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The Reduced Canonical Form of a Game

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Abstract. Cooling by $*$, followed by the elimination of the stars, is used to define an operator $G \to \overline{G}$ on short games, having the following properties: $\overline{G}$ is the simplest game infinitesimally close to G; the operator is a homomorphism; it can be used for recursive calculations, provided that the games involved are not in a "strictly cold" form.

1. Introduction

We will use the classical definitions and facts about two-person, perfect information combinatorial games with the normal winning convention, as developed in *Winning Ways* [Berlekamp et al. 1982] and *On Numbers and Games* [Conway 1976]. We recapitulate them briefly.

Formally, *games* are constructed recursively as ordered pairs $\{\Gamma^L \,|\, \Gamma^R\}$, where Γ^L and Γ^R are sets of games, called, respectively, the *set of Left options* and the *set of Right options* from G. We will restrict ourselves to *short games*, that is, games where the sets of options Γ^L and Γ^R are required to be finite in this recursive definition. The basis for this recursion is the game $\{\varnothing \,|\, \varnothing\}$, which is called 0. We will often let G^L and G^R represent typical Left and Right options of a game G, and write $G = \{G^L \,|\, G^R\}$.

The $\leq$ relation, defined inductively by

$$G \leq H \iff \text{there is no } H^R \text{ with } H^R \leq G \text{ and no } G^L \text{ with } H \leq G^L,$$

is a quasi-order (it is not antisymmetric). We identify G with H if $G \leq H$ and $H \leq G$; we then say that G and H *have the same value*, and write $G = H$. The relation $\leq$ becomes a partial order on the set of game values.

Two games G and H are *identical*, or *have the same form*, if they have identical sets of left options and identical sets of right options. In this case we write $G \equiv H$. Whenever the distinction between the value and the form of a game is essential, we will specify it; otherwise, by G we will mean the form of G.

In the *normal winning convention*, the player who makes the last move wins. The four possible classes of outcome for a game are determined by how the game's value compares with 0, as shown in the following table (where $G \| 0$ means that G is not comparable to 0):

$$\begin{aligned} &G \text{ is a win for Left whoever goes first} &&\iff G > 0;\\ &G \text{ is a win for Right whoever goes first} &&\iff G < 0;\\ &G \text{ is a win for whoever goes first} &&\iff G \parallel 0;\\ &G \text{ is a loss for whoever goes first} &&\iff G = 0. \end{aligned}$$

One can define an addition operation on the set of game values, making it into a group. Certain game values can be associated with real numbers; they are therefore called *numbers*. Since we only consider short games, all number values will be dyadic rationals. The *Left stop* $L(G)$ and *Right stop* $R(G)$ of a game G are the numbers recursively defined by: $L(G) = R(G) = x$ if $G = x$, where x is a number; otherwise $L(G) = \max_{G^L} R(G^L)$ and $R(G) = \min_{G^R} L(G^R)$. An *infinitesimal* is a game whose stops are both 0. Any infinitesimal lies strictly between all negative and all positive numbers.

Any game G admits a unique canonical form, that is, a form with no *dominated options* and no *reversible moves*. The canonical form has the *earliest birthday* among all the games that have the same value as G. (For details and proofs, see *Winning Ways*, pp. 62–65, or *On Numbers and Games*, pp. 110–112). Actually, the canonical form of a game G has the even stronger property of minimizing the size of the edge-set for the game-tree of G. For the purposes of this paper, a game G will be called *simpler than* H if the size of the edge-set of the game-tree of G is less than or equal to the size corresponding to H, so the canonical form is the simplest (in this sense) among all games with a given value.

The aim of this paper is to introduce a yet simpler form, called the *reduced canonical form*, by relaxing the condition that it should have the same value as the initial game to the condition that it should be infinitesimally close to the initial game. This new form should be the simplest possible subject to this condition, and the transformation $G \to \overline{G}$ should be linear. Algebraically, we will show that the reduced canonical forms form a subgroup $\mathbf{Rcf}$, and the group of games (with the disjunctive compound operation) is the direct sum $\mathbf{I} \oplus \mathbf{Rcf}$, where $\mathbf{I}$ is the subgroup of infinitesimals. Often, the information provided by the $\mathbf{Rcf}$-component of a game G is enough to decide the outcome class of G. For games of this type, it is important to know when knowledge of the reduced canonical forms of the options of G would imply knowledge of the reduced canonical form of G. This will be answered by Theorem 5.

2. Construction of the Reduced Canonical Form

The operation of cooling a game by a positive number is essential for the Theory of Temperature in the World of Games (see *Winning Ways*, Chapter 6, or *On Numbers and Games*, chapter 9). Similarly, one can define cooling by a non-number, and specifically by $* = \{0|0\}$:

DEFINITION. Given $G = \{G^L \,|\, G^R\}$, we recursively define a new game G_*, called G *cooled by* $*$, as follows:

$$G_* = \begin{cases} G & \text{if } G \text{ is a number,} \\ \{G_*^L{+}* \,|\, G_*^R{+}*\} & \text{otherwise,} \end{cases}$$

where, as usual, G^L and G^R are generic Left and Right options of G, and we write G_*^L for $(G^L)_*$.

It is easy to check that, if $G - H$ is a zero game, so is $G_* - H_*$. Thus the definition above is independent of the form of G, and G_* is well-defined for any game value G.

DEFINITION. If $H_0 \equiv \{H_0^L \,|\, H_0^R\}$ is the canonical form of a game H, we recursively define $p(H)$, the $*$-*projection* of H, as follows:

$$p(H) = \begin{cases} x & \text{if } H = x \text{ or } x + *, \text{ where } x \text{ is a number,} \\ \{p(H_0^L)|p(H_0^R)\} & \text{otherwise.} \end{cases}$$

Because of the uniqueness of the canonical form, the definition of $p(H)$ is independent of the form of H.

DEFINITION. The *reduced canonical form* $\overline{G}$ of G is defined as $p(G_*)$.

Observe that $p(G_*)$ is a canonical form, because p is defined in terms of the canonical form of G_*, and it follows by induction that $p(G_*)$ is in canonical form as well.

EXAMPLE. Let $G = \{\{2\,|\,0\}, 1 \,\|\, 0\}$. This game is in canonical form. Then

$$G_* = \{\{2\,|\,0\}_*{+}*, 1{+}* \,\|\, *\} = \{\{2{+}* \,|\, *\}{+}*, 1{+}* \,\|\, *\}.$$

Now we use the translation principle for stars (*Winning Ways*, p. 123), which says that, for any numbers x, y, we have $\{x\,|\,y\}{+}* = \{x* \,|\, y*\}$ if $x \geq y$ and $\{x\,|\,y*\}{+}* = \{x* \,|\, y\}$ if $x > y$. We obtain $G_* = \{\{2\,|\,0\}, 1* \,\|\, *\}$. Since $G_* > 0$ (the game is a win for Left no matter who starts), the Left option $\{2\,|\,0\}$ is reversible, and can be replaced by all the Left options from 0. There are no Left options from 0, so $G_* = 1*|*$. It is easy to check that $1*|*$ has no reversible moves, so it is the canonical form of G_*. Hence

$$p(G_*) = \{p(1*)\,|\,p(*)\} = \{1\,|\,0\}.$$

We see that, in this example, the reduced canonical form of G is strictly simpler than the canonical form of G.

3. Properties of the Reduced Canonical Form

Theorem 1. *The transformation $G \to \bar{G}$ is a homomorphism.*

Proof. We will show that $G \to G_*$ and $H \to p(H)$ are homomorphisms, hence their composition is a homomorphism. It is a straightforward inductive check that none of the players can win going first in the game $(G{-}H)_*{-}G_*{+}H_*$; therefore $(G{-}H)_* = G_*{-}H_*$. If we consider first the more general case when G, H and $G{-}H$ are not numbers, we have:

$$(G{-}H_*){-}G_*{+}H_* = \{(G^L{-}H)_*{+}*, (G{-}H^R)_*{+}* \,\|\, (G^R{-}H)_*{+}*, (G{-}H^L)_*{+}*\} \\ +\{(-G^R)_*{+}* \,\|\, (-G^L)_*{+}*\}+\{H^L_*{+}* \,\|\, H^R_*{+}*\}.$$

We can then see that, assuming the property true for pairs such as (G^L, H), (G^R, H), (G, H^L), and (G, H^R), every move in $(G-H)_* - G_* + H_*$ has an exact counter, that is, for any move of one player, there is a reply by the other player that brings the position to a value of 0.

If at least two of G, H, and $G-H$ are numbers, then all of them are numbers and the equality to be proved is trivial since we are in the first case of the definiton of cooling by $*$.

If precisely one of G, H is a number, say $H = x$, we need to show that $(G-x)_* = G_* - x$. We are in the second case of the definiton of cooling, so this is equivalent to

$$\{(G^L{-}x)_*{+}* \,|\, (G^R{-}x)_*{+}*\} = \{G^L_*{+}* \,|\, G^R_*{+}*\}{-}x.$$

Applying the translation principle (with x) one more time, we are done, because $(G^L{-}x)_* = G^L_*{-}x$ and $(G^R{-}x)_* = G^R_*{-}x$ by the induction hypothesis.

The proof that $H \to p(H)$ is a homomorphism is very similar: it is enough to show that $p(G){+}p(H){+}p(K) = 0$ if $G{+}H{+}K = 0$ and G, H, K are in canonical form.

Suppose that none of G, H, K is of the form x or $x*$ for some number x. If Left moves first in $p(G){+}p(H){+}p(K)$, he will leave for Right a position like $p(G^L){+}p(H){+}p(K)$. Since G, H and K were in canonical form and $G{+}H{+}K = 0$, there is a Right reply, H^R say, in a different component, so that $G^L{+}H^R{+}K \le 0$ (otherwise G^{LR} would be reversible). Applying the induction hypothesis to this, we obtain $p(G^L){+}p(H^R){+}p(K) \le 0$, which means that, going second in $p(G){+}p(H){+}p(K)$, Right wins. Thus $p(G){+}p(H){+}p(K) \le 0$. Note that we have only assumed inductively (and proved) the inequality $p(G){+}p(H){+}p(K) \le 0$. Because of symmetry, the opposite inequality can be obtained in the same way.

Finally, if exactly one of G, H, and K is of the form x or $x*$ for some number x, the implication $G{+}H{+}K = 0 \Rightarrow p(G){+}p(H){+}p(K) = 0$ is immediate, given the observation that if $\{G^L \,|\, G^R\}$ is a game in canonical form, so is $\{(G^L{+}x*)^c \,|\, (G^R{+}x*)^c\}$, where we are denoting by M^c the canonical form of a game M. □

The following lemma shows one sense in which $\bar{G}$ approximates G.

LEMMA 2. *If G is any game, G and $\bar{G}$ have the same stops, that is, $L(\bar{G}) = L(G)$ and $R(\bar{G}) = R(G)$. In particular, they have the same stops as G^*.*

PROOF. We need to show separately that $L(G_*) = L(G)$ and $L(p(H)) = L(H)$. The second relation can be obtained inductively: If $H \neq x$ or $x*$, and H is in canonical form, $L(p(H)) = L(\{p(H^L)|p(H^R)\}) = \max R(p(H^L)) = \max R(H^L) = L(H)$, and similarly for the Right stops.

For the first relation, we observe that $R(G + *) = R(G)$ for any game G. Hence, if G is not a number,

$$L(G_*) = \max R(G_*^L + *) = \max R(G_*^L) = \max R(G^L) = L(G). \qquad \square$$

THEOREM 3. *A game G is an infinitesimal if and only if $\bar{G} = 0$.*

PROOF. The "if" direction follows from the lemma. Next, we will prove that, if G is an infinitesimal, then $G_* = 0$ (and hence $\bar{G} = 0$). We will do so by showing inductively that

$$L(G) \leq 0 \text{ and } R(G) \leq 0 \quad \text{imply} \quad G_* \leq 0,$$
$$L(G) \geq 0 \text{ and } R(G) \geq 0 \quad \text{imply} \quad G_* \geq 0.$$

Because of the symmetry of the definition of G_*, it is enough to prove the first of these implications. Thus we assume that $L(G) \leq 0$ and $R(G) \leq 0$. Suppose G_* is a number. This is an easy case because, from the lemma, G_* has the same stops as G, so $G_* \leq 0$ and we are done. Suppose G_* is not a number. Then, Left's move in G_* will lead to a position $G_*^L + *$. Now, if G_*^L is a number, we apply the lemma again to conclude that $G_*^L \leq 0$; hence Right's move from $G_*^L + *$ to G_*^L will force a loss for Left, so $G_* \leq 0$. Suppose now that G_*^L is not a number. Then

$$G_*^L + * = \{G_*^{LL} + * | G_*^{LR} + *\} + *.$$

Since $L(G) \leq 0$, there exists a Right-option G^{LR_0} in G^L such that $L(G^{LR_0}) \leq 0$. Therefore, Right can move from $G_*^L + *$ to $G_*^{LR_0} + * + * = G_*^{LR_0}$ and, applying the lemma one more time, we conclude that $L(G_*^{LR_0}) \leq 0$. Now, if $G_*^{LR_0}$ is a number, it cannot be strictly positive, so Left will lose going first in $G_*^{LR_0}$. Finally, if $G_*^{LR_0}$ is not a number, we have $R(G_*^{LR_0}) \leq 0$ (because we already know that $L(G_*^{LR_0}) \leq 0$), so $G_*^{LR_0}$ satisfies the conditions of the induction hypothesis, so $G_*^{LR_0} \leq 0$, so Left will lose going first in G_* in any case, so $G_* \leq 0$ and the proof is completed. $\square$

THEOREM 4. *The reduced canonical form $\bar{G}$ is infinitesimally close to G.*

PROOF. We will show first that G_* is infinitesimally close to G and then that $p(H)$ is infinitesimally close to H. We will establish inductively that $G_* - G - x \leq 0$ for every positive number x. This will be enough to ensure that $\bar{G}$ is infinitesimally close to G, because applying the induction assumption to $-G$ yields $(-G)_*+G-x \leq 0$, so $-(-G)_*-G+x \geq 0$. Using the fact that $-(-G)_* = G_*$ (since G_* is a homomorphism), we get $G_*-G+x \geq 0$, so G_*-G will be greater than all negative numbers and smaller than all positive numbers.

We only have to consider the case when G is not a number. In this case,

$$G_*-G-x = \{G_*^L+*\,|\,G_*^R+*\}+\{-G^R-x\,|-G^L-x\}.$$

After Left makes his first move in this, Right can reply to one of the following:

$$G_*^L+*-G^L-x = (G_*^L-G^L-\tfrac{1}{2}x)+(*-\tfrac{1}{2}x),$$
$$-G^R-x+G_*^R+* = (G_*^R-G^R-\tfrac{1}{2}x)+(*-\tfrac{1}{2}x).$$

The induction hypothesis applies to $(G_*^L-G^L-\frac{1}{2}x)$ and $(G_*^R-G^R-\frac{1}{2}x)$, so they are both negative. Since $(*-\frac{1}{2}x)$ is also negative, Left loses, and we obtain $G_*-G-x \leq 0$, as desired.

The proof that $p(G)-G-x \leq 0$ follows precisely the same steps if we choose G to be in canonical form. □

THEOREM 5. *$\bar{G}$ is the simplest game infinitesimally close to G.*

PROOF. Let H be infinitesimally close to G. We need to show that $\bar{G}$ is at least as simple as H. Since $G-H$ is an infinitesimal, we have $\overline{G-H} = 0$, and therefore $\bar{G} = \bar{H}$. Since both sides of this equation are in canonical form, we have $\bar{G} \equiv \bar{H}$, so all we need to show is that $\bar{H}$ is at least as simple as H for any game H. For this purpose, we can relax the definition of $\bar{G}$ in the sense that p is not applied to the canonical form of G_*, but directly to G_* (that is, to the form obtained after cooling G by $*$, without deleting any dominated options or bypassing any reversible moves). If we denote the result by $\widetilde{G}$, then $\widetilde{G}$ will be at least as simple as G; it can be seen inductively that the only thing achieved in the process of forming $\widetilde{G}$ is to replace $x*$ by x everywhere in (the form of) G, which is clearly a "simplification". Yet, $\bar{G}$ is at least as simple as $\widetilde{G}$, for the following reason. When p is applied to the canonical form K^c of a game K, the outcome will be at least as simple as when p is applied directly to K (consider the sequence $K, K_1, K_2, ..., K_n = K^c$, where each K_{i+1} is obtained from K_i by deleting a dominated option or by bypassing a reversible move; by induction, $p(K_{i+1})$ will be at least as simple as $p(K_i)$). We have thus proved that $\widetilde{G}$ is at least as simple as G, and $\bar{G}$ is at least as simple as $\widetilde{G}$, which implies that $\bar{G}$ is at least as simple as G. □

NOTE. We needed this kind of argument because it can occur that G is simpler than G_*; for example, when $G = 1\,|\,*$, we have $G_* = 1*\,|\,*$. Here, the "simplification" is made by p to $p(G_*) = 1\,|\,0$.

DEFINITION. Let $G = \{G^L\,|\,G^R\}$. A number x is *permitted by* G if $G^L \not\geq x \not\geq G^R$ for every G^L and G^R.

THEOREM 6. *Let* $G = \{G^L|G^R\}$ *be such that* $\{\overline{G^L}|\overline{G^R}\}$ *permits at most one number. Then*

$$\bar{G} = \overline{\overline{G^L}|\overline{G^R}}.$$

PROOF. Suppose that at least one of G and $\{\overline{G^L}|\overline{G^R}\}$ is not a number. Then, for any positive number x, the translation principle can be applied to $H = \{G^L|G^R\}+\{-\overline{G^R}|-\overline{G^L}\}-x$, so that one of these equalities is satisfied:

$$\begin{aligned} H &= \{G^L-x|G^R-x\}+\{-\overline{G^R}|-\overline{G^L}\}, \\ H &= \{G^L|G^R\}+\{-\overline{G^R}-x|-\overline{G^L}-x\} \end{aligned}.$$

Therefore, for any Left option in H, there is a Right response in the other component that leaves a negative game (applying Theorem 3). This means that Left, going first in H, loses, so $H \le 0$. Similarly, $H' = \{G^L|G^R\}+\{-\overline{G^R}|-\overline{G^L}\}+x \ge 0$, so $G-\{\overline{G^L}|\overline{G^R}\}$ is an infinitesimal, so

$$0 = \overline{G-\{\overline{G^L}|\overline{G^R}\}} = \bar{G}-\overline{\{\overline{G^L}|\overline{G^R}\}}$$

and the result is proved in this case.

Suppose, now, that G and $\{\overline{G^L}|\overline{G^R}\}$ are both numbers. Denote by I^L and I^R the closures of the confusion intervals for $\overline{G^L}$ and $\overline{G^R}$, that is, $I^L = [L(\overline{G^L}), R(\overline{G^L})]$ and $I^R = [L(\overline{G^R}), R(\overline{G^R})]$. Since $\{\overline{G^L}|\overline{G^R}\}$ permits at most one number, and is a number itself, we must have $R(\overline{G^L}) = L(\overline{G^R}) = \{\overline{G^L}|\overline{G^R}\}$. Applying the lemma to $\overline{G^L}$ and $\overline{G^R}$ we find that G^L and G^R have the same closures of the confusion intervals as $\overline{G^L}$ and $\overline{G^R}$. Since G is a number as well, we obtain $G = \{\overline{G^L}|\overline{G^R}\}$, hence $\bar{G} = \overline{\{\overline{G^L}|\overline{G^R}\}}$. □

NOTES. 1. The reduced canonical form operator can be used to exhibit, within small errors, the recursively obtained values for games such as $2 \times n$ Domineering [Berlekamp 1988], where the main complications are due to increasingly complex infinitesimals.

2. David Wolfe has implemented this approximation operator in his Gamesman's Toolkit [Wolfe 1996]: if the user types $G[e]$, the program will return the reduced canonical form of G.

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