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Injectivity Radius Estimates and Sphere Theorems

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Abstract. We survey results about the injectivity radius and sphere theorems, from the early versions of the topological sphere theorem to the authors' most recent pinching below-$\frac{1}{4}$ theorems, explaining at each stage the new ideas involved.

Introduction

Injectivity radius estimates and sphere theorems have always been a central theme in global differential geometry. Many tools and concepts that are now fundamental for comparison geometry have been developed in this context.

This survey of results of this type reaches from the early versions of the topological sphere theorem to the most recent pinching below-$\frac{1}{4}$ theorems. Our main concern is to explain the new ideas that enter at each stage. We do not cover the differentiable sphere theorem and sphere theorems based on Ricci curvature.

In Sections 1–3 we give an account of the entire development from the first sphere theorem of H. E. Rauch to M. Berger's rigidity theorem and his pinching below-$\frac{1}{4}$ theorem. Many of the main results depend on subtle injectivity radius estimates for compact, simply connected manifolds.

In Section 4 we present our recent injectivity radius estimate for odd-dimensional manifolds M^n with a pinching constant below $\frac{1}{4}$ that is independent of n [Abresch and Meyer 1994]. With this estimate the restriction to even-dimensional manifolds can be removed from the hypotheses of Berger's pinching below-$\frac{1}{4}$ theorem.

Additional work is required in order to get a sphere theorem for odd-dimensional manifolds M^n with a pinching constant $< \frac{1}{4}$ independent of n. This result and the basic steps involved in its proof are presented in Sections 5–7; details can

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be found in [Abresch and Meyer a]. The essential step in the geometric part of the argument is to establish Berger's horseshoe conjecture for simply connected manifolds. For this purpose we need the new Jacobi field estimates in Section 6.

1. On the Topological Sphere Theorem

The topological sphere theorem was one of the first results in Riemannian geometry where the topological type of a manifold M^n is determined by inequalities for its sectional curvature K_M and some mild global assumptions. Building on earlier work of Rauch and Berger, the final version of this theorem was obtained by W. Klingenberg [1961]:

THEOREM 1.1 (TOPOLOGICAL SPHERE THEOREM). *Let M^n be a complete, simply connected Riemannian manifold with strictly $\frac{1}{4}$-pinched sectional curvature. Then M^n is homeomorphic to the sphere $\mathbb{S}^n$.*

Note that a positively curved manifold M^n is said to be *strictly δ-pinched* if and only if $\inf K_M > \delta \sup K_M$.

The first version of the theorem, for manifolds that are approximately 0.74-pinched, had been obtained by Rauch [1951]. Several years passed until Berger and Klingenberg managed to improve the result. The proofs of all these theorems are based on direct methods in comparison geometry. We shall describe the basic ideas later in this section.

First we discuss the hypotheses of the topological sphere theorem and explain in what sense the theorem is optimal.

REMARKS 1.2. (i) Strict $\frac{1}{4}$-pinching implies in particular that $\inf K_M \geq \lambda > 0$, and by Myers' theorem [1935; 1941] M^n is a compact manifold with diameter $\leq \pi/\sqrt{\lambda}$. For this reason the theorem can equivalently be stated for compact rather than complete manifolds.

(ii) By Synge's lemma [1936] any compact, oriented, even-dimensional manifold with $K_M > 0$ is simply connected. Thus, in the even-dimensional case it is enough to assume that the manifold M^n is orientable rather than simply connected.

(iii) The hypothesis $\min K_M > \frac{1}{4} \max K_M$ is optimal provided that the dimension of M^n is even and ≥ 4. In fact, the sectional curvature of the Fubini–Study metric on the complex and quaternionic projective spaces $\mathbb{CP}^m$ and $\mathbb{HP}^m$, and on the Cayley plane $\mathrm{Ca}\mathbb{P}^2$, is weakly quarter-pinched. The dimensions of these spaces are $2m$, $4m$, and 16, and except for $\mathbb{CP}^1$ and $\mathbb{HP}^1$ these spaces are not homotopically equivalent to spheres.

(iv) Nevertheless, it is possible to write more adapted curvature inequalities and relax the hypotheses of the topological sphere theorem accordingly. Sphere theorems based on pointwise pinching conditions were established in the mid seventies [Im Hof and Ruh 1975]. The most advanced result in this direction was obtained by M. Micallef and J. D. Moore in 1988, and will be discussed at the end of this section.

For two-, three-, and four-dimensional manifolds stronger results are known. Any compact, simply connected surface is diffeomorphic to $\mathbb{S}^2$, and by the Gauss–Bonnet theorem any compact, orientable surface with strictly positive curvature is also diffeomorphic to $\mathbb{S}^2$.

Furthermore, any compact, simply connected three-manifold M^3 with strictly positive Ricci curvature is diffeomorphic to the standard three-sphere. This assertion follows from a more general result about the Ricci flow:

THEOREM 1.3 [Hamilton 1982]. *Let (M^3, g) be a compact, connected, three-dimensional Riemannian manifold with Ricci curvature* $\mathrm{ric} > 0$ *everywhere. Then g can be deformed in the class of metrics with* $\mathrm{ric} > 0$ *into a metric with constant sectional curvature K_M.*

Finally, in dimension 4 one can determine all homeomorphism types under some weaker pinching condition by combining Bochner techniques with M. Freedman's classification of simply connected, topological four-manifolds [1982]:

THEOREM 1.4 [Seaman 1989]. *Let M^4 be a compact, connected, oriented Riemannian four-manifold without boundary. Suppose that the sectional curvature K_M of M^4 satisfies* $0.188 \approx (1 + 3\sqrt{1 + 2^{5/4} \cdot 5^{-1/2}})^{-1} \leq K_M \leq 1$. *Then M^4 is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$.*

Beginning with D. Gromoll's thesis [1966], various attempts have been made to prove that M^n is diffeomorphic to the sphere $\mathbb{S}^n$ with its standard differentiable structure. The optimal pinching constant for a *differentiable sphere theorem* is not known. Except for low-dimensional special cases, the best constant obtained so far is approximately 0.68 [Grove et al. 1974a; 1974b; Im Hof and Ruh 1975]. However, not a single exotic sphere is known to carry a metric with $K_M > 0$. An exotic Milnor seven-sphere that comes with a metric of nonnegative sectional curvature has been described by Gromoll and W. T. Meyer [1974].

The most recent results in this direction are due to M. Weiss [1993]. Using sophisticated topological arguments, he has shown that any exotic sphere Σ^n that bounds a compact, smooth, parallelizable $4m$-manifold with $m \geq 2$ does not admit a strictly quarter-pinched Riemannian metric, provided that Σ^n represents an element of even order in the group Γ_n of differentiable structures on $\mathbb{S}^n$.

For the sequel it will be convenient to introduce *generalized trigonometric functions*, which interpolate analytically between the usual trigonometric and hyperbolic functions. The generalized sine sn_λ is defined as the solution of $y'' + \lambda y = 0$ with initial data $y(0) = 0$ and $y'(0) = 1$: explicitly,

$$\mathrm{sn}_\lambda(\varrho) := \begin{cases} \lambda^{-1/2} \sin(\sqrt{\lambda}\,\varrho) & \text{if } \lambda > 0, \\ \varrho & \text{if } \lambda = 0, \\ |\lambda|^{-1/2} \sinh(\sqrt{|\lambda|}\,\varrho) & \text{if } \lambda < 0. \end{cases}$$

The generalized cosine and cotangent are given by $\mathrm{cn}_\lambda := \mathrm{sn}'_\lambda$ and $\mathrm{ct}_\lambda := \mathrm{cn}_\lambda/\mathrm{sn}_\lambda$.

On Rauch's proof [1951] of the sphere theorem. The argument dispenses with the two-dimensional case by referring to the Gauss–Bonnet theorem as explained above. His basic idea for proving the theorem in dimensions $n \geq 3$ was to recover the structure of a *twisted sphere* on the manifold M^n under consideration. For this purpose he studies the exponential maps $\exp_p$ and $\exp_q$ at two points $p, q \in M^n$. This approach requires a pinching constant δ equal to the positive root of $\sin(\pi\sqrt{\delta}) = \frac{1}{2}\sqrt{\delta}$, or approximately 0.74. (Note: We always report approximate bounds by rounding to the safe side, so they sometimes differ from occurrences in the literature that are rounded to the nearest value either way.)

Without loss of generality Rauch scales the Riemannian metric on M^n to ensure that $\delta < K_M < 1$. He picks the point $p \in M^n$ and some unit tangent vector $v_0 \in T_pM$ arbitrarily and sets $q := \exp_p \pi v_0$. In order to control the geometry of the exponential maps $\exp_p$ and $\exp_q$, he then develops the by now well-known *Rauch comparison theorems* for Jacobi fields. These estimates imply in particular that

(i) the conjugate radius $\operatorname{conj} M^n$ is $> \pi$;
(ii) the image of the sphere $S(0,\pi) \subset T_pM$ under $\exp_p$ has diameter $< \pi\ \mathrm{sn}_\delta(\pi)$;
(iii) for any $\hat{\varrho} < \frac{1}{2}\pi$ the ball $B(0,\hat{\varrho}) \subset \tilde{B}_q$, where $\tilde{B}_q$ is the ball $B(0,\pi) \subset T_qM$ equipped with the metric $\exp_q^* g$, is strictly convex. Its boundary has strictly positive second fundamental form.

Specializing to $\delta \approx 0.74$, Rauch concludes that the diameter of $\exp_p\bigl(S(0,\pi)\bigr)$ is bounded by $\frac{1}{2}\pi - 2\varrho$ for $\varrho > 0$ sufficiently small. Lifting the restriction $\exp_p|_{S(0,\pi)}$ under $\exp_q$, he thus obtains an immersion ϕ of $S(0,\pi) \subset T_pM$ into the closed ball of radius $\frac{1}{2}\pi - 2\varrho$ centered at the origin in T_qM, mapping πv_0 to the origin. For this construction it is crucial that the sphere $S(0,\pi) \subset T_pM$ be simply connected, that is, that the dimension of M^n be ≥ 3. (Rauch describes the lifting under the local diffeomorphism $\exp_q$ in a more classical terminology, widely used in complex analysis when dealing with monodromy. He speaks of a "c-process" based on a "purse-string construction".)

The next step is to find for each $w \in S(0,\pi) \subset T_pM$ the smallest number $t_w \in (0,1)$ such that the lift of the geodesic $t \mapsto \exp_p((1-t)w)$ under $\exp_q$ starting at $\phi(w)$ leaves the ball $B(0,\, \frac{1}{2}\pi - \varrho) \subset T_qM$. It is not a priori clear that such a number t_w exists unless $w = \pi v_0$. On the other hand, if t_w exists, it follows from (iii) that the lifted geodesic must intersect the sphere $S(0,\, \frac{1}{2}\pi - \varrho) \subset T_qM$ transversally in $t = t_w$. This observation is the basis for an elaborate continuity argument that shows that t_w exists for all $w \in S(0,\pi) \subset T_pM$. Moreover, this continuity argument provides a homeomorphism $\theta : S(0,\, \frac{1}{2}\pi - \varrho) \subset T_qM \to \partial\Omega$, where $\partial\Omega$ denotes the boundary of the star-shaped set

$$\Omega := \{\tau w \mid 0 \leq \tau < 1 - t_w,\ w \in S(0,\pi) \subset T_pM\}.$$

When gluing $\bar{\Omega}$ and $\bar{B}(0, \frac{1}{2}\pi - \varrho) \subset T_qM$ by means of θ, one obtains a twisted sphere Σ^n. The exponential maps $\exp_p$ and $\exp_q$ fit together. They induce a local homeomorphism $\Sigma^n \to M^n$. Since M^n is simply connected this local homeomorphism must actually be a global homeomorphism. □

The starting point for improving Rauch's sphere theorem was the following injectivity radius estimate:

THEOREM 1.5 [Klingenberg 1959]. *Let M^n be an even-dimensional, compact, simply connected Riemannian manifold with strictly positive sectional curvature K_M. Then the injectivity radius* $\operatorname{inj} M^n$ *is controlled in terms of the conjugate radius* $\operatorname{conj} M^n$:

$$\operatorname{inj} M^n = \operatorname{conj} M^n \geq \pi/\sqrt{\max K_M}\,.$$

We shall discuss this result together with Theorem 1.6 below. For the moment our issue is to explain how such an estimate can be employed for the proof of the sphere theorem. Actually, as a first application of Theorem 1.5, Klingenberg obtained a sphere theorem for even-dimensional manifolds requiring only a pinching constant $\delta \approx 0.55$, the positive solution of $\sin(\pi\sqrt{\delta}) = \sqrt{\delta}$ [Klingenberg 1959, Theorem 2].

The basic advantage provided by the injectivity radius estimate is the fact that the immersions $\exp_p : B(0, \pi) \to B(p, \pi)$ and $\exp_q : B(0, \pi) \to B(q, \pi)$ are recognized as diffeomorphisms onto their images in M^n. Since the points p and $q \in M^n$ are chosen in the same way as in Rauch's original approach, a slightly modified continuity argument implies that the cut locus C_q of q lies in $B(p, \pi)$ and vice versa. Thus the open balls $B(p, \pi)$ and $B(q, \pi)$ cover the manifold. It follows that M^n is the union of two closed cells with a common boundary homeomorphic to $\mathbb{S}^{n-1}$. In other words, M^n itself is recognized as a twisted sphere. This construction avoids several lifting arguments from Rauch's proof, and thus it eliminates some constraints on the pinching constant.

Theorem 1.5 was also the starting point of Berger's work on the topological sphere theorem. Combining Klingenberg's injectivity radius estimate with Toponogov's triangle comparison theorem, which had just appeared in the literature, Berger [1960b, Théorème 1] established Theorem 1.1 for even-dimensional manifolds with the optimal pinching constant. Subsequently, he published an independent proof [Berger 1962b, Theorem 3] of the triangle comparison theorem, based on an extension of Rauch's comparison theorems for Jacobi fields rather than on Alexandrov's ideas for surfaces.

ON BERGER'S PROOF OF THE SPHERE THEOREM. Here the starting point is the observation that the choice of the points $p, q \in M^n$ in the preceding work of Rauch and Klingenberg was not optimal. Berger suggested picking p and q in such a way that the distance $d(p, q)$ in M^n is maximal. The key property of such a pair of antipodal points is the fact that for any unit tangent vector $v \in T_qM$

there exists a minimizing geodesic $c : [0,1] \to M^n$ from q to p making an acute angle with v, that is, such that $g(c'(0),\, v) \geq 0$. Assuming that $\frac{1}{4} < K_M \leq 1$, he can thus apply Toponogov's triangle comparison theorem in order to conclude that the metric balls $B(p,\pi)$ and $B(q,\pi)$ cover M^n.

Then he can proceed with some arguments from Klingenberg's proof: Theorem 1.5 reveals that $B(p,\pi)$ and $B(q,\pi)$ are diffeomorphic to the balls $B(0,\pi)$ in the tangent spaces T_pM and T_qM, respectively, and again M^n can be recognized as a twisted sphere. ☐

Berger [1960b, Théorème 2] also succeeded in analyzing the additional phenomena that occur for simply connected, weakly quarter-pinched manifolds. We shall review this result in Theorem 2.1.

With Berger's proof, the only missing ingredient for the final version of the topological sphere theorem as stated in 1.1 was a suitable injectivity radius estimate in the odd-dimensional case. Such an estimate was established shortly afterwards:

THEOREM 1.6 [Klingenberg 1961]. *Let M^n be a compact, simply connected Riemannian manifold with strictly $\frac{1}{4}$-pinched sectional curvature. Then the injectivity radius* $\operatorname{inj} M^n$ *and the conjugate radius* $\operatorname{conj} M^n$ *coincide*:

$$\operatorname{inj} M^n = \operatorname{conj} M^n \geq \pi/\sqrt{\max K_M}\,.$$

REMARKS 1.7. (i) In Theorem 1.5 it is crucial to assume that the manifold has nonnegative sectional curvature. Otherwise, there is not even a uniform lower bound on the injectivity radius for simply connected surfaces. In fact, it is easy to construct surfaces of revolution with $-1 \leq K_M \leq 1$ and arbitrarily small injectivity radius. The diameter of these surfaces, which look like hourglasses, increases without bound as the injectivity radius approaches zero.

(ii) The most significant difference between Theorems 1.5 and 1.6 is the pinching condition that appears in the hypothesis of the latter. Such a condition is necessary to get a result for odd-dimensional manifolds at all. The optimal value for the pinching constant in Theorem 1.6 is not known. Berger has shown [1962a] that a constant $< \frac{1}{9}$ is not sufficient in order to obtain even the slightly weaker inequality $\operatorname{inj} M^n \geq \pi/\sqrt{\max K_M}$. For this purpose he considers a family of Riemannian metrics g_ε on the odd-dimensional spheres $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. These metrics are defined by shrinking the standard metric in the direction of the Hopf circles $\{e^{it}p \mid t \in \mathbb{R}/2\pi\mathbb{Z}\} \subset \mathbb{S}^{2n+1}$ in such a way that their lengths with respect to g_ε become $2\pi\varepsilon$. The range of the sectional curvature of g_ε is the interval $[\varepsilon^2,\, 4 - 3\varepsilon^2]$, provided of course that $0 < \varepsilon \leq 1$. Clearly, $\pi\varepsilon < \pi/\sqrt{4 - 3\varepsilon^2}$ for $\varepsilon^2 < \frac{1}{3}$. This means that for any $\delta \in (0, \frac{1}{9})$ there exists a Berger metric g_ε whose sectional curvature is δ-pinched and whose injectivity radius is strictly less than $\pi/\sqrt{\max K_M}$. Unless $\varepsilon = 1$ there does not exist any pair consisting of a (horizontal) geodesic $\gamma : \mathbb{R} \to (\mathbb{S}^{2n+1}, g_\varepsilon)$ and a parallel unit normal field v along γ such that the sectional curvature on each plane $\operatorname{span}\{\gamma'(s), v(s)\}$ equals

$4 - 3\varepsilon^2$. Hence for any $\varepsilon \in (0,1)$ the conjugate radius of g_ε is strictly greater than $\pi/\sqrt{\max K_M}$. A direct computation reveals that the conjugate radius of g_ε arises as the first positive zero of the map $s \mapsto \varepsilon^2 + (1-\varepsilon^2)\, s \cot s$, and thus the injectivity radius of g_ε ceases to be equal to its conjugate radius as ε becomes less than 0.589. This means that the optimal value for the pinching constant in Theorem 1.6 is necessarily at least $0.117 > \frac{1}{9}$.

(iii) Even worse, in dimension 7, the Aloff–Wallach examples [1975] described in the appendix contain a sequence of simply connected, homogeneous Einstein spaces whose pinching constants converge to $\frac{1}{37}$ and whose injectivity radii converge to zero. In other words, if $\delta < \frac{1}{37}$ there does not exist any a priori lower bound for the injectivity radius of a seven-dimensional, simply connected, δ-pinched Riemannian manifold.

The proofs of Theorems 1.5 and 1.6, and of essentially any other injectivity radius estimate, begin with the observation that the injectivity radius of a compact Riemannian manifold can be related to its conjugate radius and to the length of a shortest geodesic loop. We give a refined version of [Klingenberg 1959, Lemma 4]:

Lemma 1.8 [Cheeger and Ebin 1975, Lemma 5.6]. *Let M^n be a complete Riemannian manifold, and let $p \in M^n$. Let $\ell_M(p)$ denote the minimal length of a nontrivial geodesic loop $c_0 : [0,1] \to M^n$ starting and ending at p. Then the injectivity radius of M^n at p is* $\operatorname{inj}_M(p) = \min\{\operatorname{conj} M^n, \frac{1}{2}\,\ell_M(p)\}$.

By definition, $\operatorname{inj} M^n = \inf_p \operatorname{inj}_M(p)$. For compact manifolds M^n the infimum is always achieved at some point $p_0 \in M^n$. Furthermore, if $\ell_M(p_0) < 2 \operatorname{conj} M^n$, it is easy to see that the geodesic loop $c_0 : [0,1] \to M^n$ of length $2 \operatorname{inj} M^n$ with $c_0(0) = c_0(1) = p_0$ is actually a closed geodesic $c_0 : \mathbb{R}/\mathbb{Z} \to M^n$. This means that for compact Riemannian manifolds one has $\operatorname{inj} M^n = \min\{\operatorname{conj} M^n, \frac{1}{2}\,\ell(M^n)\}$, where $\ell(M^n) = \inf_p \ell_M(p)$ is the minimal length of a nontrivial closed geodesic $c_0 : \mathbb{R}/\mathbb{Z} \to M^n$.

On the proof of Theorem 1.5. It is a standard fact that a geodesic is not minimizing beyond the first conjugate point. Hence $\operatorname{inj} M^n \le \operatorname{conj} M^n$, and it is possible to proceed indirectly.

Assuming that $\operatorname{inj} M^n < \operatorname{conj} M^n$, Lemma 1.8 asserts that there is a closed geodesic $c_0 : \mathbb{R}/\mathbb{Z} \to M^n$ of length $L(c_0) = 2 \operatorname{inj} M^n$. As in the proof of Synge's lemma [1936], the canonical form theorem for the orthogonal group $\mathsf{SO}(n-1)$ leads to a closed, parallel, unit normal field v_0 along c_0. Since $K_M > 0$, the second variation formula reveals that the nearby curves $c_t : \mathbb{R}/\mathbb{Z} \to M^n$, $s \mapsto \exp_{c_0(s)} t v_0(s)$, are strictly shorter than c_0, provided that t is nonzero and sufficiently small.

At this point Klingenberg observes that the image of such a curve c_t is contained in the closed ball with radius $\frac{1}{2} L(c_t) < \operatorname{inj} M^n$ centered at $c_t(0)$, and thus he can lift c_t under $\exp_{c_t(0)}$ to a map $\tilde{c}_t : \mathbb{R}/\mathbb{Z} \to T_{c_t(0)} M$ such that $\tilde{c}_t(0) = 0$.

Notice that the argument that guarantees the existence of $\tilde{c}_t$ is very different from the arguments that justify taking local lifts under $\exp_p$ up to the conjugate radius, encountered in Rauch's proof of the sphere theorem.

By assumption, $\operatorname{inj} M^n < \operatorname{conj} M^n$, and thus the curves $\tilde{c}_t : \mathbb{R}/\mathbb{Z} \to T_{c_t(0)}M$ define an equicontinuous map $\tilde{c} : \mathbb{R}/\mathbb{Z} \times (0,\varepsilon) \to TM$. Extending this map continuously to $\mathbb{R}/\mathbb{Z} \times [0,\varepsilon)$, one obtains a lift $\tilde{c}_0 : \mathbb{R}/\mathbb{Z} \to T_{c_0(0)}M$ of c_0 under $\exp_{c_0(0)}$, satisfying $\tilde{c}_0(0) = 0$. On the other hand the lift of the geodesic c_0 must be the line $s \mapsto s\,c'(0)$, contradicting the periodicity of $\tilde{c}_0$. □

The proof of Theorem 1.6 is much more subtle. It will be explained in Section 4 together with our recent extension of this injectivity radius estimate (Theorem 4.1).

Further progress in understanding the topological sphere theorem was made by K. Grove and K. Shiohama in the late seventies. Observing that the twisted sphere construction in the proof of Theorem 1.1 resembles the proof of Reeb's theorem in Morse theory, Grove and Shiohama investigated under what conditions the function $f_{pq} : x \mapsto \operatorname{dist}(x,p) - \operatorname{dist}(x,q)$ has only two critical points, an absolute minimum at p and an absolute maximum at q. Persuing this idea, they proved the following result:

Theorem 1.9 (Diameter sphere theorem [Grove and Shiohama 1977]). *Let M^n be a connected, complete Riemannian manifold with sectional curvature $K_M \geq \lambda > 0$ and diameter $\operatorname{diam} M^n > \pi/(2\sqrt{\lambda})$. Then M^n is homeomorphic to the sphere $\mathbb{S}^n$.*

Remark 1.10. The Fubini–Study metrics on the projective spaces $\mathbb{CP}^m$, $\mathbb{HP}^m$, and $\mathrm{Ca}\mathbb{P}^2$ have diameter $\pi/(2\sqrt{\min K_M})$ if $m \geq 2$. Thus in Theorem 1.9 the hypothesis on the diameter is optimal if the dimension of the manifold is even and ≥ 4.

Notice that the distance functions to fixed points p and q are only Lipschitz functions. They are not differentiable at the cut loci C_p and C_q. In fact, the proof of Theorem 1.9 has led to a fruitful definition of what a critical point of a distance function should be. Eventually, an elaborate *critical point theory for distance functions* was developed and successfully applied to many other problems [Grove 1993]. For instance, the proof of M. Gromov's Betti numbers theorem uses critical point theory for distance functions in a substantial way [Abresch 1985; 1987; Gromov 1981b].

With this particular critical point theory it is possible to prove the diameter sphere theorem in a conceptually straightforward way by applying Toponogov's triangle comparison theorem twice [Grove 1987; 1993; Karcher 1989; Meyer 1989]. Moreover, Theorem 1.9 ties in nicely with the injectivity radius estimate from Theorem 1.6. Thinking of the injectivity radius as a lower bound for the diameter, one obtains an alternate, and structurally more appealing, proof of the topological sphere theorem.

The pointwise pinching problem requires completely different techniques. Results of this type are typically not based on direct comparison methods but on partial differential equations on the manifold M^n [Ruh 1982] or on minimal surfaces in M^n [Micallef and Moore 1988]. Here we shall focus on this theorem:

THEOREM 1.11 [Micallef and Moore 1988]. *Let M^n be a compact, simply connected Riemannian manifold of dimension $n \geq 4$. Suppose that M^n has positive curvature on totally isotropic two-planes. Then M^n is homeomorphic to $\mathbb{S}^n$.*

REMARKS 1.12. (i) The notion of positive curvature on totally isotropic two-planes arises naturally when one looks for a condition determining the sign of the difference between the Hodge Laplacian and the rough Laplacian on two-forms.

(ii) The projective spaces $\mathbb{CP}^m$, $\mathbb{HP}^m$, and $\mathrm{Ca}\mathbb{P}^2$ have nonnegative curvature on totally isotropic two-planes.

(iii) A direct computation shows that pointwise strict $\frac{1}{4}$-pinching implies positive curvature on totally isotropic two-planes. This means that the topological sphere theorem holds in dimensions $n \geq 4$ already for compact, simply connected, *pointwise* strictly $\frac{1}{4}$-pinched manifolds.

ON THE PROOF OF THEOREM 1.11. Note that any conformal harmonic map $f : \mathbb{S}^2 \to M^n$ is a common critical point for the Dirichlet functional D and the area functional and, moreover, $\mathrm{area}(f) = D(f)$. In general, area $\leq D$, so the Dirichlet functional is an upper barrier for the area functional at the surface represented by f.

The key step for the proof of the theorem is to show that any nonconstant, branched, minimal two-sphere $f : \mathbb{S}^2 \to M^n$ in a Riemannian manifold of dimension ≥ 4 with positive curvature on totally isotropic two-planes has index $\mathrm{ind}_D(f) \geq \frac{1}{2}(n-3)$. On the other hand, Micallef and Moore prove that any compact Riemannian manifold M^n with $\pi_k(M^n) \neq 0$ for some $k \geq 2$ contains a nonconstant harmonic two-sphere $f : \mathbb{S}^2 \to M^n$ with index $\mathrm{ind}_D(f) \leq k-2$. Combining these two facts, it follows that $\pi_1(M^n) = \cdots = \pi_{[n/2]}(M^n) = 0$. Hence the Hurewicz isomorphism theorem implies that $H_1(M^n;\mathbb{Z}) = \cdots = H_{[n/2]}(M^n;\mathbb{Z}) = 0$, and by the Poincaré duality theorem M^n must be a homology sphere. Thus the result follows using S. Smale's solution of the generalized Poincaré conjecture in dimensions $n \geq 5$ [Milnor 1965, p. 109; Smale 1961] and Freedman's classification of compact, simply connected four-manifolds [1982].

The existence of nonconstant harmonic two-spheres $f : \mathbb{S}^2 \to M^n$ with index $\mathrm{ind}_D(f) \leq k-2$ is established by means of the standard saddle point arguments from Morse theory. Micallef and Moore work with a perturbed version of the α-energy introduced by S. Sacks and K. Uhlenbeck, in order to have a nondegenerate Morse functional that satisfies Condition C of Palais and Smale, and study the limit as $\alpha \to 1$.

In order to obtain the lower bound for the index of such a nonconstant, conformal, branched, minimal two-sphere $f : \mathbb{S}^2 \to M^n$, Micallef and Moore express the Hessian of the Dirichlet functional D in terms of the squared norm of the $\bar{\partial}$-operator of the complexified bundle $f^*TM \otimes_{\mathbb{R}} \mathbb{C}$, rather than in terms of the squared norm of the full covariant derivative ∇. With this modification the zero-order term in the Hessian becomes an expression in the curvatures of totally isotropic two-planes containing ∂f. In particular, the claimed estimate for the index $\operatorname{ind}_D(f)$ follows upon constructing sufficiently many isotropic, holomorphic sections in $f^*TM \otimes_{\mathbb{R}} \mathbb{C}$ whose exterior product with ∂f is nontrivial. The appropriate tool for this purpose is Grothendieck's theorem on the decomposition of holomorphic vector bundles over $\mathbb{CP}^1$. Combining this theorem with the fact that the first Chern class of $f^*TM \otimes_{\mathbb{R}} \mathbb{C}$ vanishes, the authors construct a complex linear space of dimension $\geq \frac{1}{2}(n-1)$ of isotropic holomorphic sections. □

2. Berger's Rigidity Theorem and Related Results

In this section the principal goal is to present the extensions of the topological sphere theorem and the diameter sphere theorem that hold when all strict inequalities in the hypotheses of these theorems are replaced by their weak counterparts. In particular, we shall see that the projective spaces mentioned in Remarks 1.2(iii) and 1.10 are the only other possibilities for the topological type of M^n. It is not known whether the sphere theorem by Micallef and Moore can be extended correspondingly or not.

When working on the topological sphere theorem, Berger actually studied the limiting case of simply connected, weakly quarter-pinched, even-dimensional manifolds, too:

Theorem 2.1 (Berger's rigidity theorem [Berger 1960b, Théorème 2]). *Let M^n be an even-dimensional, complete, simply connected Riemannian manifold with $\frac{1}{4} \leq K_M \leq 1$. Then either*

(i) *M^n is homeomorphic to the sphere $\mathbb{S}^n$, or*
(ii) *M^n is isometric to one of the other rank-one symmetric spaces, namely $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or $\mathrm{Ca}\mathbb{P}^2$.*

The first step in Berger's proof of the rigidity theorem was in a sense a predecessor of the diameter sphere theorem. Theorem 1.5 asserts that the manifold has injectivity radius $\operatorname{inj} M^n \geq \pi$. Refining the comparison argument from the proof of the topological sphere theorem, Berger proves that a Riemannian manifold M^n with $\frac{1}{4} \leq K_M \leq 1$, $\operatorname{inj} M^n \geq \pi$, and $\operatorname{diam} M^n > \pi$ is homeomorphic to $\mathbb{S}^n$. There remains the case where $\frac{1}{4} \leq K_M \leq 1$ and $\operatorname{diam} M^n = \operatorname{inj} M^n = \pi$; here he shows that any point $p \in M^n$ lies on a closed geodesic $c : \mathbb{R}/\mathbb{Z} \to M^n$ of length $L(c) = 2\pi$. With this additional information M^n can be recognized as a rank-one symmetric space by means of an argument that Berger had used shortly before in the proof of a weaker rigidity result [Berger 1960a, Théorème 1].

The assertion that a compact, simply connected Riemannian manifold with $\frac{1}{4} \leq K_M \leq 1$ and $\operatorname{diam} M^n = \pi$ is isometric to a symmetric space is known as Berger's minimal diameter theorem [Cheeger and Ebin 1975, Theorem 6.6(2)]. J. Cheeger and D. Ebin also give a geometrically more direct proof for this result. Their idea is to study the metric properties of the geodesic reflections $\phi_p : B(p,\pi) \to B(p,\pi)$ and prove directly that these maps are isometries that can be extended continuously to M^n. Thus M^n is recognized as a symmetric space. It is an elementary fact that any symmetric space with $K_M > 0$ has rank one.

Remark 2.2. Berger's rigidity theorem has been extended to cover odd-dimensional manifolds as well, asserting that any complete, simply connected, odd-dimensional manifold with weakly quarter-pinched sectional curvature is homeomorphic to the sphere. It should be clear from our sketch of Berger's original proof that such an extension follows immediately once the injectivity radius estimate from Theorem 1.6 has been generalized to the class of simply connected, weakly quarter-pinched manifolds. Such a generalization appeared shortly afterwards in the work of Klingenberg [1962]. However, the argument is technically extremely subtle, and complete proofs were only given much later in two independent papers by Cheeger and Gromoll [1980] and by Klingenberg and T. Sakai [1980].

The diameter sphere theorem due to Grove and Shiohama can be generalized as follows:

Theorem 2.3 (Diameter rigidity theorem [Gromoll and Grove 1987]). *Let M^n be a connected, complete Riemannian manifold with sectional curvature $K_M \geq \lambda > 0$ and diameter $\operatorname{diam} M^n \geq \pi/(2\sqrt{\lambda})$. Then*

(i) *M^n is homeomorphic to the sphere $\mathbb{S}^n$, or*
(ii) *the universal covering $\tilde{M}^n$ of M^n is isometric to $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, or $\mathbb{HP}^{n/4}$, or*
(iii) *the integral cohomology ring of $\tilde{M}^n$ is isomorphic to that of $\mathrm{Ca}\mathbb{P}^2$.*

We are discussing this theorem mainly because its proof has required an entirely new approach for recognizing rank-one symmetric spaces. The details are technically quite subtle, but the basic ideas are geometrically nice and simple.

Beforehand we mention that there are only a few possibilities for the covering maps $\tilde{M}^n \to M^n$ in assertions (ii) and (iii), since by Synge's lemma any orientable even-dimensional manifold M^n with $K_M > 0$ is simply connected. The only nontrivial quotients that can arise are the real projective spaces $\mathbb{RP}^n$, the space forms $\mathbb{S}^{2m+1}/\Gamma$ where the action of Γ preserves some proper orthogonal decomposition of $\mathbb{R}^{2m+2}$, and the spaces $\mathbb{CP}^{2m+1}/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action is given by the antipodal maps in the fibers of some standard projection $\mathbb{CP}^{2m+1} \to \mathbb{HP}^m$.

Because of the structure of their cohomology rings, the spaces $\mathbb{CP}^{2m}$, $\mathbb{HP}^{2m}$, and $\mathrm{Ca}\mathbb{P}^2$ do not admit any orientation-reversing homeomorphisms. With a little more effort one can also show that none of the spaces $\mathbb{HP}^{2m+1}$, where $m \geq 1$, admits a smooth, orientation-reversing, fixed-point-free $\mathbb{Z}_2$-action. For this purpose one verifies that the projection $\pi : \mathbb{HP}^{2m+1} \to \mathbb{HP}^{2m+1}/\mathbb{Z}_2$ onto the hypothetical quotient space induces the zero-map on the fourth integral cohomology groups. Since $T\,\mathbb{HP}^{2m+1} = \pi^* T(\mathbb{HP}^{2m+1}/\mathbb{Z}_2)$, it follows that the first Pontrjagin class $p_1(\mathbb{HP}^{2m+1})$ should vanish, contradicting the fact that $p_1(\mathbb{HP}^k) = 2(k-1)\xi$ where ξ is a generator of $H^4(\mathbb{HP}^k) \cong \mathbb{Z}$ [Greub et al. 1973, Chapter IX, Problem 31(ii)]. In the quaternionic case, the existence of quotients can alternatively be ruled out by observing that the full group of isometries $\operatorname{Isom} \mathbb{HP}^k$ is connected for $k \geq 2$ [Wolf 1977, p. 381].

On the proof of the diameter rigidity theorem. The starting point is to observe that the cut locus of a subspace $\mathbb{CP}^k \subset \mathbb{CP}^m$ is the dual subspace $\mathbb{CP}^{m-k-1} \subset \mathbb{CP}^m$. Moreover, the pairs consisting of such a $\mathbb{CP}^k \subset \mathbb{CP}^m$ and its cut locus can be characterized geometrically as pairs of dual convex sets $A, A' \subset \mathbb{CP}^n$. This means that $\operatorname{dist}(p,p') = \operatorname{diam} \mathbb{CP}^n$ for any pair of points $(p,p') \in A \times A'$.

If $k = 0$, that is, if A' consists of a single point, it is possible to recover the total space $\mathbb{CP}^m$ as the Thom space of the normal bundle of $A \subset \mathbb{CP}^m$. In particular, the total space of the corresponding unit sphere bundle is an $\mathbb{S}^{2m-1}$ that is foliated by equidistant circles. Similar structures can be found on the quaternionic projective spaces $\mathbb{HP}^m$ and on the Cayley plane $\mathrm{Ca}\mathbb{P}^2$. The only differences are that the total spaces of the unit normal bundles of A are spheres $\mathbb{S}^{4m-1}$ foliated by three-spheres and an $\mathbb{S}^{15}$ foliated by seven-spheres, respectively.

In fact, when we normalize λ to 1, the diameter sphere theorem reduces the proof of Theorem 2.3 to the study of compact Riemannian manifolds with $K_M \geq 1$ and $\operatorname{diam} M^n = \frac{1}{2}\pi$. The first step in investigating this setup is to analyze the structure of dual convex sets $A, A' \subset M^n$ by means of critical point theory, establishing more and more of the properties described above. In particular, A and A' are totally geodesic submanifolds, and for any point $p' \in A'$ the exponential map defines a Riemannian submersion π_A from the unit normal sphere in p' to A. It also follows that A is a deformation retract of $M^n \setminus A'$ and vice versa. By the latter assertion, at least one of the sets A or A' is not contractible unless M^n is homeomorphic to a sphere.

On the other hand, if A' were contractible, it would consist of a single point $p' \in M^n$. In this case the manifold M^n is the mapping cone of π_A, and the fibers of this submersion are homotopy spheres of dimensions 1, 3, or 7. Now the key point is to resort to the results about low-dimensional metric foliations of Euclidean spheres in [Gromoll and Grove 1988] to conclude that π_A is isometric to a standard Hopf fibration, except possibly when $n-1 = 15$ and $\dim A = 8$. Clearly, the isometry between π_A and such a standard Hopf fibration induces a continuous map between the corresponding mapping cones, that is, it gives rise to a map from M^n to $\mathbb{CP}^{n/2}$ or $\mathbb{HP}^{n/4}$.

In order to prove the result for simply connected manifolds M^n, it remains to show that the latter map remains an isometry and, moreover, that there exists a pair of dual convex sets $A, A' \subset M^n$ such that A' is contractible. Both steps are accomplished by an argument that uses recursively the concepts presented so far. Finally, the case where M^n is not simply connected is reduced to the preceding one by means of covering theory. □

The problems in recovering the Cayley plane in Theorem 2.3 up to isometry are due to some shortcomings in understanding metrical foliations of Euclidean spheres. Recently, F. Wilhelm [1995] has treated the case of the Cayley plane under more restrictive geometric conditions that yield better information about the structure of the family of dual convex sets in M^{16}:

THEOREM 2.4 (RADIUS RIGIDITY THEOREM). *Let M^n be a connected, complete Riemannian manifold with sectional curvature $K_M \geq \lambda > 0$ and radius* $\operatorname{rad} M^n \geq \pi/(2\sqrt{\lambda})$. *Then*

(i) *M^n is homeomorphic to the sphere $\mathbb{S}^n$, or*
(ii) *the universal covering $\tilde{M}^n$ of M^n is isometric to $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or* $\mathrm{Ca}\mathbb{P}^2$.

Recall that the radius $\operatorname{rad} M^n$ of a compact, connected Riemannian manifold is defined as the infimum of the function $p \mapsto \operatorname{rad}_M(p) := \max_{q\in M^n} \operatorname{dist}(p,q)$. Clearly, $\operatorname{inj} M^n \leq \operatorname{rad} M^n \leq \operatorname{diam} M^n$. If M^n is a compact, simply connected, rank-one symmetric space, all three quantities coincide.

3. On Berger's Pinching Below-$\frac{1}{4}$ Theorem

Since the early sixties it had been a challenging problem to find out whether there is a stability result extending Berger's rigidity theorem. An affirmative answer was only found in 1983.

THEOREM 3.1 [Berger 1983]. *For any even number n there exists a constant $\delta_n < \frac{1}{4}$ such that any n-dimensional, complete, simply connected Riemannian manifold M^n with $\delta_n \leq K_M \leq 1$ is either*

(i) *homeomorphic to the sphere $\mathbb{S}^n$, or*
(ii) *diffeomorphic to $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or* $\mathrm{Ca}\mathbb{P}^2$.

Up to now there has been no analogous theorem where the upper curvature bound is replaced by a corresponding lower bound for the diameter. Results in this direction by O. Durumeric [1987] involve some additional hypotheses.

In contrast to the pinching theorems discussed so far, the proof of Theorem 3.1 does not provide an explicit constant δ_n, because it relies on the *precompactness* of certain spaces of isometry classes of Riemannian manifolds.

Our plan is to summarize the precompactness result, explaining in particular the injectivity radius estimates that are required in this context. We conclude this section by describing how Theorem 3.1 can be deduced from a rigidity theorem that extends Theorem 2.1.

In order to discuss precompactness, one needs a topology on the class of connected Riemannian manifolds. In this context there are actually two natural topologies, the Hausdorff topology and the Lipschitz topology. Both are defined in terms of appropriate distance functions on the class of inner metric spaces. Recall that a connected Riemannian manifold (M^n, g) can be considered as an inner metric space (M, d), where d denotes the Riemannian distance function corresponding to the metric g.

The Hausdorff distance between two inner metric spaces (M_μ, d_μ) is defined as an infimum over all isometric embeddings $\iota_\mu : (M_\mu, d_\mu) \to (X, d)$ into some bigger metric space:

$$\operatorname{dist}_H\bigl((M_1, d_1), (M_2, d_2)\bigr) := \inf_{\iota_\mu : M_\mu \to X} d_H\bigl(\iota_1(M_1), \iota_2(M_2)\bigr)\,,$$

where d_H denotes the Hausdorff distance of the closed subsets $\iota_1(M_1)$ and $\iota_2(M_2)$ within the metric space (X, d):

$$d_H\bigl(\iota_1(M_1), \iota_2(M_2)\bigr) = \inf\bigl\{\varepsilon > 0 \bigm| \iota_1(M_1) \subset U_\varepsilon(\iota_2(M_2)),\ \iota_2(M_2) \subset U_\varepsilon(\iota_1(M_1))\bigr\}.$$

The Lipschitz distance of (M_1, d_1) and (M_2, d_2), on the other hand, is defined as an infimum over the class of all bijective maps $f : M_1 \to M_2$:

$$\operatorname{dist}_L\bigl((M_1, d_1), (M_2, d_2)\bigr) := \inf_{f : M_1 \to M_2} \log_+(\operatorname{dil} f) + \log_+(\operatorname{dil} f^{-1})\,,$$

where $\log_+(x) := \sup\{0, \log(x)\}$ and

$$\operatorname{dil}(f) := \sup_{p \neq q} \frac{d_2(f(p), f(q))}{d_1(p, q)}\,.$$

Note that $\operatorname{dist}_L((M_1, d_1), (M_2, d_2)) = +\infty$ unless M_1 and M_2 are homeomorphic.

In the presence of a uniform bound for the diameter it is not hard to show that any two inner metric spaces that are Lipschitz close have small Hausdorff distance, too. The converse is not true in such generality, as the example consisting of a finite graph $X \subset \mathbb{R}^3$ and its distance tubes $U_\varepsilon(X)$ shows. The graph X and its tubes $U_\varepsilon(X)$ are not homeomorphic, so $\operatorname{dist}_L(X, U_\varepsilon(X)) = +\infty$ despite the fact that $\operatorname{dist}_H(X, U_\varepsilon(X)) \to 0$ as $\varepsilon \to 0$.

However, Gromov [1981a] has obtained the following result for the space $\mathfrak{M}_{\lambda\ \cdot\ v}^{\Lambda\ D\ \cdot}(n)$ consisting of all isometry classes of compact, n-dimensional Riemannian manifolds (M^n, g) with $\lambda \le K_M \le \Lambda$, $\operatorname{diam} M^n \le D$, and $0 < v \le \operatorname{vol} M^n$ (compare also [Peters 1987]):

THEOREM 3.2 (GROMOV'S COMPACTNESS THEOREM). *Let $n \in \mathbb{N}$, $\lambda \le \Lambda$, and $v, D > 0$. Then, on $\mathfrak{M} := \mathfrak{M}_{\lambda\ \cdot\ v}^{\Lambda\ D\ \cdot}(n)$, the Hausdorff and the Lipschitz topologies coincide. Furthermore, $\mathfrak{M}$ is relatively compact in the space of isometry classes of $C^{1,1}$-manifolds with C^0-metrics g.*

REMARK 3.3. An upper bound for the number of diffeomorphism types of the Riemannian manifolds corresponding to the points in $\mathfrak{M}$ had been obtained earlier by Cheeger. This result is known as *Cheeger's finiteness theorem.*

It is possible to recover Cheeger's finiteness theorem, which is a refinement of a finiteness theorem for homotopy types due to A. Weinstein [1967], from Gromov's compactness theorem, except for the precise value of the upper bound, of course. In fact, by Shikata's work [1966] on the differentiable sphere theorem, there exists for any $n \in \mathbb{N}$ a constant $\varepsilon_n > 0$ such that any two n-dimensional Riemannian manifolds (M^n_μ, g_μ) with $\operatorname{dist}_L\big((M^n_1, d_1), (M^n_2, d_2)\big) < \varepsilon_n$ are diffeomorphic. By Gromov's compactness theorem it is possible to cover the space $\mathfrak{M}$ with finitely many dist_L-balls of radius ε_n. Thus there are only finitely many diffeomorphism types among the compact, n-dimensional Riemannian manifolds with $\lambda \le K_M \le \Lambda$, $\operatorname{diam} M^n \le D$, and $0 < v \le \operatorname{vol} M^n$.

REMARKS 3.4. (i) The lens spaces $\mathbb{S}^{2m-1}/\mathbb{Z}_k$ show that the lower volume bound is crucial for Cheeger's finiteness theorem and hence also for Gromov's compactness theorem. Moreover, in dimension 7 the Aloff–Wallach examples [1975] discussed in the appendix provide a family of counterexamples that are not just coverings of each other.

(ii) It is even easier to see that the upper diameter bound is necessary. One simply considers ladders, that is, connected sums of an unbounded number of copies of the same topologically nontrivial manifold.

(iii) Similarly, the lower bound for the sectional curvature turns out to be essential, whereas the upper bound for K_M was discarded in a later finiteness theorem by Grove, P. Petersen, and J.-Y. Wu [Grove et al. 1990].

Cheeger's finiteness theorem can indeed be viewed as an immediate predecessor of Gromov's compactness theorem. The proofs of both rely on a particular injectivity radius estimate, which, in contrast to Theorems 1.5 and 1.6, must not impose any restriction on the sign of the sectional curvature K_M. As explained in Remark 1.7, such an injectivity radius estimate requires some further geometrical hypotheses in addition to the bounds for K_M. The first result in this direction was Cheeger's propeller lemma [Cheeger and Ebin 1975, Theorem 5.8]; we give a version with improved numerical constants, due to E. Heintze and H. Karcher [1978, Corollary 2.3.2]:

PROPOSITION 3.5 (PROPELLER LEMMA). *Let M^n be a complete Riemannian manifold with $\lambda \le K_M$, $\operatorname{diam} M^n \le D$, and $\operatorname{vol} M^n \ge v > 0$. Then the injectivity radius of M^n is bounded from below by*

$$\operatorname{inj} M^n \ge \inf\left\{\operatorname{conj} M^n,\ \frac{\pi v}{\operatorname{vol}\mathbb{S}^n}\ \operatorname{sn}_\lambda\big(\min\{D,\ \pi/(2\sqrt{\lambda}\,)\}\big)^{-(n-1)}\right\},$$

where $\operatorname{vol}\mathbb{S}^n$ is the volume of the unit sphere in $(n+1)$-dimensional Euclidean space.

Proof of Proposition 3.5. The idea is to use Lemma 1.8 in order to reduce the assertion to a simple volume computation. One considers distance tubes $U_r(\gamma)$ around a closed geodesic γ of length $\ell(M^n) = \inf_p \ell_M(p)$. Their volumes can be estimated as follows:

$$\begin{aligned}\operatorname{vol} U_r(\gamma) &\leq \ell(M^n)\operatorname{vol}(\mathbb{S}^{n-2}) \int_0^r \operatorname{sn}_\lambda(\varrho)^{n-2}\operatorname{cn}_\lambda(\varrho)\, d\varrho \\ &= \ell(M^n)\operatorname{vol}(\mathbb{S}^{n-2})\frac{1}{n-1}\ \operatorname{sn}_\lambda(r)^{n-1} = \ell(M^n)\frac{1}{2\pi}\operatorname{vol}(\mathbb{S}^n)\operatorname{sn}_\lambda(r)^{n-1}\,.\end{aligned}$$

Finally, one observes that the closed tube $\bar{U}_r(\gamma)$ covers M^n for $r \geq \operatorname{diam} M^n$ or for $r \geq \pi/(2\sqrt{\lambda})$ if $\lambda > 0$. □

Roughly speaking, the propeller lemma asserts that, in the presence of a lower sectional curvature bound and an upper diameter bound, giving a lower volume bound is equivalent to giving a lower injectivity radius bound. We have followed the approach of Heintze and Karcher, since it invokes the lower volume bound in a more intuitive way than Cheeger's original approach, which gave rise to the name of the result. As a word of caution, the upper bound for $\operatorname{vol} U_r(\gamma)$ used in the proof does not remain valid when the lower bound for the sectional curvature K_M is replaced by the corresponding lower bound for the Ricci curvature [Anderson 1990].

On the proof of Cheeger's finiteness theorem. The basic idea is to cover a Riemannian manifold M^n with balls $B_i = B(p_i, 2\varrho)$ such that the concentric balls of radius ϱ are disjoint, and to consider the nerve complex corresponding to this covering. Since the conjugate radius of $M^n \in \mathfrak{M}$ is bounded below by $\pi/\sqrt{\Lambda}$ if $\Lambda > 0$ and is $+\infty$ otherwise, Proposition 3.5 provides a uniform lower bound for the injectivity radius on the whole class of Riemannian manifolds.

The idea is to work with some radius ϱ that is a small fraction of the preceding injectivity radius bound. Thus the balls B_i are actually topological balls, and the edges of the nerve complex correspond to minimizing geodesics $\gamma_{ij} : [0,1] \to M^n$ from p_i to p_j. Picking an orthogonal frame $(e_i^\nu)_{\nu=1}^n$ at the center p_i of each ball in our covering, the edges of the nerve complex can be labeled by the length of γ_{ij} and by the orthogonal transformation that maps the frame $(e_i^\nu)_{\nu=1}^n$ to the frame $(e_j^\nu)_{\nu=1}^n$, when the tangent spaces $T_{p_i}M$ and $T_{p_j}M$ are identified by means of parallel transport along γ_{ij}.

The combinatorial properties of the nerve complex can be controlled using the relative volume comparison theorem. More sophisticated arguments from comparison geometry show in addition that two Riemannian manifolds in $\mathfrak{M}$ that admit combinatorially equivalent nerve complexes are diffeomorphic if the labelings of these complexes are sufficiently close. □

In some sense, Cheeger's finiteness theorem may be regarded as a vast generalization of the topological sphere theorem. For instance, the counterpart of the nerve complex appearing in the proof of the topological sphere theorem is a complex consisting of two vertices joined by one edge. The vertices correspond to the two balls in the description of a twisted sphere.

REMARK 3.6. S. Peters [1987] has given a proof of Gromov's compactness theorem following the approach to Cheeger's finiteness theorem described above. He used harmonic coordinates to construct the diffeomorphism between two manifolds in $\mathfrak{M}$ whose labeled nerve complexes are sufficiently close. In this way he actually proved that the spaces $\mathfrak{M}$ are relatively compact in the $C^{2,\alpha}$-topology. As observed by I. Nikolaev [1983], the limiting objects are $C^{3,\alpha}$-manifolds with Riemannian metrics of class $C^{1,\alpha}$. They have curvature bounds λ and Λ in distance comparison sense.

Before we finish this section with an outline of the proof of Theorem 3.1, we present another injectivity radius estimate, which extends the basic idea behind Proposition 3.5. In fact, even for complete, noncompact manifolds, it is possible to relate a lower injectivity radius bound to some lower volume bound, provided one "localizes" the relevant geometric quantities appropriately:

THEOREM 3.7 [Cheeger et al. 1982, Theorem 4.7]. *Consider two points p_0 and p in a connected, complete Riemannian manifold M^n with $\lambda \le K_M \le \Lambda$. Furthermore, let $r_0, r > 0$. Suppose that $r < \pi/(4\sqrt{\Lambda}\,)$ if $\Lambda > 0$. Then the injectivity radius at the point p can be bounded from below as follows*:

$$\begin{aligned} \operatorname{inj}_M(p) &\ge r\,\frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(p,r) + V^n_\lambda(2r)} \\ &\ge r\,\frac{V^n_\lambda(r)\,\operatorname{vol} B(p_0,r_0)}{V^n_\lambda(r)\,\operatorname{vol} B(p_0,r_0) + V^n_\lambda(2r)\,V^n_\lambda(\hat r)}\,, \end{aligned} \tag{3.1}$$

where $\hat r := \max\{r\,, r_0 + \operatorname{dist}(p_0,p)\}$, and where $V^n_\lambda(\varrho)$ denotes the volume of a ball of radius ϱ in the n-dimensional model space M^n_λ with constant sectional curvature λ.

REMARK 3.8. In [Cheeger et al. 1982] one can find even more refined versions of the preceding theorem. Here, however, we prefer to point out one important special case: if we choose the parameter r_0 as the injectivity radius at the point $p_0 \in M^n$, inequality (3.1) turns into the following *relative injectivity radius estimate*:

$$\operatorname{inj}_M(p) \ge \sup_{0<r<\pi/(4\sqrt{\Lambda})} r\,\frac{V^n_\lambda(r)\,V^n_\Lambda(r_0)}{V^n_\lambda(r)\,V^n_\Lambda(r_0) + V^n_\lambda(2r)\,V^n_\lambda(\hat r)}\,.$$

In other words, $\operatorname{inj}_M(p) \ge \varphi_{n,\lambda,\Lambda}\big(\operatorname{inj}_M(p_0), \operatorname{dist}(p_0,p)\big)$, where $\varphi_{n,\lambda,\Lambda}$ is a universal, strictly positive function that depends only on the dimension n and on the curvature bounds λ and Λ. This is the way the result is stated in [Gromov 1981a, Proposition 8.22]. Moreover, this is the version typically used when studying degenerate limits where the dimension of the Riemannian manifolds drops.

Extending Theorem 3.2, a whole theory of *collapsing Riemannian manifolds* has been developed.

On the proof of Theorem 3.7. As far as the proof is concerned, the first inequality in (3.1) is the central assertion of the theorem, whereas the second comes almost free, as a direct consequence of the relative volume comparison theorem. Yet it is the second inequality that is crucial for controlling the injectivity radius $\operatorname{inj}_M(p)$ at points $p \in M^n$ far away from the base point p_0.

The link between $\operatorname{inj}_M(p)$ and $\operatorname{vol} B(p,r)$ described in the first inequality in (3.1) is a purely local result. By hypothesis, the conjugate radius of M^n is $\geq 4r$, and by Lemma 1.8 it is therefore sufficient to show that the length $\ell_M(p)$ of the shortest nontrivial geodesic loop at p is bounded from below as follows:

$$\ell_M(p) \geq 2r \, \frac{\operatorname{vol} B(p,r)}{\operatorname{vol} B(p,r) + V_\lambda^n(2r)} \, . \tag{3.2}$$

The idea for proving this inequality is to compare the geometry of the ball $B(p,4r) \subset M^n$ with the geometry of its local unwrapping $\tilde{B}_{4r}$, which is the ball $\tilde{B}(0,4r) \subset T_pM$ equipped with the metric $\exp_p^* g$. The exponential map provides a length-preserving local diffeomorphism $\exp_p : \tilde{B}_{4r} \to B(p,4r) \subset M^n$.

Let $\tilde{p}_1 = 0$ and let $\tilde{p}_2, \ldots, \tilde{p}_N$ be the various preimages of p in the domain $\tilde{B}_r \subset \tilde{B}_{4r}$. They correspond bijectively to the geodesic loops $\gamma_1, \ldots, \gamma_N$ of length $< r$ at p. Clearly, γ_1 is the trivial loop. Furthermore, for each point $\tilde{p}_i$ there exists precisely one isometric immersion $\varphi_i : \tilde{B}_r \to \tilde{B}_{4r}$ mapping 0 to $\tilde{p}_i$ and such that $\exp_p \circ \varphi_i = \exp_p$. Without loss of generality we may assume that $L(\gamma_2) = \ell_M(p)$ is the minimal length of a nontrivial loop at p.

Analyzing short homotopies, one can show that the maps φ_i constitute a pseudogroup of local covering transformations, hence $\varphi_i(\tilde{q}) \neq \varphi_j(\tilde{q})$ for $1 \leq i < j \leq N$ and for all $\tilde{q} \in \tilde{B}_r$. This fact has two implications:

First, $N \geq 2m+1$, where $m := [r/\ell_M(p)]$. More precisely, we claim that the points $\varphi_2^\mu(\tilde{p}_1)$, for $-m \leq \mu \leq m$, are distinct preimages of p in $\tilde{B}_r$. For otherwise φ_2 would act as a permutation on the set $\{\varphi_2^\mu(\tilde{p}_1) \mid -m \leq \mu \leq m\}$. But this set has a unique center of mass $\tilde{q} \in \tilde{B}_{2r}$. Actually, $\tilde{q}$ lies in $\tilde{B}_r$, so $\tilde{q} = \varphi_2(\tilde{q})$, in contradiction with the fact that φ_2 is a local covering transformation.

Secondly, each point in $B(p,r)$ has at least N preimages in $\bigcup_{i=1}^N B(\tilde{p}_i, r) \subset \tilde{B}_{2r}$. Hence $N \operatorname{vol} B(p,r) \leq \operatorname{vol} \tilde{B}_{2r} \leq V_\lambda^n(2r)$, and inequality (3.2) follows upon combining this estimate with the fact that $N \geq 2\,[r/\ell_M(p)] + 1$. □

On the proof of Theorem 3.1. The basic idea is to pursue an indirect approach. If the theorem were false, there would exist a sequence $(M_j^n, g_j)_{j=1}^\infty$ of complete, simply connected Riemannian manifolds with $\frac{1}{4}\frac{j}{j+1} \leq K_{M_j} \leq 1$ such that none of these manifolds is diffeomorphic to a sphere Σ^n, possibly with an exotic differentiable structure, or to one of the projective spaces $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or $\mathrm{Ca}\mathbb{P}^2$.

By Myers' theorem $\operatorname{diam} M_j^n \leq 2\pi\sqrt{1+1/j} < 3\pi$. Since we are in the even-dimensional case, we can apply Theorem 1.5 to conclude that $\operatorname{inj} M_j^n \geq \pi$. Therefore the volume of each M_j^n is bounded from below by the volume of the standard sphere $\mathbb{S}^n$. (See Proposition 3.5 and Theorem 3.7 for more information about the equivalence of a lower volume bound and a lower injectivity radius bound.)

Hence Theorem 3.2 asserts that the manifolds M_j^n in our sequence belong to finitely many diffeomorphism types. One of these types must appear infinitely often; restricting ourselves to the corresponding subsequence, we are dealing in fact with a sequence of Riemannian metrics $\phi_j^* g_j$ on a fixed compact manifold M^n. Furthermore, the compactness theorem asserts that there exists a subsequence $(\phi_{j_\nu}^* g_{j_\nu})_{\nu=1}^{\infty}$ that converges in the $C^{1,\alpha}$-topology. As a limit space we thus obtain a compact, simply connected $C^{3,\alpha}$-manifold M^n that is neither homeomorphic to a sphere nor diffeomorphic to one of the projective spaces $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or $\mathrm{Ca}\mathbb{P}^2$, and that carries a Riemannian metric g of class $C^{1,\alpha}$ with $\frac{1}{4} \leq K \leq 1$ in distance comparison sense.

Thus it remains to extend Berger's rigidity theorem so it holds under the weak regularity properties of the limit spaces that appear in Gromov's compactness theorem. It is by no means a priori clear whether or not such an extension of Theorem 2.1 exists, since the smoothness of the Riemannian metric has been used in the arguments in [Berger 1960a] in a significant way. For instance, the regularity properties of the metric really matter in the theory of Riemannian manifolds with $-1 \leq K_M \leq 0$ and $\operatorname{vol} M^n < \infty$ where the smooth category exhibits vastly more phenomena than the real analytic category. Concerning Berger's rigidity theorem, however, the alternate proof given by Cheeger and Ebin [1975] is much more robust than the original proof. Following their approach to some extent, Berger succeeded in eliminating all arguments that still required smoothness, replacing them by purely metric constructions [Berger 1983].

More precisely, he is able to recognize the cut locus C_p of an arbitrary point $p \in M^n$ as a k-dimensional, totally geodesic submanifold. At the same time the projective lines are recovered as totally geodesically embedded spheres of curvature 1 and of the complementary dimension $n-k$. Furthermore, the cut locus C_p is shown to have the property that any closed geodesic $c : \mathbb{R}/\mathbb{Z} \to C_p$ of length $L(c) = 2\pi$ spans a totally geodesic $\mathbb{RP}^2 \subset M^n$ of curvature $\frac{1}{4}$. Combining all this information, he can then construct the geodesic symmetries $\phi_p : M^n \to M^n$ directly, recognizing the limit space M^n as a symmetric space and thus a posteriori as a smooth Riemannian manifold. □

4. An Improved Injectivity Radius Estimate

Already before Berger discussed the metrics g_ε of Remark 1.7(ii) on odd-dimensional spheres, it had been considered an interesting question whether or not the pinching constant in Klingenberg's injectivity radius estimate for simply connected, odd-dimensional manifolds could be improved. The extension of Theorem 1.6 to weakly quarter-pinched manifolds was summarized in our discussion of Berger's rigidity theorem in Remark 2.2.

With Berger's pinching below-$\frac{1}{4}$ theorem the problem became even more intriguing. Nevertheless, the first result in this direction was achieved only very recently:

THEOREM 4.1 (INJECTIVITY RADIUS ESTIMATE [Abresch and Meyer 1994]). *There exists a constant* $\delta_{\mathrm{inj}} \in (0.117, 0.25)$ *such that the injectivity radius* $\operatorname{inj} M^n$ *and the conjugate radius* $\operatorname{conj} M^n$ *of any compact*, *simply connected Riemannian manifold* M^n *with* δ_{inj}*-pinched sectional curvature coincide*:

$$\operatorname{inj} M^n = \operatorname{conj} M^n \geq \pi / \sqrt{\max K_M}\,.$$

The pinching constant δ_{inj} in this result is explicit and independent of the dimension. In fact, the theorem holds for $\delta_{\mathrm{inj}} = \frac{1}{4}(1+\varepsilon_{\mathrm{inj}})^{-2}$, where $\varepsilon_{\mathrm{inj}} = 10^{-6}$. Its proof is based on direct comparison methods, not involving the concept of convergence of Riemannian manifolds. Yet the constant δ_{inj} obtained by this method is by no means optimal, since the argument involves several curvature-controlled estimates that are not simultaneously sharp. Currently there is not even a natural candidate for the optimal value of the pinching constant δ_{inj} in the preceding theorem. The Berger metrics described in Remark 1.7(ii) merely show that the number must be at least $0.117 > \frac{1}{9}$.

Notice that the conclusion $\operatorname{inj} M^n = \operatorname{conj} M^n$ is best possible, and in this respect the result can be considered as a natural generalization of Klingenberg's injectivity radius estimate in Theorem 1.6. In particular, the preceding estimate can be used not only to justify the extension of Berger's rigidity theorem to odd-dimensional manifolds, but to yield a corresponding extension of the pinching below-$\frac{1}{4}$ theorem:

THEOREM 4.2 (SPHERE THEOREM [Abresch and Meyer 1994]). *For any odd integer* $n > 0$ *there exists some constant* $\delta_n \in (0, \frac{1}{4})$ *such that any complete*, *simply connected Riemannian manifold* M^n *with* δ_n*-pinched sectional curvature* K_M *is homeomorphic to the sphere* $\mathbb{S}^n$.

Here, as in Berger's pinching below-$\frac{1}{4}$ theorem, the pinching constants $\delta_n \in (0, \frac{1}{4})$ are not explicit, and there is no reason why they should not approach $\frac{1}{4}$ as the dimension n gets large. The proof of Theorem 4.2 relies on the same convergence methods as the proof of Berger's pinching below-$\frac{1}{4}$ theorem, discussed in the previous section. The details for the odd-dimensional case have been worked out by Durumeric [1987]. His result requires some uniform lower bound for the injectivity radius as an additional hypothesis. Such a bound is now provided by Theorem 4.1.

In the remainder of this section we *explain the proofs* of the injectivity radius estimates for odd-dimensional, simply connected manifolds with $\delta \leq K_M \leq 1$, as stated in Theorems 1.6 and 4.1. By Myers' theorem one has a diameter bound in terms of the positive lower bound on the sectional curvature. Nevertheless, in the absence of a lower volume bound, the arguments for establishing Theorems 1.6 and 4.1 must be very different from the proofs of Cheeger's propeller lemma (Proposition 3.5) and Theorem 3.7. As in the even-dimensional case, the hypothesis that the manifold M^n be simply connected must be used in a significant way, and, as in the proof of Theorem 1.5, the starting point is to conclude from Lemma 1.8 that it is sufficient to rule out the existence of a closed geodesic $c_0 : \mathbb{R}/\mathbb{Z} \to M^n$ of length $L(c_0) < 2 \operatorname{conj} M^n$.

However, it is not possible to use Synge's lemma. Much more sophisticated global arguments, based on a combination of lifting constructions and Morse theory, are needed. The following result essentially goes back to [Klingenberg 1962, p. 50].

Lemma 4.3 (Long homotopy lemma). *Let (M^n, g) be a compact Riemannian manifold, and let $c_t : \mathbb{R}/\mathbb{Z} \to M^n$, for $0 \le t \le 1$, be a continuous family of rectifiable, closed curves such that*

(i) *c_0 is a nontrivial geodesic digon of length $L(c_0) < 2 \operatorname{conj} M^n$, and*
(ii) *$c_1 : \mathbb{R}/\mathbb{Z} \to \{c_1(0)\} \subset M^n$ is a constant curve.*

Then this family contains a curve c_τ of length $L(c_\tau) \ge 2 \operatorname{conj} M^n$.

Here, in contrast to Klingenberg's original version of the lemma, the family $(c_t)_{t\in[0,1]}$ can be any free null homotopy of c_0.

Proof. The idea is to proceed indirectly and assume that $L(c_t) < 2 \operatorname{conj} M^n$ for all $t \in [0, 1]$. Without loss of generality we may suppose that the curves c_t are parametrized proportional to arclength.

We consider the family of curves as a continuous map $c : \mathbb{R} \times [0, 1] \to M^n$ such that $c(s + 1, t) = c(s, t) = c_t(s)$ and such that $c(0, 0)$ is a vertex of the geodesic digon. Then the segments $c_t|_{[-\frac{1}{2},0]}$ and $c_t|_{[0,\frac{1}{2}]}$ are strictly shorter than $\operatorname{conj} M^n$, and hence they can be lifted under $\exp_{c_t(0)}$. In this way one obtains a continuous map $\tilde{c} : [-\frac{1}{2}, \frac{1}{2}] \times [0, 1] \to TM$ such that $\exp \circ \tilde{c}(s, t) = c(s, t)$ and $\tilde{c}(0, t) = 0 \in T_{c_t(0)}M$. Since $\tilde{c}(s, 1) = 0$, it follows in particular that $\tilde{c}(-\frac{1}{2}, t) = \tilde{c}(\frac{1}{2}, t)$ for all $t \in [0, 1]$.

Since c_0 is a geodesic digon, it is clear that one of its arcs $c_0|_{[-\frac{1}{2},0]}$ or $c_0|_{[0,\frac{1}{2}]}$ lifts to a radial straight line segment in $T_{c_0(0)}$ of length $\frac{1}{2}L(c_0)$. The other arc lifts to a curve of equal length with the same end points, and hence their images coincide, contradicting the hypothesis that c_0 is a nontrivial digon. □

Proof of Theorem 1.6. Proceeding indirectly, we assume that $\operatorname{inj} M^n < \operatorname{conj} M^n$. By Lemma 1.8 one finds a closed geodesic $c_0 : \mathbb{R}/\mathbb{Z} \to M^n$ with $L(c_0) < 2 \operatorname{conj} M^n$. Since M^n is simply connected, there exist piecewise smooth free homotopies $c = (c_t)_{0\le t\le 1}$ beginning at c_0 and ending at some constant curve c_1.

Recall that the energy functional and the length functional are related by the inequality $E(c_t) = \frac{1}{2}\int_{\mathbb{R}/\mathbb{Z}} |c_t'(s)|^2\, ds \geq \frac{1}{2} L(c_t)^2$. Thus it follows from Lemma 4.3 that for any free homotopy c from c_0 to a constant curve the map $t \mapsto E(c_t)$ achieves its maximum at some $t_0 \in (0,1)$ and that, moreover,

$$E_{\min}(c_0) := \inf_c \max_{0\leq t\leq 1} E(c_t) \geq 2\bigl(\operatorname{conj} M^n\bigr)^2 .$$

Let $(c^j)_{j=1}^\infty$ be a minimizing sequence of such homotopies, and let $t_{0,j}$ be parameters such that $E\bigl(c^j_{t_{0,j}}\bigr) = \max_{0\leq t\leq 1} E(c^j_t)$. Clearly, $E\bigl(c^j_{t_{0,j}}\bigr) \to E_{\min}(c_0)$ as $j \to \infty$. Since the energy functional E on the free loop space satisfies Condition C of Palais and Smale, a subsequence of the curves $c^j_{t_{0,j}}$ converges towards a closed geodesic $\bar{c}_0 : \mathbb{R}/\mathbb{Z} \to M^n$ of length $L(\bar{c}_0) = \sqrt{2E_{\min}(c_0)} \geq 2 \operatorname{conj} M^n$ and Morse index $\operatorname{ind}_E(\bar{c}_0) \leq 1$. In this generality, the assertion about the Morse indices of the limiting geodesics obtained by the preceding minimax construction requires the degenerate Morse lemma from [Gromoll and Meyer 1969], since we have not made any attempt to perturb the Riemannian metric and to ensure that the energy functional E is a nondegenerate Morse function.

The contradiction appears when we look at the index form of the closed geodesic $\bar{c}_0$. The idea is to evaluate

$$I(Y,Y) = \int_{\mathbb{R}/\mathbb{Z}} \left|\frac{\nabla}{ds} Y\right|^2 - \langle R(Y, \bar{c}_0')\bar{c}_0',\, Y\rangle\, ds$$

on closed unit vector fields $v_1, \dots, v_n$ that rotate with constant angular velocity and that are pairwise orthogonal. Since $K_M > \frac{1}{4}$ and $L(\bar{c}_0) \geq 2 \operatorname{conj} M^n \geq 2\pi$, it is indeed not hard to see that $\operatorname{ind}_E(\bar{c}_0) \geq n-1 \geq 2$. □

In the preceding proof we have used the degenerate Morse lemma, avoiding the bumpy metrics theorem [Abraham 1970], since we want to explain the additional difficulties that arise in the weakly quarter-pinched case. It should be pointed out that the standard minimax construction used in the proof yields a limiting geodesic $\bar{c}_0$ but nothing like a limiting homotopy $\bar{c}$.

Remark 4.4. A similar argument can be given in the space $\Omega_{pp}M$ of loops with base point $p \in M^n$. In this case the minimax construction leads to a geodesic loop $\bar{c}_0$ of length $L(\bar{c}_0) \geq 2\pi$, and in order to estimate the index form one must use test fields Y that vanish at the initial and end points of $\bar{c}_0$. In this context it is customary to test the index form with vector fields obtained as the product of a parallel normal field and an appropriately transformed sine function. One concludes that the Morse index of $\bar{c}_0$ in $\Omega_{pp}M$ is $\geq n-1$, too.

This approach is actually pretty close to Klingenberg's original proof of Theorem 1.6. However, neither the degenerate Morse lemma nor the bumpy metrics theorem were known in 1961. Klingenberg's way out was to work in some path space $\Omega_{pq}M$, picking an end point $q \in M^n$ close to p such that in particular the energy functional E has only nondegenerate critical points on $\Omega_{pq}M$. This approach relies on the fact that the long homotopy lemma can be applied to nontrivial digons c_0 rather than merely to closed geodesics.

ON THE EXTENSION OF THE PROOF TO WEAKLY $\frac{1}{4}$-PINCHED MANIFOLDS. In the limiting case, where the simply connected, odd-dimensional manifold M^n has only weakly quarter-pinched sectional curvature, it has turned out to be necessary to work on the free loop space ΩM. Nevertheless, the index computation for the long geodesic $\bar{c}_0$ does not yield an immediate contradiction, but it leads to the following additional information:

(a) the long geodesic $\bar{c}_0$ has length $L(\bar{c}_0) = 2\pi$, and $\operatorname{conj} M^n = \pi$;
(b) the holonomy action on the normal bundle of $\bar{c}_0$ is the map $-\mathrm{id}$;
(c) $K_M(\sigma) = \frac{1}{4}$ for any tangent plane σ of M^n containing $\bar{c}_0'$.

Properties (b) and (c) mean that in some sense $\bar{c}_0$ looks like a primitive closed geodesic in the real projective space $\mathbb{RP}^n_{1/4}$ with constant sectional curvature $\frac{1}{4}$.

The argument given in [Cheeger and Gromoll 1980] is based on the observation that the lifting construction in the proof of the long homotopy lemma actually proves a little more. The authors conclude that the long geodesic $\bar{c}_0$ is *nonliftable* in a suitable sense [Cheeger and Gromoll 1980, Lemma 1]. This property, which is technically fairly delicate to deal with, turns out to be quite strong since $L(\bar{c}_0) \le 2\pi$ in the weakly quarter-pinched case. One concludes that for any $s_0 \in \mathbb{R}$ the first conjugate points of $\bar{c}_0$ on either side of $s = s_0$ appear at $s = s_0 \pm \frac{1}{2}$, an assertion much stronger than the statement about the conjugate radius in (a). It implies that there exists a closed unit normal field v_0 along $\bar{c}_0$ such that

$$K_M\big(\operatorname{span}\{\bar{c}_0', v_0\}\big) \equiv 1,$$

contradicting property (c) and thus obstructing the special long geodesic $\bar{c}_0$ as required. □

The properties of the parallel unit vector field v_0 in the preceding proof assert that the long geodesic $\bar{c}_0$ looks in some sense like the boundary of a totally geodesically immersed hemisphere Σ of constant curvature $K_\Sigma = 1$. Loosely speaking, the contradiction that concludes the proof in [Cheeger and Gromoll 1980] is due to the fact that this picture of $\bar{c}_0$ is very different from the appearance of a primitive closed geodesic in $\mathbb{RP}^n_{1/4}$.

Nevertheless, any straightforward attempt to extend the argument of Cheeger and Gromoll to a proof of Theorem 4.1 fails badly.

REMARKS 4.5. (i) If the pinching constant δ is $< \frac{1}{4}$, that is, if $\delta = 1/(4(1+\varepsilon)^2)$ for some $\varepsilon > 0$, it is still possible to follow the approach in the proof of Theorem 1.6 for a while. The assumption $\operatorname{inj} M^n < \operatorname{conj} M^n$ still implies the existence of a long geodesic $\bar{c}_0$, which in this case has the following properties:

$$L(\bar{c}_0) \ge 2 \operatorname{conj} M^n \ge 2\pi, \quad \operatorname{ind}_E(\bar{c}_0) \le 1. \tag{4.1}$$

The preceding upper bound for the Morse index of $\bar{c}_0$ yields additional information about the length and the *first holonomy angle* of the long geodesic:

$$L(\bar{c}_0) \leq 2\pi(1+\varepsilon), \quad \psi_1(\bar{c}_0) \leq \frac{\pi}{1+\varepsilon}.$$

By definition the *first holonomy angle* $\psi_1(c)$ of a closed curve $c : \mathbb{R}/\mathbb{Z} \to M^n$ is the angle of smallest absolute value among the rotation angles corresponding to the 2×2 blocks D_i in the expression of the holonomy matrix $U_c = \operatorname{diag}(1, D_1, \dots, D_{(n-1)/2}) \in \mathsf{SO}(n)$ in canonical form. The other holonomy angles $0 \leq \psi_1(c) \leq \cdots \leq \psi_{(n-1)/2}(c) \leq \pi$ are defined similarly.

Refining the minimax construction, it is furthermore possible to obtain a long geodesic $\bar{c}_0$ that is *shortly null-homotopic*. This means that there exists a free homotopy $\bar{c}$ from $\bar{c}_0$ to a constant curve $\bar{c}_1$ consisting of closed curves $\bar{c}_t$ such that $L(\bar{c}_t) < L(\bar{c}_0)$ for all $t \in (0,1]$.

(ii) By (i) the set of shortly null-homotopic closed geodesics $\bar{c}_0$ that obey (4.1) is nonempty and compact, and thus it contains an element of minimal length.

However, in contrast to the setup in the weakly quarter-pinched case, it is not possible to ensure that the curves $\bar{c}_t$ in a free null homotopy of the long geodesic $\bar{c}_0$ are strictly shorter than $2 \operatorname{conj} M^n$ for any $t \in (0,1]$. Thus $\bar{c}_0$ cannot be recognized as the limit of a family of *liftable* curves. This is precisely the point where any direct attempt to generalize the argument from [Cheeger and Gromoll 1980] seems to fail.

The idea for the proof of Theorem 4.1 as given in [Abresch and Meyer 1994] is to consider a minimal closed geodesic $\bar{c}_0 : \mathbb{R}/\mathbb{Z} \to M^n$ as described in the preceding remark and to gain further geometric information by normalizing the short null homotopy $\bar{c}$ in three basic steps. Figure 1 is a Morse theory–type picture of the last two normalization steps.

On the tail of short null homotopies. We define the *tail* of a short null homotopy $\bar{c}$ as the family $(\bar{c}_t)_{t_0<t\leq 1}$, where $t_0 := \inf\{t \mid L(\bar{c}_t) < 2\pi\}$. It will be convenient to introduce the space $\Omega M_{<2\pi}$ consisting of all closed curves in M^n of length $< 2\pi$ and the connected component $\Omega M_{<2\pi,0} \subset \Omega M_{<2\pi}$ containing the constant curves.

Since $2\pi \leq 2 \operatorname{conj} M^n \leq L(\bar{c}_0)$, it follows from the long homotopy lemma that a curve $\bar{c}_t$ in the short null homotopy of $\bar{c}_0$ is contained in $\Omega M_{<2\pi,0}$ if and only if $L(\bar{c}_t) < 2\pi$. The space $\Omega M_{<2\pi,0}$ in turn is sufficiently special to admit the following additional lifting construction:

LEMMA 4.6 [Abresch and Meyer 1994, Theorem 6.8]. *For any complete Riemannian manifold M^n with $K_M \leq 1$ there is a unique continuous map*

$$h : \Omega M_{<2\pi,0} \to M^n \times \Omega TM, \quad h(c_0) := (m_0, \tilde{c}_0),$$

such that the following conditions hold:

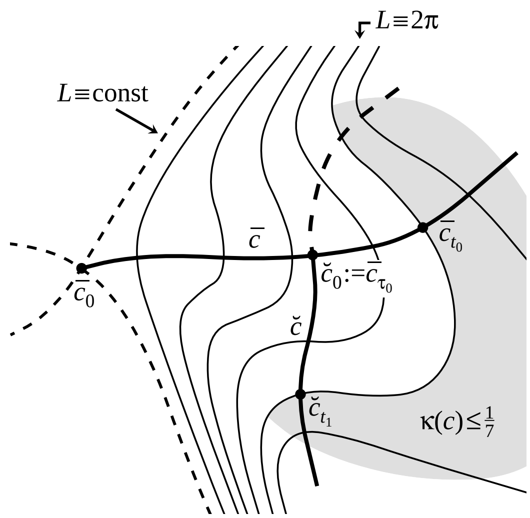

Figure 1. The initial segments $(\bar{c}_t)_{0\le t\le t_0}$ and $(\check{c}_t)_{0\le t\le t_1}$ of a normalized short null homotopy.

(i) $\tilde{c}_0$ *is a lift of* c_0 *under the exponential map* $\exp_{m_0} : B_{\pi/2}T_{m_0}M \to M^n$, *where* $B_{\pi/2}T_{m_0}M$ *denotes the ball of radius* $\frac{1}{2}\pi$ *centered at* $0 \in T_{m_0}M$, *and*
(ii) *the origin in* $T_{m_0}M$ *is the center of the circumscribed ball around the image of* $\tilde{c}_0$ *in* $B_{\pi/2}T_{m_0}M$.

It should be clear how to define the map h on a space of very short curves, say much shorter than the injectivity radius of M^n. The next step is to observe that by properties (i) and (ii) such partially defined maps h admit locally unique continuations. The existence of these continuations depends on the fact that in the standard sphere $\mathbb{S}^n$ of curvature 1 any curve of length $< 2\pi$ is contained in an open hemisphere, that is, in an open ball of radius $\frac{1}{2}\pi$. For the uniqueness part one should observe that the condition on the center of the circumscribed ball determines the tangent space containing the image of $\tilde{c}_0$. Finally, the long homotopy lemma asserts that there are no problems with monodromy.

Clearly, the map h extends continuously to the closure $\overline{\Omega M_{<2\pi,0}} \subset \Omega M$. Applying the preceding lemma with $c_0 = \bar{c}_{t_0}$, we can replace the tail of the short null homotopy $\bar{c}$ up to some reparametrization in t by the map

$$\hat{c} : \mathbb{R}/\mathbb{Z} \times [0,1] \to M^n, \quad \hat{c}(s,t) := \exp_{m_0} t\tilde{c}_0(s). \tag{4.2}$$

The image of this map has remarkable geometric properties if the total absolute curvature

$$\kappa(c_0) = \int_{\mathbb{R}/\mathbb{Z}} \left| \frac{\nabla}{ds} \frac{c_0'(s)}{|c_0'(s)|} \right| ds$$

is sufficiently small. For $\kappa(c_0) < \frac{1}{2}\pi$ it can be shown that $\hat{c}$ describes an immersed ruled surface $\Sigma \subset M^n$ with a conical singularity at $m_0 = \hat{c}(s,0)$. Clearly, the sectional curvature at the regular points of this ruled surface satisfies $K_\Sigma \le K_M \le 1$. The same upper bound holds for the curvature at the conical singularity, since by condition (ii) in Lemma 4.6 the curve $\tilde c_0$ does not lie in any open half-space in $T_{m_0}M$.

If the total absolute curvature $\kappa(c_0)$ approaches zero, the ruled surface Σ looks more and more like a totally geodesically immersed hemisphere of constant curvature $K_\Sigma = 1$. For the proof of Theorem 4.1 we only need a slightly weaker result:

LEMMA 4.7 [Abresch and Meyer 1994, Proposition 6.15 and Theorem 6.1]. *Let M^n be a complete Riemannian manifold with sectional curvature bounded by $0 < K_M \le 1$. Let c_0, $\tilde c_0$, and $\hat c$ be as in* (4.2), *and suppose that $\kappa(c_0) \le \frac{1}{6}\pi$. Then the unit vector field v along c_0 obtained by normalizing $s \mapsto (\partial \hat c/\partial t)|_{(s,1)}$ has total absolute rotation*

$$\int_{\mathbb{R}/\mathbb{Z}} \left| \frac{\nabla}{ds} v \right| ds \le \kappa(c_0) + 2\sqrt{\kappa(c_0)}\,.$$

Furthermore, the first holonomy angle $\psi_1(c_0)$ is bounded as follows:

$$\psi_1(c_0) \le \begin{cases} \kappa(c_0) & \text{if } n \text{ is even,} \\ \dfrac{\sqrt{2}\,\kappa(c_0) + 2\sqrt{\kappa(c_0)}}{\sqrt{1 - \sin\kappa(c_0)}} & \text{if } n \text{ is odd.} \end{cases}$$

This implies in particular that the initial curve $\bar c_{t_0}$ of the tail of the short null homotopy $\bar c$ has first holonomy angle $\psi_1(\bar c_{t_0}) < \frac{1}{3}\pi$ provided that the total absolute curvature of $\bar c_{t_0}$ satisfies $\kappa(\bar c_{t_0}) < \frac{1}{7}$.

A first normalization of the initial segment of the short null homotopy. The purpose of this normalization is to guarantee that the total absolute curvature of $\bar c_{t_0}$ is so much less than $\frac{1}{7}$ that the inequality $\kappa(\bar c_{t_0}) < \frac{1}{7}$ persists even after the second normalization step. This leads to the required contradiction, since the second normalization step will ensure that $\psi_1(\bar c_{t_0}) > \frac{1}{3}\pi$, provided that the pinching constant $\delta = 1/(4(1+\varepsilon)^2)$ is sufficiently close to $\frac{1}{4}$.

Recall that $\bar c_0$ is a closed geodesic of length $\eta := L(\bar c_0) \in [2\pi, 2\pi(1+\varepsilon)]$. Hence $\kappa(\bar c_0) = 0$ and $\bar c_t \in \Omega M_{<\eta,0}$ for $0 < t \le 1$. But there is no bound for the length of the initial segment $(\bar c_t)_{0 \le t \le t_0}$ when considered as a curve in the free loop space ΩM. Nevertheless, it is possible to get away with some bounds for the differential $d\kappa$ on a suitable domain in $\Omega M_{<\eta,0}$. The key observation is that the total absolute curvature κ can be interpreted as the L^1-norm of the L^2-gradient $\operatorname{grad}_{L^2} L$ of the length functional L. Recall that the L^2-norm of a vector field X along $\bar c_t$ is defined by $\|X\|_{L^2}^2 = \int_{\mathbb{R}/\mathbb{Z}} |X|^2\,|\bar c_t'|\,ds$. Since the L^2-Hessian of the length functional at some curve $\bar c_t$ is bounded from below by $-\max K_M \ge -1$, one obtains the following lemma:

LEMMA 4.8 (CURVE SHORTENING). *Let M^n, $\bar{c}_0$, and $\eta := L(\bar{c}_0)$ be as before. Suppose that the initial segment of the short null homotopy $\bar{c}$ beginning at $\bar{c}_0$ is a solution of the rescaled curve-shortening flow*

$$\frac{\partial \bar{c}_t}{\partial t} = -p(t)\,\mathrm{grad}_{L^2} L|_{\bar{c}_t}\,,$$

except at finitely many $\theta_j \in [0, t_0]$ where the weight function $p : [0, t_0] \to (0, \infty]$ is unbounded. Then for any $t \in [0, t_0]$ the total absolute curvature of $\bar{c}_t$ satisfies

$$\kappa\big(\bar{c}_t\big)^2 \le 2\,\big(L(\bar{c}_0) - L(\bar{c}_t)\big) \le 4\pi\varepsilon.$$

REMARKS 4.9. (i) Clearly, $\theta_0 = 0$, and each $\theta_j > 0$ means that the trajectory approaches a closed geodesic $\bar{c}_{\theta_j}$. By our choice of the initial curve $\bar{c}_0$ such a geodesic must have Morse index $\mathrm{ind}_E\big(\bar{c}_{\theta_j}\big) \ge 2$, and thus it is possible to restart the curve-shortening flow below the critical level after performing some tiny, explicit deformation of $\bar{c}_{\theta_j}$ by means of the degenerate Morse lemma. By Condition C of Palais and Smale the closed interval $[2\pi^2, \frac{1}{2}\eta^2]$ contains only finitely many critical values of the energy functional E, and thus after at most finitely many restarts the flow passes through a curve $\bar{c}_{t_0}$ of length $L\big(\bar{c}_{t_0}\big) = 2\pi$.

(ii) In [Abresch and Meyer 1994, Proposition 5.1] we are actually working in some fixed, finite-dimensional subspace $\Omega^k_\ell M_{<\eta}$ of broken geodesics rather than on the Hilbert manifold ΩM, whose metric is induced by the H^1-inner products on the tangent spaces. This is to avoid the analytical difficulties concerning the long-time existence of the curve-shortening flow. A second benefit of working in a fixed finite-dimensional approximation space is the fact that two-sided bounds for the Hessians of the length and the energy functionals are available. These bounds will be used in a significant way in the next normalization step.

A second normalization of the initial segment of $\bar{c}$. From now on we suppose that the initial segment $(\bar{c}_t)_{0\le t\le t_0}$ of the short null homotopy $\bar{c}$ has been normalized by means of the curve-shortening flow as discussed in the preceding step. Moreover, we assume that $\varepsilon \le \frac{1}{64\,000}$. With this constraint on the pinching constant δ, Lemma 4.8 implies that $\kappa(\bar{c}_t) < \frac{1}{70}$ for all $t \in [0, t_0]$, and by Lemma 4.7 the first holonomy angle of the curve $\bar{c}_{t_0}$ is much smaller than $\frac{1}{3}\pi$. Since the map $t \mapsto \psi_1(\bar{c}_t)$ is continuous, we conclude that there exists some $\tau_0 \in (0, t_0)$ such that $\bar{c}_{\tau_0}$ has first holonomy angle equal to $\frac{1}{2}\pi$ and total absolute curvature $< \frac{1}{70}$. Our goal is to replace the segment $(\bar{c}_t)_{\tau_0\le t\le t_0}$ of the short null homotopy $\bar{c}$ by some alternate arc $\check{c} : [0, t_1] \to \Omega M_{<\eta,0}$ from $\check{c}_0 = \bar{c}_{\tau_0}$ to some curve $\check{c}_{t_1}$ representing an element of the boundary of $\Omega M_{<2\pi,0}$. The construction of $\check{c}$ shall imply that the total absolute curvature of $\check{c}_{t_1}$ is still bounded by $\frac{1}{7}$ and that its first holonomy angle is still $> \frac{1}{3}\pi$, contradicting the assertion in Lemma 4.7.

The key observation for constructing $\check{c}$ is to realize that because of the equation $\psi_1(\check{c}_0) = \frac{1}{2}\pi$ there exists a subspace $W \subset T_{\check{c}_0}\Omega M$ of dimension ≥ 2 consisting of vector fields v along $\check{c}_0$ such that

$$\frac{\nabla}{\partial s} v \perp v \qquad \text{and} \qquad \left|\frac{\nabla}{\partial s} v\right| = \tfrac{1}{2}\pi\,|v|\,.$$

In other words, each of these fields v rotates with constant speed. On the one hand, the angular velocity is sufficiently large in comparison to the total absolute curvature $\kappa(\check{c}_0)$ to allow us to conclude that v and the tangent field $\check{c}_0'$ are almost perpendicular. On the other hand, the angular velocity of v is already sufficiently small for us to conclude that

$$\operatorname{hess}_{L^2}(L)_{|\check{c}_0}(v,v) \lessapprox \left(\frac{\pi^2}{4L(\check{c}_0)^2} - \frac{1}{4(1+\varepsilon)^2}\right)\|v\|_{L^2}^2 \approx -\tfrac{3}{16}\|v\|_{L^2}^2 < 0. \tag{4.3}$$

It is customary to control the path dependence of parallel transport in terms of the norm of the Riemannian curvature tensor of M^n and the area of a spanning homotopy. A similar argument can be used to estimate the change in the first holonomy angle along $\check{c}$:

$$\bigl|\psi_1(\check{c}_{t_1}) - \psi_1(\check{c}_0)\bigr| \le \tfrac{4}{3}\operatorname{area}(\check{c})\,. \tag{4.4}$$

The corresponding statements for the finite-dimensional approximation spaces $\Omega_\ell^k M_{<\eta}$ used in [Abresch and Meyer 1994] are somewhat more technical. Even if the number k of corners of the broken geodesics is chosen to be large, the constants appearing in Lemma 4.7 and in inequalities (4.3)–(4.4) differ significantly from the constants in the corresponding statements for $\Omega_\ell^k M_{<\eta}$ in [Abresch and Meyer 1994]. The reason is that the natural metrics g_k on the approximation spaces converge to the normalized L^2-inner product on ΩM rather than to the standard L^2-inner product used in the preceding discussion. Moreover, the estimates in [Abresch and Meyer 1994] are stated in terms of the energy functional E rather than the length functional L.

The advantage of working in the space $\Omega_\ell^k M_{<\eta}$ is that a two-sided bound for the Hessian of L is available, so there is a bound for the norm of the differential $d\kappa$. This means that the required bounds for the first holonomy angle and for the total absolute curvature of $\check{c}_{t_1} \in \partial\Omega_\ell^k M_{<2\pi,0}$ can be guaranteed if the length of $\check{c}$ with respect to the metric g_k on $\Omega_\ell^k M_{<\eta}$ is bounded by a small constant $t_{\max} > 0$ and if ε is sufficiently small.

In the finite-dimensional setup, we are dealing with a broken geodesic $\check{c}_0 = \bar{c}_{\tau_0} \in \Omega_\ell^k M_{<\eta,0}$. The tangent space $T_{\check{c}_0}\Omega_\ell^k M_{<\eta}$ also contains a subspace W_k of dimension at least 2 on which the Hessian of the length functional is bounded from above by an explicit negative constant. Now the idea is to pick the homotopy $\check{c}$ as a normal g_k-geodesic in $\Omega_\ell^k M_{<\eta}$ of length $\le t_{\max}$ starting at $\check{c}_0 = \bar{c}_{\tau_0}$ with initial vector $\frac{\partial}{\partial t}\check{c}_t|_{t=0} = v \in W_k$. Since $\dim W_k \ge 2$, we may assume that

$$dL\Bigl(\frac{\partial}{\partial t}\check{c}_t\Big|_{t=0}\Bigr) \le 0.$$

As we decrease $\varepsilon > 0$, the difference $L(\breve{c}_0) - 2\pi$ becomes as small as we please, so the proof can be finished establishing a negative upper bound for

$$\frac{d^2}{dt^2} L(\breve{c}_t) = \mathrm{hess}_{L^2}(L)\Big(\frac{\partial}{\partial t}\breve{c}_t, \frac{\partial}{\partial t}\breve{c}_t\Big) \tag{4.5}$$

for all $t \in [0, t_{\max}]$, and not just at $t = 0$ as in inequality (4.3). For this purpose we cannot refer to any modulus of continuity for the Hessian $\mathrm{hess}_{L^2}(L)$ itself, since this would require a bound for the covariant derivative of the curvature tensor. However, the bounds for

$$\int_{\mathbb{R}/\mathbb{Z}} \left|\frac{\nabla}{\partial s}\frac{\partial}{\partial t}\breve{c}_t\right|^2 \left|\frac{\partial}{\partial s}\breve{c}_t\right|^{-1} ds \qquad \text{and} \qquad \left|\frac{\partial}{\partial s}\breve{c}_t \wedge \frac{\partial}{\partial t}\breve{c}_t\right|,$$

which have been used to prove inequality (4.3) for $\breve{c}_0$, can be extended continuously along the g_k-geodesic $t \mapsto \breve{c}_t$, and thus we obtain a uniform negative upper bound for the second derivative of the map $t \mapsto L(\breve{c}_t)$ on $[0, t_{\max}]$, provided that $t_{\max}$ and ε are sufficiently small.

This concludes our sketch of the proof of Theorems 1.6 and 4.1.

5. A Sphere Theorem with a Universal Pinching Constant Below $\frac{1}{4}$

The injectivity radius estimate in Theorem 4.1 was the first result with a pinching constant below $\frac{1}{4}$ and independent of the dimension. In this context it is natural to ask whether the assertions in Berger's pinching below-$\frac{1}{4}$ theorem and the sphere theorem in the preceding section (Theorems 3.1 and 4.2) remain valid for some universal pinching constants. In the odd-dimensional case the answer is affirmative:

THEOREM 5.1 (SPHERE THEOREM [Abresch and Meyer a]). *There exists a constant $\delta_{\mathrm{odd}} \in (0, \frac{1}{4})$ such that any odd-dimensional, compact, simply connected Riemannian manifold M^n with δ_{odd}-pinched sectional curvature is homeomorphic to the sphere $\mathbb{S}^n$.*

In fact, the constant δ_{odd} is explicit. Our proof works for $\delta_{\mathrm{odd}} = \frac{1}{4}(1 + \varepsilon_{\mathrm{odd}})^{-2}$, where $\varepsilon_{\mathrm{odd}} = 10^{-6}$. In the even-dimensional case, however, our methods are not yet sufficient to generalize Berger's pinching below-$\frac{1}{4}$ theorem accordingly. So far, there is only the following partial result [Abresch and Meyer a]:

THEOREM 5.2 (COHOMOLOGICAL PINCHING BELOW-$\frac{1}{4}$ THEOREM). *There exists a constant $\delta_{\mathrm{ev}} \in (0, \frac{1}{4})$ such that for any even-dimensional, compact, simply connected Riemannian manifold M^n with δ_{ev}-pinched sectional curvature the cohomology rings $H^*(M^n; R)$ with coefficients $R \in \{\mathbb{Q}, \mathbb{Z}_2\}$ are isomorphic to the corresponding cohomology rings of one of the compact, rank-one symmetric spaces $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, or $\mathrm{Ca}\mathbb{P}^2$, or the rings $H^*(M^n; R)$ are truncated polynomial rings generated by an element of degree 8.*

Again, the constant δ_{ev} is explicit and independent of the dimension. The proof works for $\delta_{\text{ev}} = \frac{1}{4}(1+\varepsilon_{\text{ev}})^{-2}$, where $\varepsilon_{\text{ev}} = \frac{1}{27\,000}$.

Recall that $H^*(\mathrm{Ca}\mathbb{P}^2; R) = R[\xi_R]/(\xi_R^3)$, where $\deg \xi_R = 8$. However, we cannot exclude the possibility that $H^*(M^n; R) = R[\xi_R]/(\xi_R^{m+1})$, where $\deg \xi_R = 8$ and $m > 2$. For instance, we cannot apply J. Adem's result [1953, Theorem 2.2], which is based on Steenrod's reduced third power operations, since we do not have enough control on the cohomology ring of M^n with coefficients $\mathbb{Z}_3$.

Notice that in dimensions 2, 3, and 4 the more special results mentioned in the discussion of the topological sphere theorem in Section 1 are still much stronger than the assertions in Theorems 5.1 and 5.2. In fact, we even need to refer to Hamilton's result, since our proof for the sphere theorem depends on Smale's solution of the Poincaré conjecture in dimensions ≥ 5. We use Theorem 1.3 in order to handle the three-dimensional case.

The starting point for the proofs of both theorems is to establish the horseshoe conjecture of Berger [1962a], which had remained open until recently:

THEOREM 5.3 (HORSESHOE INEQUALITY [Abresch and Meyer a, Theorem 2.4]). *There exists a constant $\varepsilon_{\text{hs}} > 0$ such that, for any complete Riemannian manifold M^n satisfying*

$$\delta_{\text{hs}} := \tfrac{1}{4}\,(1+\varepsilon_{\text{hs}})^{-2} \leq K_M \leq 1$$

and

$$\pi \leq \operatorname{inj} M^n \leq \operatorname{diam} M^n \leq \pi\,(1+\varepsilon_{\text{hs}}),$$

the following statement holds: For any $p_0 \in M^n$ and any $v \in \mathbb{S}^{n-1} \subset T_{p_0}M$, the distance between the antipodal points $\exp_{p_0}(-\pi v)$ and $\exp_{p_0}(\pi v)$ is less than π.

If M^n is one of the projective spaces $\mathbb{CP}^m$, $\mathbb{HP}^m$, and $\mathrm{Ca}\mathbb{P}^2$ with its Fubini–Study metric, the two points $\exp_{p_0}(-\pi v)$ and $\exp_{p_0}(\pi v)$ coincide. The horseshoe inequality asserts that their distance is less than the injectivity radius of M^n if the relevant geometric invariants of M^n do not deviate too much from the corresponding quantities of the projective spaces. Notice in particular that the pinching constant δ_{hs} is explicit and independent of the dimension. In fact, the assertion of Theorem 5.3 holds for $\varepsilon_{\text{hs}} = \frac{1}{27\,000}$.

A horseshoe inequality for manifolds with nontrivial fundamental group had been established earlier by Durumeric [1984, Lemma 6]. His argument was based on a detailed investigation of the geometry of Dirichlet cells in the universal covering of M^n, and relied on the hypothesis $\pi_1(M^n) \neq 0$. The proof of Theorem 5.3 is very different. It requires some refined Jacobi field estimates, which might be useful in other contexts as well. Our plan is to explain these Jacobi field estimates in the next section and describe the proof of the horseshoe inequality in Section 7.

We conclude this section by explaining how to deduce Theorems 5.1 and 5.2 for manifolds M^n of dimension $n \geq 4$ by combining the injectivity radius estimate of Theorem 4.1, the diameter sphere theorem, and the horseshoe inequality of

Theorem 5.3 with some results from algebraic topology. This reduction had already been known to Berger [1962a]. Complete proofs are given in [Abresch and Meyer a]. Here we list only the basic steps.

In fact, Theorem 5.3 does not come into the proofs of Theorems 5.1 and 5.2 directly. Instead we need the following corollary:

COROLLARY 5.4 [Berger 1962a, Proposition 2]. *Let* $\delta_{\mathrm{hs}} \in (0, \frac{1}{4})$ *be the constant of Theorem* 5.3. *Then any compact Riemannian manifold* M^n *satisfying* $\delta_{\mathrm{hs}} \le K_M \le 1$ *and*

$$\pi \le \operatorname{inj} M^n \le \operatorname{diam} M^n \le \frac{\pi}{2\sqrt{\delta_{\mathrm{hs}}}}$$

admits a continuous, *piecewise smooth map* $f : \mathbb{RP}^n \to M^n$ *of degree* 1.

Here $\deg f$ denotes the standard integral mapping degree if M^n is odd-dimensional and orientable. Otherwise $\deg f$ has to be understood as the $\mathbb{Z}_2$-mapping degree.

REMARK 5.5. If $M^n = \mathbb{CP}^m$, with $n = 2m > 2$, the map $f : \mathbb{RP}^{2m} \to \mathbb{CP}^m$ obtained from the preceding corollary can be visualized in terms of the standard cell decompositions. Recall that $\mathbb{RP}^{2m} = \mathbb{RP}^{2m-1} \cup_\varphi e_{2m}$ and $\mathbb{CP}^m = \mathbb{CP}^{m-1} \cup_{\bar\varphi} e_{2m}$. Moreover, the fibers of the attaching map $\bar\varphi : \partial e_{2m} \to \mathbb{CP}^{m-1}$ are the Hopf circles in $\mathbb{S}^{2m-1} = \partial e_{2m}$. They are invariant under the antipodal map, and thus there is an induced map $\psi : \mathbb{RP}^{2m-1} \to \mathbb{CP}^{m-1}$ such that $\bar\varphi$ factors as $\psi \circ \varphi$. Hence the identity map on the $2m$-cell e_{2m} induces a map $f : \mathbb{RP}^{2m} \to \mathbb{CP}^m$ with $\deg_{\mathbb{Z}_2} f = 1$. This map coincides with the map constructed in the proof of the corollary.

It turns out that the mere existence of such a map $f : \mathbb{RP}^n \to M^n$ is a strong constraint for the topology of the manifold M^n:

THEOREM 5.6. *Let* M^n *be a compact*, *simply connected*, *odd-dimensional manifold. Suppose that there exists a continuous map* $f : \mathbb{RP}^n \to M^n$ *with* $\deg_{\mathbb{Z}} f = 1$. *Then* M^n *is a homology sphere.*

THEOREM 5.7. *Let* M^n *be a compact*, *simply connected*, *even-dimensional manifold. Suppose that there exists a continuous map* $f : \mathbb{RP}^n \to M^n$ *with* $\deg_{\mathbb{Z}_2} f = 1$. *Then the cohomology rings of* M^n *with coefficients* $R \in \{\mathbb{Q}, \mathbb{Z}_2\}$ *are isomorphic to truncated polynomial rings generated by an element* ξ_R *of degree* 2, 4, 8, *or* n.

Both theorems follow essentially from standard computations in algebraic topology, based on the Poincaré duality theorem, the Steenrod squares, and the universal coefficient theorem. The proof of Theorem 5.7 refers in addition to J. F. Adams's results on secondary cohomology operations and the Hopf invariant one problem [Adams 1960]. A summary of this work is given in [Milnor and Stasheff 1974, page 134]. Historically, Theorem 5.6 goes back to H. Samelson [1963], whereas Theorem 5.7 can be extracted from [Berger 1965, p. 135ff].

Theorem 5.7 is far from recognizing the manifold M^n up to homeomorphism. It determines the integral cohomology ring $H^*(M^n;\mathbb{Z})$ only up to the Serre class of torsion groups of odd order. This ambiguity reflects the fact that we can only work with the modulo-2 mapping degree of f. Furthermore, the *fake projective spaces* discovered by J. Eells and N. H. Kuiper [1962] show that it would not even be sufficient to recover the integral cohomology rings in order to recognize the manifold M^n up to homeomorphism.

On the proof of Theorem 5.1 in dimensions $n \geq 5$. We set $\delta_{\text{odd}} := \max\{\delta_{\text{inj}}, \delta_{\text{hs}}\}$, where δ_{inj} and δ_{hs} are the constants from Theorems 4.1 and 5.3. It is convenient to scale the metric on M^n such that $\delta_{\text{odd}} \leq K_M \leq 1$. Because of the diameter sphere theorem of Grove and Shiohama it is sufficient to consider manifolds with $\operatorname{diam} M^n \leq \pi/(2\sqrt{\delta_{\text{odd}}})$. By Theorem 4.1 we have $\operatorname{inj} M^n \geq \pi$, so Corollary 5.4 yields a continuous, piecewise smooth map $f : \mathbb{RP}^n \to M^n$ of degree $\deg_{\mathbb{Z}} f = 1$. With Theorem 5.6 we conclude that the manifold M^n is a homology sphere. Since by hypothesis M^n is simply connected, Smale's solution of the Poincaré conjecture in dimensions $n \geq 5$ can be applied [Milnor 1965, p. 109; Smale 1961]. □

On the proof of Theorem 5.2 in dimensions $n \geq 4$. The argument is very similar. We estimate the injectivity radius of M^n by means of Theorem 1.5 rather than Theorem 4.1, so we may set $\delta_{\text{ev}} := \delta_{\text{hs}}$. Again we obtain a continuous, piecewise smooth map $f : \mathbb{RP}^n \to M^n$. We still have $\deg_{\mathbb{Z}_2} f = 1$, so we can apply Theorem 5.7. In order to conclude the proof, we observe that the truncated polynomial rings $R[\xi_R]/(\xi_R^{m+1})$ where $m = n/\deg \xi_R$ are precisely the cohomology rings of $\mathbb{S}^n$, $\mathbb{CP}^{n/2}$, or $\mathbb{HP}^{n/4}$, if the degree of the generator is n, 2, or 4, respectively. □

6. New Jacobi Field Estimates

In this section the issue is to obtain precise control over normal Jacobi fields Y with $Y(0) = 0$ along any geodesic $\gamma : [0, r_2] \to M^n$. We are interested in mixed estimates for Y at some point $r_1 \in (0, r_2)$, which depend on information about the size of the initial derivative $\frac{\nabla}{dr}Y(0)$ and the boundary value $Y(r_2)$, and which refine the standard estimate provided by the Rauch comparison theorems. For this purpose it is essential to work with two-sided bounds for the sectional curvature K_M of the Riemannian manifold M^n. The basic estimates have been established in [Abresch and Meyer a, Theorem 5.6].

Theorem 6.1 (Mixed Jacobi field estimates). *Let $\lambda < \Lambda$ and let $0 < r_1 \leq r_2$. Suppose that $r_2 \leq \pi/\sqrt{\Lambda}$ if $\Lambda > 0$. Then there exists a continuous function $\Psi_{r_1 r_2} : [0,\infty)\times[0,\infty) \to [0,\infty)$ such that, for any Riemannian manifold M^n with sectional curvature bounded by $\lambda \leq K_M \leq \Lambda$ and for any geodesic $\gamma : [0, r_2] \to M^n$ with $|\gamma'| \equiv 1$, there is the following lower bound for a normal*

Jacobi field Y along γ with initial value $Y(0) = 0$:

$$|Y(r_1)| \geq \Psi_{r_1 r_2}\left(\left|\frac{\nabla}{dr}Y(0)\right|, |Y(r_2)|\right).$$

Furthermore, the function $\Psi_{r_1 r_2}$ has the following properties:

(i) *it is weakly convex and positively homogeneous of degree* 1;
(ii) *it is nondecreasing with respect to both variables*;
(iii) *it is locally of class $C^{1,1}$ except at $(\alpha, \eta) = (0,0)$*;
(iv) $\Psi_{r_1 r_2}(\alpha, \eta) \geq \max\{\alpha\,\mathrm{sn}_\Lambda(r_1),\ \eta\,\mathrm{sn}_\lambda(r_1)/\mathrm{sn}_\lambda(r_2)\}$ *for all $(\alpha, \eta) \in [0, \infty)^2$.*

Remark 6.2. The lower bounds for $\Psi_{r_1 r_2}$ in (iv) reflect the standard Jacobi field estimates. The term $\alpha\,\mathrm{sn}_\Lambda(r_1)$ is due to the first Rauch comparison theorem, whereas the term $\eta\,\mathrm{sn}_\lambda(r_1)/\mathrm{sn}_\lambda(r_2)$ follows from the monotonicity of the map $s \mapsto \mathrm{sn}_\lambda(s)^{-1}\,|Y(s)|$ asserted by the infinitesimal Rauch comparison theorem. In fact,

$$\Psi_{r_1 r_2}(\alpha, \eta) = \begin{cases} \alpha\,\mathrm{sn}_\Lambda(r_1) & \text{if } \eta \leq \alpha\,\mathrm{sn}_\Lambda(r_2), \\ \eta\,\mathrm{sn}_\lambda(r_1)/\mathrm{sn}_\lambda(r_2) & \text{if } \alpha\,\mathrm{sn}_\lambda(r_2) \leq \eta. \end{cases} \tag{6.1}$$

The continuity of the first derivatives of $\Psi_{r_1 r_2}$ as asserted in (iii) implies that inequality (iv) is strict, provided that $\alpha\,\mathrm{sn}_\Lambda(r_2) < \eta < \alpha\,\mathrm{sn}_\lambda(r_2)$ and r_1 is sufficiently close to r_2. This is the range of parameters in which Theorem 6.1 improves the classical estimates.

The homogeneity property of $\Psi_{r_1 r_2}$ corresponds to the linearity of the Jacobi field equation. The convexity asserted in (i) is essential in order to apply Jensen's inequality when integrating the Jacobi field estimate from Theorem 6.1 over a family of geodesics. In particular, it enables us to deduce:

Theorem 6.3 [Abresch and Meyer a, Theorem 5.4]. *Let $\lambda < \Lambda$ and let $0 < r_1 \leq r_2$. Suppose that $r_2 \leq \pi/\sqrt{\Lambda}$ if $\Lambda > 0$. Consider a Riemannian manifold M^n with sectional curvature bounded by $\lambda \leq K_M \leq \Lambda$ and a ruled surface $\gamma : [0, r_2] \times [0,1] \to M^n$ generated by normal geodesics $\gamma_\theta = \gamma(\,\cdot\,, \theta)$ emanating from a fixed point $p_0 \in M^n$. Then the lengths $\ell(r_i)$ of the circular arcs $\theta \mapsto \gamma(r_i, \theta)$ and the total angle*

$$\varphi_0 := \int_0^1 \left|\frac{\nabla}{\partial r}\frac{\partial \gamma}{\partial \theta}(0, \theta)\right| d\theta$$

satisfy the inequality

$$\ell(r_1) \geq \Psi_{r_1 r_2}\big(\varphi_0, \ell(r_2)\big)\,,$$

where $\Psi_{r_1 r_2}$ is the comparison function introduced in Theorem 6.1.

For computational purposes it is necessary to have a more explicit description of the comparison functions $\Psi_{r_1 r_2}$ that appear in the preceding theorems. In fact, the proof of Theorem 6.1 provides the following information:

On the comparison functions $\Psi_{r_1 r_2}$. The values at all pairs (α, η) outside the cone $\alpha\,\mathrm{sn}_\Lambda(r_2) < \eta < \alpha\,\mathrm{sn}_\lambda(r_2)$ are given by formula (6.1). By homogeneity it is sufficient to define $\Psi_{r_1 r_2}(1, \eta)$ for $\mathrm{sn}_\Lambda(r_2) < \eta < \mathrm{sn}_\lambda(r_2)$. For this purpose we introduce the functions $\bar{y} : [0, r_2] \times [0, r_2] \to [0, \infty)$ by means of

$$\bar{y}(r_0, r) := \begin{cases} \mathrm{sn}_\Lambda(r) & \text{if } r \le r_0\,, \\ \mathrm{sn}_\Lambda(r_0)\,\mathrm{cn}_\lambda(r - r_0) + \mathrm{cn}_\Lambda(r_0)\,\mathrm{sn}_\lambda(r - r_0) & \text{if } r_0 \le r \text{ and } n = 2, \\ \mathrm{sn}_\lambda(r)\,\bar{w}(r_0, r)^{1/2} & \text{if } r_0 \le r \text{ and } n > 2, \end{cases} \tag{6.2}$$

where sn_λ and cn_λ are the generalized trigonometric functions defined on page 3, and where $\bar{w}(r_0, r)$ is the nonnegative number

$$\bar{w}(r_0, r) := \frac{\mathrm{sn}_\Lambda^2(r_0)}{\mathrm{sn}_\lambda^2(r_0)} - 2\det\begin{pmatrix} \mathrm{cn}_\lambda(r_0) & \mathrm{cn}_\Lambda(r_0) \\ \mathrm{sn}_\lambda(r_0) & \mathrm{sn}_\Lambda(r_0) \end{pmatrix} \int_{r_0}^{r} \frac{\mathrm{sn}_\Lambda(\varrho)}{\mathrm{sn}_\lambda^3(\varrho)}\,d\varrho\,.$$

The graphs of the functions $\bar{y}_{r_0} : r \mapsto \bar{y}(r_0, r)$ with $0 \le r_0 < r_2$ foliate the domain $\{(r, y) \mid \mathrm{sn}_\Lambda(r) < y \le \mathrm{sn}_\lambda(r),\ 0 < r \le r_2\}$, as shown in Figure 2. In particular, for any $\eta \in [\mathrm{sn}_\Lambda(r_2), \mathrm{sn}_\lambda(r_2)]$ there is precisely one $r_0 \in [0, r_2]$ such that $\bar{y}(r_0, r_2) = \eta$, and $\Psi_{r_1 r_2}(1, \eta)$ is defined implicitly by the equation

$$\Psi_{r_1 r_2}\bigl(1,\ \bar{y}(r_0, r_2)\bigr) = \bar{y}(r_0, r_1)\,. \tag{6.3}$$

Remark 6.4. For $\lambda = \frac{1}{4}$ and $\Lambda = 1$ it is possible to evaluate the integral in the expression for $\bar{w}$ in terms of trigonometric functions. Moreover, the solution of the equation $\bar{y}(\hat{r}_0, \pi) = 1$, which is needed in order to compute $\Psi_{r_1 \pi}(1, 1) =$

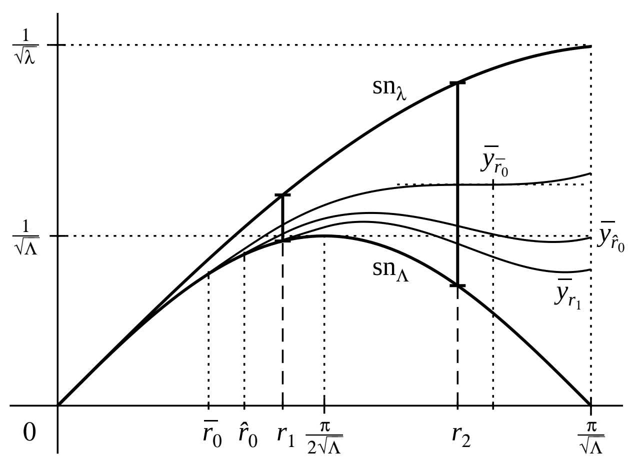

Figure 2. The functions $\bar{y}_{r_0}$ and the map $\eta \mapsto \Psi_{r_1 r_2}(1, \eta)$. The graphs of the functions $\bar{y}_{r_0}$, $\bar{y}_{\hat{r}_0}$, and $\bar{y}_{r_1}$ are only qualitative pictures with the correct number of local maxima, local minima, and saddle points, provided that $0 < \lambda < \frac{1}{4}\Lambda$. In an actual plot the qualitative properties of these functions would be almost invisible.

$\bar{y}(\hat{r}_0, r_1)$ for $r_1 \in (0,\pi)$, is given by

$$\hat{r}_0 = \begin{cases} 2\,\arccos\bigl(2^{-1/3}\bigr) & \approx 0.416\,304\,\pi \quad \text{if } n=2,\\ 2\,\arcsin\bigl(\frac{1}{2}+\sin(\frac{1}{18}\pi)\bigr) & \approx 0.470\,548\,\pi \quad \text{if } n>2. \end{cases}$$

In particular, if $r_1 = \frac{59}{120}\pi \in \bigl(\hat{r}_0, \frac{1}{2}\pi\bigr)$, it follows that $\Psi_{r_1\pi}(1,1) \geq (1+a_0)\,\sin(r_1)$ for $a_0 \approx 0.001\,663 > 0$. By continuity this inequality persists with a slightly smaller constant $a_\varepsilon > 0$ if $\lambda = \frac{1}{4}(1+\varepsilon)^{-2}$ and $\varepsilon > 0$ is sufficiently small. A numerical computation shows that $a_\varepsilon \approx 0.001\,661$ for $\varepsilon = \varepsilon_{\mathrm{hs}} = \frac{1}{27\,000}$.

With the preceding definition of the functions $\Psi_{r_1 r_2}$ in equations (6.1)–(6.3) it is straightforward, but tedious, to verify all the analytical properties listed as assertions (i)–(iv) in Theorem 6.1. Here we shall rather concentrate on the geometric ideas leading to the claimed lower bound for $|Y(r_1)|$.

On the proof of Theorem 6.1 in dimension $n=2$. In this case the argument is quite easy. The normal Jacobi field Y can be written as a product yE of a nonnegative function $y : [0,r_2] \to [0,\infty)$ with a parallel unit normal field along the geodesic γ. The Jacobi field equation reduces to the scalar differential equation $y'' + K_{M\,|\gamma}\, y = 0$, and the Rauch comparison theorems assert that

$$y'(0)\,\mathrm{sn}_\Lambda(r_1) \leq y(r_1) \leq y'(0)\,\mathrm{sn}_\lambda(r_1)$$

for all $r_1 \in [0,r_2]$. The infinitesimal version of the Rauch comparison theorems provides the inequality $y(r_2)\,\mathrm{sn}_\lambda(r_1)/\mathrm{sn}_\lambda(r_2) \leq y(r_1)$. The latter inequality can be improved by applying the maximum principle to the differential inequality $y'' + \lambda y \leq 0$. We conclude that for any $r_0 \in [0,r_2)$ the restriction of y to the interval $[r_0,r_2]$ is bounded from below by any solution $\bar{y}_{r_0 r_2}$ of the differential equation $z'' + \lambda z = 0$ with boundary data $z(r_0) = y'(0)\,\mathrm{sn}_\Lambda(r_0)$ and $z(r_2) \leq y(r_2)$. Maximizing over r_0 leads to the functions $\bar{y}_{r_0} = \bar{y}(r_0,.)$ introduced in formula (6.2). One finds that $y(r_1) \geq y'(0)\,\bar{y}_{r_0}(r_1)$ for any $r_1 \in [0,r_2]$ provided that $y'(0)\,\bar{y}_{r_0}(r_2) \leq y(r_2)$. ☐

In dimensions $n > 2$ there is no way to apply the maximum principle directly. Nevertheless, it is still possible to write the normal Jacobi field Y as a product yE of a nonnegative function $y = |Y|$ and a unit normal field E along the geodesic γ. The difficulty is that the unit normal field E needs not be parallel.

Yet, the idea is to reduce to a two-dimensional situation by considering the ruled surface Σ defined by a variation of the geodesic γ which corresponds to the Jacobi field Y. The intrinsic sectional curvature K_Σ of this surface can be determined by means of the Gauss equations

$$K_\Sigma|_\gamma = K_M\bigl(T\Sigma_{|\gamma}\bigr) - \Bigl|\frac{\nabla}{dr}E\Bigr|^2,$$

so $\lambda - \bigl|\frac{\nabla}{dr}E\bigr|^2 \leq K_\Sigma \leq \Lambda$. We are able to proceed, since the angular velocity $\frac{\nabla}{dr}E$ of the Jacobi field Y can be bounded as follows:

LEMMA 6.5 [Abresch and Meyer a, Proposition 5.12]. *Let* $\gamma : [0, r_2] \to M^n$ *be a normal geodesic in a Riemannian manifold with sectional curvature bounded by* $\lambda \le K_M \le \Lambda$. *Suppose that* $r_2 \le \pi/\sqrt{\Lambda}$ *if* $\Lambda > 0$. *Then on the interval* $(0, r_2)$ *the angular velocity* $\frac{\nabla}{dr}E$ *of a nontrivial normal Jacobi field* $Y = yE$ *(where* $y = |Y|$*), with initial value* $Y(0) = 0$, *can be estimated in terms of the function* $u := y^{-1}\, y'$ *as follows*:

$$\left|\frac{\nabla}{dr}E\right|^2 \le \tfrac{1}{4}\,(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)^2 - \big(u - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\big)^2 = -\,\mathrm{ct}_\lambda\,\mathrm{ct}_\Lambda + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\,u - u^2.$$

PROOF. Since $Y(0) = 0$, the Jacobi field equation for Y can be expressed as a Riccati equation for the Hessian A of a local distance function along γ:

$$\frac{\nabla}{dr}Y = AY \qquad \text{and} \qquad \frac{\nabla}{dr}A + A^2 + R(\,\cdot\,, \gamma')\gamma' = 0\,.$$

Since E is a unit normal field, the standard comparison results for the Riccati equation assert that $\mathrm{ct}_\Lambda\ P_\gamma \le A \le \mathrm{ct}_\lambda\ P_\gamma$, where ct_λ and ct_Λ are the generalized cotangent functions introduced on page 3 and where $P_\gamma := \mathrm{id} - \langle\,\cdot\,, \gamma'\rangle\gamma'$. On the other hand it is easy to see that $\frac{\nabla}{dr}E + u\,E = AE$, or, equivalently:

$$\frac{\nabla}{dr}E + \big(u - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\big)\,E = A\,E - \tfrac{1}{2}(\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\,E.$$

The lemma follows, since $\frac{\nabla}{dr}E$ is orthogonal to the unit vector field E and since $\frac{1}{2}(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)$ is an upper bound for the norm of the right hand side. □

The preceding lemma means that the lower bound for the curvature K_Σ of the ruled surface Σ is a function of the parameter r along the geodesic γ, and that this function depends on the logarithmic derivative u of $y = |Y|$. Expressing the Jacobi field equation in Σ in terms of u instead of y, we obtain the differential inequality

$$u' = -K_\Sigma - u^2 \le -\lambda - \mathrm{ct}_\lambda\,\mathrm{ct}_\Lambda\ + (\mathrm{ct}_\lambda + \mathrm{ct}_\Lambda)\,u - 2u^2\,. \tag{6.4}$$

This differential inequality is still of Riccati type. However, because of the factor 2 in front of the quadratic term it corresponds to a linear differential inequality of second order for the function y^2, or more appropriately for $z := y^2/\,\mathrm{sn}_\lambda$, rather than for y itself. A straightforward computation shows that z satisfies the inequality

$$z'' + (\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)\,z' + \big(\lambda - \mathrm{ct}_\lambda\,(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)\big)\,z \le 0.$$

By construction, sn_λ is a solution of the corresponding differential equation. Since $\mathrm{sn}_\lambda > 0$ on $(0, r_2]$, the maximum principle implies that for any interval $[r_0, r_2] \subset (0, r_2]$ the boundary value problem

$$z'' + (\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)\,z' + \big(\lambda - \mathrm{ct}_\lambda\,(\mathrm{ct}_\lambda - \mathrm{ct}_\Lambda)\big)\,z = 0$$

with

$$z(r_0) = \frac{\mathrm{sn}_\Lambda(r_0)^2}{\mathrm{sn}_\lambda(r_0)}\, y'(0)^2 \quad \text{and} \quad z(r_2) = \frac{\mathrm{sn}_\Lambda(r_2)^2}{\mathrm{sn}_\lambda(r_2)}$$

has a unique solution $\bar z_{r_0 r_2} \geq 0$, and that $y|_{[r_0,r_2]} \geq \sqrt{\mathrm{sn}_\lambda\, \bar z_{r_0 r_2}}$. By means of the Wronskian and the special solution sn_λ it is possible to compute the functions $\bar z_{r_0 r_2}$ explicitly. As in the two-dimensional case it remains to maximize over r_0, in order to arrive at the expression for $\bar y$ given in the third line in (6.2).

7. On the Proof of Berger's Horseshoe Conjecture

In this section we explain the proof of Theorem 5.3. Again we shall concentrate on the geometric ideas. For brevity we write ε rather than $\varepsilon_{\mathrm{hs}}$, and we always assume that ε is sufficiently small. Details can be found in [Abresch and Meyer a, § 4].

The hypothesis on the diameter of M^n implies that the distance between the points $p_1 := \exp_{p_0}(-\pi v)$ and $p_2 := \exp_{p_0}(\pi v)$ does not exceed $\pi\,(1{+}\varepsilon)$, and the geodesic $s \mapsto \exp_{p_0}(sv)$, for $s \in [-\pi, \pi]$, connecting p_1 to p_2 looks like a horseshoe. In order to use the upper bound on the diameter of M^n more efficiently, we consider the intermediate points $q_1^\varepsilon := \exp_{p_0}\bigl(-\frac{1}{2}(1+\varrho_\varepsilon)\pi v\bigr)$ and $q_2^\varepsilon := \exp_{p_0}\bigl(\frac{1}{2}(1+\varrho_\varepsilon)\pi v\bigr)$, where $\varrho_\varepsilon \approx \frac{4}{\pi}\,\varepsilon^{2/3}$ is defined as the solution of the equation

$$\sin\bigl(\tfrac{1}{2}\varrho_\varepsilon\pi\bigr) = \sin\bigl(\tfrac{1}{4}\varepsilon^{1/3}\pi\bigr)^{-1}\sin\bigl(\tfrac{1}{2}\varepsilon\pi\bigr).$$

Let $c^\varepsilon : [0,1] \to M^n$ be a minimizing geodesic from q_1^ε to q_2^ε. Since $\operatorname{diam} M^n \leq \pi\,(1{+}\varepsilon)$, it is clear that c^ε does not pass through p_0. The configuration described so far is depicted in Figure 3.

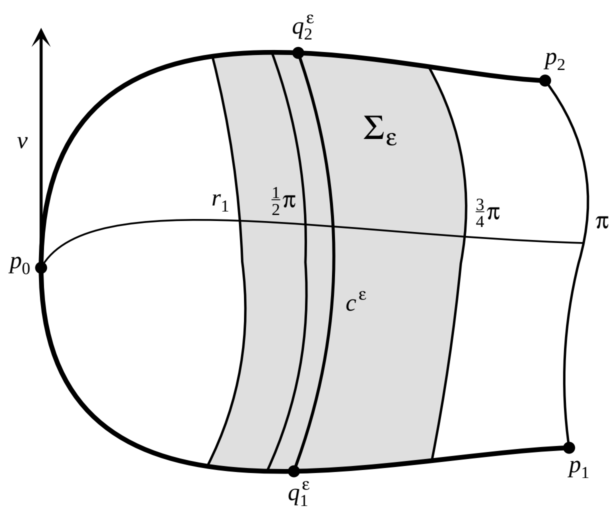

Figure 3. The horseshoe $p_1p_0p_2$, the geodesic c^ε, and the spherical ribbon Σ_ε.

We proceed indirectly and assume, contrary to the assertion of Theorem 5.3, that $d(p_1, p_2) \geq \pi$. This assumption enables us to control various properties of the geodesic c^ε, which finally lead to the contradiction $L(c^\varepsilon) > \pi\,(1+\varepsilon) \geq \operatorname{diam} M^n$.

Lemma 7.1. *If* $0 < \varepsilon \leq \frac{1}{64}$ *and* $d(p_1, p_2) \geq \pi$, *there are the following bounds for the distance between* p_0 *and the geodesic* c^ε *constructed above*:

$$\tfrac{7}{16}\pi \leq \tfrac{1}{2}\bigl(1 - \tfrac{1}{2}\varepsilon^{1/3}\bigr)\pi \leq d\bigl(p_0,\, c^\varepsilon(t)\bigr) < \tfrac{3}{4}\pi \qquad \textit{for all } t \in [0,1].$$

Notice that the lower bound converges to $\frac{1}{2}\pi$ as ε approaches zero.

Sketch of proof. The idea for proving the lower bound is to consider the weak contraction $\Phi : M^n \to \mathbb{S}^n$ onto the sphere of constant curvature 1 which is induced by a linear isometry $T_{p_0}M^n \to T_{\bar{p}_0}\mathbb{S}^n$ and by the corresponding exponential maps. The map Φ collapses the entire complement of the ball $B(p_0, \pi)$ to the antipodal point of $\bar{p}_0$. Clearly,

$$d\bigl(p_0, c^\varepsilon(t)\bigr) \geq \inf\bigl\{d(\bar{p}_0, \bar{q}) \bigm| \bar{q} \in \mathbb{S}^n,\, d(\bar{q}, \bar{q}_1^\varepsilon) + d(\bar{q}, \bar{q}_2^\varepsilon) \leq \pi\,(1+\varepsilon)\bigr\},$$

where $\bar{q}_i^\varepsilon := \Phi(q_i^\varepsilon)$. In view of the fact that the infimum is achieved at some point $\bar{q}^\varepsilon \in \mathbb{S}^n$ with $d(\bar{q}^\varepsilon, \bar{q}_1^\varepsilon) = d(\bar{q}^\varepsilon, \bar{q}_2^\varepsilon) = \frac{1}{2}\pi\,(1+\varepsilon)$, it is straightforward to compute the numerical value of this lower bound using the law of cosines.

For $0 \leq t \leq \frac{1}{2}$ the upper bound for the distance between p_0 and $c^\varepsilon(t)$ is obtained by applying Toponogov's triangle comparison theorem three times. Our model space is always the sphere $\mathbb{S}^2_\delta$ of constant curvature $\delta = \frac{1}{4}(1+\varepsilon)^{-2}$. The first step is to consider the triangle $p_1p_0p_2$. By hypothesis $d(p_1, p_0) = d(p_0, p_2) = \pi$ and $d(p_1, p_2) \geq \pi$, so we get a lower bound for the length $d(p_1, q_2^\varepsilon)$ of the secant from p_1 to q_2^ε, which converges to π if $\varepsilon \to 0$. The next step is to consider the triangle $p_1q_1^\varepsilon q_2^\varepsilon$. Since the lengths of the edges $p_1q_1^\varepsilon$ and $q_1^\varepsilon q_2^\varepsilon$ are bounded by $\frac{1}{2}\pi$ and $\pi\,(1+\varepsilon)$ respectively, the angle at q_1^ε is bounded from below by some number approaching $\frac{1}{2}\pi$ if $\varepsilon \to 0$. This angle is the exterior angle for the hinge $p_0q_1^\varepsilon c^\varepsilon(t)$. Because of the inequality $0 \leq t \leq \frac{1}{2}$ we know that $d(p_0, q_1^\varepsilon) \leq \frac{1}{2}\,(1+\varrho_\varepsilon)\pi$ and $d\bigl(q_1^\varepsilon, c^\varepsilon(t)\bigr) \leq \frac{1}{2}(1+\varepsilon)\pi$. These data yield an upper bound for $d\bigl(p_0, c^\varepsilon(t)\bigr)$ that converges to $\frac{2}{3}\pi$ if ε approaches zero.

For $\frac{1}{2} \leq t \leq 1$ the upper bound is established in a similar way. One merely needs to employ the symmetry of the horseshoe and exchange the points p_1 and p_2 as well as q_1^ε and q_2^ε. $\square$

The preceding lemma guarantees that for $\varepsilon \leq \frac{1}{27\,000}$ the geodesic c^ε can be lifted under the exponential map $\exp_{p_0}$ to a curve

$$\tilde{c}^\varepsilon : [0,1] \to \overline{B(0, \tfrac{3}{4}\pi)} \setminus B(0, r_1) \subset T_{p_0}M^n,$$

where $r_1 := \frac{59}{120}\pi$. Furthermore, the formula $\gamma^\varepsilon(r,t) := \exp_{p_0}\bigl(r\,|\tilde{c}^\varepsilon(t)|^{-1}\,\tilde{c}^\varepsilon(t)\bigr)$ defines a differentiable map $\gamma^\varepsilon : [0,\pi] \times [0,1] \to M^n$. The restriction of γ^ε to $[0,\pi) \times [0,1]$ describes an immersed ruled surface in M^n with a conical singularity at p_0. The geodesic c^ε lies in the image of this ruled surface. Its lift to the domain of γ^ε is the graph of the function $\tilde r^\varepsilon = |\tilde c^\varepsilon| : [0,1] \to [r_1, \frac{3}{4}\pi]$.

By the infinitesimal version of the Rauch comparison theorem the pullback metric $(\gamma^\varepsilon)^* g$ on the rectangular domain $[r_1, \frac{3}{4}\pi] \times [0,1]$ can be bounded from below in terms of the rescaled arclength function

$$\tilde\varphi^\varepsilon : t \mapsto \frac{1}{\sin(r_1)} \int_0^t \Big| \frac{\partial}{\partial\theta} \gamma^\varepsilon(r_1, \theta) \Big| \, d\theta$$

of the circular arc $t \mapsto \gamma^\varepsilon(r_1, t)$ as follows:

$$(\gamma^\varepsilon)^* g \geq dr^2 + \sin(r)^2 \, (d\tilde\varphi^\varepsilon)^2 .$$

Hence the map $\mathrm{id} \times \tilde\varphi^\varepsilon$ yields a weak contraction from $\big([r_1, \frac{3}{4}\pi] \times [0,1], (\gamma^\varepsilon)^* g\big)$ to the spherical ribbon $\Sigma_\varepsilon = \big([r_1, \frac{3}{4}\pi] \times [0, \tilde\varphi^\varepsilon(1)], \bar g\big)$ where $\bar g := dr^2 + \sin(r)^2 \, d\varphi^2$ denotes the standard metric of constant curvature 1. In particular, the length of the geodesic c^ε in M^n is bounded from below by the distance of the points $\tilde q_1^\varepsilon := \big(\tilde r^\varepsilon(0), 0\big)$ and $\tilde q_2^\varepsilon := \big(\tilde r^\varepsilon(1), \tilde\varphi^\varepsilon(1)\big)$ in the inner metric space Σ_ε:

$$\operatorname{diam} M^n \geq L\big(c^\varepsilon\big) \geq \operatorname{dist}_{\Sigma_\varepsilon}\big(\tilde q_1^\varepsilon, \tilde q_2^\varepsilon\big) . \tag{7.1}$$

By construction, $\frac{\partial}{\partial r}\gamma^\varepsilon(0,1) = -\frac{\partial}{\partial r}\gamma^\varepsilon(0,0) = v$, so the total angle

$$\varphi_0^\varepsilon := \int_0^1 \Big| \frac{\nabla}{\partial r} \frac{\partial}{\partial t} \gamma^\varepsilon(0,t) \Big| \, dt$$

at the conical singularity of the ruled surface described by γ^ε is bounded from below by π. The arc $t \mapsto \gamma^\varepsilon(\pi, t)$ connects the points $p_1 = \exp_{p_0}(-\pi v)$ and $p_2 = \exp_{p_0}(\pi v)$, and thus its length is $\geq d(p_1, p_2) \geq \pi$.

After these preparations we apply Theorem 6.3 with $\lambda = \delta_{\mathrm{hs}}$, $\Lambda = 1$, and $r_2 = \pi$, and use the monotonicity and homogeneity properties of $\Psi_{r_1\pi}$ asserted in Theorem 6.1(i) and (ii) to conclude that

$$\sin(r_1)\, \tilde\varphi^\varepsilon(1) \geq \Psi_{r_1\pi}(\varphi_0^\varepsilon, \pi) \geq \pi \, \Psi_{r_1\pi}(1,1) .$$

As explained in Remark 6.4, it follows that $\tilde\varphi^\varepsilon(1) \geq (1 + a_\varepsilon)\pi$ for $a_\varepsilon \approx 0.001\,661$. Notice that a_ε is a monotonically decreasing function of ε, and thus the preceding lower bound for $\tilde\varphi^\varepsilon(1)$ holds uniformly for all $\varepsilon \leq \varepsilon_{\mathrm{hs}}$.

Geometrically, this bound means that the length of the equatorial arc joining the two longitudinal segments in the boundary of the spherical ribbon Σ_ε exceeds π by a certain fixed amount. Since the points $\tilde q_1^\varepsilon$ and $\tilde q_2^\varepsilon$ on these longitudinal boundary arcs approach the equator if ε gets small, we conclude that their distance in the inner geometry of the ribbon is not only greater than π but even greater than $(1+\varepsilon)\pi$, provided ε is sufficiently small. But the inequality $\operatorname{dist}_{\Sigma_\varepsilon}\big(\tilde q_1^\varepsilon, \tilde q_2^\varepsilon\big) > (1+\varepsilon)\pi \geq \operatorname{diam} M^n$ contradicts (7.1).

8. Final Remarks

The injectivity radius estimate in Theorem 4.1, the sphere theorem and the cohomological pinching below-$\frac{1}{4}$ theorem stated in 5.1 and 5.2, and the horseshoe inequality in Theorem 5.3 have this feature in common: In each theorem the pinching constant is an explicit number $< \frac{1}{4}$, independent of the dimension. Nevertheless, we do not know the optimal value for any of the pinching constants δ_{inj}, δ_{odd}, δ_{ev}, or δ_{hs}. The current proofs combine several curvature controlled estimates in such a way that they do not become sharp simultaneously.

The values provided by these proofs are only slightly smaller than $\frac{1}{4}$, and thus they differ significantly from the values that are obstructed by the counterexamples known today. Yet, these lower bounds cannot be considered any less artificial than the numbers provided by the current proofs. For instance, because of the Berger spheres described in Remark 1.7(ii) it is necessary that $\delta_{\mathrm{inj}} \geq 0.117 > \frac{1}{9}$. Furthermore, the examples in Table 1 show that $\delta_{\mathrm{ev}} \geq \frac{1}{64}$ and $\delta_{\mathrm{odd}} \geq \frac{1}{37}$. But the significance of the latter two numbers is impaired by the fact that only in low dimensions do we know any examples of compact, simply connected Riemannian manifolds with strictly positive sectional curvature that are not homeomorphic to spheres or projective spaces.

Appendix: Compact Manifolds of Positive Curvature

Table 1 lists all simply connected manifolds that are known to carry metrics with positive sectional curvature. To begin with, there are the symmetric spaces with $K_M > 0$. They must have rank one, and thus one is left with the spheres $\mathbb{S}^n$ and the projective spaces $\mathbb{CP}^n$, $\mathbb{HP}^n$, and $\mathrm{Ca}\mathbb{P}^2$ equipped with the Fubini–Study metric.

Further examples of compact manifolds with $K_M > 0$ are already quite scarce. Currently, non-symmetric examples appear only in very few dimensions. They have been discovered as follows: First, Berger [1961] classified the simply connected, normal homogeneous spaces with $K_M > 0$. The only new examples that appeared in this classification are the two odd-dimensional spaces $\mathsf{Sp}(2)/\mathsf{SU}(2)$ and $M^{13} = \mathsf{SU}(5)/(\mathsf{Sp}(2) \times \mathsf{S}^1)$. Their pinching constants $\delta_M := \min K_M / \max K_M$ were calculated later by H. Eliasson [1966] and Heintze [1971], respectively.

The next step was the classification by Wallach [1972] of simply connected, even-dimensional, homogeneous spaces with $K_M > 0$. Besides spheres and projective spaces there are only the three flag manifolds $M^6 = \mathsf{SU}(3)/\mathsf{T}^2$, $M^{12} = \mathsf{Sp}(3)/(\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2))$, and $M^{24} = \mathsf{F}_4/\mathsf{Spin}(8)$. They are, respectively, an $\mathbb{S}^2$-bundle over $\mathbb{CP}^2$, an $\mathbb{S}^4$-bundle over $\mathbb{HP}^2$, and an $\mathbb{S}^8$-bundle over $\mathrm{Ca}\mathbb{P}^2$. As shown in [Valiev 1979], the pinching constants for the three flag manifolds are $\frac{1}{64}$; compare also [Grove 1989].

An infinite series of homogeneous, odd-dimensional examples has been found by S. Aloff and N. Wallach [1975]. They are quotient spaces $M^7_{k,l} = \mathsf{SU}(3)/\mathsf{S}^1_{k,l}$, where the integers k and l label the various embeddings of S^1 into a maximal torus $\mathsf{T}^2 \subset \mathsf{SU}(3)$. H.-M. Huang [1981] computed that the pinching constants induced by a particular left-invariant metric defined in [Aloff and Wallach 1975] approach $\frac{16}{29\cdot 37}$ as $\frac{k}{l} \to 1$. The manifolds $M^7_{k,l}$ are not only interesting for their geometric properties. M. Kreck and S. Stolz [1991] have found out that this sequence of examples contains seven-manifolds that are homeomorphic but not diffeomorphic.

M	dim	$\min K_M / \max K_M$	
symmetric spaces			
$\mathbb{S}^n$	n	1	
$\mathbb{CP}^n$	$2n$	$\frac{1}{4}$	
$\mathbb{HP}^n$	$4n$	$\frac{1}{4}$	
$\mathrm{Ca}\mathbb{P}^2$	16	$\frac{1}{4}$	
normal homogeneous spaces [Berger 1961]			
$\mathsf{Sp}(2)/\mathsf{SU}(2)$	7	$\frac{1}{37}$	[Eliasson 1966]
$\mathsf{SU}(5)/(\mathsf{Sp}(2)\times\mathsf{S}^1)$	13	$\frac{16}{29\cdot 37}$	[Heintze 1971]
ditto with a nonnormal metric		$\frac{1}{37}$	[Püttmann 1996]
flag manifolds [Wallach 1972]			
$\mathsf{SU}(3)/\mathsf{T}^2$	6	$\frac{1}{64}$	[Valiev 1979]
$\mathsf{Sp}(3)/(\mathsf{SU}(2)^3)$	12	$\frac{1}{64}$	[Valiev 1979]
$\mathsf{F}_4/\mathsf{Spin}(8)$	24	$\frac{1}{64}$	[Valiev 1979]
Aloff–Wallach examples [1975]			
$\mathsf{SU}(3)/\mathsf{S}^1_{k,l}$	7	$\to \frac{16}{29\cdot 37}$	[Huang 1981]
ditto with certain Einstein metrics		$\to \frac{1}{37}$	[Püttmann 1996]
inhomogeneous orbit spaces [Eschenburg 1982; Bazaĭkin 1995]			
$\mathsf{SU}(3)/(\mathsf{T}^2\text{-action})$	6	?	
$\mathsf{SU}(3)/(\mathsf{S}^1_{klpq}\text{-action})$	7	$\to \frac{1}{37}$	[Püttmann 1996]
$\mathsf{S}^1_{p_1\dots p_5}\backslash\mathsf{U}(5)/(\mathsf{Sp}(2)\times\mathsf{S}^1)$	13	$\to \frac{1}{37}$	[Püttmann 1996]

Table 1. Compact, simply connected manifolds with $K_M > 0$. The arrows in front of some of the pinching constants indicate that the given values appear as limits for properly chosen subsequences.

L. Bérard-Bergery [1976] showed that there exist no other simply connected, odd-dimensional, homogeneous spaces of positive curvature. This result finishes the classification of the simply connected, homogeneous spaces with $K_M > 0$.

So far, nonhomogeneous examples have only been obtained as inhomogeneous orbit spaces, where a subgroup $H \subset G \times G$ acts on a simply connected Lie group G. Analyzing the two-sided T^2- and S^1-actions on $\mathsf{SU}(3)$, J. Eschenburg [1984; 1982; 1992] has found a six-dimensional inhomogeneous space of positive curvature and an infinite family of seven-dimensional inhomogeneous orbit spaces. These examples resemble the Aloff–Wallach examples in many respects.

Following Eschenburg's approach, A. Bazaĭkin [1995] has recently constructed an infinite sequence of 13-dimensional, simply connected, pairwise nonhomeomorphic orbit spaces with strictly positive sectional curvature. These examples are biquotients that are closely related to the second one of the normal homogeneous spaces discovered by Berger. A complete classification of all two-sided actions which lead to simply connected Riemannian manifolds with positive curvature has not been accomplished as yet.

Recently, I. Taimanov [1996] has discovered an isometric, totally geodesic embedding of the Aloff–Wallach space $M^7_{1,1}$, equipped with the metric considered by Huang, into the Berger space M^{13}, so explaining to some extent the curious coincidence of the pinching constants determined by Heintze and Huang, respectively. In Bazaĭkin's work the manifold $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times \mathsf{S}^1)$ appears with a deformed metric that is homogeneous but not normal homogeneous. Under Taimanov's embedding $M^7_{1,1} \hookrightarrow \mathsf{SU}(5)/(\mathsf{Sp}(2) \times \mathsf{S}^1)$ the corresponding one-parameter deformation induces the Aloff–Wallach metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ with a slightly different parametrization. Th. Püttmann [1996] has computed the curvature tensors of these one-parameter deformations of metrics on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$ and $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times \mathsf{S}^1)$ in a systematic way, finding that $\frac{1}{37}$ is the optimal pinching constant in each case. The coincidence of the two constants is not a complete surprise, since Taimanov's embedding stays totally geodesic for the entire deformation; however, we do not have any explanation why this value also coincides with the pinching constant of the seven-dimensional Berger space $\mathsf{Sp}(2)/\mathsf{SU}(2)$. As yet, the curvature computations cover all homogeneous metrics on $\mathsf{Sp}(2)/\mathsf{SU}(2)$ and $\mathsf{SU}(5)/(\mathsf{Sp}(2) \times \mathsf{S}^1)$, but it is still an open question whether or not $\frac{1}{37}$ remains optimal for the entire nine-parameter family of homogeneous metrics that exists on $\mathsf{SU}(3)/\mathsf{S}^1_{1,1}$.

As a consequence of Püttmann's improvments, any subsequence of the Aloff–Wallach examples $M^7_{k,l}$ where $\frac{k}{l} \to 1$ admits metrics whose pinching constants approach $\frac{1}{37}$. An analogous statement holds for the seven-dimensional inhomogeneous spaces of Eschenburg and the 13-dimensional spaces of Bazaĭkin.

It is a curious coincidence that the optimal, $\frac{1}{37}$-pinched metric on the Aloff–Wallach space $M^7_{1,1}$ is one of the Einstein metrics discovered by M. Wang [1982].

Furthermore, it can be shown that any $M^7_{k,l}$ with $\frac{k}{l}$ sufficiently close to 1 carries a homogeneous Einstein metric whose pinchimg constant is close to $\frac{1}{37}$. In contrast, the optimal metric on the 13-dimensional Berger space M^{13} is not Einstein.

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