is a C such that $\|u\|_p \leq C\|Lu\|_p$. These two inequalities clearly establish our result.

Write $L = \partial_i(a^{ij}\partial_j) = a^{ij}\partial_i\partial_j + b^j\partial_j = \tilde{L} + b^j\partial_j$. We then have, from the previous L^p estimates,

$$\begin{aligned}\|u\|_{p,2,\Omega} &\leq C_1\left(\|\tilde{L}u\|_{p,\Omega} + \|u\|_{p,\Omega}\right) \\ &\leq C_1\left(\|Lu\|_{p,\Omega} + \|b^j\partial_j u\|_{p,\Omega} + \|u\|_{p,\Omega}\right) \\ &\leq C_1\left(\|Lu\|_{p,\Omega} + \|b^j\|_{p,\Omega}\, \|\partial_j u\|_{C^0,\Omega} + \|u\|_{p,\Omega}\right) \\ &\leq C_1\left(\|Lu\|_{p,\Omega} + Q\,\|\partial_j u\|_{C^0,\Omega} + \|u\|_{p,\Omega}\right).\end{aligned}$$

Thus we need to bound $\|\partial_j u\|_{C^0}$ in terms of $\|Lu\|_p$ and $\|u\|_p$. This is done by a bootstrap method beginning with an L^2 estimate for $\partial_j u$. To get this L^2 estimate, we use

$$u\,Lu = u\,\partial_i(a^{ij}\partial_j u) = \partial_i u\, a^{ij}\,\partial_j u - \partial_i(u\,a^{ij}\partial_j u) \geq e^{-Q}\partial_i u \delta^{ij}\partial_j u - \partial_i(u\,a^{ij}\partial_j u).$$

Integration over $B(0,r)$, together with the fact that $u = 0$ on $\partial B(0,r)$, then yields

$$\int_{B(0,r)} u\, Lu \geq e^{-Q} \int_{B(0,r)} \sum_i (\partial_i u)^2 - \int_{B(0,r)} \partial_i (u\, a^{ij} \partial_j u) = e^{-Q} \int_{B(0,r)} \sum_i (\partial_i u)^2.$$

Thus $\|\nabla u\|_2^2 \leq e^Q \|u\|_2 \|Lu\|_2 \leq e^Q c(n,r) \|u\|_p \|Lu\|_p$. In particular $\|\nabla u\|_2 \leq \max\{\|u\|_p, e^Q C(n,r) \|Lu\|_p\} \leq C_2(\|Lu\|_p + \|u\|_p)$. This implies that

$$\begin{aligned} \|u\|_{r_1,2} &\leq C_1(\|\tilde{L}u\|_{r_1} + \|u\|_{r_1}) \leq C_1(\|Lu\|_{r_1} + \|b^j\|_p \|\partial_j u\|_2 + \|u\|_{r_2}) \\ &\leq C_3(\|Lu\|_p + \|u\|_p), \end{aligned}$$

where $1/r_1 = 1/p + 1/2$. Now recall that we have a Sobolev inequality $\|\partial_j u\|_{q_1} \leq C(r_1, q_1, n)\|u\|_{r_1,2}$ where $1/q_1 = 1/r_1 - 1/n = 1/2 + (1/p - 1/n) > 0$; also $q_1 > 2$, so we now have a better bound on $\partial_j u$. Doing this over again we get $\|u\|_{r_2,2} \leq C_4\,(\|Lu\|_p + \|u\|_p)$, where $1/r_2 = 1/p + 1/q_1 = (1/p - 1/n) + 1/p + 1/2$ and $\|\partial_j u\|_{q_2} \leq C\,\|u\|_{r_2,2}$, where $1/q_2 = 1/r_2 - 1/n = 1/2 + 2(1/p - 1/n)$. After l iterations of this type, we end up with

$$\|u\|_{r_l,2} \leq C_{2+l}\,(\|Lu\|_p + \|u\|_p), \|\partial_j u\|_{q_l} \leq C\,\|u\|_{r_l,2}$$

where $1/r_l = l(1/p - 1/n) + 1/p + 1/2$ and $1/q_l = 1/2 + (l+1)(1/p - 1/n)$. This will end as soon as $1/2 + (l+1)(1/p - 1/n) < 0$, in which case we can use the other type of Sobolev inequality: $\|\partial_j u\|_{C^0} \leq C\,\|u\|_{r_l,2}$ if $r_l > n$. Since the number of iterations we need to go through in order for this to happen only depends on p and n, we get the desired C^0 estimate on $\partial_j u$.

We now need to show that there is a constant $C(p,n,r)$ such that $\|u\|_p \leq C\,\|Lu\|_p$. If such a C does not exist we can find a sequence u_l with $\|u_l\|_p = 1$, $u_l = 0$ on $\partial\Omega$, and $L_l = \partial_i(a_l^{ij}\partial_j)$ with $\|L_l u_l\|_p \to 0$ as $l \to \infty$. Since all the operators L_l have the same bounds we get

$$\|u_l\|_{p,2} \leq C\,(\|L_l u_l\|_p + \|u_l\|_p) \leq 2C \quad \text{as } l \to \infty.$$

So we can, in addition, assume that u_l converges in the $C^{1+\alpha}$ topology, $\alpha < 1 - n/p$, to a function u with $\|u\|_p = 1$. On the other hand, we also know that

$$\|\nabla u_l\|_2^2 \leq e^Q\,\|u_l\|_2 \|L_l u_l\|_2 \to 0 \quad \text{as } l \to \infty.$$

The C^1 function u must therefore satisfy $\nabla u = 0$ and $u = 0$ on $\partial\Omega$, which implies that $u \equiv 0$ on Ω. This contradicts $\|u\|_p = 1$. □

Elliptic regularity results. Let's return to the case where $L = a^{ij}\partial_j\partial_i$ and a^{ij} is a symmetric matrix whose eigenvalues lie in $[e^{-Q}, e^Q]$. Suppose $u \in L^{p,2}$ solves the equation $Lu = f, a^{ij} \in C^{l+\alpha}$ and $f \in L^{p,l}$ (or $f \in C^{l+\beta}$). Then $u \in L^{p,l+2}$ (or $C^{l+2+\beta}$).

Appendix B: On R and the coefficients of Δ

Suppose that we have a sequence of metrics on a fixed domain $B(0,r)$. How will the curvature operator R and the coefficients of Δ vary when these metrics are assumed to converge in the $L^{p,k}$ topology? We will show below that $\|R\|_p$ varies continuously in the $L^{p,2}$-topology if $p > n/2$ and that the coefficients of Δ vary continuously in the L^p-norm if the metrics converge in the $L^{p,1}$-topology, for $p > n$.

To make the formulation easier, suppose $G : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a positive definite matrix whose entries g_{ij} are functions on some bounded domain $\Omega \subset \mathbb{R}^n$.

If we write $\Delta = g^{ij}\partial_{ij} + b^i\partial_i$, $b^i = \partial_j g^{ij} + (g^{ij}/g)\partial_j g$, then clearly $g^{ij} = G^{-1}$ and $b^i = \partial_j g^{ij} + \frac{1}{2} g^{ij}\partial_j(\log g)$. So we must check how G^{-1} and $\log G$ depend on G. Now $\partial_i G^{-1} = -G^{-1}(\partial_i G)G^{-1}$, so if we vary G in the $L^{p,1}$-topology, for $p > n$, then G, G^{-1}, $\det G$, and $\det G^{-1}$ vary continuously in the C^0 topology $\partial_i G$, $\partial_i G^{-1}$ vary continuously in the L^p topology. To see that b^i varies continuously in the L^p topology, it suffices to check that $\partial_j g = \partial_j \det G$ varies continuously in the L^p topology. But $\partial_j \det G = F(G, \partial_j G)$ for some algebraic function which is linear in the second matrix variable. This observation leads to the desired result.

To check the continuity of R, recall that $R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \sum_r \Gamma^r_{jk}\Gamma^l_{ir} - \Gamma^r_{ik}\Gamma^l_{jr}$, where

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l g^{il}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}).$$

So if G varies continuously in $L^{p,2}$, for $p > n/2$, then ∂G varies continuously in L^{2p}; and from the above we conclude that Γ^i_{jk} varies continuously in $L^{1,p} \subset L^{2p}$, so R varies continuously in L^p.

If we vary G in $L^{p,k}$, for $k \geq 2$ and $p > n/2$, we can also easily check that R varies continuously in $L^{p,k-2}$, that G^{-1} varies continuously in $L^{p,k}$, and that b^i (first order term in Δ) varies continuously in $L^{p,k-1}$.

From the remarks about the coefficients of Δ, we see that conditions (h2) and (h3) imply that we can use the results from Appendix A without further ado.

Appendix C: The Eguchi–Hanson metric

(See also [Petersen and Zhu 1994].)

Let σ_1, σ_2 and σ_3 be the standard left- or right-invariant one-forms on $S^3 = \mathrm{SU}(2)$ with $d\sigma_i = -2\sigma_{i+1} \wedge \sigma_{i+2}$ (indices taken modulo 3). The canonical metric of curvature 1 can be written $ds^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. We shall consider metrics on $I \times S^3$ of the form $dr^2 + \varphi^2(r)\sigma_1^2 + \psi^2(r)(\sigma_2^2 + \sigma_3^2)$, where $r \in I$ and I is an interval. We think of φ as being a function adjusting the length of the Hopf fiber in S^3. The volume form of this metric is $dr \wedge \varphi(r)\sigma_1 \wedge \psi(r)\sigma_2 \wedge \psi(r)\sigma_3$. Using this choice, we can define the Hodge $*$ operator, which, in particular, maps Λ^2 to Λ^2 and satisfies $*^2 = 1$. The two-forms $dr \wedge \varphi\sigma_1 \pm \psi^2\sigma_2 \wedge \sigma_3$, $dr \wedge \psi\sigma_2 \pm \psi\sigma_3 \wedge \varphi\sigma_1$, and $dr \wedge \psi\sigma_3 \pm \varphi\sigma_1 \wedge \psi\sigma_2$ have eigenvalues ± 1 for $*$, and form a basis for $\Lambda^2 T_p M$

for all $p \in M$. Normalizing these vectors by $1/\sqrt{2}$ we get an orthonormal basis for $\Lambda^2 T_p M$. In this basis, the curvature operator R has the form

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix},$$

where A and D are the parts of the curvature tensor on the eigenspaces corresponding to the eigenvalues 1 and -1, respectively. It is a simple algebraic exercise to see that a four-dimensional manifold is Einstein if and only if $\sec(\pi) = \sec(\pi^\perp)$ for all planes π and their orthogonal components $\pi^\perp$. Using this, one can check that the metric is Einstein if and only if $B \equiv 0$. In our case, A, B, and D are diagonal matrices. If we set $S_{ij} = \sec(\sigma_i \wedge \sigma_j)$, where $\sigma_0 = dr$ etc., and further set $M = (\dot{\varphi}\psi - \dot{\psi}\varphi)/\psi^3$, the diagonal matrices bemoce

$$\begin{aligned} B &= \tfrac{1}{2}\operatorname{diag}(-S_{01} + S_{23}, -S_{02} + S_{13}, -S_{03} + S_{12}), \\ A &= \tfrac{1}{2}\operatorname{diag}(S_{01} + S_{23} - 4M, S_{02} + S_{13} + 2M, S_{03} + S_{12} + 2M), \\ D &= \tfrac{1}{2}\operatorname{diag}(S_{01} + S_{23} + 4M, S_{02} + S_{13} - 2M, S_{03} + S_{12} - 2M). \end{aligned}$$

The sectional curvatures can be computed using the tube formula, Gauss equations, and Codazzi–Maimardi equations, if we think of S^3 as sitting in $I \times S^3$ as a hypersurface for each r. We get

$$\begin{aligned} S_{23} &= \frac{4\psi^2 - 3\varphi^2}{\psi^4} - \frac{\dot{\psi}^2}{\psi^2}, & S_{j0} &= -\frac{\ddot{\psi}}{\psi} & &\text{for } j = 2, 3, \\ S_{10} &= -\frac{\ddot{\varphi}}{\varphi}, & S_{1i} &= \frac{\varphi^2}{\psi^4} - \frac{\dot{\varphi}\dot{\psi}}{\varphi\psi} & &\text{for } i = 2, 3. \end{aligned}$$

Hence the metric is Einstein if

$$-\frac{\ddot{\varphi}}{\varphi} = \frac{4\psi^2 - 3\varphi^2}{\psi^4} - \frac{\dot{\psi}^2}{\psi^2}, \qquad -\frac{\ddot{\psi}}{\psi} = \frac{\varphi^2}{\psi^4} - \frac{\dot{\varphi}\dot{\psi}}{\varphi\psi}.$$

One can explicitly solve these equations, but we will only be interested in the special case where $\varphi = \psi\dot{\psi}$ and $\dot{\psi}^2 = 1 - (\alpha/6)\psi^2 + k\psi^{-4}$, where α is the Einstein constant and k is just some integration factor. When $\alpha = 6$ and $k = 0$, we see that $\psi = \sin r$ and $\varphi = \sin r \cos r$; this corresponds to the Fubini–Study metric on $\mathbb{C}P^2$. When $\alpha = 0$ and $k < 0$, we get a family of Ricci-flat metrics. These are the examples of Eguchi and Hanson. You may want to check that changing k in this case is the same as scaling the metric.

To get a smooth complete metric, we need the initial condition $\psi(0) = (-k)^{-1/4}$. This implies $\dot{\psi}(0) = 0$, $\varphi(0) = 0$, and $\dot{\varphi}(0) = 2$. Since $\dot{\varphi}(0)$ is twice as large as we would like, we divide out by the antipodal maps on S^3, which is still an isometry of $dr^2 + \varphi^2\sigma_1^2 + \varphi^2(\sigma_2^2 + \sigma_3^2)$. Then our metric lives on $I \times \mathbb{RP}^3 = I \times SO(3)$, and the Hopf fiber now has length π rather than 2π. As $r \to 0$, we can then check that the above metric defines a smooth metric on $TS^2 = \big([0, \infty) \times \mathbb{RP}^3\big)/(0 \times \mathbb{RP}^3)$, where $0 \times \mathbb{RP}^3 \sim S^2$ by the Hopf fibration.

The level set $r = 0$ is a totally geodesic S^2 of constant curvature $4(-k)^{1/2}$, while the level sets $r = r_0$ are SO(3) with the metric $\varphi^2(r_0)\sigma_1^2 + \psi^2(r_0)(\sigma_2^2 + \sigma_3^2)$.

The differential equation $\dot\psi^2 = 1 + k\psi^{-4}$ shows that ψ is on the order of r as $r \to \infty$. Thus $\dot\psi \simeq 1$, $\ddot\psi = -4k\psi^{-5} \simeq 0$ as $r \to \infty$. Using this, one can easily see that the norm of the curvature operator, $|R|$, is of order r^{-6} at infinity.

The volume form satisfies

$$\begin{aligned} dr \wedge \varphi(r)\sigma_1 \wedge \psi^2(r)\sigma_2 \wedge \sigma_3 &= \psi^3\,(1 + k\psi^{-4})^{1/2}\, dr \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &\simeq r^3\, dr \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3. \end{aligned}$$

Thus the metric at ∞ is very close to the cone over $\mathbb{RP}^3$, namely $dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3)$ on $[0,\infty) \times SO(3)$. This implies that $\operatorname{vol} B(p,r) \geq \frac{1}{2}\omega_4 r^4$ for all r and p.

Also, $\|R\|_p = \int |R|^p\, d\operatorname{vol} \leq C \int_1^\infty r^{-3p} r^3 dr = C \int_1^\infty r^{3-3p} dr < \infty$ for $p > \frac{4}{3}$. In particular, $\|R\|_{n/2} = \|R\|_2$ is finite.

As $k \to -\infty$, the level set $r = 0$ will degenerate to a point and the whole metric converges to the cone metric $dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ on $[0,\infty) \times SO(3)$. At the same time, $\|R\|_2$ stays bounded, but will go to zero on sets $r \geq \varepsilon > 0$. What happends is that the collapse of S^2 absorbs all of the $\|R\|_2$.

We have now constructed some interesting complete Ricci-flat metrics. To get compact examples, we can just use the fact that these examples as $k \to -\infty$ look like cones, and then glue them into compact examples which locally have such cones. We illustrate this as done in [Anderson 1990a]: The graphs for φ and ψ look approximately like r at ∞. The larger k is, the faster they get to look linear. For large k, take φ, ψ on [0, 5] and then bend them to have zero derivatives at $r = 5$. This can be done keeping $\operatorname{Ric} \geq 0$ and without changing the volume much. Next reflect φ and ψ in the line $r = 5$. This yields metrics on $TS^2 \cup TS^2 = S^2 \times S^2$ that, near $r = 0$ and $r = 10$, look like the Eguchi–Hanson metric, and otherwise don't change much. As $k \to -\infty$, these compact examples degenerate at $r = 0, 10$ and the space in the limit becomes a suspension over $\mathbb{RP}^3$. One can see that this process can happen through metrics with $0 \leq \operatorname{Ric} \leq 1000$, $\operatorname{vol} \geq \frac{1}{5}$, $\operatorname{diam} \leq 10$, and $\|R\|_2 \leq 1000$.

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Peter Petersen
Department of Mathematics
University of California
Los Angeles, CA 90095